Lecture 6 8.05. 2018

Newton's spacetime

Recall:

Tensors e.g. $T^\mu{}_\nu(e)$ s.t. 0-forms

$$\boxed{T^\mu{}_\nu(ae) = a^\mu{}_\rho T^\rho{}_\sigma(e)\, a^{-1\sigma}{}_\nu}$$

Connections $\omega^\mu{}_\nu(e)$ s.t. 1-forms and

$$\boxed{\omega^\mu{}_\nu(ae) = a^\mu{}_\rho\, \omega^\rho{}_\sigma(e)\, a^{-1\sigma}{}_\nu - da^\mu{}_\rho\, a^{-1\rho}{}_\nu}$$

Covariant exterior differential transforms k-forms of type ρ to (k+1) forms of type ρ; e.g.

$$\boxed{DT^\mu{}_\nu(e) = dT^\mu{}_\nu(e) + \omega^\mu{}_\rho(e)\wedge T^\rho{}_\nu(e) - \omega^\rho{}_\nu(e)\wedge T^\mu{}_\rho(e)}$$

Curvature 2-form

$$\boxed{\Omega^\mu{}_\nu(e) = d\omega^\mu{}_\nu(e) + \omega^\mu{}_\rho(e)\wedge\omega^\rho{}_\nu(e)}$$

2-form of representation type Ad !

Canonical 1-form of vector type (type id).

$$\boxed{\theta^\mu(e) := e^\mu}$$ || Note $\boxed{e_\rho \lrcorner\, \theta^\alpha(e) = e_\rho \lrcorner\, e^\alpha = \delta_\rho{}^\alpha}$

Torsion 2-form

$$\boxed{\Theta^\mu(e) := D\theta^\mu(e) = d\theta^\mu(e) + \omega^\mu{}_\nu(e)\wedge\theta^\nu(e)}$$

2-form of type id.

For tensors and a vector field:

$$\nabla_X T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}(e) := X \lrcorner\, DT^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}(e)$$

$$X = X^\mu(e)\, e_\mu\, ; \quad \nabla_X T^{\cdots}{}_{\cdots} = X^\mu \nabla_\mu T^{\cdots}{}_{\cdots}$$

$\boxed{\text{Autoparallels}}$

$]-\varepsilon,\varepsilon[\ \ni t \longrightarrow \gamma(t) \in M$ parameterized curve in M

Tangent vector to $\gamma(t)$

$$X(t) = \frac{d}{dt} f\circ\gamma(t) = X^\mu \frac{\partial}{\partial x^\mu}(t)$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$X(x^\mu) = \frac{d}{dt}x^\mu(t) \qquad\qquad X^\mu \frac{\partial x^\mu}{\partial x^\nu} = X^\nu \delta^\mu{}_\nu = X^\nu$$

$$\boxed{X = \frac{dx^\mu}{dt}\frac{\partial}{\partial x^\mu}}\boxed{= \frac{d}{dt}} \qquad e_\mu = \frac{\partial}{\partial x^\mu}, \qquad \boxed{X^\mu(e) = \frac{dx^\mu}{dt}(e)}$$

A curve $\gamma(t)$ is autoparallel iff

$$\boxed{\nabla_X X^\mu = \lambda X^\mu} \qquad \lambda = \lambda(t).$$

Note that $\omega^\mu{}_\nu(e)$ is a 1-form. It can be decomposed onto the basis of 1-forms θ^μ

$$\omega^\mu{}_\nu(e) = \Gamma^\mu{}_{\nu\rho}(e)\,\theta^\rho(e)$$

Let us calculate:

$$\nabla_X X^\mu = X \lrcorner DX^\mu = X \lrcorner \left(dX^\mu + \omega^\mu{}_\nu X^\nu\right) =$$

$$= X\lrcorner dX^\mu + (X\lrcorner\omega^\mu{}_\nu)X^\nu = X(X^\mu) + X\lrcorner(\Gamma^\mu{}_{\nu\rho}\theta^\rho)X^\nu =$$

$$= \frac{d}{dt}\frac{dx^\mu}{dt} + \omega^\mu{}_{\nu\rho}\, X^\alpha(e_\alpha\lrcorner\theta^\rho)X^\nu =$$

$$= \frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\rho}X^\nu X^\rho =$$

$$= \boxed{\frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = \lambda\frac{dx^\mu}{dt}}$$

Affine parametrization

Can we, by changing parametrization, put $\lambda(t)=0$ or $\gamma(t)$?

$$t = f(\bar{t})$$

$$\frac{d}{dt} = \frac{d}{f' d\bar{t}}$$

$$\frac{d^2}{dt^2} = \frac{d}{f' d\bar{t}}\,\frac{d}{f' d\bar{t}} = \frac{1}{f'^2}\frac{d^2}{d\bar{t}^2} - \frac{1}{f'^3} f''\frac{d}{d\bar{t}}$$

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = \lambda\frac{dx^\mu}{dt}$$

$$\Downarrow$$

$$\frac{1}{f'^2}\frac{d^2x^\mu}{d\bar{t}^2} - \frac{f''}{f'^3}\frac{dx^\mu}{d\bar{t}} + \frac{1}{f'^2}\Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{d\bar{t}}\frac{dx^\rho}{d\bar{t}} = \lambda\frac{1}{f'}\frac{dx^\mu}{d\bar{t}}$$

$$\frac{d^2x^\mu}{d\bar{t}^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{d\bar{t}}\frac{dx^\rho}{d\bar{t}} = \left(\lambda f' + \frac{f''}{f'}\right)\frac{dx^\mu}{d\bar{t}}$$

$\Longrightarrow$ $f'' + \lambda f'^2 = 0$ locally always have a solution.

Parameter t s.t. $\boxed{\nabla_X X^\mu = 0}$ is an *affine parameter* for the autoparallel γ

Note affine parameter is defined up to

$$t \to at + t_0 \qquad a, t_0 = \text{const}$$

t_0: changing the start moment of the clock

a: scaling time.

$$\boxed{\frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0}$$

equation for an autoparallel in affine parametrization.

Inertial and gravitational mass.

In Newton's theory mass appears in 3 different context.

(1) as inertial mass, m_I, in 2^{nd} law of dynamics

$$m_I \cdot \vec{a} = \vec{F}$$

(2) as a passive gravitational mass, m_{GP}, in the expression ~~f~~or a force $\vec{F}$ exerted on mass m_{GP} in a gravitational field comming from gravitational potential φ:

$$\vec{F} = - m_{GP} \operatorname{grad} \varphi$$

(3) as an active gravitational mass ~~producing~~, m_{GA}, ~~p~~roducing gravitational potential φ:

$$\varphi = - \frac{k\, m_{GA}}{r}$$

~~Third~~ Now the 3r law of Newton's dynamics says: that the force exerted on the passive body is the same in value as the one exerted on the active body so we have

$$m_{1GP}\, m_{2GA} = m_{2GP}\, m_{1GA}$$

$$\Rightarrow \quad \frac{m_{1GP}}{m_{1GA}} = \frac{m_{2GP}}{m_{2GA}} \Rightarrow \quad \frac{m_{GP}}{m_{GA}} = \text{const}$$

may be put to 1 including it in gravitational constant k.

$$\Rightarrow \quad \boxed{m_{GP} = m_{GA}} = m_G$$

charge for gravity.

Galileo: $\boxed{m_G \equiv m_I}$ law of physics

experimentally checked with accuracy 10^{-8}
(Eötvös), 10^{-10} (Dicke, Brovs)
1894 1964

$\Rightarrow \quad m_I \vec{a} = - m_G \operatorname{grad} \varphi \qquad \Big| : m_I = m_G$

$$\Rightarrow \quad \boxed{\frac{d^2 \vec{r}}{dt^2} + \vec{\nabla} \varphi = 0} \qquad (1)$$

It is a property of spacetime!

free fall equation of motion

does Not depend on the moving particle! It depends on φ, which solves field equation

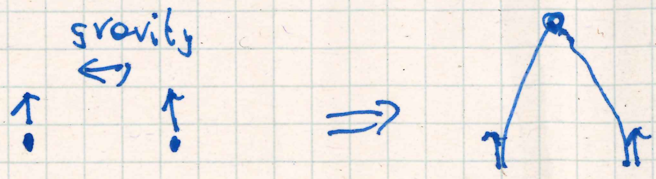
$$\boxed{\Delta \varphi = +4\pi G \rho} \qquad (2)$$

(1) and (2) is an essence of Newton's theory of gravity.
How to change the model of spacetime to include gravity?

One has to abandon 1-st Law of dynamics!
Free falls are trajectories of particles in free motion!

It is rather impossible to find affine space in which free falls would be straight lines
Parallel straight lines do not intersect.

But



Idea: Interpret (1) as an equation for autoparallels on a 4-dim manifold with connection.

Newton's gravitational spacetime

(1) Spacetime is a 4-dim manifold

(2) Spacetime is endowed with torsion-free affine connection

(3) world lines of free falls are autoparallels in affine parametrization.

(4) a) in every tangent space ~~to~~ the manifold there is a tensor τ_μ

b) and a symmetric tensor $h^{\mu\nu}$ of signature $(+,+,+,0)$ such that: $\boxed{h^{\mu\nu}\tau_\nu = 0}$

$\cancel{D h^{\mu\nu} = 0}$

$\cancel{D \tau_\mu = 0}$

(5) $D h^{\mu\nu} = 0$

$D \tau_\mu = 0$

Consequences: of (2)

$$\Theta^\mu = d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu = 0$$

in coordinate frame $\theta^\mu(e) = dx^\mu$
holonomic

$$0 = \cancel{d^2 x^\mu} + \omega^\mu{}_\nu \wedge dx^\nu = 0$$

$$\Gamma^\mu{}_{\nu\rho}\, dx^\rho \wedge dx^\nu = 0 \quad \Rightarrow \quad \boxed{\Gamma^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\nu\mu}}$$

in holonomic frame $\frac{\partial}{\partial x^\mu} = e^\mu$

Consequences of 5

$$0 = D\tau_\mu = d\tau_\mu - \omega^\rho{}_\mu \tau_\rho \overset{\text{Vielo}}{=}$$

$$= \frac{\partial \tau_\mu}{\partial x^\nu} dx^\nu - \Gamma^\rho{}_{\mu\nu} dx^\nu \tau_\rho = 0$$

$$\Rightarrow \quad \frac{\partial \tau_\mu}{\partial x^\nu} = \tau_\rho \Gamma^\rho{}_{\mu\nu}$$

$$\frac{\partial \tau_\mu}{\partial x^\nu} - \frac{\partial \tau_\nu}{\partial x^\mu} = \tau^\rho (\Gamma^\rho{}_{\mu\nu} - \Gamma^\rho{}_{\nu\nu}) = 0$$

$$\tau = \tau_\mu dx^\mu$$

$$d\tau = d\tau_\mu \wedge dx^\nu = \frac{\partial \tau_\mu}{\partial x^\rho} dx^\rho \wedge dx^\nu = 0$$

$$\Rightarrow \quad \tau = dt = t_{,\mu} dx^\mu$$

defines local time

3' t is ~~[illegible]~~ assumed to be an affine parameter for free falls,

$\gamma(t)$ – free fall world-line

$$\gamma(t) = \begin{pmatrix} t \\ x^i(t) \end{pmatrix} = x^\mu(t)$$

$$\frac{dx^0}{dt} = 1, \quad \frac{dx^i}{dt}$$

$$0 = \frac{d^2x^0}{dt^2} + \Gamma^0{}_{00}\frac{dx^0}{dt}\frac{dx^0}{dt} + 2\Gamma^0{}_{0i}\frac{dx^0}{dt}\frac{dx^i}{dt} + \Gamma^0{}_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0$$

$$0 = \frac{d^2x^i}{dt^2} + 2\Gamma^i{}_{0j}\frac{dx^0}{dt}\frac{dx^j}{dt} + \Gamma^i{}_{00}\frac{dx^0}{dt}\frac{dx^0}{dt} + \Gamma^i{}_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0$$

$$\left\{\begin{array}{l}\Gamma^0{}_{00} + 2\Gamma^0{}_{0i}\frac{dx^i}{dt} + \Gamma^0{}_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0 \\ \frac{d^2x^i}{dt^2} + \Gamma^i{}_{00} + 2\Gamma^i{}_{0j}\frac{dx^j}{dt} + \Gamma^i{}_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0\end{array}\right.$$

Compare with free falls:

$$\left\{\frac{d^2x^i}{dt^2} + \frac{\partial\varphi}{\partial x^i} = 0\right.$$

$$\Rightarrow \boxed{\begin{array}{ll}\Gamma^0{}_{0i} = 0 & \Gamma^0{}_{00} = 0 \\ \Gamma^0{}_{ij} = 0 & \\ \Gamma^i{}_{0j} = 0 & \Gamma^i{}_{00} = \frac{\partial\varphi}{\partial x^i} \\ \Gamma^i{}_{jk} = 0 & \end{array}} \quad (3)$$

Introduce connection, ~~which~~ which in a coordinate $x^\mu = (t, x^i)$ frame looks like (3). Then (1) is a geodesic equation.

<u>Curvature of this connection</u>

$$\omega^\mu{}_\nu = \Gamma^\mu{}_{\nu\rho}\, dx^\rho$$

$$\left\{\begin{array}{l}\omega^0{}_0 = 0 \\ \omega^0{}_i = 0 \\ \omega^i{}_0 = \frac{\partial\varphi}{\partial x^i}\, dt \\ \omega^i{}_j = 0\end{array}\right\} \Rightarrow \begin{array}{l}\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu \\ \Omega^i{}_0 = d\left(\frac{\partial\varphi}{\partial x^i}\, dt\right) + 0 = \\ \qquad = \frac{\partial^2\varphi}{\partial x^i \partial x^j}\, dx^j \wedge dt\end{array}$$

$$\Rightarrow \Omega^\mu{}_\nu = \frac{1}{2} R^\mu{}_{\nu\rho\sigma}\, dx^\rho \wedge dx^\sigma$$

$$\Rightarrow \boxed{R^i{}_{0j0} = \frac{\partial^2\varphi}{\partial x^i \partial x^j}}$$

Note that

$$\Delta\varphi = R^{\mu}{}_{0\mu0}$$

Poisson equation

$$R^{\mu}{}_{0\mu0} = 4\pi G\rho$$

<u>Ricci tensor</u>

$$\Omega^{\mu}{}_{\nu} = \frac{1}{2} R^{\mu}{}_{\nu\rho\sigma}\, \theta^{\rho} \wedge \theta^{\sigma}$$

$$R_{\nu\sigma} := R^{\mu}{}_{\nu\mu\sigma}$$

$$\begin{cases} R_{00} = 4\pi G\rho \\ R_{0i} = 0 \\ R_{ij} = 0 \end{cases}$$

$$\boxed{R_{\mu\nu} = 4\pi G\rho\, \tau_{\mu}\tau_{\nu}}$$