Lecture 7 11.05.2018

Newton's spacetime (continued)

① M – differentiable manifold of dimension 4

② Spacetime is endowed with torsion-free affine connection ω

③ World lines of free falls are ~~geodesics~~ selfparallels

④ ~~In~~ every tangent space to M there is given

a) a non-zero 1form τ (τ_μ)

b) a symmetric contravariant tensor h $(h^{\mu\nu})$ of signature $0,+,+,+$ such that

$$h^{\mu\nu}\tau_\nu = 0$$

$$\nabla_\rho h^{\mu\nu} = 0 \iff Dh^{\mu\nu} = 0$$

$$\nabla_\mu \tau_\nu = 0 \iff D\tau_\nu = 0$$

The last equation implies that $\tau = dt$

Tangent vectors V^μ are spacelike if $V^\mu \tau_\mu = 0$, otherwise they are timelike

③a free falls are ~~timelike geodesics~~ autoparallels which are affinely parametrized by t

Ideal clocks measure t.

⑤ Curvature conditions:

$$\tau_{[\mu} R^\nu{}_{\rho]\sigma\gamma} = 0$$

$$h^{\alpha\delta} R^\beta{}_{\gamma\delta\varepsilon} + h^{\beta\delta} R^\alpha{}_{\varepsilon\delta\gamma} = 0$$

⑥ Condition for empty space $R_{\mu\nu} = 0$

Condition for absence of gravitational field $R^\mu{}_{\nu\rho\sigma} = 0$.

Ad ②

$$\theta^\mu,\ \omega^\mu{}_\nu \implies \Theta^\mu = d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu$$

$$\omega^\mu{}_\nu = \Gamma^\mu{}_{\nu\rho}\theta^\rho$$

$$d\theta^\mu = -\tfrac{1}{2} c^\mu{}_{\nu\rho}\,\theta^\nu\wedge\theta^\rho \iff [e_\mu, e_\nu] = c^\rho_{\mu\nu} e_\rho$$

$$\Theta^\mu = -\tfrac{1}{2} c^\mu{}_{\nu\rho}\,\theta^\nu\wedge\theta^\rho + \Gamma^\mu{}_{\nu\rho}\,\theta^\rho\wedge\theta^\nu =$$

$$= \left(-\tfrac{1}{2} c^\mu{}_{\nu\rho} - \Gamma^\mu{}_{\rho\nu}\right)\theta^\nu\wedge\theta^\rho$$

$$\implies \Gamma^\mu{}_{[\rho\nu]} = -\tfrac{1}{2} c^\mu{}_{\nu\rho}$$

$$\implies \Gamma^\mu{}_{\rho\nu} - \Gamma^\mu{}_{\nu\rho} = -c^\mu{}_{\nu\rho}$$

$$\boxed{\Gamma^\mu{}_{\nu\rho} - \Gamma^\mu{}_{\rho\nu} = c^\mu{}_{\nu\rho}}$$

In holonomic frame $c^\mu{}_{\nu\rho} = 0$

$$\implies \boxed{\Gamma^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\rho\nu}}$$ ← connection is "symmetric"!

Ad ④ absolute time.

$$0 = D\tau_\mu = d\tau_\mu - \omega^\nu{}_\mu \wedge \tau_\nu \underset{\substack{\uparrow \\ \text{holonomic} \\ \text{frame}}}{=} \frac{\partial\tau_\mu}{\partial x^\rho}dx^\rho - \Gamma^\nu{}_{\mu\rho}\,dx^\rho\,\tau_\nu =$$

$$= \left(\frac{\partial\tau_\mu}{\partial x^\rho} - \Gamma^\nu{}_{\mu\rho}\tau_\nu\right)dx^\rho.$$

$$\frac{\partial\tau_\mu}{\partial x^\rho} = \Gamma^\nu{}_{\mu\rho}\tau_\nu$$

|| ← connection symmetric.

$$\frac{\partial\tau_\rho}{\partial x^\mu} = \Gamma^\nu{}_{\rho\mu}\tau_\nu$$

$$\implies \boxed{\tau_{\mu,\rho} - \tau_{\rho,\mu} = 0}$$

$$d\tau = d(\tau_\mu dx^\mu) = \tau_{\mu,\rho}\,dx^\rho\wedge dx^\mu = \tau_{[\mu,\rho]}\,dx^\rho\wedge dx^\mu = 0.$$

$d\tau = 0 \implies \tau = dt = t_{,\mu}dx^\mu \implies t$ - absolute time.

(3a)

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0$$

$$x^\mu = \begin{pmatrix} t \\ x^i \end{pmatrix}$$

$$\frac{d^2x^i}{dt^2} + \nabla^i\varphi = 0$$

$\Gamma^\mu{}_{\nu\rho}$ s.t.
$$\begin{cases} \omega^0{}_0 = 0 \\ \omega^0{}_i = 0 \\ \omega^i{}_0 = \frac{\partial\varphi}{\partial x^i}\,dt \\ \omega^i{}_j = 0 \end{cases}$$

$\Rightarrow$ 5 because

$$\begin{cases} \Omega^0{}_0 = 0 \\ \Omega^0{}_i = 0 \\ \Omega^i{}_0 = \frac{\partial^2\varphi}{\partial x^i\partial x^j}\,dx^j\wedge dt \\ \Omega^i{}_j = 0 \end{cases} \Rightarrow R^i{}_{0j0} = \frac{\partial^2\varphi}{\partial x^i\partial x^j}$$

$\Rightarrow$ 6 $\Rightarrow$ $R_{\mu\nu} = 0 \Leftrightarrow \Delta\varphi = 0$ (non-pend $R_{\mu\nu} = 4\pi G\rho\, t_\mu t_\nu$)

$R^\mu{}_{\nu\rho\sigma} = 0 \Leftrightarrow \frac{\partial^2\varphi}{\partial x^i\partial x^j} = 0$

Comment on ~~[illegible]~~ free falls

In Newton's physics

$$\frac{d^2x^\mu}{dt^2} = F^\mu \text{ — force}$$

$$\frac{d^2x^\mu}{dt^2} = \underbrace{-\Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt}}_{\text{force}}$$

$\Gamma^\mu{}_{\nu\rho} \sim \operatorname{grad}\varphi$

but $\Gamma^\mu{}_{\nu\rho}$ is NOT tensorial

$\Rightarrow$ passing to another frame will produce new terms

$-\Gamma^{\mu}{}_{\nu\rho}\, u^{\nu} u^{\rho} \sim$ gravity + Coriolis force + centrifugal force +

+ other inertial forces

"inseparability of the forces of gravitation and inertial forces" is ~~implic~~ built in the theory $\equiv$

$\equiv$ nontensorial character of connection.

— * —

Elements of Riemannian geometry

1) Riemannian manifold (M, g) $\qquad \dim M = n$

$$g : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathcal{F}(M)$$

1) $g(X,Y) = g(Y,X)$

2) $\big(g(X,Y) = 0 \quad \forall X\big) \Rightarrow Y = 0.$

$$g_{\mu\nu}(e) := g(e_\mu, e_\nu)$$

$$g_{\mu\nu} = g_{\nu\mu}$$

At every point one can be brought to

$$g_{\mu\nu} = \mathrm{diag}(\underbrace{-1, \dots, -1}_{p}, \underbrace{1, \dots 1}_{q}) \text{ by } GL(n, \mathbb{R})$$

$\to$ signature (p,q).

Signature does not change (by continuity)

2) Geodesics $\quad t \to \gamma(t)$ curve of class C^2 in M.

$\gamma \to$ $X = \frac{d}{dt}$ – tangent vector $X^{\mu}(t) = \frac{dx^{\mu}}{dt}$.

Arc length $\quad s = \int_{t_0}^{t_1} \sqrt{|g(X,X)|}\, dt$

$$s = \int_{t_0}^{t_1} \sqrt{\left| g_{\mu\nu} \frac{dx^{\mu}}{dt} \frac{dx^{\nu}}{dt} \right|}\, dt.$$

note $\;$ s does not depends on parametrization.

Definition

$t \to \gamma(t)$ is a geodesic iff ~~it~~ satisfies neccessary condition for extremalizing s:

$$\begin{pmatrix} t \to \gamma(t) \text{ is a} \\ \text{geodesic} \end{pmatrix} \iff \begin{pmatrix} \delta \int_{t_0}^{t_1} \sqrt{\left| g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \right|}\, dt = 0 \\ \delta x^\mu(t_0) = \delta x^\nu(t_0) = 0 \end{pmatrix}$$

Euler-Lagrange equations

$$\mathcal{L} = \mathcal{L}(x^\mu, \dot{x}^\mu) = \sqrt{|g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu|}$$

$$\boxed{\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0} \quad \text{E-L.}$$

$$\frac{d}{dt} \frac{\partial}{\partial \dot{x}^\mu} \sqrt{|g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu|} - \frac{\partial}{\partial x^\mu} \sqrt{|g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu|} = 0$$

$\frac{\partial}{\partial x^\mu} =: \partial_\mu$

$$\frac{d}{dt} \frac{2 g_{\mu\nu} \dot{x}^\nu}{2\sqrt{\quad}} - \frac{1}{2} \frac{g_{\alpha\beta,\mu} \dot{x}^\alpha \dot{x}^\beta}{\sqrt{\quad}} = 0 \qquad (*)$$

Formulation is parametrization independent.
We can take ANY parameter. We take arc length.

$$s = \int \sqrt{|\ |}\, dt \implies ds = \sqrt{\quad}\, dt$$

$$\frac{d}{ds} = \frac{d}{\sqrt{\quad}\, dt}$$

$(*) \cdot \frac{1}{\sqrt{\;}}$

$$\frac{d}{ds}\left(g_{\mu\nu}\frac{dx^\nu}{ds}\right) - \frac{1}{2} g_{\alpha\beta,\mu}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0$$

$$g_{\mu\nu}\frac{d^2x^\nu}{ds^2} + g_{\mu\nu,\alpha}\frac{dx^\alpha}{ds}\frac{dx^\nu}{ds} - \frac{1}{2} g_{\alpha\nu,\mu}\frac{dx^\alpha}{ds}\frac{dx^\nu}{ds} = 0$$

$$g_{\mu\nu}\frac{d^2x^\nu}{ds^2} + \frac{1}{2}\left(g_{\mu\nu,\alpha} + g_{\mu\alpha,\nu} - g_{\alpha\nu,\mu}\right)\frac{dx^\alpha}{ds}\frac{dx^\nu}{ds} = 0$$

$$\frac{d^2x^\rho}{ds^2} + \underbrace{\frac{1}{2} g^{\rho\mu}\left(g_{\mu\alpha,\nu} + g_{\nu\mu,\alpha} - g_{\alpha\nu,\mu}\right)}_{\left\{ {\rho \atop \alpha\nu} \right\}}\frac{dx^\alpha}{ds}\frac{dx^\nu}{ds} = 0$$

$$\boxed{\frac{d^2x^\rho}{ds^2} + \left\{ {\rho \atop \alpha\nu} \right\}\frac{dx^\alpha}{ds}\frac{dx^\nu}{ds} = 0}$$

$$\boxed{\left\{ {\rho \atop \alpha\nu} \right\} = \frac{1}{2} g^{\rho\mu}\left(g_{\mu\alpha,\nu} + g_{\nu\mu,\alpha} - g_{\alpha\nu,\mu}\right)}$$ Christoffel symbols

This defines a connection, ~~which~~ $\omega^\alpha{}_\beta$, which in coordinate frame is

$$\cancel{\omega^\rho{}_\alpha = \left\{ {\rho \atop \alpha\nu} \right\} dx^\nu}$$

$$\omega^\rho{}_\alpha(dx) = \cancel{}\left\{ {\rho \atop \alpha\nu} \right\} dx^\nu$$

<u>Note</u> $\left\{ {\rho \atop \alpha\nu} \right\} = \left\{ {\rho \atop \nu\alpha} \right\} \Rightarrow$ connection is symmetric in coordinate frame

$\Rightarrow$ it has ZERO TORSION.

Geodesics $\equiv$ autoparallels in this connection !

$$\omega^\mu{}_\nu(dx) = \left\{ {\mu \atop \nu\rho} \right\} dx^\rho =$$

$$= \tfrac{1}{2} g^{\mu\alpha} \left(g_{\alpha\nu,\rho} + g_{\rho\alpha,\nu} - g_{\nu\rho,\alpha} \right) dx^\rho =$$

$$= \tfrac{1}{2} g^{\mu\alpha} dg_{\alpha\nu} + \tfrac{1}{2} g^{\mu\alpha} \left(g_{\rho\alpha,\nu} - g_{\nu\rho,\alpha} \right) dx^\rho$$

Let us calculate

$$(Dg)_{\alpha\nu} = dg_{\alpha\nu} - \omega^\rho{}_\alpha\, g_{\rho\nu} - \omega^\rho{}_\nu\, g_{\alpha\rho} =$$

$$= \cancel{dg_{\alpha\nu}} - g_{\rho\nu}\left[\tfrac{1}{2} \cancel{g^{\rho\beta} dg_{\beta\alpha}} + \tfrac{1}{2} g^{\rho\beta} \left(\cancel{g_{\sigma\beta,\alpha}} - g_{\alpha\sigma,\beta} \right) dx^\sigma \right]$$
$$- g_{\alpha\rho}\left[\tfrac{1}{2} \cancel{g^{\rho\beta} dg_{\beta\nu}} + \tfrac{1}{2} g^{\rho\beta} \left(\cancel{g_{\sigma\beta,\nu}} - g_{\nu\sigma,\beta} \right) dx^\sigma \right]$$
$$= 0.$$

The connection defined in terms of Christoffel symbols in coordinate frame is

$$\boxed{\begin{array}{lll} (1) & \text{torsion free} & D\theta^\mu = \Theta^\mu \equiv 0 \\ (2) & \text{metric} & Dg_{\mu\nu} = 0 \end{array}}$$

Fact Θ and Dg uniquely determin connection.

Fact On Riemannian manifold there is a distinguished connection, called Levi-Civita connection, which is (1) torsion-free
(2) metric.

$$\begin{cases} A^\mu(e) = \frac{1}{2} A^\mu{}_{\nu\rho}(e)\, e^\nu \wedge e^\rho \\ B_{\mu\nu}(e) = B_{\mu\nu\rho}(e)\, e^\rho \end{cases} \qquad B_{\mu\nu} = B_{\nu\mu}.$$

Q Does there exist a connection ω on (M, g) s.t.

$$\begin{cases} \Theta^\mu = A^\mu \\ D g_{\mu\nu} = B_{\mu\nu} \end{cases} \qquad ?$$

We look for $\omega^\mu{}_\nu$.

Define

$$\omega_{\mu\nu} = g_{\mu\rho}\,\omega^\rho{}_\nu$$

$$B_{\mu\nu} = D g_{\mu\nu} = d g_{\mu\nu} - \omega^\rho{}_\mu\, g_{\rho\nu} - \omega^\rho{}_\nu\, g_{\mu\rho} =$$

$$= d g_{\mu\nu} - \omega_{\nu\mu} - \omega_{\mu\nu}$$

$$\omega_{\mu\nu} = \Gamma_{\mu\nu\rho}\,\theta^\rho$$

$$(\Gamma_{\mu\nu\rho} + \Gamma_{\nu\mu\rho})\,\theta^\rho = d g_{\mu\nu} - B_{\mu\nu} \qquad | \, e_\rho \lrcorner$$

$$\boxed{\Gamma_{\mu\nu\rho} + \Gamma_{\nu\mu\rho} = e_\rho \lrcorner \,(d g_{\mu\nu} - B_{\mu\nu})} \qquad (1)$$

$$A^\mu = \Theta^\mu = d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu$$

$$\omega^\mu{}_\nu \wedge \theta^\nu = A^\mu - d\theta^\mu$$

$$\omega_{\mu\nu} \wedge \theta^\nu = g_{\mu\rho}\,(A^\rho - d\theta^\rho)$$

$$\Gamma_{\mu\nu\rho}\,\theta^\rho \wedge \theta^\nu = g_{\mu\alpha}\,(A^\alpha - d\theta^\alpha) \qquad | \, e_\rho \lrcorner\, e_\nu \lrcorner$$

$$\underbrace{\Gamma_{\mu\rho\nu} - \Gamma_{\mu\nu\rho} = e_\rho \lrcorner\, e_\nu \lrcorner\, (A^\alpha - d\theta^\alpha)\; g_{\alpha\mu}} \qquad (2)$$

(1) + (2)

$$\Gamma_{\mu\rho\nu} + \Gamma_{\nu\mu\rho} = \underbrace{e_\rho \lrcorner \left[dg_{\mu\nu} - B_{\mu\nu} + e_\nu \lrcorner \left(A^\alpha - d\theta^\alpha\right) g_{\alpha\mu} \right]}_{H_{\nu\mu\rho} \;\longleftarrow\; \text{GIVEN !}}$$

$$\begin{array}{l} + \\ + \\ - \end{array} \left\{ \begin{array}{l} \cancel{\Gamma_{\mu\rho\nu}} + \Gamma_{\nu\mu\rho} = H_{\nu\mu\rho} \\ \Gamma_{\nu\mu\rho} + \cancel{\Gamma_{\rho\nu\mu}} = H_{\rho\nu\mu} \\ \cancel{\Gamma_{\rho\nu\mu}} + \cancel{\Gamma_{\mu\rho\nu}} = H_{\mu\rho\nu} \end{array} \right.$$

$$\boxed{\Gamma_{\nu\mu\rho} = \frac{1}{2}\left(H_{\nu\mu\rho} + H_{\rho\nu\mu} - H_{\mu\rho\nu}\right)}$$

In particular if $A^\mu = 0$, $B_{\mu\nu} = 0$ (torsion = 0, nonmetricity = 0)

$$\Rightarrow \quad H_{\nu\mu\rho} = e_\rho \lrcorner \, dg_{\mu\nu} - e_\rho \lrcorner e_\nu \lrcorner d\theta^\alpha =$$

$$= e_\rho(g_{\mu\nu}) + \frac{1}{2} e_\rho \lrcorner e_\nu \lrcorner \left(C^\alpha{}_{\beta\gamma}\, \theta^\beta \wedge \theta^\gamma\right) g_{\alpha\mu} =$$

$$= e_\rho(g_{\mu\nu}) + \frac{1}{2}\left(C^\alpha{}_{\nu\rho} - C^\alpha{}_{\rho\nu}\right) g_{\alpha\mu} =$$

$$= e_\rho(g_{\mu\nu}) + C_{\mu\nu\rho}$$

$$\boxed{\begin{aligned} \Gamma_{\nu\mu\rho} &= \frac{1}{2}\left(e_\rho(g_{\mu\nu}) + e_\mu(g_{\nu\rho}) - e_\nu(g_{\rho\mu})\right) \\ &\quad + \frac{1}{2}\left(C_{\mu\nu\rho} + C_{\nu\rho\mu} - C_{\rho\mu\nu}\right) \end{aligned}}$$

$\nu\mu\rho$
$\rho\nu\mu$
$\mu\rho\nu$

in holonomic frame: $\Gamma_{\nu\mu\rho} = \frac{1}{2}\left(g_{\mu\nu,\rho} + g_{\rho\nu,\mu} - g_{\mu\rho,\nu}\right)$

a frame in which $g_{\mu\nu} = \text{const}$ $\quad \Gamma_{\nu\mu\rho} = \frac{1}{2}\left(C_{\mu\nu\rho} + C_{\nu\rho\mu} - C_{\rho\mu\nu}\right)$