Lecture 8 15.05.2018

Einstein's spacetime

① Spacetime – M – 4-dimensional differentiable manifold

② Spacetime is endowed with Riemannian metric of signature (1,3) ⇝ hence Levi-Civita connection

③ World lines of free falls – geodesics timelike or null
affine parameter (arc length in case of a timelike geodesic) is a proper time measured by an observer in free fall.

④ –

⑤ –

⑥ • Spacetime metric satisfies Einstein's equations

$$\underbrace{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}_{\text{trace corrected Ricci tensor called Einstein's tensor}} = \varkappa T_{\mu\nu}$$

constant

energy momentum tensor of matter.

No gravity : $R^{\mu}{}_{\nu\rho\sigma} = 0$.

$(M, g) \rightsquigarrow g_{\mu\nu}(e) = g(e_\mu, e_\nu)$

in coordinate frame

$$\omega^\mu{}_\nu = \left\{ {\mu \atop \nu\rho} \right\} dx^\rho$$

$$\left\{ {\mu \atop \nu\rho} \right\} = \frac{1}{2} g^{\mu\alpha}(g_{\alpha\nu,\rho} + g_{\rho\alpha,\nu} - g_{\nu\rho,\alpha})$$

Equation of free falls

$$\frac{d^2x^\mu}{dt^2} + \left\{ {\mu \atop \nu\rho} \right\} \frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0$$

$x^\mu = x^\mu(t)$ is timelike iff

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu < 0$$

is null iff

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$$

Correspondence principle

a) Einstein's spacetime without gravity $\Rightarrow$ Minkowski spacetime

b) Einstein's theory when $\left|\frac{\vec{v}}{c}\right| \ll 1$ and gravitational field is weak should become Newton's theory.

Ad a) In Einstein's theory NO GRAVITY means that $R^{\mu}{}_{\nu\rho\sigma} = 0$ where $R^{\mu}{}_{\nu\rho\sigma}$ is curvature tensor of the Levi-Civita connection.

$$(R^{\mu}{}_{\nu\rho\sigma} = 0) \Leftrightarrow (\text{locally in the neighbourhood } g_{\mu\nu} = \text{diag}(-1,1,1,1))$$

Proof

$\Leftarrow$ $$g_{\mu\nu} = \text{diag}(-1,1,1,1) \Rightarrow \left\{ {\mu \atop \nu\rho} \right\} \equiv 0 \Rightarrow \omega^{\mu}{}_{\nu} \equiv 0 \Rightarrow \Omega^{\mu}{}_{\nu} \equiv 0$$
$$\Rightarrow R^{\mu}{}_{\nu\rho\sigma} = 0.$$

$\Rightarrow$ L.C. connection is determined by
$$\Theta^{\mu} = d\theta^{\mu} + \omega^{\mu}{}_{\nu} \wedge \theta^{\nu} = 0 \quad (\text{no torsion})$$
$$Dg_{\mu\nu} = dg_{\mu\nu} - \omega^{\rho}{}_{\mu} g_{\rho\nu} - \omega^{\rho}{}_{\nu} g_{\mu\rho} = 0 \quad (\text{metricity})$$

$$R^{\mu}{}_{\nu\rho\sigma} \equiv 0 \Rightarrow 0 \equiv \Omega^{\mu}{}_{\nu} = \tfrac{1}{2} R^{\mu}{}_{\nu\rho\sigma}\, \theta^{\rho} \wedge \theta^{\sigma} = d\omega^{\mu}{}_{\nu} + \omega^{\mu}{}_{\alpha} \wedge \omega^{\alpha}{}_{\nu}$$
$\Rightarrow$ $\omega^{\mu}{}_{\nu} = 0$ in the neighbourhood in some frame

~~$\Rightarrow$ ...~~ in this frame

$$0 \equiv de^{\mu} + \cancel{\omega^{\mu}{}_{\nu} \wedge e^{\nu}} \Rightarrow de^{\mu} = 0 \Rightarrow \text{this frame is a coordinate frame}$$

$\Rightarrow$ $dg_{\mu\nu} = 0$ in this frame, in which
$$g = g_{\mu\nu}\, dx^{\mu} dx^{\nu}$$

$\Rightarrow$ $g_{\mu\nu} = \text{const}$

since $g_{\mu\nu} = g_{\nu\mu} \Rightarrow g_{\mu\nu} = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \ddots\ \lambda_3 \end{pmatrix}$ by a constant linear transf.

$$g = \lambda_0 dx^{0\,2} + \lambda_1 dx^{1\,2} + \lambda_2 dx^{2\,2} + \lambda_3 dx^{3\,2}$$
signature $\Rightarrow$ $\lambda_0 < 0$, $\lambda_1, \lambda_2, \lambda_3 > 0$; absorb $\Rightarrow g = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$

Ad b)

- slow motions $\left|\frac{\vec{v}}{c}\right| \ll 1$ — space velocity
- weak gravitational field $|g_{\mu\nu} - \eta_{\mu\nu}| \ll 1$

where $\eta = \eta_{\mu\nu} dx^\mu dx^\nu = -dx^{0\,2} + dx^{1\,2} + dx^{2\,2} + dx^{3\,2}$

Minkowski metric

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

- gravitational field is slowly changing

$$|\partial_0 g_{\mu\nu}| \ll |\partial_i g_{\mu\nu}|.$$

Curve of a free fall

$$x^\mu = x^\mu(t)$$

proper time = arc length

$$ds = \sqrt{|g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu|}\, dt \simeq \sqrt{|\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu|}\, dt$$

$$x^\mu = \begin{pmatrix} ct \\ x^i(t) \end{pmatrix}$$

$$\frac{dx^\mu}{dt} = \begin{pmatrix} c \\ \dot{x}^i \end{pmatrix} = \begin{pmatrix} c \\ v^i \end{pmatrix}$$

$$ds = \sqrt{|-c^2 + \vec{v}^2|}\, dt = \sqrt{1 - \frac{\vec{v}^2}{c^2}}\, c\,dt \simeq c\,dt.$$

Equation of a free fall

$$\frac{du^\mu}{ds} + \left\{ {\mu \atop rs} \right\} u^r u^s = 0 \qquad u^0 = 1, \quad u^i = \frac{v^i}{c}$$

$$\frac{du^i}{ds} + \left\{ {i \atop 00} \right\} 1 \cdot 1 + \text{terms } \frac{1}{c} = 0$$

$$\left\{ {i \atop 00} \right\} = \frac{1}{2} g^{ij} \left(g_{j0,0} + g_{0j,0} - g_{00,j} \right) \simeq -\frac{1}{2} g^{ij} g_{00,j}$$

$$\simeq -\frac{1}{2} \eta^{ij} g_{00,j} =$$

$$= -\frac{1}{2} g_{00,i}$$

$$\frac{1}{c^2}\frac{dv^i}{dt} - \frac{1}{2}\, g_{00,i} = 0$$

$$\frac{d^2\vec{r}}{dt^2} \underbrace{-\vec{\nabla}\left(\frac{c^2}{2}\, g_{00}\right)}_{\vec{\nabla}\varphi} = 0$$

$\Longrightarrow$

$$\vec{\nabla} g_{00} = \vec{\nabla}\left(-\frac{2\varphi}{c^2}\right)$$

$$\Longrightarrow \quad \boxed{g_{00} = -1 - \frac{2\varphi}{c^2}}$$

Newtonian gravitational potential in this aproximation is related to g_{00} component of the spacetime metric

Ad 6 Field equations.

In Newton's theory

$$\Delta\varphi = 4\pi G \rho .$$

So from our aproximation follows that field equations should be equations having on the left hand side 2^{nd} derivatives of the metric.

There should be a tensor 2^{nd} order in derivatives of the metric.

$\rightsquigarrow$ Riemann tensor $\equiv$ curvature tensor of L.C. connection.

Recall:

$$d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu = 0$$

$$d g_{\mu\nu} - \omega^\rho{}_\mu\, g_{\rho\nu} - \omega^\rho{}_\nu\, g_{\mu\rho} = 0$$

$$\frac{1}{2} R^\mu{}_{\nu\rho\sigma}\, \theta^\rho \wedge \theta^\sigma = \Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu$$

Riemann tensor: $\underline{R^\mu{}_{\nu\rho\sigma}}$

On the right hand side there should be tensor describing the matter content.

$\boxed{\text{Physics}}$ matter is described by energy momentum tensor SYMMETRIC $\binom{0}{2}$

- perfect fluid with 4-velocity u^μ and energy density ρ and isotropic pressure p

$$\boxed{T_{\mu\nu} = (p+\rho) u_\mu u_\nu + p\, g_{\mu\nu}} + \boxed{u^\mu u_\mu = -1}$$

In Minkowski spacetime, $g_{\mu\nu} = \eta_{\mu\nu}$, in rest frame of the fluid, $u_\mu = (1,0,0,0)$,

$$\Rightarrow T_{\mu\nu} = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$$

Special case DUST $p=0$

$$\boxed{T_{\mu\nu} = \rho u_\mu u_\nu, \; u_\mu u^\mu = -1}$$

- electromagnetic field

$$\boxed{T_{\mu\nu} = F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}}$$

- null dust

$$\boxed{T_{\mu\nu} = \Phi^2 k_\mu k_\nu \qquad k_\mu k^\nu = 0.}$$

In Special Relativity conservation of energy requires that

$$\partial^\mu T_{\mu\nu} = 0 \iff \partial_\mu T^{\mu\nu} = 0$$

Principle of minimal coupling

Equations of special theory become the same in General theory under the condition that $\partial_\mu \to \nabla_\mu$.

So in Einstein's theory we want that $\nabla_\mu T^{\mu\nu} = 0$

$$\nabla^\mu := g^{\mu\nu} \nabla_\nu . \qquad \boxed{\nabla^\mu T_{\mu\nu} = 0}$$

$$R^{\mu}{}_{\nu\rho\sigma} \;::\;::\; \propto T_{\mu\nu}$$

4 indices — 2 indices

$$\underbrace{R^{\mu}{}_{\nu\rho\sigma}}_{\rho\sigma \text{ antisymmetric}} \rightsquigarrow g_{\mu\alpha} R^{\alpha}{}_{\nu\rho\sigma} = R_{\mu\nu\rho\sigma}$$

It turns out that

Ex.
$$\begin{cases} R_{\mu\nu\rho\sigma} = - R_{\nu\mu\rho\sigma} \checkmark & \text{(metricity)}. \\ R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} & \\ R_{\mu[\nu\rho\sigma]} = 0 \checkmark & \text{(lack of torsion)} \end{cases}$$

Why?

~~Bianchi identities~~

$\cancel{d\theta^{\mu} + \omega^{\mu}{}_{\nu} \wedge \theta^{\nu} = 0}$ ~~lack of torsion~~

$$\boxed{\begin{array}{l} \text{Ricci identities} \\ \begin{cases} D^2 A^{\mu} = \Omega^{\mu}{}_{\nu} \wedge A^{\nu} \\ D^2 B_{\mu\nu} = - \Omega^{\rho}{}_{\mu} \wedge B_{\rho\nu} - \Omega^{\rho}{}_{\nu} \wedge B_{\mu\rho}. \end{cases} \end{array}}$$

Exercise.

$$\Rightarrow \quad 0 = \Theta^{\mu} = D\theta^{\mu}$$

$$0 = D\Theta^{\mu} = D^2\theta^{\mu} = \Omega^{\mu}{}_{\nu} \wedge \theta^{\nu} = \frac{1}{2} R^{\mu}{}_{\nu\rho\sigma}\, \theta^{\rho} \wedge \theta^{\sigma} \wedge \theta^{\nu}$$

$$\Rightarrow \boxed{R^{\mu}{}_{[\nu\rho\sigma]} = 0}.$$

1st Bianchi identity.

$$0 = D g_{\mu\nu}$$

$$\Rightarrow 0 = D^2 g_{\mu\nu} = -\Omega^{\rho}{}_{\mu} g_{\rho\nu} - \Omega^{\rho}{}_{\nu} g_{\mu\rho} = - \Omega_{\nu\mu} - \Omega_{\mu\nu}$$

Second Bianchi identity

$$D\Omega^\mu{}_\nu = d\Omega^\mu{}_\nu + \omega^\mu{}_\alpha \wedge \Omega^\alpha{}_\nu - \omega^\alpha{}_\nu \wedge \Omega^\mu{}_\alpha =$$

object of type Ad

$$= d\Omega^\mu{}_\nu = d(d\omega^\mu{}_\nu + \omega^\mu{}_\alpha \wedge \omega^\alpha{}_\nu) =$$

in a frame when $\omega = 0$

$$= d\omega^\mu{}_\alpha \wedge \omega^\alpha{}_\nu - \omega^\mu{}_\alpha \wedge d\omega^\alpha{}_\nu =$$

$$= 0 \qquad = 0$$

in a frame when $\omega=0$ $\qquad$ in any frame

$$\boxed{D\Omega^\mu{}_\nu = 0}$$

$$0 = D\left(\frac{1}{2} R^\mu{}_{\nu\rho\sigma}\, \theta^\rho \wedge \theta^\sigma\right) = \frac{1}{2} DR^\mu{}_{\nu\rho\sigma}\, \theta^\rho \wedge \theta^\sigma + \dots \quad \underset{0}{\overset{}{D\theta}} =$$

$$= \frac{1}{2} \nabla_\alpha R^\mu{}_{\nu\rho\sigma}\, \theta^\alpha \wedge \theta^\rho \wedge \theta^\sigma$$

$$\Rightarrow \nabla_{[\alpha} R^\mu{}_{|\nu|\rho\sigma]} = 0.$$

notation $\nabla_\alpha T = T_{;\alpha}$

$$\boxed{R^\mu{}_{\nu[\rho\sigma;\alpha]} = 0}$$

$$0 = R^\mu{}_{\nu\rho\sigma;\alpha} + R^\mu{}_{\nu\alpha\rho;\sigma} + R^\mu{}_{\nu\sigma\alpha;\rho}$$
$$- R^\mu{}_{\nu\sigma\rho;\alpha} - R^\mu{}_{\nu\alpha\sigma;\rho} - R^\mu{}_{\nu\rho\alpha;\sigma}$$

$$\Rightarrow \boxed{R^\mu{}_{\nu\rho\sigma;\alpha} + R^\mu{}_{\nu\alpha\rho;\sigma} + R^\mu{}_{\nu\sigma\alpha;\rho} = 0.}$$

$$R^{\mu}{}_{\nu\varrho\sigma} \;\vdots\vdots\vdots\; \varkappa T_{\alpha\beta}$$

Ricci tensor $\boxed{R_{\nu\sigma} = R^{\mu}{}_{\nu\mu\sigma}}$ Note that $\underline{R_{\nu\sigma} = R_{\sigma\nu}}$. !

Tempting:

$$R_{\nu\sigma} = \varkappa T_{\nu\sigma} \quad ??$$

BAD $\Rightarrow$ $\nabla^{\nu} R_{\nu\sigma} = 0$ over determined. !

Is there a tensor related to Ricci tensor s.t. $\nabla^{\nu} E_{\nu\sigma} = 0$?

II nd Bianchi:

$$R_{\nu\sigma;\alpha} - R_{\nu\alpha;\sigma} + \nabla_{\varrho} R^{\varrho}{}_{\nu\sigma\alpha} = 0. \qquad \nu \to \alpha$$

$$\nabla^{\nu} R_{\nu\sigma} - \nabla_{\sigma} R + \nabla^{\nu} R_{\nu\sigma} = 0$$

where $R = g^{\mu\nu} R_{\mu\nu}$

$$2\nabla^{\nu} R_{\nu\sigma} - \nabla^{\nu}(g_{\nu\sigma} R) = 0$$

$$\nabla^{\nu}\left(R_{\nu\sigma} - \tfrac{1}{2} g_{\nu\sigma} R\right) = 0$$

$$E_{\nu\sigma} = R_{\nu\sigma} - \tfrac{1}{2} g_{\nu\sigma} R$$

Einstein equations

$$\underline{E_{\nu\sigma} = \varkappa T_{\nu\sigma}}$$

$$\boxed{R_{\nu\sigma} - \tfrac{1}{2} g_{\nu\sigma} R = \varkappa T_{\nu\sigma}}$$

In empty space $T_{\mu\nu} = 0$

$$R_{\nu\sigma} - \tfrac{1}{2} g_{\nu\sigma} R = 0 \Rightarrow R - 2R = 0 \Rightarrow R = 0 \Rightarrow \boxed{R_{\nu\sigma} = 0}$$