Lecture 9 22.05.2018

Cosmological constant

⊙ **Summary of formulation of GR.**

- spacetime – (M, g) 4-dimensional Riemannian manifold with metric g of signature $(-+++)$
- gravity – manifests itself when the metric g has nonzero Riemannian curvature $R_{\mu\nu\rho\sigma} \neq 0$.
- newtonian concept of gravitational force: ELIMINATED!
- movements of particles in free fall:

particles move along geodesics $\gamma(s)$ in spacetime, in a coordinate chart when $\gamma(s) = x^\mu(s)$, $g = g_{\mu\nu} dx^\mu dx^\nu$

$$\boxed{\frac{d^2 x^\mu}{ds^2} + \left\{ {\mu \atop \nu\rho} \right\} \frac{dx^\nu}{ds} \frac{dx^\rho}{ds} = 0}$$

Three possibilities for a sign of $g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$:

(1) massive particles move along geodesics on which

$$g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = -1$$

s is an arc length, called proper time of a particle

(2) massles particles move along null geodesics, i.e. s.t.

$$g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

Trajectories of photons are like that. They are counterparts of light rays in Minkowski spacetime.

(3) if $g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} > 0$ then we have particles so far not observed. They would move with spatial velocity > speed of light.

Tachyons (?)

- from the Newtonian limit we get that in weak fields $g_{00} = -1 - \frac{2\varphi}{c^2}$.

- field equations (1916) — Einstein

(E.E.) $$\boxed{\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{E_{\mu\nu}} = \varkappa T_{\mu\nu}}$$

$E_{\mu\nu}$ – Einstein tensor

$\varkappa$ – constant of physics.

$T_{\mu\nu}$ – energy momentum tensor

$R = g^{\mu\nu} R_{\mu\nu}$

Contracted second Bianchi identity:

$$\nabla^\mu E_{\mu\nu} = 0. \implies \nabla^\mu T_{\mu\nu} = 0. \checkmark$$

(Note similarity to Maxwell's equations:

$$\begin{cases} \partial^\mu F_{\mu\nu} = 4\pi j_\nu & \longrightarrow \quad E_\nu = 4\pi j_\nu \to \partial^\nu E_\nu = \partial^\nu \partial^\mu F_{\mu\nu} = 0 \\ \partial^\mu {*F_{\mu\nu}} = 0 \quad (\text{B.I.}) & \qquad\qquad \Downarrow \; \partial^\nu j_\nu = 0 \end{cases}$$
)

- But: $\nabla^\rho g_{\mu\nu} = 0 \implies \nabla^\mu g_{\mu\nu} = 0$

so one can modify equations (E.E.) by

$$\boxed{\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}\; + \Lambda g_{\mu\nu} = \underbrace{\varkappa T_{\mu\nu}}}$$

Λ – constant – COSMOLOGICAL CONSTANT

Einstein (1917).

Why?

$$g = -c^2 dt^2 + dx^{1\,2} + dx^{2\,2} + dx^{3\,2} \qquad \text{Minkowski metric}$$

$\Downarrow$

Universe ~ has gravity $\implies R_{\mu\nu\rho\sigma} \neq 0$.

Einstein believed that Universe is STATIC.

Had no beginning ~ no end.

What could be more perfect than

$$M = \mathbb{R} \times S^3$$

$\mathbb{R}$ – time

S^3 – space – is finite (has finite volume) but unrestricted (without boundary)

Metric : product of a standard metric on $\mathbb{R}$ and on a sphere of radius R:

$$g = -c^2 dt^2 + R^2 g_{S^3}$$

$$S^3 = \{ \mathbb{R}^4 : (y^1, y^2, y^3, y^4) \text{ s.t. } {y^1}^2 + {y^2}^2 + {y^3}^2 + {y^4}^2 = 1 \}$$

$$\left.\begin{aligned} y^1 &= \sin\chi \sin\theta \sin\varphi \\ y^2 &= \sin\chi \sin\theta \cos\varphi \\ y^3 &= \sin\chi \cos\theta \\ y^4 &= \cos\chi \end{aligned}\right\} \to dy^\mu = \ldots$$

$$g_{S^3} = {dy^1}^2 + \ldots + {dy^4}^2 \Big|_{S^3} =$$

$$= \underline{d\chi^2 + \sin^2\chi \, (d\theta^2 + \sin^2\theta \, d\varphi^2)}.$$

$$\boxed{g = -c^2 dt^2 + R^2 \left(d\chi^2 + \sin^2\chi \, (d\theta^2 + \sin^2\theta \, d\varphi^2) \right)}$$

Orthonormal frame:

$$g = -{e^0}^2 + {e^1}^2 + {e^2}^2 + {e^3}^2 = g_{\mu\nu} e^\mu e^\nu$$

$$(*) \begin{cases} e^0 = c\,dt \\ e^1 = R\,d\chi \\ e^2 = R \sin\chi \, d\theta \\ e^3 = R \sin\chi \sin\theta \, d\varphi \end{cases} \qquad g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \text{const!}$$

We will calculate Levi-Civita connection $\omega^\mu{}_\nu(e)$ in frame $(*)$

$$\boxed{de^\mu + \omega^\mu{}_\nu \wedge e^\nu = 0}$$

$$\cancel{dg_{\mu\nu}} - \omega^\rho{}_\mu g_{\rho\nu} - \omega^\rho{}_\nu g_{\mu\rho} = 0$$

$$\Downarrow$$

$$\boxed{\omega_{\mu\nu} + \omega_{\nu\mu} = 0.} \qquad \boxed{\omega_{\mu\nu} = g_{\mu\rho}\omega^\rho{}_\nu}$$

$$de^0 = 0$$
$$de^1 = 0$$
$$de^2 = R\cos\chi\, d\chi \wedge d\vartheta = e^1\wedge e^2 \frac{\mathrm{ctg}\,\chi}{R} = \frac{\mathrm{ctg}\,\chi}{R} e^1\wedge e^2$$
$$de^3 = \frac{\mathrm{ctg}\,\chi}{R} e^1\wedge e^3 + \frac{\mathrm{ctg}\,\vartheta}{R\sin\chi} e^2\wedge e^3,$$

$$A = \mathrm{ctg}\,\chi / R\,, \quad B = \mathrm{ctg}\,\vartheta / R\sin\chi$$

$$de^0 = 0 \qquad = -\omega^0{}_\mu\wedge e^\mu = \omega_{0\mu}\wedge e^\mu$$
$$de^1 = 0 \qquad = -\omega^1{}_\mu\wedge e^\mu = -\omega_{1\mu}\wedge e^\mu$$
$$de^2 = Ae^1\wedge e^2 \qquad = -\omega^2{}_\mu\wedge e^\mu = -\omega_{2\mu}\wedge e^\mu = -\omega_{20}\wedge e^0 - \omega_{2i}\wedge e^i$$
$$de^3 = Ae^1\wedge e^3 + Be^2\wedge e^3 = -\omega^3{}_\mu\wedge e^\mu = -\omega_{3\mu}\wedge e^\mu$$

$$\omega_{0i} = 0$$
$$-\omega_{21} = -Ae^2 \quad \Rightarrow \quad \omega_{21} = Ae^2 \qquad \omega_{12} = -Ae^2$$
$$-\omega_{31} = -Ae^3 \quad \Rightarrow \quad \omega_{31} = Ae^3 \qquad \omega_{13} = -Ae^3$$
$$-\omega_{32} = -Be^3 \quad \Rightarrow \quad \omega_{32} = Be^3 \qquad \omega_{23} = -Be^3$$

all other equal to zero.

$$\Rightarrow \begin{cases} \omega_{0\mu} = \omega_{\mu 0} = 0 \\ \omega_{12} = -\omega_{21} = -Ae^2 \\ \omega_{13} = -\omega_{31} = -Ae^3 \\ \omega_{23} = -\omega_{32} = -Be^3. \end{cases}$$

Unique! so we're found it!

Now curvature:

$$\underline{\Omega_{\mu\nu} = d\omega_{\mu\nu} + \omega_{\mu}{}^{\alpha}\wedge\omega_{\alpha\nu}}$$

$$\Omega_{0\mu} = d\omega_{0\mu} + \omega_{0\alpha}\wedge\omega^{\alpha}{}_{\mu} = 0$$

$$\Omega_{ij} = d\omega_{ij} + \omega_{i\alpha}\wedge\omega^{\alpha}{}_{j} = d\omega_{ij} + \omega_{ik}\,\omega^{k}{}_{j} =$$

$$= d\omega_{ij} + \sum_{k=1}^{3} \omega_{ik}\wedge\omega_{kj}$$

$$d\omega_{ij} = ?$$

$$dA = -\frac{e^1}{R^2\sin^2\chi}$$

$$dB = -\frac{\mathrm{ctg}\,\chi\,\mathrm{ctg}\,\theta}{R^2\sin\chi}\,e^1 - \frac{e^2}{R^2\sin^2\chi\,\sin^2\theta}$$

$$\begin{cases} \Omega_{12} = \dfrac{e^1\wedge e^2}{R^2} \\[2mm] \Omega_{13} = \dfrac{e^1\wedge e^3}{R^2} \\[2mm] \Omega_{23} = \dfrac{e^2\wedge e^3}{R^2} \end{cases} \qquad \begin{aligned} R_{1212} &= \frac{1}{R^2} \\ R_{1313} &= \frac{1}{R^2} \\ R_{2323} &= \frac{1}{R^2} \end{aligned}$$

$$\cancel{R_{ij} = R^{k}{}_{ikj} = \dots} \qquad R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} = \dots$$

$$R_{\mu\nu} = \begin{pmatrix} 0 & & & \\ & \frac{2}{R^2} & & \\ & & \frac{2}{R^2} & \\ & & & \frac{2}{R^3} \end{pmatrix}$$

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p\,g_{\mu\nu}$$

$$T = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$$

$$u^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad u_\mu = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$R = g^{\mu\nu}R_{\mu\nu} = \frac{6}{R^2}$$

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \begin{pmatrix} 3/R^2 & & & \\ & -\frac{1}{R^2} & & \\ & & -\frac{1}{R^2} & \\ & & & -\frac{1}{R^2} \end{pmatrix}$$

problem!

Negative pressure!

$$E_{\mu\nu} + \Lambda g_{\mu\nu} = \begin{pmatrix} \frac{3-\Lambda}{R^2} & \\ & -\frac{(1-\Lambda)}{R^2}\,\delta_{ij} \end{pmatrix}$$

$$\Lambda = +1$$

$$E_{\mu\nu} + \Lambda g_{\mu\nu} = \begin{pmatrix} \frac{2}{R^2} & & \\ & 0 & \\ & & 0 \end{pmatrix} = T_{\mu\nu} \text{ of dust with } \rho = \frac{2}{R^2}$$

Introduction of Λ term in the equations ~~says~~ enables to interpret Einstein's Universe as a solution of Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \underset{\substack{\| \\ 1}}{\lambda} g_{\mu\nu} = \begin{pmatrix} \rho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

$$\rho = \text{const} = \frac{2}{R^2}$$

Linearization of Einstein's equations

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$g = g_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} dx^\mu dx^\nu + h_{\mu\nu} dx^\mu dx^\nu =$$

$$= \underbrace{\left(-c^2 dt^2 + dx^2 + dy^2 + dz^2\right)}_{\text{Minkowski}} + \underbrace{h_{\mu\nu} dx^\mu dx^\nu}_{h\text{ - small perturbation}}$$

$$|h_{\mu\nu}| \ll 1$$

We can simplify $h_{\mu\nu}$ by an appropriate change of coordinates.

$$x^\mu \rightarrow x^\mu = x'^\mu + \xi^\mu$$

we assume small perturb of Minkowski coords: $|\xi^\mu| \ll 1$

$$g = g_{\mu\nu} dx^\mu dx^\nu = \underbrace{g_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta}} dx'^\alpha dx'^\beta$$

$$g'_{\alpha\beta} = g_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta}$$

$$\cancel{a^\alpha_\beta = \frac{\partial x^\alpha}{\partial x'^\beta}} \qquad \underline{(a^{-1})^\alpha{}_\beta = \frac{\partial x^\alpha}{\partial x'^\beta}}$$

$$g'_{\alpha\beta} = g_{\mu\nu}\left(\delta^\mu_\alpha + \frac{\partial\xi^\mu}{\partial x'^\alpha}\right)\left(\delta^\nu_\beta + \frac{\partial\xi^\nu}{\partial x'^\beta}\right) \cong$$

$$\shortparallel$$

$$\eta_{\alpha\beta} + h'_{\alpha\beta} -$$

$$\simeq (\eta_{\mu\nu} + h_{\mu\nu})\left(\delta^\mu_\alpha \delta^\nu_\beta + \delta^\mu_\alpha \frac{\partial\xi^\nu}{\partial x'^\beta} + \delta^\nu_\beta \frac{\partial\xi^\mu}{\partial x'^\alpha}\right)$$

$$= \eta_{\alpha\beta} + \xi_{\alpha,\beta} + \xi_{\beta,\alpha} + h_{\alpha\beta}$$

$$\boxed{h'_{\alpha\beta} = h_{\alpha\beta} + \xi_{\alpha,\beta} + \xi_{\beta,\alpha}}$$

Gauge transformation for linearized grav. field.

<u>Define</u> $\boxed{h = g^{\mu\nu} h_{\mu\nu} \approx \eta^{\mu\nu} h_{\mu\nu}}$

$$\boxed{\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h}$$

then $\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = \eta^{\mu\nu}\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h\right) = h - 2h = -h$

$\Rightarrow \quad h_{\mu\nu} = \bar{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu} h = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}$

$$\boxed{h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}}$$

We want to calculate Einstein tensor for

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$\Rightarrow \quad \omega^\mu{}_\nu = \left\{ {\mu \atop \nu\rho} \right\} dx^\rho = \frac{1}{2}\eta^{\mu\sigma}\left(g_{\sigma\nu,\rho} + g_{\rho\sigma,\nu} - g_{\nu\rho,\sigma}\right) dx^\rho$$

$$= \frac{1}{2}\eta^{\mu\sigma}\left(h_{\sigma\nu,\rho} + h_{\rho\sigma,\nu} - h_{\nu\rho,\sigma}\right) dx^\rho =$$

$$= \frac{1}{2}\left(h^\mu{}_{\nu,\rho} + h^\mu{}_{\rho,\nu} - \partial^\mu h_{\nu\rho}\right) dx^\rho$$

$$\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \cancel{\omega^\mu{}_\rho \wedge \omega^\rho{}_\nu} =$$

quadratic in ω

$$= \frac{1}{2}\left(\cancel{h^\alpha{}_{\nu,\rho\alpha}} + h^\mu{}_{\beta,\nu\alpha} - \partial_\alpha \partial^\mu h_{\nu\beta}\right) dx^\alpha \wedge dx^\beta$$

$$R^\mu{}_{\nu\sigma\rho} = e_\rho \lrcorner\, e_\sigma \lrcorner\, \Omega^\mu{}_\nu =$$

$$= \frac{1}{2}\left(h^\mu{}_{\rho,\nu\sigma} - h^\mu{}_{\sigma,\nu\rho} - \partial_\sigma \partial^\mu h_{\nu\rho} + \partial_\rho \partial^\mu h_{\nu\sigma}\right)$$

$\Rightarrow$ Ricci

$$R_{\nu\rho} = \frac{1}{2}\left(h^\mu{}_{\rho,\nu\mu} - h^\mu{}_{\mu,\nu\rho} - \Box h_{\nu\rho} + h_\nu{}^\mu{}_{,\mu\rho}\right) =$$

$$= \frac{1}{2}\Big[\partial_\nu \partial_\mu \big(\bar{h}^\mu{}_\rho - \cancel{\tfrac{1}{2}\bar{h}\eta^\mu{}_\rho}\big) + \cancel{\partial_\nu \partial_\rho \bar{h}} - \Box\big(\bar{h}_{\nu\rho} - \tfrac{1}{2}\bar{h}\eta_{\nu\rho}\big) + \partial^\mu \partial_\rho \big(\bar{h}_{\nu\mu} - \cancel{\tfrac{1}{2}\eta_{\mu\nu}\bar{h}}\big)\Big]$$

$$\boxed{R_{\nu\rho} = \frac{1}{2}\partial_\nu \partial_\mu \bar{h}^\mu{}_\rho + \frac{1}{2}\partial^\mu \partial_\rho \bar{h}_{\nu\mu} - \frac{1}{2}\Box \bar{h}_{\nu\rho} + \frac{1}{4}\eta_{\nu\rho}\Box \bar{h}}$$

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}$$

$\Rightarrow$ We can impose $\boxed{\partial_\mu \bar{h}^\mu{}_\nu = 0}$ DeDonder gauge!

$$\bar{h}'_{\mu\nu} = h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} - \frac{1}{2}\eta_{\mu\nu}\left(h + 2\partial_\rho \xi^\rho\right) =$$

$$= \bar{h}_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} - \eta_{\mu\nu}\partial_\rho \xi^\rho$$

$$0 = \partial^\mu \bar{h}'_{\mu\nu} = 0 + \cancel{\partial^\mu \partial_\nu \xi_\mu} + \partial^\mu \partial_\mu \xi_\nu - \cancel{\partial_\nu \partial_\mu \xi^\mu}$$

$\Box \xi_\nu = 0$ still freedom!

$$R_{\nu\varrho} \overset{\text{DeDonder}}{=} -\frac{1}{2}\Box \bar{h}_{\nu\varrho} + \frac{1}{4}\eta_{\nu\varrho}\Box\bar{h} =$$

$$= -\frac{1}{2}\Box\left(\bar{h}_{\nu\varrho} - \frac{1}{2}\eta_{\nu\varrho}\bar{h}\right) = -\frac{1}{2}\Box h_{\nu\varrho}$$

<u>Einstein</u>

$$E_{\nu\varrho} = R_{\nu\varrho} - \frac{1}{2}g_{\nu\varrho}R =$$

$$= -\frac{1}{2}\Box h_{\nu\varrho} - \frac{1}{2}\eta_{\nu\varrho}\left(-\frac{1}{2}\Box h\right) =$$

$$= -\frac{1}{2}\Box\left(h_{\nu\varrho} - \frac{1}{2}\eta_{\nu\varrho}h\right) = -\frac{1}{2}\Box\bar{h}_{\nu\varrho}.$$

Linearized Einstein eqs:

$$\boxed{-\frac{1}{2}\Box\bar{h}_{\nu\varrho} + \Lambda(\eta_{\nu\varrho} + h_{\nu\varrho}) = \varkappa T_{\nu\varrho}}$$

$\Lambda = 0$

$|\partial_0^2| \ll |\partial_i^2|$

$\Rightarrow$

$$-\frac{1}{2}\Delta \bar{h}_{\nu\rho} = \varkappa T_{\nu\rho}$$

$T_{00} = \rho c^2 \qquad T_{0i} = 0 = T_{ij}$

$-\Delta \bar{h}_{00} = 2\rho c^2 \varkappa \qquad \bar{h}_{\text{other}} = 0$

$\Delta \bar{h}_{\text{others}} = 0$

$\bar{h} = \eta^{\mu\nu} \bar{h}_{\mu\nu} = -\bar{h}_{00}$

$h_{00} = \bar{h}_{00} - \frac{1}{2}\eta_{00}\bar{h} = \bar{h}_{00} + \frac{1}{2}\bar{h} = \bar{h}_{00} - \frac{1}{2}\bar{h}_{00} = \frac{1}{2}\bar{h}_{00}$

$\bar{h}_{00} = 2h_{00} = 2\,(g_{00} - \eta_{00}) = 2\left(-1 - \frac{2\varphi}{c^2} + 1\right) = -\frac{4\varphi}{c^2}$

$$-\Delta\left(-\frac{4\varphi}{c^2}\right) = 2\rho c^2 \varkappa$$

$$\Delta\varphi = \frac{1}{2}\rho c^4 \varkappa = 4\pi G\rho$$

$$\boxed{\varkappa = \frac{8\pi G}{c^4}}$$