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# Short introduction
# to
# standard cosmology

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# General relativity

## Principle of equivalence (S. Weinberg):

**At every space–time point in an arbitrary gravitational field it is possible to choose a "locally inertial coordinate frame" such that, within a sufficiently small region of the point in question, the laws of nature take the same form as in unaccelerated Cartesian coordinate systems in the absence of gravitation.**

## Practical prescription:

- **Take equations which describe a system in the absence of gravity**
- **Generalize them in such a way that they are generally covariant (preserve their form under a general coordinate transformation) and reduce to the starting equations in flat Minkowski space-time.**

# Example:

**Particle in the absence of any (non-gravitational) forces**

**locally inertial coordinates:** $\xi^\alpha$

**proper time:** $\mathrm{d}\tau^2 = -\mathrm{d}s^2 = -\eta_{\alpha\beta}\,\mathrm{d}\xi^\alpha\,\mathrm{d}\xi^\beta$

**equation of motion:**

$$\frac{\mathrm{d}^2\xi^\alpha}{\mathrm{d}\tau^2} = 0$$

**in arbitrary coordinates** $x^\mu$ **we get**

$$\mathrm{d}\tau^2 = -g_{\alpha\beta}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

$$\frac{\mathrm{d}^2x^\rho}{\mathrm{d}\tau^2} + \Gamma^\rho_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0$$

**geodesic equation**

## The whole effect of gravity is encoded in the metric tensor $g_{\alpha\beta}$ and the affine connection $\Gamma^\rho_{\mu\nu}$!

**affine connection**

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left( g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma} \right)$$

**curvature tensor**

$$R^{\mu}{}_{\nu\rho\sigma} = \Gamma^{\mu}_{\nu\sigma,\rho} - \Gamma^{\mu}_{\nu\rho,\sigma} + \Gamma^{\mu}_{\rho\kappa}\Gamma^{\kappa}_{\nu\sigma} - \Gamma^{\mu}_{\sigma\kappa}\Gamma^{\kappa}_{\nu\rho}$$

**Ricci tensor and curvature scalar**

$$R_{\mu\nu} = R^{\kappa}{}_{\mu\kappa\nu} \qquad\qquad R = g^{\mu\nu} R_{\mu\nu}$$

**covariant derivatives**

$$V^{\mu}{}_{;\nu} = V^{\mu}{}_{,\nu} + \Gamma^{\mu}_{\nu\sigma} V^{\sigma} \qquad\qquad V_{\mu;\nu} = V_{\mu,\nu} - \Gamma^{\sigma}_{\mu\nu} V_{\sigma}$$

**geometrical interpretation:**
**(covariant) vector $A_\mu$ after parallel transport along infinitesimally small closed loop changes by**

$$\Delta A_\mu = \frac{1}{2} R^{\nu}{}_{\mu\rho\sigma} A_\nu \oint x^\rho \mathrm{d}x^\sigma$$

# Einstein equations

**In the Newtonian limit (slowly moving bodies in weak, static gravitational field):**

$$g_{00} = -(1 + 2\phi) \qquad \nabla^2\phi = 4\pi G_N \rho$$

**we get**

$$-\nabla^2 g_{00} = 8\pi G_N T_{00}$$

**This can be generalized to**

$$\tilde{G}_{\mu\nu} = 8\pi G_N T_{\mu\nu}$$

$\tilde{G}_{\mu\nu}$ **is a covariantly conserved tensor with an appropriate Newtonian limit of its** $00$ **component**

**There is only one such tensor build from the metric tensor and its derivatives (bilinear in derivatives):**

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}$$

**In general the l.h.s. of the Einstein equations could contain terms with arbitrary number of derivatives**

$$\tilde{G}_{\mu\nu} = \sum_{n=0} \tilde{G}^{(n)}_{\mu\nu}$$

$$\tilde{G}^{(n)}_{\mu\nu} \sim c_n \nabla^n \sim \left( \frac{l_n}{l_{phys}} \right)^n$$

**Terms with small (large) number of derivatives are more important at large (small) distances (as compared to the standard $n = 2$ term)**

**Cosmology – large distances**

**There is only one tensor with $n < 2$: the metric tensor itself**
**Its coefficient is called the cosmological constant**

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}$$

$n > 2$
**Small distances – very early Universe (?)**
**quantum theory of gravity (superstings)**

# Robertson–Walker metric

**The observed (part of the) Universe seems to be isotropic and homogeneous in very large scales**

**Space–time which at every time ($t =$ const) is homogeneous and isotropic can be described by the Robertson–Walker metric**

$$
\begin{aligned}
\mathrm{d}s^2 = & -\mathrm{d}t^2 \\
& + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right]
\end{aligned}
$$

**where $r$, $\theta$, $\phi$ are comoving coordinates**

**3–dimensional spaces for fixed $t$ are spaces with constant curvature** ${}^{(3)}R = 6k/a^2(t)$

$\mathbf{a(t)}$ **is the time–dependent cosmic scale factor**

$k = +1$ **: finite volume** $2\pi^2 a^3(t)$

$k = 0$ **: infinite flat space**

$k = -1$ **: infinite hyperbolic space**

# Red-shift in Robertson-Walker metric

**Light,** $\mathrm{d}s^2 = 0$**, emitted at** $r = r_1$**,** $t = t_1$
**received by as at** $r = 0$**,** $t = t_0$

$$\int_{t_1}^{t_0} \frac{\mathrm{d}t}{a(t)} = \int_0^{r_1} \frac{\mathrm{d}r}{\sqrt{1 - kr^2}} = \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{\mathrm{d}t}{a(t)}$$

$$\frac{\delta t_1}{a(t_1)} = \frac{\delta t_0}{a(t_0)}$$

**Received radiation has frequency different from the frequency of emission**

$$\frac{\nu_1}{\nu_0} = \frac{\lambda_0}{\lambda_1} = \frac{a(t_0)}{a(t_1)} = z + 1$$

$$z = \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{a(t_0)}{a(t_1)} - 1$$

**Almost all observed galaxies has red-shifted spectrum** $z > 0$
$\Rightarrow$ **the cosmic scale factor at the time of emission was** smaller **than it is today**
$\Rightarrow$ **the Universe expands**

## Friedmann-Robertson-Walker models

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}$$

**Energy–momentum tensor for perfect fluid**

$$T_{\mu\nu} = p g_{\mu\nu} + (p + \rho) u_\mu u_\nu$$

**For the Robertson–Walker metric one gets the Friedmann equations:**

$$3 \left( \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) = 8\pi G_N \rho$$

$$\left( 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) = -8\pi G_N p$$

**Their combination gives**

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3} (3p + \rho)$$

**Today** $\dot{a} > 0$
$(3p + \rho) > 0 \Rightarrow \ddot{a} < 0$

$\Rightarrow a(t = 0) = 0 -$ **Big Bang**

# Dynamics

**Energy–momentum has contributions from radiation, (nonrelativistic) matter and cosmological constant**

$$\rho = \rho_R + \rho_M + \rho_\Lambda \qquad\qquad p = p_R + p_M + p_\Lambda$$

**Equations of state** $p_i = w_i \rho_i$

$$w_R = \frac{1}{3} \qquad w_M = 0 \qquad w_\Lambda = -1$$

**Energy density changes with the cosmic scale factor**

$$\rho_R = \rho_{R0}\left(\frac{a_0}{a}\right)^4 \qquad \rho_M = \rho_{M0}\left(\frac{a_0}{a}\right)^3 \qquad \rho_\Lambda = \rho_{\Lambda 0}$$

**Friedmann equation determines the time dependence of the cosmic scale factor**

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G_N}{3}\left(\frac{\rho_{R0}a_0^4}{a^4} + \frac{\rho_{M0}a_0^3}{a^3} + \rho_\Lambda\right) - \frac{k}{a^2}$$

**radiation domination (RD):** $a \sim t^{1/2}$

**matter domination (MD):** $a \sim t^{2/3}$

**cosmological const. domin. ($\Lambda$D):** $a \sim \exp(\sqrt{\frac{\Lambda}{3}}\, t)$

# Curvature (problem)

**Friedmann equation** $\Rightarrow$ $\Omega - 1 = \frac{k}{a^2 H^2}$

$$\Omega = \frac{\rho}{\rho_c} \qquad \rho_c \approx 10^{-29} \frac{\mathrm{g}}{\mathrm{cm}^3}$$

**(MD):** $|\Omega - 1| \sim a^1$

**(RD):** $|\Omega - 1| \sim a^2$

$|\Omega - 1|$ **increases with time in expanding, radiation or matter dominated Universe**

$|\Omega_0 - 1| < 10^{-1}$

$|\Omega(t_{\mathrm{EQ}}) - 1| < 10^{-5}$ $\qquad t_{\mathrm{EQ}} = \mathcal{O}(10^4 \mathrm{yr})$

$|\Omega(t_{\mathrm{nucl}}) - 1| < 10^{-17}$ $\qquad t_{\mathrm{nucl}} = \mathcal{O}(1\mathrm{s})$

**Natural solution**

**($\Lambda$D):** $|\Omega - 1| \sim e^{-Ht}$

# Horizon (problem)

**Distance to the particle horizon (distance which massless particles could travel from time $t = 0$ till time $t$)**

$$d_h(t) = a(t) \int_0^{r_h(t)} \frac{\mathrm{d}r}{\sqrt{1 - kr^2}} = a(t) \int_0^t \frac{\mathrm{d}t'}{a(t')}$$

**It is finite for radiation or matter dominated Universe**

**$\Rightarrow$ only finite parts of the Universe are causally connected**

**The part of the Universe we observe today consists of about $10^5$ parts which were causally disconnected at the time of recombination**

**Why the observed Cosmic Microwave Background Radiation is so isotropic?**

# Future of the Universe

- $\Lambda = 0$, $k = -1$ ($\rho < \rho_c$)
  the curvature terms will dominate Friedmann equation giving $a(t) \sim t \to \infty$

- $\Lambda = 0$, $k = 0$ ($\rho = \rho_c$)
  $a(t) \sim t^{2/3} \to \infty$

- $\Lambda = 0$, $k = +1$ ($\rho > \rho_c$)
  expansion stops at the maximal
  $a_{\max} = (8/3)\pi G_N \rho_0 a_0^3$
  and collapse begins leading to Big Crunch

- $\Lambda < 0$, arbitrary $k$ ($\rho$)
  like above with $a_{\max}$ decreasing with increasing $|\Lambda|$

- $\Lambda > 0$, $k \leq 1$ ($\rho \leq \rho_c$)
  eternal expansion, exponential at late times

- $\Lambda > 0$, $k = +1$ $(\rho > \rho_c)$

  Let us define $\Lambda_E = \left(4\pi G_N \rho_{M0} a_0^3\right)^{-2}$

  - $\Lambda < \Lambda_E$
    expansion stops at the maximal
    $a_{\max} > (8/3)\pi G_N \rho_0 a_0^3$
    and collapse begins leading to Big Crunch

  - $\Lambda = \Lambda_E$
    expansion asymptotically stops at the
    $a_E = 4\pi G_N \rho_0 a_0^3$

  - $\Lambda > \Lambda_E$
    expansion slows down until
    $a(t) \approx a_E$
    then speeds up becoming exponential at late times

Static Universe proposed by Einstein:

$$a = a_E \qquad \Lambda = \Lambda_E$$

# Thermodynamics of the Universe

**We assume that Universe was close to thermodynamical equilibrium during most of its evolution.**

**In equilibrium energy density and pressure are dominated by relativistic particles**

$$\rho_R(T) = g_* \frac{\pi^2}{30} T^4 \qquad p_R(T) = g_* \frac{\pi^2}{90} T^4$$

$$g_* = \sum_{\text{bos.}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{\text{fer.}} g_j \left(\frac{T_j}{T}\right)^4$$

**The effective number of relativistic degrees of freedom $g_*$ at a given temperature is determined by the elementary particle spectrum**

# Time vs. temperature

**The expansion rate depends on $g_*$. For (RD)**

$$H^2 = \frac{8\pi^3}{90} g_* T^4$$

**On the other hand**

$$H^2 = \frac{\dot{a}}{a} = \frac{1}{2t}$$

$$\left(\frac{t}{\mathrm{s}}\right) \approx \frac{2.4}{\sqrt{g_*}} \left(\frac{T}{\mathrm{MeV}}\right)^{-2}$$

**At time** $t \approx 1\,\mathrm{s}$
**temperature was** $T \approx 1\,\mathrm{MeV}$

**Neutrinos decouple and primordial nucleosynthesis starts at about that time**

# Entropy

**Entropy density is given by**

$$s = \frac{S}{V} = g_{*s} \frac{2\pi^2}{45} T^3$$

$$g_{*s} = \sum_{\text{bos.}} g_i \left( \frac{T_i}{T} \right)^3 + \frac{7}{8} \sum_{\text{fer.}} g_j \left( \frac{T_j}{T} \right)^3$$

**Entropy in fixed comoving volume is conserved**

$$S = g_{*s} \frac{2\pi^2}{45} a^3 T^3 = \text{const}$$

**Temperature decreases as the inverse of the cosmic scale factor**

$$T \sim a^{-1}$$

**when $g_{*s}$ is constant and slower when $g_{*s}$ decreases i.e. when some species becomes nonrelativistic**

**Decaying particles "reheat" the Universe (particles which are still in thermal equilibrium)**

# Decoupling

**Particles stay in thermodynamical equilibrium only when their interactions are strong enough**

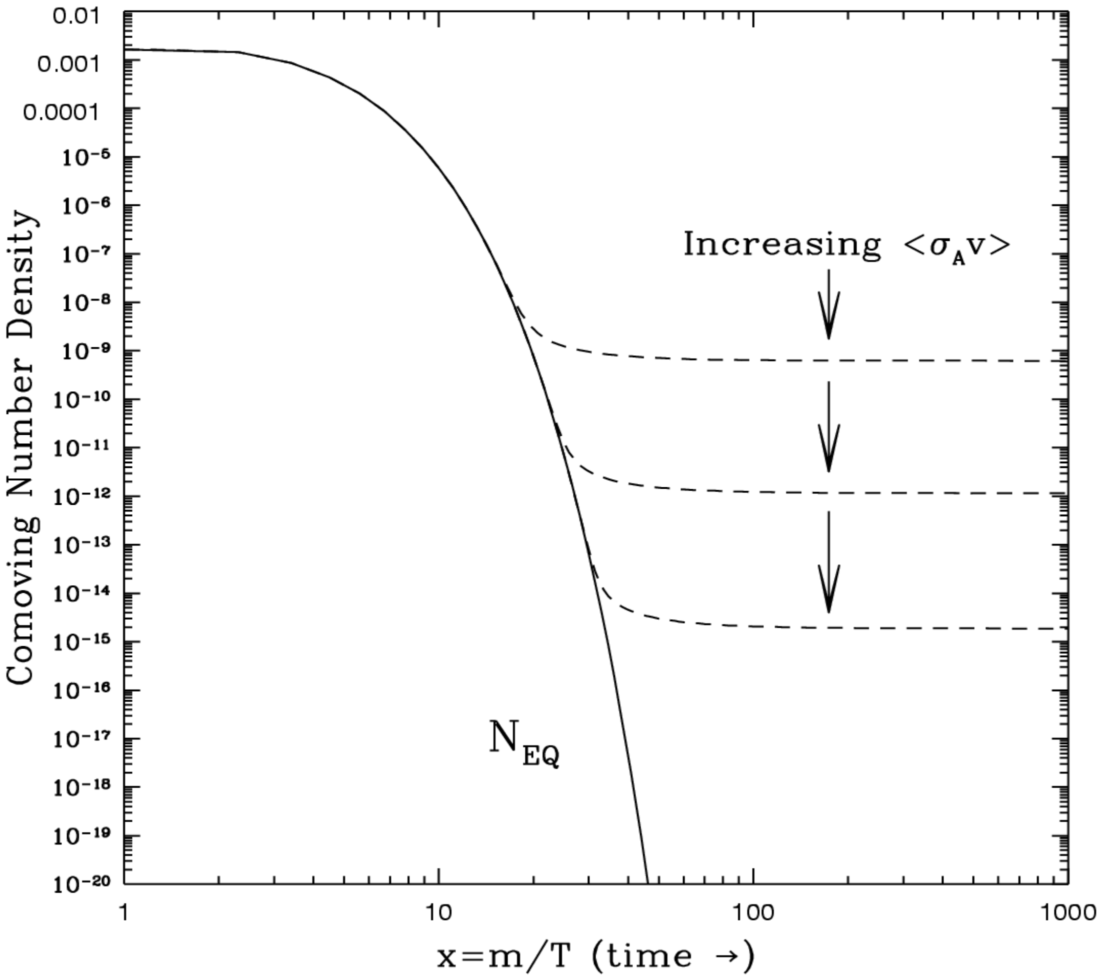
**Number of interactions per particle of given type after time $t$ (for $\Gamma \sim T^n$)**

$$N_{\rm int} \approx \int_t^{\infty} \Gamma(t')\mathrm{d}t' \sim \left(\frac{\Gamma}{H}\right)\Big|_t$$

**Particle of a given type stay in equilibrium only when $\Gamma \gg H$**

**They decouple and freeze out at time when $\Gamma \approx H$ (or decay if are unstable)**

**The details of this decoupling are described by the Boltzmann equation**




**Particles with stronger interactions:**

- **stay longer in equlibrium in which their number decreases as** exp(-m/T)
- **have smaller relic density**

**Stable, weakly interactiong particles with masses of order GeV-TeV are very good candidates for the DARK MATTER**

# Decoupling of neutrinos

**At temperature** $T \gtrsim 2\,\mathrm{MeV}$ **in equilibrium were photons, electrons and neutrinos**

**At temperature** $0.5\,\mathrm{MeV} \lesssim T \lesssim 2\,\mathrm{MeV}$ **in equilibrium were photons and electrons:**

**At temperature** $T \approx 0.5\,\mathrm{MeV}$ **electrons annihilated reheating photons by a factor** $(11/4)^{1/3}$

$$g_*(\gamma, e^{\pm}) = \frac{11}{2} \qquad g_*(\gamma) + = 2$$

**"Temperature" of particles after decoupling decreases as"**

- **"$T$"** $\sim a^{-1}$ **for hot relics**
- **"$T$"** $\sim a^{-2}$ **for cold relics**

**Today the "temperatures" are**

$$T_{\gamma 0} = 2.725 \pm 0.002\,\mathrm{K} \qquad T_{\nu 0} \approx 1.95\,\mathrm{K}$$

# Primordial nucleosynthesis

**For nonrelativistic nuclei ($A$, $Z$) in equilibrium**

$$n_A = g_A \left(\frac{m_A T}{2\pi}\right)^{3/2} \exp\left(\frac{\mu_A - m_A}{T}\right)$$

$$\mu_A = Z\mu_p + (A - Z)\mu_n$$

**For mixture of different nuclei**

$$X_A = g_A A^{5/2} 2^{(3A-5)/2} \pi^{-(A-1)/2} \zeta(3)^{A-1} \cdot$$

$$\cdot \left(\frac{T}{m_A}\right)^{3(A-1)/2} \eta^{A-1} X_p^Z X_n^{A-Z} \exp(B_A/T)$$

$$X_A = \frac{A n_A}{n_B} \qquad B_A = \text{ binding energy}$$

$$\eta = \frac{n_B}{n_\gamma} = (6.1 \pm 0.3) \cdot 10^{-10}$$

**Nuclei of type $A$ should give important contribution to the total barion mass at temperature of order**

$$\frac{B_A}{(A-1)}\left[\frac{3}{2}\ln\left(\frac{m_A}{T}\right)+\ln\left(\frac{1}{\eta}\right)\right]^{-1} \ll \frac{B_A}{(A-1)}$$

**For $^4$He this temperature is about 0.28 MeV instead of $\mathcal{O}$(10 MeV)**

$\eta \gg 1 \quad \Rightarrow$ **photons from the Maxwell tail disintegrate heavy nuclei**

**In a slowly cooling universe most of the barions should build the most strongly bounded nuclei – Fe**

**The real primordial nucleosynthesis was very much different because the Universe expands**

# Timetable of primordial nucleosynthesis

- $T = \mathcal{O}(10\,\mathrm{MeV}) \qquad t = \mathcal{O}(10^{-2}\,\mathrm{s})$ **:**
  $X_p = X_n = \frac{1}{2}$ **,** $\qquad X_2 < 10^{-11}$

- $T = \mathcal{O}(1\,\mathrm{MeV}) \qquad t = \mathcal{O}(1\,\mathrm{s})$ **:**
  **neutrons decouple with**
  $X_p \approx \frac{6}{7}$ **,** $\qquad X_n \approx \frac{1}{7}$ **,** $\qquad X_2 < 10^{-11}$

- $T \lesssim 1\,\mathrm{MeV} \qquad t \gtrsim 1\,\mathrm{s}$ **:**
  **neutrons decay with** $\tau_N \approx 10.5\,\mathrm{min}$
  $\Rightarrow X_n(0.3\,\mathrm{MeV}) \approx 0.12$
  **in equilibrium it would be only about** $\mathcal{O}(\mathbf{0.01})$ **neutrons**

- $T = \mathcal{O}(0.3\,\mathrm{MeV}) \qquad t = \mathcal{O}(1\,\mathrm{min})$ **:**
  $^4$**He should dominate but it does not because**
  - **there are not enough lighter nuclei:** $^2$**H,** $^3$**H,** $^3$**He**
  - **Coulomb barrier becomes more important**

- $T = \mathcal{O}(0.1\,\mathrm{MeV})$ $\quad t = \mathcal{O}(3\,\mathrm{min})$ **:**
  **There is enough light nuclei ($^2$H, $^3$H, $^3$He) and practically all neutrons are used to produce $^4$He**

$$\left(\frac{n_n}{n_p}\right) \approx \frac{1}{7}$$

$$X_4 = \frac{4n_4}{n_B} = \frac{4(n_n/2)}{n_n + n_p} = \frac{2(n_n/n_p)}{1 + (n_n/n_p)} \approx \frac{2/7}{8/7} = \frac{1}{4}$$

- $T \lesssim \mathcal{O}(0.1\,\mathrm{MeV})$ $\quad t \gtrsim \mathcal{O}(3\,\mathrm{min})$ **:**
  **No production of heavier nuclei ($^{12}$C, $^{16}$O, ...) because**
  - **Coulomb barrier very strong**
  - **no stable nuclei with** $A = 5,\ 8$
  - **too small density for** $3\,{}^3\mathrm{He} \to {}^{12}\mathrm{C} + \gamma$

**The Universe after primordial nucleosynthesis contained 25% of helium and 75% of hydrogen (instead of being dominated by heavy nuclei like iron)**

# How primordial nucleosynthesis depends on particle physics

- **bigger number of neutrino species $N_\nu > 3$**
  **$\Rightarrow$ bigger $g_*$**
  **$\Rightarrow$ bigger Hubble constant ($\sim \sqrt{g_*}$)**
  **$\Rightarrow$ higher neutron freeze out temperature**
  **$\Rightarrow$ more neutrons at $T = \mathcal{O}(0.1\,\mathrm{MeV})$**
  **$\Rightarrow$ more $^4$He produced**

- **shorter neutron lifetime $\tau_n$**
  **$\Rightarrow$ faster neutron decays between decoupling and $^4$He production**
  **$\Rightarrow$ (exponentially) less $^4$He produced**

- **stronger weak interactions (smaller $M_W$ or bigger coupling)**
  **$\Rightarrow$ neutrons stay longer in equilibrium**
  **$\Rightarrow$ exponentially smaller number of neutrons at $T = \mathcal{O}(0.1\,\mathrm{MeV})$**
  **$\Rightarrow$ (exponentially) less $^4$He produced**