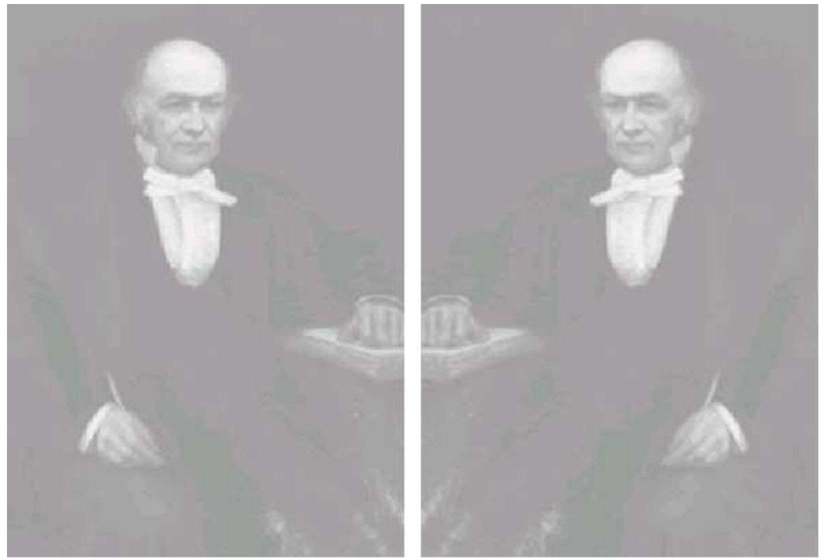

# Bihamiltonian structures from the point of view of symmetries

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# Introduction: the Gelfand–Zakharevich decomposition

**Theorem:** (Gelfand–Zakharevich, 1989) Let $V$ be a linear space over $\mathbb{C}$, $\dim V < \infty$. Then for each pair of 2-forms $(b^{(1)}, b^{(2)}),\ b^{(i)} \in \bigwedge^2 V^*$, there exists a decomposition (i.e. $V = \oplus_{j=1}^k V_j, b^{(i)} = \sum_{j=1}^k b_j^{(i)}$, $b_j^{(i)} \in \bigwedge^2 V_j^*,\ i = 1, 2$) to "irreducible blocks"$(V_j, (b_j^{(1)}, b_j^{(2)}))$ of the following types:

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1. ("Jordan block"): $\dim V_j = 2n_j$ and in some basis of $V_j$ the matrices of the pair $(b_j^{(1)}, b_j^{(2)})$ are equal

$$\begin{pmatrix} 0 & \mathbf{I}_{n_j} \\ -\mathbf{I}_{n_j} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathbf{J}_{n_j}(\lambda) \\ -(\mathbf{J}_{n_j}(\lambda))^T & 0 \end{pmatrix}$$

where $\mathbf{J}_{n_j}(\lambda)$ is the standard $n_j \times n_j$-Jordan block with the eigenvalue $\lambda$.

# Introduction: the Gelfand–Zakharevich decomposition

2. ("Kronecker block"): $\dim V_j = 2n_j + 1$ and in some basis of $V_j$ the matrices of the pair $(b_j^{(1)}, b_j^{(2)})$ are equal

$$\begin{pmatrix} 0 & K_1 \\ -K_1^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & K_2 \\ -K_2^T & 0 \end{pmatrix},$$

where

$$K_1 = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \ K_2 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

($n_j \times (n_j + 1)$-matrices).

# Introduction: bihamiltonian structures and integrable systems

**Def:** Let $M$ be a smooth manifold and $b^{(1)}, b^{(2)} \in \Gamma(\bigwedge^2 TM)$. We say that $(b^{(1)}, b^{(2)})$ is a *Poisson pair* if $b^t := t^{(1)}b^{(1)} + t^{(2)}b^{(2)}$ is a Poisson bivector field for any $t = (t^{(1)}, t^{(2)}) \in \mathbb{K}^2, \mathbb{K} = \mathbb{R}, \mathbb{C}$. The whole family

$$B := \{b^t\}_{t\in\mathbb{K}^2}$$

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**Def:** Let $B := \{b^t\}$ be a bihamiltonian structure. Put

$$E_B(x) = \{t \in \mathbb{C}^2 \mid \operatorname{rank} b_x^t < \max_{t \in \mathbb{C}^2} \operatorname{rank} b_x^t\}, x \in M$$

(this set is called *exceptional* for $B$ at $x$). It is clear that either

$$E_B(x) = \{0\} \text{ or } E_B(x) = \operatorname{Span}\{t_1\} \cup \cdots \cup \operatorname{Span}\{t_n\},$$

where $t_i$ are pairwise nonproportional. We put also

$$Z_x^B := \operatorname{Span}\left(\bigcup_{t \notin E_B(x)} \ker b_x^t\right).$$

# Introduction: bihamiltonian structures and integrable systems

**Main Lemma of the theory of bihamiltonian structures** Let $\{b^t\}$ be a bihamiltonian structure on $M$. Fix $x \in M$. Assume $E_B(x) = \mathrm{Span}\{t_1\} \cup \cdots \cup \mathrm{Span}\{t_n\}$. Then

1. for any $t \in \mathbb{C}^2$ and any linearly independent elements $t', t'' \in \mathbb{C}^2$ we have $b^t_x(\ker b^{t'}_x, \ker b^{t''}_x) = 0$;
2. for any $t \in \mathbb{C}^2$ and any $t' \in \mathbb{C}^2 \setminus E_B(x)$ we have $b^t_x(\ker b^{t'}_x, \ker b^{t'}_x) = 0$; in particular $b^t_x(Z^B_x, Z^B_x) = 0$.

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**Corollary:** Let $I_x^i \subset \ker b_x^{t_i}, i = 1, \dots, n$, be an isotropic subspace with respect to the restriction of $b_x^t, t \notin E_B(x)$, to $\ker b_x^{t_i}$. Then

$$Z_x^B + I_x^1 + \cdots + I_x^n$$

is also isotropic.

# Introduction: bihamiltonian structures and integrable systems

## Classical ways of constructing functions in involution

**First way** (Uses only $I^1 + \cdots + I^n$.) Corresponds to the situation when only Jordan blocks are present in the G–Z decomposition and, moreover, they are of dimension 2 ($n_j = 1, \mathbf{J}_{n_j}(\lambda)$ semisimple). Classically it exploits the eigenfunctions of the so-called recursion operator $b^{(1)} \circ (b^{(2)})^{-1}$.

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**Second way** (Uses only $Z^B$.) Corresponds to the situation when only Kronecker blocks are present in the G–Z decomposition. Exploits the Casimir functions of Poisson bivectors of the pencil.

**Third way (less classical one)** (Uses the whole space $Z^B + I^1 + \cdots + I^n$.) Was developed by Bolsinov in the context of Lie pencils, i.e. pencils of Lie algebras on a vector space. Bolsinov (1995) gives conditions on a Lie pencil sufficient for the maximality of the isotropic subspace $Z^B + I^1 + \cdots + I^n$. However, these conditions are not necessary and imply that only semisimple-type Jordan blocks appear in the G–Z decomposition.

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Let a pencil $B$ of bivectors on a vector space $V$ be given and $E_B = \mathrm{Span}\{t_1\} \cup \ldots \cup \mathrm{Span}\{t_n\}$, $t_i$ being pairwise nonproportional. Assume that for any $i \in \{1, \ldots, n\}$ a subspace $Z^i \subset \ker b^{t_i}$ is chosen.

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Fix an element $t_0 \in \mathbb{C}^2 \setminus E_B$ and introduce the subspaces

$$Z^{0,i} := \{z \in Z^i \mid \exists w \in V^* \colon b^{t_0}(z) = b^{t_i}(w)\}, i = 1, \ldots, n,$$

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and the 2-forms

$$\gamma^i : \bigwedge^2 Z^{0,i} \to \mathbb{C}, \;\; \gamma^i(z_1, z_2) := b^{t_i}(w_1, w_2),$$

where $w_j \in V^*$ are any elements such that $b^{t_0}(z_j) = b^{t_i}(w_j)$, $j = 1, 2$. Note that these forms are correctly defined. Indeed, if $w_j'$ are another elements with $b^{t_0}(z_j) = b^{t_i}(w_j')$, we have$v_j := w_j - w_j' \in Z^{t_i}$ and $b^{t_i}(w_1', w_2') = b^{t_i}(w_1 + v_1, w_2 + v_2) = b^{t_i}(w_1, w_2)$.

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**Theorem:** Let a pencil $B$ of bivectors on $V$ be given such that $E_B = \mathrm{Span}\{t_1\} \cup \ldots \cup \mathrm{Span}\{t_n\}$, $t_i$ being pairwise nonproportional. Let $Z^i \subset \ker b^{t_i}$ be any subspace, $\mathcal{J}^i : Z^i \hookrightarrow V^*$ the natural embedding and $J^i : V \to (Z^i)^*$ the dual map. Fix $t_0 \in \mathbb{C}^2 \setminus E_B$, and assume that $I^i \subset Z^i, i = 1, \ldots, n$, is a maximal isotropic subspace with respect to $((Z^i)^*, J^i_* b^{t_0})$. Then the subspace

$$Z^B + I^1 + \cdots + I^n \subset V^*$$

is maximal isotropic with respect to $(V, b^{t_0})$ if and only if the following condition holds for any $i \in \{1, \ldots, n\}$:

$$\begin{aligned} \mathrm{corank}\, J^i_* b^{t_0} + \mathrm{corank}\, b^{t_0} &= \\ 2 \dim Z^{0,i} &- \dim Z^i - \mathrm{rank}\, \gamma^i + \mathrm{corank}\, b^{t_i}. \end{aligned}$$

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**Remark:** If $Z^i = \ker b^{t_i}$ the above conditions reduce to

$$\mathrm{corank}\, J^i_* b^{t_0} + \mathrm{corank}\, b^{t_0} = 2 \dim Z^{0,i} - \mathrm{rank}\, \gamma^i$$

and are equivalent the absence of the Jordan blocks of dimension $> 4$ in the G–Z decomposition.

# Illustration: a single Jordan block

$$V := \mathbb{C}^{2m}, [b^{(1)}] := \begin{bmatrix} 0 & \mathbf{I}_m \\ -\mathbf{I}_m & 0 \end{bmatrix}, [b^{(2)}] := \begin{bmatrix} 0 & \mathbf{J}_m(\lambda) \\ -(\mathbf{J}_m(\lambda))^T & 0 \end{bmatrix},$$

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$$E_B = \mathrm{Span}\{t_1\}, t_1 = (-\lambda, 1), [b^{t_1}] = \begin{bmatrix} 0 & \mathbf{N}_m \\ -\mathbf{N}_m^T & 0 \end{bmatrix},$$

$$[b^{t_0}] = \begin{bmatrix} 0 & r\mathbf{I}_m + s\mathbf{N}_m \\ -(r\mathbf{I}_m + s\mathbf{N}_m)^T & 0 \end{bmatrix},$$

$$\mathbf{N}_m := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, r \neq 0$$

# Illustration: a single Jordan block

Put $Z^1 := Z^{t_1} = \ker b^{t_1} = \{[0,\dots,x_1,x_2,\dots,0]^T\}$. We have

$$\begin{aligned} [b^{t_0}][0,\dots,0,x_1,x_2,0,\dots,0]^T &= [rx_2,\dots,0,0,\dots,-rx_1]^T \\ [b^{t_1}][0,\dots,rx_1,0,0,rx_2,\dots,0]^T &= [rx_2,\dots,0,0,\dots,-rx_1]^T \end{aligned}$$

$$\Longrightarrow Z^{0,1} = \begin{cases} 0, & m = 1 \\ Z^1, & m > 1 \end{cases}$$

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For the 2-form $\gamma^1 : \bigwedge^2 Z^{0,1} \to \mathbb{C}$ we have $\gamma^1([x_1,x_2],[y_1,y_2]) =$

$$\begin{aligned}[0,\ldots,rx_1,0,0,rx_2,\ldots,0][b^{t_1}][0,\ldots,ry_1,0,0,ry_2,\ldots,0]^T &=\\ [0,\ldots,rx_1,0,0,rx_2,\ldots,0][ry_2,\ldots,0,0,\ldots,-ry_1]^T &=\end{aligned}$$

$$= \begin{cases} r^2(x_1y_2-x_2y_1), & m=2\\ 0, & m>2\end{cases}$$

# Corollaries

**Corollary:** In the hypotheses of the above theorem, the subspace

$$Z^B + I^1 + \cdots + I^n \subset V^*$$

is maximal isotropic with respect to $(V, b^{t_0})$ if one of the following condition holds:

1. $\operatorname{corank} b^{t_0} = \dim Z^{0,i} - \dim Z^i - \operatorname{rank} \gamma^i + \operatorname{corank} b^{t_i}, i \in \{1, \ldots, n\}$;
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**Def:** Let $B = \{b^t\}$ be a bihamiltonian structure on $M$. If there exists an open dense set $U \subset M$ such that the exceptional set $E_B(x) =: E_B$ is *independent of* $x \in U$ we call $B$ *admissible*.

# Corollaries

Let $B = \{b^t\}$ be admissible, $E_B = \mathrm{Span}\{t_1\} \cup \ldots \cup \mathrm{Span}\{t_n\}$. Put

$$\mathcal{Z}^t := \{\text{Casimir functions of } b^t\},\ \mathcal{Z}^B := \mathrm{Span}\Big(\bigcup_{t \notin E_B} \mathcal{Z}^t\Big).$$

**Lemma:** Fix $t_0 \in \mathbb{C}^2 \setminus E_B$. Then $\mathcal{Z}^{t_i}$ is a Lie algebra with respect to the Poisson bracket $\{,\}^{t_0}$ related to $b^{t_0}$.

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**Def:** Let $\mathfrak{z}^i \subset \mathcal{Z}^{t_i}, i = 1, \ldots, n$, be a finite-dimensional subalgebra. Define the *action* $\rho^i : \mathfrak{z}^i \to \mathcal{V}ect(M)$ of $\mathfrak{z}^i$ on $M$ by

$$\rho^i(z) := b^{t_0}(\mathcal{J}^i(z)),$$

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**Remark:** The action $\rho^i$ is hamiltonian with respect to any Poisson structure $b^t, t \notin E_B$. The functions from $\mathfrak{z}^i$ can be interpreted as the Noether integrals related to this symmetry. But $\rho^i$ in general is not hamiltonian with respect to the exceptional Poisson structures $b^{t_i}, i = 1, \ldots, n$. The two-forms $\gamma^i$ are related to the nonequivariance cocycles.

# Corollaries

**Theorem:** Assume that $\mathcal{I}^i \subset \mathcal{F}un((\mathfrak{z}^i)^*), i = 1, \ldots, n$, is a maximal involutive set of functions with respect to $((\mathfrak{z}^i)^*, J^i_* b^{t_0})$, where $J^i : M \to (\mathfrak{z}^i)^*$ is the momentum map of the action $\rho^i$. Then the set of functions

$$\mathcal{Z}^B + (J^1)^* \mathcal{I}^1 + \cdots + (J^n)^* \mathcal{I}^n \subset \mathcal{F}un(M)$$

is maximal involutive with respect to the Poisson bracket $\{,\}^{t_0}$ if and only if there exists a point $x \in M$ such that for any $i \in \{1, \ldots, n\}$:

$$\begin{aligned} \operatorname{corank} [J^i_* b^{t_0}]_{J^i(x)} + \operatorname{corank} b^{t_0}_x &= \\ 2 \dim \mathfrak{z}^{0,i}_x &- \dim \mathfrak{z}^i_x - \operatorname{rank} \gamma^i_x + \operatorname{corank} b^{t_i}_x, \end{aligned}$$

where

$$\mathfrak{z}^i_x := \{d_x f \mid f \in \mathfrak{z}^i\},\ \mathfrak{z}^{0,i}_x := \{z \in \mathfrak{z}^i_x \mid \exists y \in T^*_x M \colon b^{t_0}_x z = b^{t_i}_x y\},$$

and $\gamma^i_x$ is a 2-form on $\mathfrak{z}^{0,i}_x$ defined similarly to the 2-form $\gamma^i$ on $Z^{0,i}$.

# Lie pencils

**Def:** Let $[,]^{(1)}, [,]^{(2)} : \bigwedge^2 \mathfrak{g} \to \mathbb{C}$, where $\mathfrak{g}$ is a vector space. Assume that for any $t = (t_1, t_2) \in \mathbb{C}^2$ the bracket $[,]^t := t_1[,]^{(1)} + t_2[,]^{(2)}$ is a Lie bracket on $\mathfrak{g}$. Then we say that the family of Lie algebras $\{\mathfrak{g}^t\}_{t\in\mathbb{C}^2}, \mathfrak{g}^t := (\mathfrak{g}, [,]^t)$ is a *Lie pencil*.

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**Remark:** Given a Lie pencil $\Lambda = \{\mathfrak{g}^t\}$, we obtain a Poisson pencil $B_\Lambda$ on $\mathfrak{g}^*$ consisting of the corresponding Lie-Poison structures. If $\Lambda$ is such that $H^2(\mathfrak{g}^t, \mathfrak{g}^t) = 0$ (eg. $\mathfrak{g}^t$ semisimple) for some $t$, then $B_\Lambda$ is admissible. Moreover, there exist natural finite-dimensional subalgebras $\mathfrak{z}^i \subset \mathcal{Z}^{t_i}, i = 1, \dots, n$, equal to the centres of the "exceptional" Lie algebras $\mathfrak{g}^{t_i}$.

# Examples

**Example 1** Let $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$, where $\lambda_1 = \ldots = \lambda_k = \alpha$, $\lambda_{k+1} = \ldots = \lambda_m = \beta$, a $[x, y]_D = xDy - yDx$. Consider the Lie pencil $\mathfrak{g}^t := (\mathfrak{so}(m), [,]^t), [,]^t := t^{(1)}[,] + t^{(2)}[,]_D$. The exceptional algebras are $\mathfrak{g}^{t_1}, \mathfrak{g}^{t_2}$, where $t_1 = (-\alpha, 1), t_2 = (-\beta, 1)$. The Lie algebra $\mathfrak{g}^{t_1}$ is the semidirect product of $\mathfrak{so}(k)$ and the solvable ideal "$\mathfrak{so}(m-k) \times \mathfrak{m}$", similarly $\mathfrak{g}^{t_2}$. We have $\mathfrak{z}^{t_1} = \mathfrak{so}(k)$, $\mathfrak{z}^{t_2} = \mathfrak{so}(m-k)$.

| | |
|---|---|
| $\mathfrak{so}(k)$ | $\mathfrak{m}$ |
| $\mathfrak{m}$ | $\mathfrak{so}(m-k)$ |

For this pencil the Bolsinov conditions are satisfied. The corresponding hamiltonian system is the "n-dimensional free rigid body system" on $\mathfrak{so}(n) \cong \mathfrak{so}(n)^*$. The $\mathcal{I}^i$-families of functions in involution correspond to the Noetherian integrals of the system coming from the symmetries of the body whose "inertia matrix" $D$ has the spectrum with multiplicities.

# Examples

**Example 2** Let $\mathfrak{g} = \mathfrak{gl}(m, \mathbb{C})$ and let

$$N := \mathbf{N}_m := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

be the standard nilpotent matrix. Then for any $k = 1, \ldots, m-1$ we have the Lie pencil

$$(\mathfrak{g}, [,]^t), [,]^t := t^{(1)}[,] + t^{(2)}[,]_{N^k}, [X,Y]_{N^k} := XN^kY - YN^kX.$$

The only exceptional algebra is $\mathfrak{g}^{t_1} = (\mathfrak{g}, [,]_{N^k})$. For $1 < k \leq n/2$ this pencil does not satisfy the Bolsinov condition but satisfies our more general condition (the corresponding 2-form $\gamma^1$ is nontrivial).

# Further applications

*The argument translation method*

Let $\mathfrak{g}$ be a Lie algebra. Put: $b^{(1)} := b_{\mathfrak{g}}$, the Lie-Poisson structure on $\mathfrak{g}^*$; $b^{(2)} := b_{\mathfrak{g}}(a)$, where $a \in \mathfrak{g}^*$. Then $\left(b^{(1)}, b^{(2)}\right)$ is a Poisson pair and the Poisson pencil $B = \left\{b^t := t^{(1)}b^{(1)} + t^{(2)}b^{(2)}\right\}$ is admissible.

$\mathcal{Z}^B$ is generated by $f_1(x + \lambda a), \ldots, f_m(x + \lambda a), \lambda \in \mathbb{K}$, where $f_1, \ldots, f_m$ are invariants of the coadjoint action. The families of functions $\mathcal{I}_i$ come into consideration when $a$ is a singular element.

*Nonadmissible Poisson pencils...*

# Many thanks for your attention!