Explicit solution for the Zhukovski–Volterra gyrostat

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This is an analytical study of the simple classical generalization of the Euler top problem: *the Zhukovski–Volterra (ZV) system* describing the free motion of a gyrostat (a rigid body carrying a rotator inside), which was first investigated by N. Zhukovski [2] and, independently, by V. Volterra [1] (1899).

In contrast to the Euler top, the equations of motion of ZV are not homogeneous, which makes their integration technically more complicated.

We revise the solutions for the angular momentum first obtained by Volterra and present alternative solutions based on an algebraic parametrization of the invariant curves (Proposition 3).
This also enables us to derive an effective description of the motion of the body in space. The proposed construction is completely explicit and requires resolving three quartic algebraic equations.

Equations of motion. The evolution of the angular velocity ω of the gyrostat is described by the equations

$$\frac{d}{dt}(J\omega) = (J\omega + d) \times \omega,$$

where J is the tensor of inertia and d is a constant vector characterizing the motion of the rotator. Let $M = J\omega + d \in \mathbb{R}^3$ be the vector of the angular momentum. By setting

$$\omega = aM - g, \quad g = (g_1, g_2, g_3)^T, \quad a = J^{-1} = \operatorname{diag}(a_1, a_2, a_3).$$

one can rewrite the above system in the form

$$\boxed{\dot{M} = M \times (aM - g)} \tag{1}$$

It possesses 2 first integrals

$$f_1(M) = M_1^2 + M_2^2 + M_3^2 = k^2, \tag{2}$$

$$f_2(M) = a_1M_1^2 + a_2M_2^2 + a_3M_3^2 - 2M_1g_1 - 2M_2g_2 - 2M_3g_3 = l$$

$$k,\ l = \text{const}.$$

The complex invariant manifold $S = \{f_1(M) = k^2, f_2(M) = l\}$ $\in \mathbb{C}^3$ is isomorphic to the plane elliptic curve

$$\mathcal{E} = \left\{w^2 = P_4(z)\right\},\ P_4 = \begin{vmatrix} z - a_1 & 0 & 0 & g_1 \\ 0 & z - a_2 & 0 & g_2 \\ 0 & 0 & z - a_3 & g_3 \\ g_1 & g_2 & g_3 & l - k^2 z \end{vmatrix} \tag{3}$$

$$= -k^2(z - \lambda_1)(z - \lambda_2)(z - \lambda_3)(z - \lambda_4),$$

the roots $\lambda_1, \ldots, \lambda_4$ being dependent on the parameters a_i, g_i and the constants of motion k, l.

This indicates that the generic solution of (1) is given in terms of elliptic functions of $\mathcal{E}$.

- **Volterra's solution of the ZV system (1891)**

Theorem 1. *The explicit complex solution of the ZV equations (2) with the constants of motion k, l has the form*

$$\boxed{M_i(t) = \frac{\sum_{\alpha=1}^{3} A_{i\alpha}\mu_\alpha\sigma_\alpha(u) + A_{i4}\mu_4\sigma(u)}{\sum_{\alpha=1}^{3} A_{4\alpha}\mu_\alpha\sigma_\alpha(u) + A_{44}\mu_4\sigma(u)},} \tag{4}$$

$$i = 1, 2, 3, \qquad u = \delta t + u_0,$$

where $\sigma_\alpha(u), \sigma(u)$ are the Weierstrass sigma-functions with quasi-periods $2\omega_1, 2\omega_3$ associated with the curve $\mathcal{E}$, and $A_{i\alpha}, \mu_\alpha$ are constants depending on the initial conditions:

$$A_{i\alpha} = \frac{g_i}{(a_i - \lambda_\alpha)\sqrt{\Delta_\alpha}}, \quad A_{4\alpha} = \frac{\sqrt{-k}}{\sqrt{\Delta_\alpha}}, \quad \alpha = 1, 2, 3, 4, \tag{5}$$

$$\Delta_\alpha = \sum_{i=1}^{3} \frac{g_i^2}{(a_i - \lambda_\alpha)^2} - k. \tag{6}$$

Remark. It follows that M_i are elliptic functions of u with the minimal periods $4\omega_1, 4\omega_3$ and that they have the same poles $q_1, \ldots, q_4$ in the corresponding parallelogram of periods.

The solution (4) does not provide the information about position of poles q_i and zeros $p_i^{(j)}$ of $M_i(t)$ in the parallelogram of periods.

Objectives:

- *To express the momenta M_i in terms of the coordinates z, w on $\mathcal{E}$;*

- *To solve the Poisson equations, describing the motion of the top in space $\dot{\gamma} = \gamma \times \omega \equiv \gamma \times a\bar{M}(t)$, where γ be a unit vector fixed in space.*

Relation between generic solutions of the ZV system and of the Euler top.

The classical Euler equations, describing the free rotation of a rigid body with the inertia tensor J are

$$\dot{\bar{M}} = \bar{M} \times a\bar{M}\,, \qquad a = J^{-1} = \mathrm{diag}(a_1, a_2, a_3), \tag{7}$$

where $\bar{M} = (\bar{M}_1, \bar{M}_2, \bar{M}_3)^T = J\omega \in \mathbb{R}^3$ be the vector of the angular momentum. They have two independent integrals

$$\langle \bar{M}, a\bar{M} \rangle = l\,, \qquad \langle \bar{M}, \bar{M} \rangle = k^2\,, \qquad l, k = \mathrm{const}\,.$$

Proposition 2. *Let $M(t)$ be a solution of the ZV system with constants of motion k, l and the corresponding roots $\lambda_1, \dots, \lambda_4$ defined in (3), and let $\bar{M}(t)$ be the solution of the Euler equations (7) with the parameters $a_\alpha = \lambda_\alpha$ and $k = 1, l = \lambda_4$. Then these solutions are related by the projective transformations*

$$M_i = g_i \frac{\sum_{\alpha=1}^{3} \dfrac{\bar{M}_\alpha}{(a_i - \lambda_\alpha)\sqrt{\Delta_\alpha}} + \dfrac{1}{(a_i - \lambda_4)\sqrt{-\Delta_4}}}{\sum_{\alpha=1}^{3} \bar{M}_\alpha / \sqrt{\Delta_\alpha} + 1/\sqrt{-\Delta_4}}\,, \tag{8}$$

where Δ_i are defined in (6).

Alternative parametrization of the ZV solution

Proposition 3. *1). The components of momenta M_i has the following natural parametrization in terms of the coordinates z, w on the elliptic curve $\mathcal{E} = \{w^2 = P_4(z)\}$:*

$$M_i = \frac{\alpha_i w + U_i\,(z)}{w + U_0(z)}, \quad U_i = u_{i2}z^2 + u_{i1}z + u_{i0}, \tag{9}$$

where α_i, u_{ij} are certain constants depending only on the values of the integrals (2);

2). The evolution of z is described by the quadrature

$$\frac{dz}{\sqrt{-k(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)(z-\lambda_4)}} = dt. \tag{10}$$

The right hand sides of (9) have precisely 4 simple zeros and poles on the curve $\mathcal{E}$, as required by the structure of the Volterra solution (4).

Proposition 4. *The momenta $\bar{M}_i$ of the Euler top system (7) admit the following rational parameterizations in terms of the coordinates z, w on the degree 4 curve $\mathcal{E}$*

$$\bar{M}_i = \frac{k}{2w\sqrt{-(a_i-a_j)(a_i-a_k)}}\big[(a_j+a_k-a_i-c)z^2 + 2z(ca_i-a_ja_k)+c(a_2a_3-a_1a_3-a_1a_2)+a_1a_2a_3\big]. \qquad (11)$$

$$(i,j,k) = \text{a cyclic permutation of } (1,2,3),$$

$$w=\sqrt{-k(z-a_1)(z-a_2)(z-a_3)(z-c)}, \qquad c=l/k^2.$$

- **Real part of the above parametrization:**

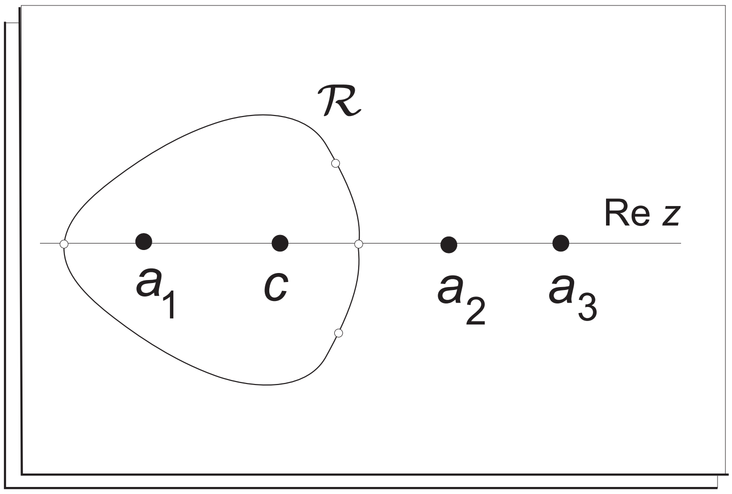

Figure 1: The curve $\mathcal{E}$ as a 2-fold ramified covering of $\mathbb{C} = \{z\}$ and the ovals $\mathcal{R}$, which correspond to 2 real trajectories $\bar{M}(t) \in \mathbb{R}^3$ for the case $a_1 < c < a_2 < a_3$.
The white dots on the ovals stand for pairs of real zeros of $\bar{M}_2(z)$ and $\bar{M}_3(z)$. The component $M_1(z)$ does not have real zeros.

Using this parametrization and Proposition 2 one obtain explicit expressions for the coefficients u_{i2}, u_{i1}, u_{i0} in the parametrization (9) for the momenta M_i of the ZV system.

Determination of the motion of the gyrostat in space

Let us choose a *fixed* in space orthonormal frame $O\,\mathbf{e}_1\mathbf{e}_2, \mathbf{e}_3$ such that the third axis is directed along the constant momentum vector M of the gyrostat, and θ, ψ, ϕ be the Euler angles of nutation, precession, and rotation with respect to this frame.

Then, according to the definition of the angles,

$$M_1 = -|M| \sin\theta \sin\phi, \quad M_2 = |M| \sin\theta \cos\phi, \quad M_3 = |M| \cos\theta.$$

These expressions allow to determine trigonometric functions of θ and ψ in terms of M_i and, in view of the solution (4), as functions of time t.

Next, as follows from the Euler kinematical equations

$$\dot{\psi} = \frac{-\omega_1 \cos\phi + \omega_2 \sin\phi}{\sin\theta} \equiv -k \frac{\omega_1 M_1 + \omega_2 M_2}{M_1^2 + M_2^2}\,.$$

Setting here $\omega_i = a_i M_i - g_i$, $i = 1, 2$ and fixing the value l of the energy integral in (2), we obtain

$$\dot{\psi} = -k \frac{l - a_3 k^2 + g_1 M_1 + g_2 M_2 + 2 g_3 M_3}{k^2 - M_3^2} - k a_3.$$

This form suggests introducing new angle $\tilde{\psi} = \psi + k a_3 t$. In view of the relation (10) between dt and dz, we then get

$$d\tilde{\psi} = -\frac{l - a_3 k^2 + g_1 M_1 + g_2 M_2 + 2 g_3 M_3}{(k - M_3)(k + M_3)} \frac{dz}{\sqrt{P_4(z)}}. \tag{12}$$

Now, in view of the parametrization (9) for M_i in terms of z, w, we see that (12) *is a meromorphic differential of the third kind* on the elliptic curve $\mathcal{E}$ with 4 pairs of simple poles $\mathcal{D}_1^{\pm}, \dots, \mathcal{D}_4^{\pm}$ given by equations $M_3 = \pm k$. That is, their z-coordinates are the solutions of

$$(\alpha_3 \mp k)\sqrt{P_4(z)} = U_3(z) \mp kU_0(z), \tag{13}$$

which is equivalent to a quartic equation.

Proposition 5. *The residues of $d\tilde{\psi}$ at $\mathcal{D}_i^{\pm}$ equal $\pm\sqrt{-1}$.*

- The final step: Consider the Abel map

$$\mathcal{A} : \mathcal{E} \mapsto \mathbb{C}, \quad \mathcal{A}(P) = \int_{P_0}^{P} \frac{dz}{\sqrt{P_4(z)}}, \quad P_0 = (\lambda_4, 0)$$

and set $u = \mathcal{A}(z, w)$, $d_i^{\pm} = \mathcal{A}(\mathcal{D}_i^{\pm})$.

Integrating the meromorphic differential (12), we express the new angle $\tilde{\psi}$ as the following function of u and t:

$$\tilde{\psi} = \sqrt{-1} \log \frac{\sigma(u - d_1^+) \cdots \sigma(u - d_4^+)}{\sigma(u - d_1^-) \cdots \sigma(u - d_4^-)} + Vu + C,$$

where V, C are certain constants and $u = kt + u_0$.

As a result, for the original precession angle ψ we obtain

$$e^{\sqrt{-1}\psi} = \frac{\sigma(u - d_1^-) \cdots \sigma(u - d_4^-)}{\sigma(u - d_1^+) \cdots \sigma(u - d_4^+)} \, e^{(V - ka_3)u + C}. \tag{14}$$

This allows to express $\cos\psi$, $\sin\psi$ *as meromorphic functions of the complex variable u.*

Jointly with the expressions for $\cos\phi$, $\sin\phi$, $\cos\theta$, $\sin\theta$, they give the components of the unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ in the body as functions of time t. Thus we obtain a complete analytic description of the motion of the gyrostat in space.

References

[1] Volterra V. Sur la théorie des variations des latitudes. *Acta Math.* **22** (1899), 201–357

[2] Zhykovsky N. E. On the motion of a rigid body with cavities filled with a homogeneous fluid. Collected works, 1, Moscow-Leningrad, Gostekhisdat, 1949 (Russian)