Bi-hamiltonian property and related class of separation relations

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Introduction

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There are some milestones of that theory in the works of Stäckel, Levi-Civita, Eisenhart, Woodhouse, Klanins, Miller, Benenti, Rauch-Wojciechowski and others.
The first constructive theory of separated coordinates was made by Sklyanin. He adapted the method of Lax representation and r-matrix theory to derive separated coordinates. In that approach involutive functions appear as coefficients of characteristic equation (*spectral curve*) of Lax matrix.

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Content:

1. Separation relations
2. Classification
3. Bi-hamiltonian extension
3. Generalized Stäckel transform
4. Multi-times reciprocal transformations

Separation relations

Let us consider Liouville integrable system on $2n$ dimensional phase space M, given in canonical representation:
$M \ni u = (q_1, ..., q_n, p_1, ..., p_n)^T$ and n functions $H_i(q,p)$ in involution with respect to the canonical Poisson tensor π

$$\{H_i, H_j\}_\pi = \pi(dH_i, dH_j) = < dH_i, \pi\, dH_j > = 0, \quad i, j = 1, ..., n,$$

where $< \cdot, \cdot >$ is the duality map between TM and T^*M.

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where $< \cdot, \cdot >$ is the duality map between TM and T^*M.
Functions H_i generate n Hamiltonian dynamic systems

$$u_{t_i} = \pi\, dH_i = X_{H_i}, \quad i = 1, ..., n,$$

where X_{H_i} are called the *Hamiltonian vector fields*.

Separation relations

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$W(q,a)$ is a complete integral of related *Hamilton-Jacobi equations* (HJ)

$$H_i\left(q_1,...,q_n,\frac{\partial W}{\partial q_1},...,\frac{\partial W}{\partial q_n}\right) = a_i, \qquad i = 1,...,n.$$

Separation relations

In (b,a) representation the t_i-dynamic is trivial

$$(a_j)_{t_i} = 0, \quad (b_j)_{t_i} = \delta_{ij}$$

hence,

$$b_j(q,a) = \frac{\partial W}{\partial a_j} = t_j + const_j, \qquad j = 1, ..., n.$$

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Where are the difficulties?

(λ, μ): distinguished canonical coordinates in which there exist n relations:

Separation relations

$$\varphi_i(\lambda_i, \mu_i; a_1, ..., a_n) = 0, \qquad i = 1, ..., n, \;\; a_i \in \mathbb{R}, \;\; \det\left[\frac{\partial \varphi_i}{\partial a_j}\right] \neq 0,$$

each containing one pair of canonical coordinates.

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each containing one pair of canonical coordinates.
If the functions $W_i(\lambda_i, a)$ are solutions of a system of n decuple ODE's

$$\varphi_i\left(\lambda_i, \mu_i = \frac{dW_i(\lambda_i, a)}{d\lambda_i}, a_1, \ldots, a_n\right) = 0, \ \ i = 1, ..., n,$$

then the function $W(\lambda, a) = \sum_{i=1}^{n} W_i(\lambda_i, a)$ is an additively separable solution of the above system and *simultaneously* it is a solution of all Hamilton-Jacobi equations.

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Consider a generic case of separation relations linear in H_i:

$$\sum_{k=1}^{n} S_i^k(\lambda_i, \mu_i) H_k = \psi_i(\lambda_i, \mu_i), \qquad i = 1, ..., n,$$

called generalized Stäckel separation relations, while $S = (S_i^k)$ is generalized Stäckel matrix.

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called generalized Stäckel separation relations, while $S = (S_i^k)$ is generalized Stäckel matrix.
If additionally $S_i^k = S^k$ and $\psi_i = \psi$ then separation conditions can be represented by n copies of some curve:

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Degenerations: assume that in (λ,μ) coordinates H_1 separates, i.e. $H_1 = H_{1,1} + H_{2,1}$, then we get separation relations for two sub-systems

$$\sum_{k=1}^{n_1} f^{1,k}(\lambda,\mu)H_{1,k} = \psi_1(\lambda,\mu)$$

$$\sum_{k=1}^{n_2} f^{2,k}(\lambda,\mu)H_{2,k} = \psi_2(\lambda,\mu), \quad n_1 + n_2 = n.$$

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In order to get the bi-Hamiltonian extension we further restrict to:

Separation relations

$$\sum_{k=1}^{n} \mu^{\alpha'_k} \lambda^{\beta'_k} H_k = \psi(\lambda, \mu), \qquad \alpha'_k, \beta'_k \in \mathbb{N},$$

with the following normalization:
$\alpha'_1 \geq \alpha'_2 \geq ... \geq \alpha'_n = 0, \beta'_n = 0.$

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$\alpha'_1 \geq \alpha'_2 \geq ... \geq \alpha'_n = 0, \beta'_n = 0.$
For further purpose let us collect terms from the l.h.s. in the following form

$$\sum_{k=1}^{m} \mu^{\alpha_k} \lambda^{\beta_k} H^{(k)}(\lambda) = \psi(\lambda, \mu), \qquad m \leq n, \qquad \alpha_k, \beta_k \in \mathbb{N},$$

where

$$H^{(k)}(\lambda) = \sum_{i=1}^{n_k} \lambda^{n_k - i} H_i^{(k)}, \qquad n_1 + ... + n_m = n$$

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For separation curves under consideration S is uniquelly determined by fixed sequences:

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Example 1. $m = 1 : (0), (0), (n)$ and functions ψ being quadratic in momenta (Benenti class):

$$\sum_{i=1}^{n} \lambda^{n-i} H_i = \frac{1}{2} f(\lambda)\mu^2 + \gamma(\lambda)$$

classification

Example 2. $m = 1 : (0), (0), (n)$ and functions ψ being exponential in momenta

$$\sum_{i=1}^{n} \lambda^{n-i} H_i = f_1(\lambda) \exp(a\mu) + f_2(\lambda) \exp(-b\mu) + \gamma(\lambda).$$

(periodic Toda lattice, KdV dressing chain, Rujsenaar-Schneider system,...).

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(periodic Toda lattice, KdV dressing chain, Rujsenaar-Schneider system,...).
Example 3. $m = 2 : (1, 0), (0, 0), (n_1, n_2)$ and functions ψ being qubic in momenta

$$\mu(\sum_{i=1}^{n_1} \lambda^{n-i} H_i^{(1)}) + \sum_{i=1}^{n_2} \lambda^{n-i} H_i^{(2)} = \mu^3 + \gamma_1(\lambda)\mu + \gamma_2(\lambda)$$

stationary Businesq hierarchy - $(n_1 = n - 2, n_2 = 2)$,
dynamic on loop algebra $\mathfrak{sl}(3)$ - $(n_1 = 2, n_2 = 4)$.

Example 4. Systems from classes with $1 < m \leq n$, $\alpha_i = 0$ and ψ quadratic in momenta

$$\sum_{i=1}^{n} \lambda^{\beta_i} H_i = \frac{1}{2} f(\lambda)\mu^2 + \gamma(\lambda).$$

Bi-hamiltonian representation

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Stäckel Hamiltonians fulfil the following quasi-bi-Hamiltonian representation (Tondo,Falqui,Pedroni):

$$\pi_1 dH_i = \sum_{j=1}^{n} F_{ij}\, \pi_0\, dH_j, \qquad i = 1, ..., n,$$

where

$$\pi_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad \pi_1 = \begin{pmatrix} 0 & \Lambda_n \\ -\Lambda_n & 0 \end{pmatrix}, \quad \Lambda_n = diag(\lambda_1, ..., \lambda_n),$$

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$$F_{ij} = (S^{-1}\Lambda_n S)_{ij},$$

where $S_k^l = \mu_k^{\alpha_l'} \lambda_k^{\beta_l'}$.

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Observation.

$$F_{ij} = (J^{-1}\Lambda_n J)_{ij} = \frac{W_{ij}}{W},$$

where $W = \det S,\ W_{ki} = \det U_{ki}$ and U_{ki} is matrix S with the k-th column replaced by that $(\mu_1^{\alpha_i'}\lambda_1^{\beta_i'+1}, ...,$ $\mu_n^{\alpha_i'}\lambda_n^{\beta_i'+1})^T$.

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Now, the important question is: which $F_{ij} \neq 0$. In other words, for which i, j determinant W_{ij} has no repeating columns.

Bi-hamiltonian representation

First, for the following separation curve

$$\sum_{k=1}^{m} \mu^{\alpha_k} \lambda^{\beta_k} H^{(k)}(\lambda) = \psi(\lambda,\mu), \qquad m \le n, \qquad \alpha_k, \beta_k \in \mathbb{N},$$

where

$$H^{(k)}(\lambda) = \sum_{i=1}^{n_k} \lambda^{n_k - i} H_i^{(k)}, \qquad n_1 + ... + n_m = n$$

we rewrite the quasi-bi-Hamiltonian chain in the form

$$\pi_1 dH_i^{(k)} = \sum_{l=1}^{m} \sum_{j=1}^{n_l} F_{i,j}^{k,l}\, \pi_0\, dH_j^{(l)}, \qquad k = 1, ..., m, \quad i = 1, ..., n_k$$

Bi-hamiltonian representation

A simple inspection shows that

$$F^{k,k}_{i,i+1} = 1, \qquad F^{k,l}_{i,1} \neq 0.$$

Hence, a quasi-bi-Hamiltonian representation takes the form

$$\pi_1 dH^{(k)}_i = \pi_0\, dH^{(k)}_{i+1} + \sum_{l=1}^{m} F^{k,l}_{i,1}\, \pi_0\, dH^{(l)}_1.$$

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$M \to \mathcal{M}, dimM = 2n, dim\mathcal{M} = 2n + m,$ with additional coordinates $c_i, i = 1, ..., m.$

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$M \to \mathcal{M}, dim M = 2n, dim \mathcal{M} = 2n + m,$ with additional coordinates $c_i, i = 1, ..., m.$

Then we extend Hamiltonians:

Bi-hamiltonian representation

$$H_i^{(k)}(\lambda,\mu) \rightarrow h_i^{(k)}(\lambda,\mu,c) = H_i^{(k)}(\lambda,\mu) - \sum_{l=1}^{m} F_{i,1}^{k,l}(\lambda,\mu)\, c_l.$$

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separation relations for $h_i^{(k)}$ are given by

$$\sum_{k=1}^{m} \mu^{\alpha_k} \lambda^{\beta_k} h^{(k)}(\lambda) = \psi(\lambda,\mu), \qquad m \le n, \qquad \alpha_k, \beta_k \in \mathbb{N},$$

where

$$h^{(k)}(\lambda) = \sum_{i=0}^{n_k} \lambda^{n_k - i} h_i^{(k)}, \qquad h_0^{(k)} = c_k, \qquad n_1 + \ldots + n_m = n.$$

Bi-hamiltonian representation

On $\mathcal{M}$ Poisson tensors π_0 and π_1 will be denoted by Π_0 and Π_{1D}, respectively. Both are degenerated with common Casimirs $c_i, i = 1, ..., m$

$$\Pi_0 = \left(\begin{array}{c|c} \pi_0 & 0 \\ \hline 0 & 0 \end{array}\right), \qquad \Pi_{1D} = \left(\begin{array}{c|c} \pi_1 & 0 \\ \hline 0 & 0 \end{array}\right).$$

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Quasi-bi-Hamiltonian representation on $\mathcal{M}$:

$$\Pi_{1D}\, dh_i^{(k)} = \Pi_0\, dh_{i+1}^{(k)} + \sum_{l=1}^{m} F_{i,1}^{k,l}\, \Pi_0\, dh_1^{(l)}, \qquad F_{0,1}^{k,l} = -\delta_{kl}.$$

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Let us introduce the following bivector

$$\Pi_1 := \Pi_{1D} + \sum_{k=1}^{m} X_1^{(k)} \wedge Z_k, \quad X_1^{(k)} = \Pi_0 dh_1^{(k)}, \quad Z_k := \frac{\partial}{\partial c_k}.$$

Bi-hamiltonian representation

Theorem.
1. Bivector Π_1 is Poisson.
2. Poisson bivectors Π_0 and Π_1 are compatible.
3. $h^{(k)}(\lambda)$ are Casimir functions of the Poisson pencil $\Pi_\lambda = \Pi_1 - \lambda\Pi_0$: functions $h_i^{(k)}$ form bi-Hamiltonian chains with respect to Π_0, Π_1.

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Example. $n = 2$, Benenti class.

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The transformation to cartesian coordinates (q, p) takes the form:

Bi-hamiltonian representation

$$q^1 = \lambda^1 + \lambda^2, \quad q^2 = 2\sqrt{-\lambda^1\lambda^2},$$

$$p_1 = \frac{\lambda^1\mu_1}{\lambda^1 - \lambda^2} + \frac{\lambda^2\mu_2}{\lambda^2 - \lambda^1}, \quad p_2 = \sqrt{-\lambda^1\lambda^2}\left(\frac{\mu_1}{\lambda^1 - \lambda^2} + \frac{\mu_2}{\lambda^2 - \lambda^1}\right)$$

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$$H_1 = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + (q^1)^3 + \frac{1}{2}q^1(q^2)^2,$$

$$H_2 = \frac{1}{2}q^2p_1p_2 - \frac{1}{2}q^1p_2^2 + \frac{1}{16}(q^2)^4 + \frac{1}{4}(q^1)^2(q^2)^2,$$

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The quasi-bi-Hamiltonian chain is

$$\pi_1 dH_r = \pi_0 dH_{r+1} - \rho_r\pi_0 dH_1, \qquad r = 1,2,$$

where $\rho_1 = -\lambda_1 - \lambda_2 = -q^1$, $\rho_2 = \lambda_1\lambda_2 = -\frac{1}{4}(q^2)^2$ and

Bi-hamiltonian representation

$$\pi_1 = \begin{pmatrix} 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \\ -\lambda_1 & 0 & 0 & 0 \\ 0 & -\lambda_2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & q^1 & \frac{1}{2}q^2 \\ 0 & 0 & \frac{1}{2}q^2 & 0 \\ -q^1 & -\frac{1}{2}q^2 & 0 & \frac{1}{2}p_2 \\ -\frac{1}{2}q^2 & 0 & \frac{1}{2}p_2 & 0 \end{pmatrix}.$$

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$$\pi_1 = \begin{pmatrix} 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \\ -\lambda_1 & 0 & 0 & 0 \\ 0 & -\lambda_2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & q^1 & \frac{1}{2}q^2 \\ 0 & 0 & \frac{1}{2}q^2 & 0 \\ -q^1 & -\frac{1}{2}q^2 & 0 & \frac{1}{2}p_2 \\ -\frac{1}{2}q^2 & 0 & \frac{1}{2}p_2 & 0 \end{pmatrix}.$$

In $\mathcal{M} \ni (q^1, q^2, p_1, p_2, c)$ the extended Hamiltonians

$$h_0 = c, \quad h_1 = H_1 - cq^1, \quad h_2 = H_2 - \frac{1}{4}c(q^2)^2$$

form one bi-Hamiltonian chain

$$\begin{aligned} \Pi_0 dh_0 &= 0 \\ \Pi_0 dh_1 &= X_1 = \Pi_1 dh_0 \\ \Pi_0 dh_2 &= X_2 = \Pi_1 dh_1 \\ 0 &= \Pi_1 dh_2 \end{aligned}$$

Bi-hamiltonian representation

where

$$\Pi_0 = \left(\begin{array}{c|c} \pi_0 & 0 \\ \hline 0 & 0 \end{array}\right) \ , \ \Pi_1 = \left(\begin{array}{c|c} \pi_1 & \pi_0 dh_1 \\ \hline -(\pi_0 dh_1)^T & 0 \end{array}\right)$$

and separation curve takes the form

$$c\lambda^2 + h_1\lambda + h_2 = \frac{1}{2}\lambda\mu^2 + \lambda^4.$$

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Example. $n = 2$, non-Benenti class.

Bi-hamiltonian representation

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Separation curve:

$$H_1^{(1)}\lambda^2 + H_1^{(2)} = \frac{1}{2}\lambda\mu^2 + \lambda^4.$$

Bi-hamiltonian representation

The quasi-bi-Hamiltonian chain takes the form

$$\pi_1 dH_1^{(k)} = F_{1,1}^{k,1}\pi_0 dH_1^1 + F_{1,1}^{k,2}\pi_0 dH_1^{(2)},\ k = 1, 2.$$

and

$$F_{1,1}^{k,1} = -\rho_k + \rho_{k-1}\rho_2\rho_1^{-1}, \quad F_{1,1}^{k,2} = -\rho_{k-1}\rho_1^{-1}.$$

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In extended phase space $\mathcal{M} \ni (q^1, q^2, p_1, p_2, c_1, c_2)$

$$h_0^{(1)} = c_1,$$

$$h_1^{(1)} = \frac{1}{2}\frac{1}{q^1}p_1^2 + \frac{1}{2}\frac{1}{q^1}p_2^2 + (q^1)^2 + \frac{1}{2}(q^2)^2 - c_1[q_1 + \frac{1}{4}\frac{1}{q^1}(q^2)^2] - c_2\frac{1}{q^1}$$

$$h_0^{(2)} = c_2,$$

$$h_1^{(2)} = -\frac{q_2^2}{8q^1}p_1^2 + \frac{1}{2}q^2 p_1 p_2 - \frac{1}{2}q^1 p_2^2 - \frac{q_2^2}{8q^1}p_2^2 - \frac{(q^2)^4}{16} + c_1\frac{(q^2)^4}{16q^1} + c_2\frac{(q^2)}{4q^1}$$

Bi-hamiltonian representation

$$\Pi_0 = \left(\begin{array}{c|cc} \pi_0 & 0 & 0 \\ \hline 0 & & \\ 0 & \multicolumn{2}{c}{0} \end{array} \right), \quad \Pi_1 = \left(\begin{array}{c|cc} \pi_1 & \pi_0 dh_1^{(1)} & \pi_0 dh_1^{(2)} \\ \hline -(\pi_0 dh_1^{(1)})^T & & \\ -(\pi_0 dh_1^{(2)})^T & \multicolumn{2}{c}{0} \end{array} \right).$$

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Two bi-Hamiltonian sub-chains are

$$\begin{aligned} \Pi_0 dh_0^{(1)} &= 0 \\ \Pi_0 dh_1^{(1)} &= \Pi_1 dh_0^{(1)} \\ 0 &= \Pi_1 dh_1^{(1)} \end{aligned} \qquad \begin{aligned} \Pi_0 dh_0^{(2)} &= 0 \\ \Pi_0 dh_1^{(2)} &= \Pi_1 dh_1^{(2)} \\ 0 &= \Pi_1 dh_3 \ , \end{aligned}$$

Bi-hamiltonian representation

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$$\begin{aligned} \Pi_0 dh_0^{(1)} &= 0 & \qquad \Pi_0 dh_0^{(2)} &= 0 \\ \Pi_0 dh_1^{(1)} &= \Pi_1 dh_0^{(1)} & \qquad \Pi_0 dh_1^{(2)} &= \Pi_1 dh_1^{(2)} \\ 0 &= \Pi_1 dh_1^{(1)} & \qquad 0 &= \Pi_1 dh_3 \ , \end{aligned}$$

Separation curve for extended system is

$$\lambda^2(c_1\lambda + h_1^{(1)}) + c_2\lambda + h_1^{(2)} = \frac{1}{2}\lambda\mu^2 + \lambda^4.$$

Generalized Stäckel transform

Generalized Stäckel transform

Consider Liouville integrable system with n involutive Hamiltonians H_i which depend linearly on $k \leq n$ parameters $\alpha_1, ..., \alpha_k$:

$$H_i = H_{i,0} + \sum_{j=1}^{k} \alpha_j H_{i,j}, \quad i = 1, \dots, n.$$

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$$H_i = H_{i,0} + \sum_{j=1}^{k} \alpha_j H_{i,j}, \quad i = 1, \ldots, n.$$

Let us define n new Hamiltonians $\tilde{H}_i$ in the following way:
from $(1, 2, ..., n)$ fix a k-tuple $(s_1, \ldots, s_k)$, then

$$H_{s_i,0} + \sum_{j=1}^{k} \tilde{H}_{s_j} H_{s_i,j} = \tilde{\alpha}_i, \quad i = 1, \ldots, k,$$

$$\tilde{H}_i = H_{i,0} + \sum_{j=1}^{k} \tilde{H}_{s_j} H_{i,j}, \quad i \neq s_j \quad \text{for} \quad j = 1, \ldots, k.$$

Generalized Stäckel transform

Notice that new Hamiltonians are of the form:

$$\tilde{H}_i = \tilde{H}_{i,0} + \sum_{j=1}^{k} \tilde{\alpha}_j \tilde{H}_{i,j}, \quad i = 1, \ldots, n.$$

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$$\tilde{H}_i = \tilde{H}_{i,0} + \sum_{j=1}^{k} \tilde{\alpha}_j \tilde{H}_{i,j}, \quad i = 1, \ldots, n.$$

We shall refer to the above transformation from H_i, to $\tilde{H}_i$, as to the k-parameter *generalized Stäckel transform* generated by $H_{s_1}, .., H_{s_k}$ (Błaszak,Sergeyev). One-parameter case was considered by Hieterinta, Grammaticos, Dorizi, Ramani and by Boyer, Kalnins, Miller.

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Theorem. Hamiltonians $\tilde{H}_i, i = 1, ..., n$ form Liouville integrable systems.

Multi-times reciprocal transformations

Multi-times reciprocal transformations

Assume that $\tilde{H}_i$, $i = 1, \ldots, n$, are related to H_i, $i = 1, \ldots, n$, through the k-parameter Stäckel transform generated by $H_{s_1}, \ldots, H_{s_k}$.

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Consider simultaneously the equations of motion for the Hamiltonians H_{s_i} with the times t_{s_i} and for $\tilde{H}_{s_i}$ with the times $\tilde{t}_{s_i}$:

$$dx^b/dt_{s_i} = (X_{H_{s_i}})^b, \quad b = 1, \dots, \dim M, \quad i = 1, \dots, k,$$

$$dx^b/d\tilde{t}_{s_i} = (X_{\tilde{H}_{s_i}})^b, \quad b = 1, \dots, \dim M, \quad i = 1, \dots, k.$$

where x^b are local coordinates on M

Theorem.(Błaszak,Sergeyev)
Consider the equations of motion for H_{s_i}, $i = 1, \ldots, k$, restricted onto the common level surface $N_{\tilde{\alpha}}$ of H_{s_i}:

$$N_{\tilde{\alpha}} = \{x \in M | H_{s_i}(x, \alpha_1, \ldots, \alpha_k) = \tilde{\alpha}_i, \quad i = 1, \ldots, k\}.$$

Then the reciprocal transformation

$$d\tilde{t}_{s_i} = -\sum_{j=1}^{k} H_{s_j,i} dt_{s_j}, \quad i = 1, \ldots, k$$

is well defined on these restricted equations of motion and sends them into the equations of motion for $\tilde{H}_{s_i}$, $i = 1, \ldots, k$, restricted onto the common level surface $\tilde{N}_\alpha$ of $\tilde{H}_{s_i}$, where

$$\tilde{N}_\alpha = \{x \in M | \tilde{H}_{s_i}(x, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k) = \alpha_i, \quad i = 1, \ldots, k\},$$

moreover $\tilde{N}_\alpha = N_{\tilde{\alpha}}$.

Stäckel equivalent separable systems

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Separation relations under consideration are as follows

$$\sum_{k=1}^{m} \mu^{\alpha_k} \lambda^{\beta_k} H^{(k)}(\lambda) = \psi(\lambda, \mu), \qquad m \leq n, \qquad \alpha_k, \beta_k \in \mathbb{N},$$

where

$$H^{(k)}(\lambda) = \sum_{i=1}^{n_k} \lambda^{n_k - i} H_i^{(k)}, \qquad n_1 + ... + n_m = n$$

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The classes for which $\alpha_1 > \alpha_2 > ... > \alpha_m$ will be called seed classes.

Theorem. Any separable system under consideration belongs either to some seed class or is Stäckel equivalent to an appropriate system from some seed class.

Stäckel equivalent separable systems

Consider classical Stäckel systems and related classes of separation relations:

$$\sum_{i=1}^{n} \lambda^{\beta_i} H_i = \frac{1}{2} f(\lambda)\mu^2 + \gamma(\lambda).$$

Benenti class, where $(\beta_1, ..., \beta_n) = (n-1, n-2, ..., 0)$, is the only seed class. All other classes are Stäckel related to the Benenti one.

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Consider a seed class of separation curves qubic in momenta:

$$\mu(\sum_{i=1}^{n_1} \lambda^{n-i} H_i^{(1)}) + \sum_{i=1}^{n_2} \lambda^{n-i} H_i^{(2)} = \mu^3 + \gamma_1(\lambda)\mu + \gamma_2(\lambda).$$

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All other classes of the form

Stäckel equivalent separable systems

$$(\sum_{i=1}^{n_1} \lambda^{\alpha_i} H_i^{(1)})\mu + \sum_{i=1}^{n_2} \lambda^{\beta_i} H_i^{(2)} = \mu^3 + \gamma_1(\lambda)\mu + \gamma_2(\lambda)$$

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THE END

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