Gaudin systems and their limits: classical and quantum cases. [1]

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[1] Based on Joint work(s) with A. Chervov, L. Rybnikov - ITEP-Moscow and F. Musso –Roma III

Outline

The Gaudin model was introduced by M. Gaudin as a spin model related to the Lie algebra sl_2, and later generalized to the case of arbitrary semisimple Lie algebras.
The Hamiltonian is

$$H_G = \sum_{i \neq j,\, a} x_a^{(i)} x_a^{(j)}.$$

$\{x_a\},\ a = 1, \ldots, \dim \mathfrak{g}$, is an orthonormal basis of $\mathfrak{g}$ with respect to the Killing form.
These objects are regarded as elements of the polynomial algebra $\mathcal{S}(\mathfrak{g}^*)^{\otimes N}$ in the classical case, and as elements of the universal envelopping algebra $U(\mathfrak{g})^{\otimes N}$ in the quantum case, as

$$x_a^{(i)} = 1 \otimes \cdots \otimes \underbrace{x_a}_{i-th\ factor} \otimes 1 \cdots \otimes 1.$$

Gaudin himself found that the quadratic Hamiltonians

$$H_i = \sum_{k \neq i} \sum_{a=1}^{\dim \mathfrak{g}} \frac{x_a^{(i)} x_a^{(k)}}{z_i - z_k}. \tag{1}$$

provide a set of "constants of the motion" for H_G, , and $z_1, \dots, z_N$ are pairwise distinct complex numbers. Later it was shown (Jurco) that - in the classical case - the spectral invariants of the Lax matrix

$$L_G(z) = \sum_{i,a} \frac{x_a^{(i)}}{z - z_i}$$

encode a (basically complete) set of invariant quantities.
Feigin Frenkel and Reshetikhin proved the existence of a large commutative subalgebra $\mathcal{A}(z_1, \dots, z_N) \subset U(\mathfrak{g})^{\otimes N}$ containing H_i.
For $\mathfrak{g} = sl_2$, the algebra $\mathcal{A}(z_1, \dots, z_N)$ is generated by H_i and the central elements of $U(\mathfrak{g})^{\otimes N}$.

In other cases, the algebra $\mathcal{A}(z_1, \ldots, z_N)$ has also some new generators known as higher Gaudin Hamiltonians. Their explicit construction for $\mathfrak{g} = gl_r$ was obtained in 2004 by D. Talalaev and further discusse in papers by A. Chervov and D. Talalaev.
In the present talk we will consider – both form the classical and from the quantum point of view – the problem of discussing what happens when the arbitrary points $z_1, \ldots, z_N$ appearing in the Lax matrix, and in the quadratic Hamiltonians H_i glue together.
In particular, we will concentrate on the "extreme" case, when in some sense all points collide .

The Lax matrix

$$L_G(z) = \sum_{i,a} \frac{x_a^{(i)}}{z - z_i} = \sum_{i=1}^{n} \frac{X_i}{z - z_i}$$

satisfies the Poisson algebra

$$\{L(z) \otimes 1, 1 \otimes L(u)\} = [\frac{\Pi}{z-u}, L(z) \otimes 1 + 1 \otimes L(u)]$$

where Π is the permutation matrix $\Pi(X \otimes Y) = Y \otimes X$.
This linear r-matrix structure (together with suitable reductions) is associated with a huge number of classical integrable systems: e.g:

- The Neumann type systems and the gl_n- Manakov tops.
- Finite gap reductions of KdV (and n-GD) systems.
- Hitchin's system on singular rational curves.

As it is well known, the r-matrix Poisson brackets presented before are a shorthand notation for the following situation:

- The phase space[2] of the N-site Gaudin model is $\mathfrak{g}^{*\otimes N}$
- The "physical" Hamiltonian is a mean field spin–spin interaction,

$$H_G = \frac{1}{2}\sum_{i\neq j=1}^{N} \mathrm{Tr}(X_i \cdot X_j), \quad X_j =\in \mathfrak{g}^*(\simeq \mathfrak{g}).$$

- The Poisson brackets defind by the r-matrix formula are just product Lie-Poisson brackets on $\mathfrak{g}^{*\otimes N}$.
- The definition of the (classical) Lax matrix as $L_G = \sum_{i=1}^{N} \frac{X_i}{z-z_i}$ defines an embedding of our phase space $\mathfrak{g}^{\otimes N}$ into a Loop space $L\mathfrak{g} \simeq \mathfrak{g}((z))$.

[2] In this talk we will always deal with metric (or reductive) Lie algebras (and, in particular, with gl_r), so that we will tacitly identify $\mathfrak{g}$ with $\mathfrak{g}^*$

The interplay between classical Lax matrices, Loop algebras and Poisson manifolds has a long history. It was initiated in a (somewhat underestimated) paper by Magri and Morosi (1986), and later settled and expounded in the works of the Leningrad's school (see, e.g. Reyman - Semenov-Tyan-Shanski).
It can be encoded in the notion of R-operator. On the space $\mathfrak{g}^*((z))$ of Laurent polynomials with values in (the dual of) a Lie algebra $\mathfrak{g}$, there is a family of mutually compatible Poisson brackets, $\{\cdot,\cdot\}_k$ associated with a family of classical R–operators

$$R_k(X(z)) = (z^k X(z))_+ - (z^k X(z))_-. \tag{2}$$

via the formula

$$\{F,G\}_k(X) = \langle X, [R_k(\nabla(F)), \nabla(G)]\rangle - \langle X, [R_k(\nabla(G)), \nabla(F)]\rangle$$

Spectral invariants of a polynomial Lax matrix form an abelian Poisson subalgebra (w.r.t. any of the brackets $\{\cdot,\cdot\}_k$).

Let us consider the space $\mathfrak{g}^*_{N,A}$ of matrices of the form

$$X(z) = z^{N+1}A + \sum_{i=0}^{N} z^i X_i, \tag{3}$$

where A is a *fixed* element of $\mathfrak{g}$.

- The brackets associated with $R_0, \dots, R_N$ on $\mathfrak{g}^*((z))$ restrict to the affine subspace $\mathfrak{g}^*_{N,A}$, and thus on $\mathfrak{g}^*_N = \mathfrak{g}^*_{N,\mathbf{0}}$which is the case we will henceforth consider).
- These brackets are mutually compatible; in particular, we can associate with any degree N polynomial $\mathcal{Q}(z) = \sum_{i=0}^{N} \kappa_i z^{N-i}$ a Poisson bracket (of the "polynomial" family) via:

$$\{\cdot,\cdot\}_{\mathcal{Q}} = \sum_{i=0}^{N} \kappa_i \{\cdot,\cdot\}_i .$$

The standard (product) Lie–Poisson structure on $\mathfrak{g}^{\otimes N}$ defined by the standard r-matrix recalled above, via the "Lax map"

$$L_G = \sum_{i=1}^{N} \frac{X_i}{z - z_i}$$

is realized, in this framework, by the bracket

$$\{\cdot,\cdot\}_{\mathcal{S}} = \{\cdot,\cdot\}_N + \sum_{i=0}^{N-1} (-1)^i \sigma_i \{\cdot,\cdot\}_i.$$

the σ_i being the elementary symmetric polynomials in the quantities $z_1, \ldots, z_N$, that is, is associated with the polynomial

$$S(z) = \prod_{i=1}^{N} (z - z_i) = z^N + \sum_{i=0}^{N-1} (-1)^i \sigma_i z^{N-1}$$

This gives the possibility of constructing bihamiltonian structures for the classical Gaudin case. Indeed the structure defined by the polynomial

$$\left(\frac{S(z)}{z}\right)_+$$

provides, together with the standard one, a “good” bihamiltonian structure for the Gaudin model: the coefficients of the characteristic polynomial of $L_G(z)$ fill in recursion relations of GZ (LM) type.

Also,this gives the opportunity of studying, from this point of view, the limits of (suitable combinations of) the "polynomial" Poisson structures when some (eventually, all) poles of the Lax matrix $z_1, \ldots, z_N$ "glue together".
This means the following: we pull back, via the map

$$L_G \to L_{G,pol}, \quad L_{G,pol} = \prod_{i=1}^{N}(z - z_i)\Big(\sum_{i=1}^{N} \frac{X_i}{z - z_i}\Big))$$

the fundamental polynomial Poisson brackets $\{\cdot,\cdot\}_k, k = 0, \ldots, N$; we obtain some (complicated) rational expressions depending on $(z_1, \ldots, z_N)$, suitable linear combinations thereof admit non trivial limits when some (ev., all) of the z_i's glue together.

To the best of my knowledge, this problem was studied, in the $e(3)$ case, by the late Vadim Kuznetsov.

On general (Lie – Poisson theoretic) grounds when points glue together we pass from a semi-simple Lie Poisson structure to a non semisimple ones. (Musso-Petrera-Ragnisco, 05 and 08)

In particular, when all z_i glue, we obtain the bracket ($\mathfrak{g} = gl_r$))

$$\{F, G\}_{limit} = \sum_{i,j,k} r_{ijk} \mathrm{Tr}\left(\nabla F_i \left[X_k, \nabla G_j\right]\right)$$

$$r_{ijk} = (k-1)\delta_{ij}\delta_{jk} - \theta_{(i-k)}\delta_{ij} + \theta_{(j-i)}\delta_{ik} + \theta_{(i-j)}\delta_{jk}$$

where $\nabla F_i, (\nabla G_i)$ are the differentials of $F\,(G)$ w.r.t. the i–th entry.
These Poisson brackets are still compatible with the standard r-matrix ones.

Pictorially (and in the 3-site case) we have the following representation – by Poisson operators – of the brackets as follows:

$$P_{\text{standard } r} = \begin{pmatrix} [X_1,\cdot] & & \\ & [X_2,\cdot] & \\ & & [X_3,\cdot] \end{pmatrix}$$

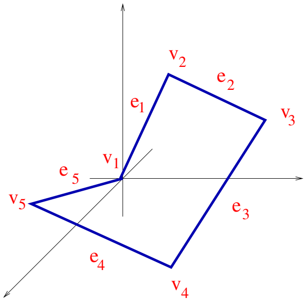
$$P_{\text{limit}} = \begin{pmatrix} 0 & [X_1,\cdot\,] & [X_1,\cdot\,] \\ [X_1,\cdot\,] & [X_2 - X_1,\cdot\,] & [X_2,\cdot\,] \\ [X_1,\cdot\,] & [X_2,\cdot\,] & [2X_3 - X_2 - X_1,\cdot\,] \end{pmatrix}$$

According to the (Gel'fand–Zakharevich version of the) Lenard-Magri scheme, $P_{\text{standard } r}$ and P_{limit} define the gl_r (as well as $\mathfrak{g}$) - valued generalization of the so called *Bending Flows*, introduced in the case corresponding to $\mathfrak{g} = \mathfrak{gl}(2)$ by Kapovich and Millson (1996).

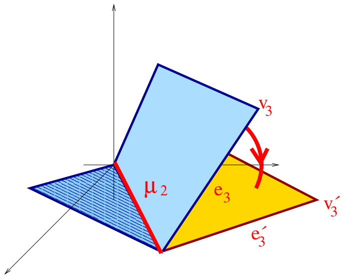
Bending flows are defined on the moduli space $\mathcal{M}_{\mathbf{r}}$ of (N)–gons with fixed sides lengths $\mathbf{r} = (r_1, \ldots, r_N),\ r_i > (\geq) 0$.

- $\mathcal{M}_{\mathbf{r}}$ is a smooth $2N - 6$–dimensional manifold;
- $\mathcal{M}_{\mathbf{r}}$ is carry a natural symplectic structure, since it is a symplectic quotient of products of spheres:

$$\{(e_1,\ e_2,\ \cdots,\ e_N) \in\ S^2_{r_1} \times S^2_{r_2} \times \cdots S^2_{r_N}\}//SO(3)(\text{diag. action})$$



This moduli space of polygons carries a (A)CIHS, whose action variables are the lenghts of diagonals stemming from one vertex, and whose angle variables are (indeed) the dihedral angles. Geometrically, the flow *bend* one part of the polygon around the corresponding diagonal, keeping the rest fixed.



$\mathfrak{g}$–Bending flows can be defined on the same phase space of the $\mathfrak{g}$–Gaudin system, the link between the two (in the KM N-gon case) being the fact that S_r^2 is a symplectic leaf of $\mathfrak{su}(2)$.

- $\mathfrak{g}$-Bending flows admit a set of $N-1$ Lax matrices:

$$L_k(z) = zX_k + \sum_{i=k+1}^{N} X_i, k = 1, \dots, N-1.$$

  whose spectral invariants (together with the integral of the motion associated with global $\mathfrak{g}$-invariance) provide a complete set of Hamiltonians

- Hamiltonians come in "clusters", each cluster being associated with the corresponding Lax matrix. For $\mathfrak{g} = gl_r$

$$H^a_{k,m} = \mathrm{res}_{z=0} \frac{1}{z^{a+1}} \mathrm{Tr} L_k^m(z).$$

The ring of Hamitonians of the $\mathfrak{g}$-Bending flows comprises the "physical" Hamiltonian of the Gaudin model. Thus, bending Hamiltonians are an alternative set of Integral of the Motion for the Gaudin system.

Separation of variables can be performed in this scheme, with "clusters" of canonical conjugated variables associated, via the Sklyanin magic recipe, to each of the Lax matrices. (G.F, F. Musso, MPAG2006).

In the standard picture, the Jacobi separation relations live (for $\mathfrak{g} = sl(r)$) on a curve of genus $g_{standard} = \frac{r-1}{2}(N-r-2)$.

In the "Bending picture", the genus of each spectral curve is $g_{Bend} = \frac{(r-1)(r-2)}{2}$, independent of the *number* N of sites.

The existence of a large quantum commutative subalgebra $\mathcal{A}(z_1, \ldots, z_N) \subset U(\mathfrak{g})^{\otimes N}$ containing the Gaudin Hamiltonians H_i was shown (FFR94). In general $\mathcal{A}(z_1, \ldots, z_N)$ has new generators, besides the quadratic ones, known as *higher Gaudin Hamiltonians*. The definition of $\mathcal{A}(z_1, \ldots, z_N)$ is via identifiying a commutative subalgebra of the enveloping algebra $U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}])$. To any collection $z_1, \ldots, z_n$ of pairwise distinct complex numbers, one can naturally assign the evaluation map $U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) \to U(\mathfrak{g})^{\otimes N}$. The image of the center under the composition of the above homomorphisms is $\mathcal{A}(z_1, \ldots, z_n)$, and is called *(Quantum) Gaudin Algebra*.
The problem of finding explicit representatives for generators of $\mathcal{A}(z_1, \ldots, z_N)$ was solved some ten years later by D. Talalaev.

The Gaudin algebra defined by Feigin, Frenkel and Reshetikhin – and concretely identified by Talalaev's formula for N-spin $\mathfrak{gl}_r$ Gaudin systems – depends in general on the position of points $z_1, \ldots, z_N$.

Fact

$\mathcal{A}(z_1, \ldots, z_N)$ is invariant under permutations and under simultaneous rescalings $z_i \to \alpha\, z_i + \beta, \alpha, \beta \in \mathbb{C}$; thus the "two site" algebra $\mathcal{A}(z_1, z_2)$ is independent of z_1, z_2.

We consider limits of the Gaudin algebras when some of the points $z_1, \ldots, z_N$ glue together. (A. Chervov, G.F, L. Rybnikov, arXiv:0710.4971).
This means that we can set $z_1, \ldots, z_k$ "fixed", and let the remaining $N - k$ points glue to z, via

$$z_{k+i} = z + s\, u_i,\ \ i = 1, \ldots, N-k,\ z_i \neq z_j; u_i \neq u_j,\ s \to 0.$$

To describe this limit, we consider the maps

$$D_{k,N} := \mathrm{id}^{\otimes k} \otimes \mathrm{diag}_{N-k} : U(\mathfrak{g})^{\otimes(k+1)} \hookrightarrow U(\mathfrak{g})^{\otimes N},$$
$$D_{k,N}(X_1 \otimes \cdots \otimes X_k \otimes X_{k+1}) = X_1 \otimes \cdots X_k \otimes \underbrace{X_{k+1} \otimes \cdots \otimes X_{k+1}}_{N-k \text{ times}}$$

and

$$I_{k,N} := 1^{\otimes k} \otimes \mathrm{id}^{\otimes(N-k)} : U(\mathfrak{g})^{\otimes(N-k)} \hookrightarrow U(\mathfrak{g})^{\otimes N},$$
$$I_{k,N}(X_1 \otimes \cdots X_{N-k}) = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{k \text{ times}} \otimes X_1 \otimes \cdots \otimes X_{N-k}$$

and define

$$\mathcal{A}_{(z_1,\ldots,z_k,z),(u_1,\ldots,u_{N-k})} = D_{k,N}(\mathcal{A}(z_1,..,z_k,z)) \cdot I_{k,N}(\mathcal{A}(u_1,..,u_{N-k}))$$

Theorem

The algebra $\mathcal{A}_{(z_1,\ldots,z_k,z),(u_1,\ldots,u_{N-k})}$ *is commutative;*

$$\lim_{s\to 0} \mathcal{A}(z_1,\ldots,z_k,z+su_1,\ldots,z+s\,u_{N-k}) = \mathcal{A}_{(z_1,\ldots,z_k,z),(u_1,\ldots,u_{N-k})}.$$

In words

In some sense we arrive at a kind of factorization by "adding" one point. Indeed, The commutative algebra $\mathcal{A}_{(z_1,\ldots,z_k,z),(u_1,\ldots,u_{N-k})}$ involves the FFR algebra associated with the points $(z_1,\ldots,z_k,z)$, and a FFR algebra related with the points $u_1,\ldots,u_{N-k}$.
Iterating this limiting procedure described above we can obtain some new commutative subalgebras in $U(\mathfrak{g})^{\otimes N}$.

Namely, we can iterate the procedure

$$\mathcal{A}(z_1, z_2, \ldots, z_N) \to \mathcal{A}_{(z_1,z_2)\,(z_2,z_3,\ldots,z_N)}$$

(Here we have to use translation invariance of $\mathcal{A}(z_1, \ldots, z_M)$), to obtain the subalgebra

$$\mathcal{A}_{\text{lim}} \equiv \mathcal{A}_{(z_1,z_2),\ldots,(z_1,z_2)} \subset U(\mathfrak{g})^{\otimes n}$$

(Here we have to use that the two site Gaudin Algebra $\mathcal{A}(u, v)$ is independent of (u, v)).
This limit algebra is generated by

$$D_{1,N}(\mathcal{A}(z_1, z_2)),\ 1 \otimes D_{1,N-1}(\mathcal{A}(z_1, z_2)),\ \ldots,\ 1^{\otimes(N-2)} \otimes \mathcal{A}(z_1, z_2).$$

Proposition

The subalgebra $\mathcal{A}_{\lim}$ is generated by elements $H_{l,k}^{(\alpha)} \in U(gl_r)^{\otimes n}$ such that their classical "limits" $\overline{H}_{l,k}^{(\alpha)}$, are given by

$$\overline{H}_{l,k}^{(\alpha)}(X_1, \ldots, X_n) := Res_{z=0} \frac{1}{z^{\alpha+1}} Tr(X_k + z(\sum_{i=k+1}^{n} X_i))^l,$$

Summing up:

The classical limits of the (still unspecified) "Quantum Bending Hamiltonians" do coincide with the spectral invariants of the Lax operators $L_k(z) = z\,X_k + \sum_{i=k+1}^{N} X_i, i = 2, \ldots, N$.
"Quantum Bending Hamiltonians" associated with Lax matrices $L_k, L_{k'},\ k \neq k'$ commute.
Finally, we are left with the determination of the correct (quantum) Hamiltonian defined by a Lax matrix of the form $L = zA + B$

Theorem (Talalaev,2004)

Let $L(z)$ be the Lax matrix of the gl_r-Gaudin model, and consider the differential operator in the variable z

$$"DET"\left(\partial_z - L(z)\right) = \sum_{i=0,\dots,r} QH_i(z)\partial_z^i;$$

Then:

$$\forall i,j \in 0,\dots,r, \text{ and } u,v \in \mathbb{C}, \quad [QH_i(z)|_{z=u}, QH_j(z)|_{z=v}] = 0.$$

The $QH_i(z)$'s provide a full set generators of quantum mutually commuting conserved quantities.

Problem:

We cannot apply directly Talalaev's scheme to our matrices $L = zA + B$.

Way out: Classical Bending Hamiltonians can be encoded in a set of generators of the ring

$$\mathrm{Tr}\left(\frac{X_k}{z_1} + \frac{1}{z_2}\left(\sum_{i=k+1}^{n} X_i\right)\right)^l, \text{ for any } z_1, z_2 \in \mathbb{C},\ k = 1, \ldots, N-1.$$

We can use the fact that classical Lax matrices are "invariant" by rational changes of the spectral parameter and multiplication by functions thereof, to consider, instead of $L = z\,A + B$, a matrix of the form

$$L_{pole} = \frac{A}{z+1} + \frac{B}{z},$$

This matrix satisfies standard R matrix commutation rules; we are left with the determination of the quantum Hamiltonians associated with L_{pole}, i.e., to the determination of $\mathcal{A}_{(0,-1)}$

Two ways:

- We use Talalev's results briefly reminded before, considering the operators $QH^{(\alpha)}_{l,k}$ defined as

$$\sum_{l=0,\ldots,N} QH^{(\alpha)}_{l,k}(\partial_z)^l := \mathrm{Res}_{z=0} z^{\alpha-1} det^c(\partial_z - \hat{L}_{pole,\,k}(z)), \quad (4)$$

  (actually, this would quantize the coefficients of the characteristic polynomials of the Bending flows Lax matrices) or
- We can consider a suitable quantization of the traces of the powers of the Lax matrix.

We consider the (differential operator valued) matrix considered in Talalev's theorem,

$$\mathcal{M}(z) = \partial_z - L(z)$$

The standard linear r-matrix commutation

$$[L(z) \otimes 1, 1 \otimes L(u)] = [\frac{\Pi}{z-u}, L(z) \otimes 1 + 1 \otimes L(u)]$$

imply that the matrix elements of $\mathcal{M}$ satisfy special commutation relations that turns it into a "Manin" matrix (A. Chervov, G.F, J.Phys. A. 08, arXiv:0711.2236, and work in progress with V. Rubtsov).
The name originates from a seminal paper of Yu. I Manin on quantum groups (1988).

Manin Matrices

Formally: matrices in a sense associated with linear maps between commutative rings.
Operative definition: Let M_{ij} be a matrxi with elemnts in a (unital) ring $\mathcal{R}$; we call it a (column) Manin Matrix if:

- Elements in the same column commute among themselves;
- Commutators of the cross terms in any 2×2 submatrix are equal:

$$[M_{ij}, M_{kl}] = [M_{kj}, M_{il}] \quad \text{e.g. } [M_{11}, M_{22}] = [M_{21}, M_{12}].$$

Let M be a Manin matrix. Define "a" determinant of M by column expansion:

$$detM = det^{column} M = \sum_{\sigma \in S_n} (-1)^\sigma \overset{\curvearrowright}{\prod_{i=1,\ldots,n}} M_{\sigma(i),i}, \tag{5}$$

where S_n is the group of permutations of n letters, and the symbol $\curvearrowright$ means that in the product $\prod_{i=1,\ldots,n} M_{\sigma(i),i}$ one writes at first the elements from the first column, then from the second column and so on and so forth.

Proposition

The determinant of a Manin matrix does not depend on the order of the columns in the column expansion, i.e.,

$$\forall p \in S_n \quad det^{column} M = \sum_{\sigma \in S_n} (-1)^\sigma \overset{\curvearrowright}{\prod_{i=1,\ldots,n}} M_{\sigma(p(i)),p(i)} \tag{6}$$

Proposition

- $\partial_z - L_{gl_r-Gaudin}(z)$ *is Manin, where* $L_{gl_r-Gaudin}(z)$ *is the Lax matrix for the Lie algebra* gl_r *(as well as its generalization to the affine algebra* $gl_r[t]$*).*
- $e^{-\partial_z} T_{gl_r-Yangian}(z)$ *is Manin, where* $T_{gl_r-Yangian}(z)$ *is the Lax (or "transfer") matrix for the Yangian algebra* $Y(gl_n)$ *(quadratic* r*-matrix commutation relations.*

The Talalev theorem about the commutation of the quantum Gaudin Hamiltonians holds for such matrices (Talalaev, 2006). In particular, a good notion of "*DET*" in the formulation of Talalaev's quantum IM is given by the column determinant. Manin matrices enter other topics in (quantum) integrability.

Main properties of MMs *a* vol d'oiseau:

- The inverse of a Manin matrix M is again Manin.
- The (left) Cramer formula holds:

$$M^{adj}M = det^{column}(M)\mathbf{1}$$

- Schur's formula for the determinant of block matrices holds:

$$det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = det(A)det(D - CA^{-1}B) =$$
$$det(D)det(A - BD^{-1}C)$$

.

- The Cayley-Hamilton theorem: $det(t - M)|_{t=M} = 0$;
- The Newton identities between TrM^k and coefficients of $det(t + M)$).

The quantization of the traces of powers of the Lax matrices can be defined according to the last property reminded above and the following considerations. We (re)consider the (classical) quantities

$$\overline{K}_{n,k}^{(\alpha)}(X_1,\ldots,X_N) := \mathrm{Res}_{z=0} z^{\alpha-1} \mathrm{Tr} \left(\frac{X_k}{z+1} + \frac{\sum_{i=k+1}^{n} X_i}{z} \right)^n .$$

The corresponding quantum Hamiltonians can be defined as:

$$\hat{K}_{n,k}^{(\alpha)} = \mathrm{Res}_{z=0} z^{\alpha-1} \mathrm{Tr} \hat{L}_k(z)^{[n]}, \qquad (7)$$

Here $\hat{L}_k(z)$ are the "quantum" Lax matrices

$$\hat{L}_k(z) = \frac{\hat{X}_k}{z+1} + \frac{\sum_{i=k+1}^{n} \hat{X}_i}{z},$$

corrsponding to $L_{k,pole}$, and $\hat{L}_k(z)^{[n]}$ are the corresponding "quantum powers" defined by

$$\hat{L}_k^{[0]}(z) = Id, \qquad \hat{L}_k^{[i]}(z) = \hat{L}_k^{[i-1]}(z)\hat{L}_k(z) - \frac{\partial}{\partial z}(\hat{L}_k^{[i-1]}(z)).$$

The Quantum counterparts of the traces of powers of the "Bending flows Lax matrices" are obtained taking (ordinary) traces of these object.

Thanks for the attention!