Continuous and discrete Neumann systems on Stiefel varieties

Yuri Fëdorov

In collaboration with Bozidar Jovanovic (Belgrade, Serbia)

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• **Classical Neumann system** on $T^* S^{n-1} = \left\{ \langle q, q \rangle = 1,\ \langle q, p \rangle = 0 \right\} \subset \mathbb{R}^{2n}$.

$$H = \frac{1}{2}\langle p, p \rangle + \frac{1}{2}\langle Aq, q \rangle, \qquad A = \mathrm{diag}(a_1, \dots, a_n),$$

Hamilton equations (with respect to the Dirac bracket on $\mathbb{R}^{2n}$)

$$\dot{q} = p, \quad \dot{p} = -Aq + \nu q, \qquad \nu = \langle p, p \rangle - \langle q, Aq \rangle$$

Big ($n \times n$) Lax pair by Moser, Small (2×2) Lax pair by Mumford:

$$\dot{L}(\lambda) = [L(\lambda), A(\lambda)],$$

$$L(\lambda) = \begin{pmatrix} \sum_{i=1}^n \frac{q_i p_i}{\lambda - a_i} & \sum_{i=1}^n \frac{q_i^2}{\lambda - a_i} \\ -1 - \sum_{i=1}^n \frac{p_i^2}{\lambda - a_i} & -\sum_{i=1}^n \frac{q_i p_i}{\lambda - a_i} \end{pmatrix} = Y + \sum_{i=1}^n \frac{N_i}{\lambda - a_i}, \quad A(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda + \nu(p, q) & 0 \end{pmatrix}$$

First integrals

$$F_i = q_i^2 + \sum_{j \neq i} \frac{(p_i q_j - p_j q_i)^2}{a_j - a_i}, \quad i = 1, \dots, n$$

Real generic tori $\mathbb{T}^{n-1} \mapsto$ complex tori $\mathbb{T}_{\mathbb{C}}^{n-1} =$ open subsets of $\mathrm{Jac}(\Gamma)$, Γ being the hyperelliptic genus $n-1$ spectral curve of $L(\lambda)$.

• **Generalizations to higher-order potentials**

(S. Wojciechowski, O. Bogoyavlenski, P. Saksida)

• **Discrete Neumann system** (Bäcklund transformation) $\mathcal{B}_{\lambda^*} : (p,q) \mapsto (\tilde{p}, \tilde{q})$ (Veselov, Kuznetsov, Vanhaecke, Suris)

$$\tilde{q} = A^{-1/2}(\lambda^*)(\beta q + p), \quad \tilde{p} = -A^{1/2}(\lambda^*)q + A^{-1/2}(\lambda^*)(\beta^2 q + \beta p),$$
$$A(\lambda^*) = \lambda^* \mathbf{I} - A \quad \beta = \langle \tilde{q}, A^{1/2}(\lambda^*) q \rangle,$$

$\lambda^* \in \mathbb{C}$ being arbitrary step parameter. $\lambda^* \to \infty$ gives the continuous limit.

Alternative form

$$p = A^{1/2}(\lambda^*)\tilde{q} - \beta q, \quad \tilde{p} = -A^{1/2}(\lambda^*)q + \beta \tilde{q}.$$

Theorem 1. *1). Up to the action of the group of reflections* $(p_i, q_i) \to (-p_i, -q_i)$, *the map* $\mathcal{B}_{\lambda^*}$ *is equivalent to the discrete Lax pair*

$$\tilde{L}(\lambda) M(\lambda|\lambda^*) = M(\lambda|\lambda^*) L(\lambda), \qquad M(\lambda|\lambda^*) = \begin{pmatrix} -\beta & 1 \\ \lambda - \lambda^* + \beta^2 & -\beta \end{pmatrix}.$$

2) (Veselov) $\mathcal{B}_{\lambda^*}$ *is given by a shift on* $Jac(\Gamma)$ *by* $\mathcal{A}(\lambda^*) - \mathcal{A}(\infty)$

The Stiefel variety $V(n,r) = SO(n)/SO(n-r)$, $(r < n)$ the set of $n \times r$ matrices

$$X = (e_1 \cdots e_r), \quad e_s \in \mathbb{R}^n, \quad X^T X = \mathbf{I}_r,$$

The cotangent bundle $T^*V(n,r)$, the set of $n \times r$ pairs (X,P), $P = (p_1 \cdots p_r)$, $p \in \mathbb{R}^n$

$$X^T X = \mathbf{I}_r, \quad X^T P + P^T X = 0,$$

giving $r(r+1)$ independent scalar constraints.

- Reduction to the oriented Grassmannian $G(r,n) = V(n,r)/SO(r)$

The canonical symplectic structure ω on $T^*V(n,r)$ is the restriction of the canonical 2-form in the ambient space $T^*M_{n,r}(\mathbb{R})$,

$$\omega_0 = \sum_{i=1}^{n} \sum_{s=1}^{r} dp_s^i \wedge de_s^i$$

Given $H(X,P)$, the Hamiltonian equations with respect to the Dirac Poisson structure or involve $r \times r$ symmetric matrix Lagrange multipliers Λ, Π,

$$\dot{X} = \frac{\partial H}{\partial P} - X\Pi, \quad \dot{P} = -\frac{\partial H}{\partial X} + X\Lambda + P\Pi$$

• **Two natural $SO(n)$-invariant metrics** on $V(n,r)$:

The Euclidean metric

$$L_E(X,\dot{X}) = \frac{1}{2}\operatorname{Tr}(\dot{X}^T\dot{X}) \;\mapsto H_E(X,P) = \frac{1}{2}\operatorname{Tr}(P^TP)$$

The normal metric

$$H_0(X,P) = \frac{1}{2}\langle\Phi,\Phi\rangle = \frac{1}{2}\operatorname{Tr}(P^TP) - \frac{1}{2}\operatorname{Tr}((X^TP)^2)$$

There are other $SO(n)$-invariant metrics (....)

• The potential

$$V = \frac{1}{2}\operatorname{Tr}(X^TAX) = \frac{1}{2}\sum_{i=1}^{r}(e_i, Ae_i)$$

• The Neumann system with the normal metric

$$\dot{X} = P - XP^TX, \quad \dot{P} = -AX - XP^TP + PX^TP + XX^TAX,$$

• The Neumann system with the Euclidean metric

$$\dot{X} = P, \quad \dot{P} = -AX - XP^TP + XX^TAX$$

Theorem 2. *The above equations admit* <u>*the same*</u> $n \times n$ *matrix Lax representation (by Reyman and Semonov-Tian-Shanski)*

$$\frac{d}{dt}\mathcal{L}(\lambda) = [\mathcal{A}(\lambda), \mathcal{L}(\lambda)],$$
$$\mathcal{L}(\lambda) = \lambda M + XX^T - \lambda^2 A, \quad M = PX^T - XP^T, \quad \mathcal{A}(\lambda) = \Phi - \lambda A,$$

This yields commuting (with respect to ω) integrals

$$\mathfrak{F} = \left\{ \operatorname{Tr} (\lambda(PX^T - XP^T) + XX^T - \lambda^2 A)^k \mid k = 1, \dots, n, \ \lambda \in \mathbb{R} \right\}$$

Theorem 3. *If all the eigenvalues of* A *are distinct, then the Neumann systems are completely integrable in the* <u>*non-commutative*</u> *sense with the set of integrals* $\mathfrak{F}$ *and by the* $r(r-1)/2$ *components of the* $SO(r)$*-momentum mapping* $\boxed{\Psi(X, P) = X^T P - P^T X}$. *The generic motions of the system are quasi-periodic over the isotropic tori of dimension*

$$\boxed{d = \frac{1}{2}\left(2r(n-r) + \frac{r(r-1)}{2} - \left[\frac{r}{2}\right]\right) + \left[\frac{r}{2}\right]}, \qquad \boxed{d < \frac{1}{2} Dim\, T^* V(n,r), \ (r > 1)}$$

Bi-Hamiltonian description

$$\Theta\ : T^*V(n,r) \mapsto gl(n,\mathbb{R}) = so(n) + Symm(n)$$
$$(X,P) \mapsto (XP^T - P^T X) + XX^T$$

On $gl^*(n)$, the pair of of compatible Poisson brackets given by Poisson tensors (Bolsinov)

$$\Lambda_1(\xi+\eta,\zeta+\theta)|_x = \langle x, [\xi,\zeta] + [\xi,\theta] + [\eta,\zeta]\rangle,$$
$$\Lambda_2(\xi+\eta,\zeta+\theta)|_x = \langle x - A, [\xi+\eta,\zeta+\theta]\rangle,$$

$x \in gl^*(n)$, $\xi,\zeta \in so(n)$, $\eta,\theta \in Symm(n)$

The Casimirs of $\Lambda_1 + \lambda\Lambda_2$, $\lambda \in \mathbb{P}^1$ appear as integrals of the Lax matrix $P + \lambda M - \lambda^2 A$ (integrable tops)

$$\text{Dim of generic } \mathcal{S} \text{ in } (gl^*(n),\Lambda_1) = n^2 - n, \text{ which is less than}$$
$$\text{Dim of generic } \mathcal{S} \text{ in } \Theta^* T^*V(n,r) = 2r(n-r) + \frac{r(r-1)}{2} - \left[\frac{r}{2}\right]$$

The image of $\Theta(T^*V(n,r))$ is the union of *singular* symplectic leaves of $(gl^*(n),\Lambda_1)$.

Theorem 4. *The set of functions* $\mathfrak{F}$ *is complete on* <u>*generic*</u> *symplectic leaves in* $\Theta(T^*V(n,r)) \subset (gl^*(n),\Lambda_1)$.

Marsden–Weinstein reduction to Grassmannian $G(n,r)$,
the quotient space of $V(n,r)$ by the right $SO(r)$-action.

Assume $\Psi = X^T P - P^T X = 0$.
The reduced phase space $\Psi^{-1}(0)/SO(r)$ is symplectomorphic to the cotangent bundle $T^*G(n,r)$ with a canonical symplectic structure.

Theorem 5. *If all eigenvalues of A are distinct, then the reduced Neumann system on $T^*G(n,r)$ is completely integrable in the Liouville sense by means of the integrals $\mathfrak{F}$. The dimension of generic invariant tori $= \frac{1}{2}$ dimension of $T^*G(n,r)$.*

• "Small" $(2r \times 2r)$ matrix Lax representation generalizing the Mumford Lax pair (a modification of that in an unpublished manuscript by S. Kapustin)

Theorem 6. *Up to the action of the discrete group generated by reflections* $(X, P) \longmapsto (\pm X, \pm P)$, *the Neumann flows are* *equivalent* *to the Lax pair with*

$$\frac{d}{dt} L(\lambda) = [L(\lambda), A(\lambda)], \qquad \lambda \in \mathbb{R},$$

$$L(\lambda) = \begin{pmatrix} -X^T(\lambda \mathbf{I}_n - A)^{-1} P & -X^T(\lambda \mathbf{I}_n - A)^{-1} X \\ \mathbf{I}_r + P^T(\lambda \mathbf{I}_n - A)^{-1} P & P^T(\lambda \mathbf{I}_n - A)^{-1} X \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \mathbf{I}_r & 0 \end{pmatrix} + \sum_i \frac{\mathcal{N}_i}{\lambda - a_i},$$

where for the systems with the normal, respectively Euclidean, metric,

$$A(\lambda) = \begin{pmatrix} X^T P & \mathbf{I}_r \\ \Lambda - \lambda \mathbf{I}_r & -P^T X \end{pmatrix}, \quad \text{respectively,} \quad A(\lambda) = \begin{pmatrix} 0 & \mathbf{I}_r \\ \Lambda - \lambda \mathbf{I}_r & 0 \end{pmatrix},$$

$$\Lambda = X^T A X - P^T P.$$

(compare with 2×2 Lax matrix for the classical Neumann system

$$L(\lambda) = \begin{pmatrix} -q^T(\lambda \mathbf{I}_n - A)^{-1} p & -q^T(\lambda \mathbf{I}_n - A)^{-1} q \\ 1 + p^T(\lambda \mathbf{I}_n - A)^{-1} p & p^T(\lambda \mathbf{I}_n - A)^{-1} q \end{pmatrix}$$

It provides integrals $\mathfrak{F}$, as well as Casimirs $\operatorname{Tr} \Psi^k$.

• *What are generic complex tori* ?
The spectral curve $\mathcal{S}$ of $L(\lambda)$

$$F(\lambda, w) = |(\lambda - a_1)\cdots(\lambda - a_n)\mathcal{L}(\lambda) - w\mathbf{I}_n| = 0$$

1). The regularized spectral curve $\mathcal{S}'$ has genus $g > d$,
infinite points $\infty_1, \ldots, \infty_r$ and admits the involution $\sigma : (\lambda, w) \to (\lambda, -w)$;
Let $\mathcal{C} = \mathcal{S}'/\sigma$, "small" curve of genus g_0.

2). σ extends to $\mathrm{Jac}(\mathcal{S}') = \mathbb{C}^g/\Lambda \cong \mathrm{Jac}(\mathcal{C}) \oplus \mathrm{Prym}(\mathcal{S}'/\sigma)$
Dym. (Prym $(\mathcal{S}'/\sigma)) = g - g_0 < d$;

3). $\exists$ precicely $[r/2]$ meromorphic differentials Ω_i with pairs of simple poles at ∞
such that $\sigma^*\Omega_i = -\Omega_i$

Generalized Jacobian $\widetilde{\mathrm{Jac}}(\mathcal{S}', \Omega_i) = \mathbb{C}^{g+[r/2]}/\tilde{\Lambda} \cong \mathrm{Jac}(\mathcal{S}') \times \underbrace{\mathbb{C}^* \times \cdots \times \mathbb{C}^*}_{[r/2]}$.

σ extends also to $\widetilde{\mathrm{Jac}}(\mathcal{S}', \Omega_i)$.

Theorem 7. *1). Generic complex tori of the Neumann system on* $T^*V(n, r)$ *are open subsets of* $\widetilde{\mathrm{Prym}}(\mathcal{S}'/\sigma, \Omega_i)$.

2). Generic complex tori of the Neumann system on $T^*G(n, r)$ *are open subsets of* $\mathrm{Prym}(\mathcal{S}'/\sigma)$.

Some integrable generalizations.

- The Neumann system on the Grassmannian $G(n,r)$ with *quartic* potential:

$$H(X,P) = \frac{1}{2}\,\mathrm{Tr}(P^T P) + \mathrm{Tr}(X^T A^2 X) - \mathrm{Tr}(X^T A X X^T A X),$$

which for $r = 1$, takes the form

$$H = \frac{1}{2}(p,p) + \sum_{i=1}^{n} a_i^2 e_i^2 - \left(\sum_{i=1}^{n} a_i e_i^2 \right)^2$$

(S. Wojciechowski, P. Saksida).

• Neumann Systems on Complex Stiefel Manifolds $W(n,r) \cong U(n)/U(n-r)$, the set of $n \times r$ matrices $Z \in M_{n,r}(\mathbb{C})$ satisfying $\bar{Z}^T Z = \mathbf{I}_r$.

The real Stiefel manifold $V(n,r)$ is a submanifold of $W(n,r)$ given by $Z = \bar{Z}$.

The Euler-Lagrange equations with multipliers

$$\ddot{Z} = -AZ + Z\Lambda, \quad \ddot{\bar{Z}} = -A\bar{Z} + \bar{Z}\bar{\Lambda},$$
$$\Lambda = \bar{Z}^T A Z - \dot{\bar{Z}}^T \dot{Z} = \bar{\Lambda}^T.$$

Matrix $2r \times 2r$ Lax representation

$$\frac{d}{dt}\mathcal{L}^*(\lambda) = [\mathcal{L}^*(\lambda), \mathcal{A}^*(\lambda)],$$
$$\mathcal{L}^*(\lambda) = \begin{pmatrix} -Z^T(\lambda\mathbf{I}_n - A)^{-1}\dot{\bar{Z}} & -Z^T(\mathbf{I}_n - \lambda A)^{-1}\bar{Z} \\ \mathbf{I}_r + \dot{Z}^T(\mathbf{I}_n - \lambda A)^{-1}\dot{\bar{Z}} & \dot{Z}^T(\lambda\mathbf{I}_n - A)^{-1}\bar{Z} \end{pmatrix}, \quad \mathcal{A}^*(\lambda) = \begin{pmatrix} 0 & \mathbf{I}_r \\ \bar{\Lambda} - \lambda\mathbf{I}_r & 0 \end{pmatrix},$$

• **Integrable discretization of the Neumann systems on** $V(n,r)$, a family of Bäcklund tansformations $\mathcal{B}_{\lambda^*}\ : (X,P)\mapsto(\tilde X,\tilde P)$

$$P = \ A^{1/2}(\lambda^*)\,\tilde X - X\,\Gamma(\lambda^*),$$
$$\tilde P = -A^{1/2}(\lambda^*)\,X + \tilde X\,\Gamma(\lambda^*),$$
$$A(\lambda)=\lambda\mathbf{I}_n - A, \quad \Gamma(\lambda^*) = \frac{1}{2}\left(\tilde X^T A^{1/2}(\lambda^*) X + X^T A^{1/2}(\lambda^*)\tilde X\right).$$

The alternative form (discrete Lagrange equations on $V(n,r)$) ($\lambda^*=0$, Veselov–Moser)

$$X+\tilde{\tilde X} = A^{-1/2}(\lambda^*)\tilde X B, \qquad B=\frac{1}{2}\left(\tilde X^T A^{1/2}(\lambda^*)(X+\tilde{\tilde X}) + (X+\tilde{\tilde X})^T A^{1/2}(\lambda^*)\tilde X\right)$$

For generic λ^*, the *complex* map $\mathcal{B}_{\lambda^*}$ is 2^r-valued.

Why the integrals of the continuous system are preserved?

Theorem 8. *The discrete Neumann system is equivalent to the intertwining* $2r\times 2r$ *matrix relation*

$$\tilde L(\lambda) M(\lambda|\lambda^*) = M(\lambda|\lambda^*) L(\lambda),$$
$$L(\lambda)=\begin{pmatrix} X^T(\lambda\mathbf{I}_n - A)^{-1}P & X^T(\lambda\mathbf{I}_n - A)^{-1}X \\ \mathbf{I}_r - P^T(\lambda\mathbf{I}_n - A)^{-1}P & -P^T(\lambda\mathbf{I}_n - A)^{-1}X \end{pmatrix} \text{ (as in the cont. case)},$$
$$M(\lambda|\lambda^*)=\begin{pmatrix} -\Gamma(\lambda^*) & \mathbf{I}_r \\ (\lambda-\lambda^*)\mathbf{I}_r + \Gamma^2(\lambda^*) & -\Gamma(\lambda^*) \end{pmatrix},$$

How the map is described on the invariant tori ?

Theorem 9. *The* 2^r*-valued map* $\mathcal{B}_{\lambda^*}$ *is given by translations by one of the vectors in* $\mathrm{Prym}(\mathcal{S}'/\sigma, \Omega)$

$$\tilde{\mathcal{A}}(\lambda_i^*) - \tilde{\mathcal{A}}(\infty_j).$$