Partial and superintegrabilty of Hamiltonian systems

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Outline

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Outline

Hamiltonian systems

- Complex analytic symplectic manifold: (M^{2n}, ω). Our main example

$$M^{2n} = \mathbb{C}^{2n}, \qquad \omega = \sum_{i=1}^{n} \mathrm{d}q_i \wedge \mathrm{d}p_i.$$

- Holomorphic Hamiltonian: $H : M^{2n} \to \mathbb{C}$. Our main example

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(q_1, \ldots, q_n),$$

where

$$V \in \mathbb{C}(q_1, \ldots, q_n) =: \mathbb{C}(\boldsymbol{q}).$$

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Hamiltonian systems

- Hamiltonian vector field: X_H, $\omega(X_H, Y) = \mathrm{d}H \cdot Y$. In our example

$$X_H = \sum_{i=1}^{n} \left(p_i \frac{\partial}{\partial q_i} - \frac{\partial V}{\partial q_i} \frac{\partial}{\partial p_i} \right).$$

- Poisson bracket: $\{F, G\} := \omega(X_F, X_G)$. In our example

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- First integrals of X_H: $\{H, F\} = 0$.

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Definition

A Hamiltonian system is partially integrable iff

- it admits independent first integrals $F_1 = H, \ldots, F_m$, $1 < m \leq n$, and
- $\{F_i, F_j\} = 0$, for $1 < i, j < m$.

Definition

A Hamiltonian system is superintegrable iff

- it admits independent first integrals $F_1 = H, \ldots, F_{n+m}$, $1 \leq m < n$, and
- $\{F_i, F_j\} = 0$, for $1 \leq i, j \leq n$.

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Our problem

Problem

Find necessary conditions for partial and superintegrabilty for an arbitrary $1 < m < n$.

. . . method?

Hint

In the frame of the of the differential Galois approach to the integrability (Morales-Ramis theory).

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Outline

Variational equations

Main Idea

A non-linear system leaves fingerprints of its properties in variational equations.

Main assumption: for a given holomorphic system

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \boldsymbol{v}(\boldsymbol{x}), \qquad \boldsymbol{x} \in M^n, \qquad t \in \mathbb{C} \tag{1}$$

we know a non-equilibrium particular solution $\varphi(t)$, Γ is the corresponding phase curve.

Variational equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi = A(t)\xi, \qquad A(t) = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}(\varphi(t)), \qquad \xi \in T_\Gamma M^n \tag{2}$$

Leading terms

Definition

The leading term f of a holomorphic function F is the lowest order term of an expansion

$$F(\varphi(t) + \xi) = F_m(\xi) + O(\|\xi\|^{m+1}), \qquad F_m \neq 0,$$

i.e., $f(\xi) := F_m(\xi)$. Note that $f(\xi)$ is a homogeneous polynomial with respect to $\xi = (\xi_1, \ldots, \xi_n)$ of degree m and its coefficients are polynomials in $\varphi(t)$.

Definition

If F is a meromorphic function, then $F = P/Q$ for certain holomorphic functions P and Q. In this case, the leading term f of F is defined as $f = p/q$, where p and q are leading terms of P and Q, respectively. As result $f(\xi)$ is a homogeneous rational function of ξ.

First integrals of the system and its VEs

First implication

If F is a meromorphic (holomorphic) first integral of the differential system, then its leading term f is a rational (polynomial) first integral of variational equations. Similarly, If the system possesses $k \geq 2$ functionally independent meromorphic first integrals $F_1, \ldots, F_k$, then, by the Ziglin Lemma, VEs have k functionally independent rational first integrals. Warning: generally they are NOT leading terms of $F_1, \ldots, F_k$!

Second implication

If $\mathcal{G} \subset \mathrm{GL}(n, \mathbb{C})$ is the differential Galois group of (2), and $f \in \mathbb{C}(\xi)$ its rational first integral of variational equations, then
$(g^{-1} \cdot f)(\xi) := f(g(\xi)) = f(\xi)$ for every $g \in \mathcal{G}$, i.e.,
f is a rational invariant of group $\mathcal{G}$.

Correspondence between first integrals of the system and invariants of DGG

Theorem

If system has k functionally independent first integrals which are meromorphic in a connected neighbourhood of a non-equilibrium solution $\varphi(t)$, then the differential Galois group $\mathcal{G}$ of the variational equations along $\varphi(t)$ has k functionally independent rational invariants.

$$\mathbb{C}(\boldsymbol{x})^{\mathcal{G}} := \{f \in \mathbb{C}(\boldsymbol{x}) \mid \; g \cdot f = f \quad \text{for all} \quad g \in \mathcal{G}\}$$

Fact

The differential Galois group $\mathcal{G}$ of a system of linear equations is a linear algebraic group, so in particular it is also a Lie group.

Passing to Lie algebras

$\mathfrak{g} \subset \mathrm{gl}(n, \mathbb{C})$ – the Lie algebra of $\mathcal{G}$.

With a $Y \in \mathfrak{g}$ we connect a linear vector field:

$$\boldsymbol{x} \mapsto Y(\boldsymbol{x}) := Y \cdot \boldsymbol{x}$$

for $\boldsymbol{x} \in \mathbb{C}^n$.

Definition

$f \in \mathbb{C}(\boldsymbol{x})$ is an integral of $\mathfrak{g}$, iff $L_Y(f) = 0$ for all $Y \in \mathfrak{g}$.

$$\mathbb{C}(\boldsymbol{x})^{\mathfrak{g}} := \{f \in \mathbb{C}(\boldsymbol{x}) \mid \; L_Y(f) = 0 \quad \text{for all} \quad Y \in \mathfrak{g}\}$$

Lemma

If $f_1, \ldots, f_k \in \mathbb{C}(\boldsymbol{x})^{\mathcal{G}}$ are algebraically independent invariants of an algebraic group $\mathcal{G} \subset \mathrm{GL}(n, \mathbb{C})$, then $f_1, \ldots, f_k \in \mathbb{C}(\boldsymbol{x})^{\mathfrak{g}}$, where $\mathfrak{g}$ is the Lie algebra of $\mathcal{G}$.

Normal Variational Equations

Fact

$\boldsymbol{v}(\varphi(t))$ is a non-zero solution of VEs

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\xi} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}(\varphi(t))\boldsymbol{\xi}.$$

The reduction gives NVEs

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\eta} = A_{\mathrm{N}}(t)\boldsymbol{\eta}, \qquad \boldsymbol{\eta} \in \mathbb{C}^{n-1}.$$

Formally

$$N = T_{\Gamma}M^n/T\Gamma, \qquad \pi : T_{\Gamma}M^n \to F,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\eta} = \pi_{\star}(T\boldsymbol{v}(\boldsymbol{x})\pi^{-1}(\boldsymbol{\eta})), \qquad \boldsymbol{\eta} \in N$$

Main Theorem

Fact

For a Hamiltonian system the differential Galois group $\mathcal{G}$ of variatonal equations is a subgroup of $\mathrm{Sp}(2n,\mathbb{C})$ and its Lie algebra $\mathfrak{g}$ is a Lie subalgebra of $\mathrm{sp}(2n,\mathbb{C})$.

Theorem (Morales-Ramis)

Assume that a Hamiltonian system is meromorphically integrable in the Liouville sense in a neighbourhood of the analytic phase curve Γ. Then the Lie algebra $\mathfrak{g}$ of the differential Galois group $\mathcal{G}$ of the variational equations along Γ is Abelian.

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Outline of the proof

1. Commuting independent first integrals $F_1, \dots, F_n$ of X_H give rational, commuting and indepenent first integrals $f_1, \dots f_n$ of variational variational equations (Ziglin)
2. Thus, $f_1, \dots f_n \in \mathbb{C}(\boldsymbol{x}, \boldsymbol{y})^{\mathfrak{g}}$, where $\mathfrak{g}$ is the Lie algebra of the differential Galois group of variational equations.
3. Missing point:

Lemma (Key Lemma)

If a Lie algebra $\mathfrak{g} \subset \mathrm{sp}(2n, \mathbb{C})$ admits n independent and commuting first integrals, then it is Abelian.

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NVEs for Hamiltonian Systems

Double reduction

Restrict Hamilton's equations to the level $E = H(\varphi(t))$ and then reduce VEs to the normal boundle.

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta = A_{\mathrm{N}}(t)\eta, \qquad \eta \in \mathbb{C}^{2n-2}.$$

Fact I

$\mathcal{G}_{\mathrm{N}}$ is a linear algebraic subgroup of $\mathrm{Sp}(2n-2, \mathbb{C})$.

Fact II

The Morales-Ramis Theorem is true if we change *VEs* to *NVEs*.

Outline

Maximally superintegrable Hamiltonian systems

Theorem (I)

Assume that a holomorphic Hamiltonian system with n degrees admits $2n-1$ first integrals which are meromorphic in a neighbourhood U of a phase curve Γ and independent in $U\setminus\Gamma$. Then the Lie algebra $\mathfrak{g}_{\mathrm{N}}$ of the differential Galois group $\mathcal{G}_{\mathrm{N}}$ of the normal variational equations along Γ is the zero algebra, i.e., $\mathcal{G}_{\mathrm{N}}$ is a finite subgroup of $\mathrm{Sp}(2n-2,\mathbb{C})$.

Proof

- $\mathcal{G}_{\mathrm{N}}$ admits $2n-2$ independent rational first integrals $f_1, \ldots, f_{2n-2}$;
- for each $Y \in \mathfrak{g}_{\mathrm{N}} \subset \mathrm{sp}(2n-2, \mathbb{C})$, $Y(f_i) = 0$ for $i = 1, \ldots, 2n-2$, thus $Y = 0$.

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Maximally superintegrable non-Hamiltonian systems

Theorem (II)

Assume that a holomorphic n-dimensional system admits $n-1$ first integrals which are meromorphic in a neighbourhood U of a phase curve Γ and independent in $U \setminus \Gamma$. Then the Lie algebra $\mathfrak{g}_{\mathrm{N}}$ of the differential Galois group $\mathcal{G}_{\mathrm{N}}$ of the normal variational equations along Γ is the zero algebra, i.e., $\mathcal{G}_{\mathrm{N}}$ is a finite subgroup of $\mathrm{GL}(n-1, \mathbb{C})$.

Darboux Points and Particular Solutions

Assumption

Potential $V \in \mathbb{C}(\boldsymbol{q})$ is homogeneous and $\deg V = k \in \mathbb{Z}^\star$.

Definition

Darboux point $\boldsymbol{d} \in \mathbb{C}^n$ is a solution of

$$V'(\boldsymbol{d}) = \boldsymbol{d}, \qquad \boldsymbol{d} \neq \mathbf{0}.$$

Particular solution

$$\boldsymbol{q}(t) = \varphi(t)\boldsymbol{d}, \quad \boldsymbol{p}(t) = \dot{\varphi}(t)\boldsymbol{d}, \quad \text{provided} \quad \ddot{\varphi} = -\varphi^{k-1}.$$

Phase curve Γ_ε:

$$\dot{\varphi}^2 = \frac{2}{k}\left(\varepsilon - \varphi^k\right)$$

Variational equations

$$\ddot{\boldsymbol{x}} = -\varphi(t)^{k-2} V''(\boldsymbol{d})\boldsymbol{x}.$$

If $V''(\boldsymbol{d})$ is diagonalisable, then in an appropiate base

$$\ddot{y}_i = -\lambda_i \varphi(t)^{k-2} y_i, \qquad 1 \leq i \leq n, \tag{3}$$

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $V''(\boldsymbol{d})$. One of these eigenvalues, let us say λ_n is $k-1$.

Differential Galois group

$$\mathcal{G} \subset \mathcal{G}(\lambda_1) \times \cdots \times \mathcal{G}(\lambda_n) \subset \mathrm{Sp}(2n, \mathbb{C}), \qquad \mathcal{G}(\lambda_i) \subset \mathrm{Sp}(2, \mathbb{C}).$$

and

$$\mathcal{G}_{\mathrm{N}} \subset \mathcal{G}(\lambda_1) \times \cdots \times \mathcal{G}(\lambda_{n-1}) \subset \mathrm{Sp}(2n-2, \mathbb{C}), \qquad \mathcal{G}(\lambda_i) \subset \mathrm{Sp}(2, \mathbb{C}).$$

Hence

$$\mathfrak{g} \subset \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n,$$

and

$$\mathfrak{g}_{\mathrm{N}} \subset \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n-1},$$

where $\mathfrak{g}_i$ is a Lie subalgebra of $\mathrm{sp}(2, \mathbb{C})$, for $i = 1, \ldots, n$.

Transformation to hypergeometric equations

$$\ddot{\eta} = -\lambda \varphi(t)^{k-2}\eta,$$

$$\Gamma_\varepsilon : \qquad \varepsilon = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{k}\varphi^k.$$

Differential Galois group $\mathcal{G}(k,\lambda) \subset \mathrm{Sp}(2,\mathbb{C})$.
Yoshida transformation

$$z := \frac{1}{\varepsilon k}\varphi(t)^k.$$

$$\left.\begin{aligned} &z(1-z)\eta'' + [c-(a+b+1)z]\eta' - ab\eta = 0, \\ &a+b = \frac{k-2}{2k}, \qquad ab = -\frac{\lambda_i}{2k}, \qquad c = 1-\frac{1}{k}. \end{aligned}\right\} \tag{H}$$

Differenial Galois group $G(k,\lambda) \subset \mathrm{GL}(2,\mathbb{C})$.

Properties of $G(k,\lambda)^\circ$

Fact

The identity component $\mathcal{G}(k,\lambda)^\circ$ is isomorphic to $G(k,\lambda)^\circ$.

Proposition

If $G(k,\lambda)^\circ$ is solvable then it is Abelian.

Lemma (K)

The identity component $G(k,\lambda)^\circ$ of the differential Galois group of hypergeometric equation (H) *is Abelian if and only if (k,λ) belong to the following list*

Properties $G(k,\lambda)^{\circ}$

case	k	λ
1.	± 2	λ
2.	k	$p+\dfrac{k}{2}p(p-1)$
3.	k	$\dfrac{1}{2}\left(\dfrac{k-1}{k}+p(p+1)k\right)$
4.	3	$-\dfrac{1}{24}+\dfrac{1}{6}(1+3p)^2\,,\quad -\dfrac{1}{24}+\dfrac{3}{32}(1+4p)^2$ $-\dfrac{1}{24}+\dfrac{3}{50}(1+5p)^2\,,\quad -\dfrac{1}{24}+\dfrac{6}{25}(1+5p)^2$
5.	4	$-\dfrac{1}{8}+\dfrac{2}{9}(1+3p)^2$

Properties $G(k, \lambda)^{\circ}$

case	k	λ
6.	5	$-\dfrac{9}{40} + \dfrac{5}{18}(1+3p)^2, \quad -\dfrac{9}{40} + \dfrac{2}{5}(1+5p)^2$
7.	-3	$\dfrac{25}{24} - \dfrac{1}{6}(1+3p)^2, \quad \dfrac{25}{24} - \dfrac{3}{32}(1+4p)^2$
		$\dfrac{25}{24} - \dfrac{3}{50}(1+5p)^2, \quad \dfrac{25}{24} - \dfrac{6}{25}(1+5p)^2$
8.	-4	$\dfrac{9}{8} - \dfrac{2}{9}(1+3p)^2$
9.	-5	$\dfrac{49}{40} - \dfrac{5}{18}(1+3p)^2, \quad \dfrac{49}{40} - \dfrac{2}{5}(1+5p)^2$

where p is an integer and λ an arbitrary complex number.

Outline

Theorem

Assume that the Hamiltonian a natural Hamiltonian system system with a homogeneous potential $V \in \mathbb{C}(\boldsymbol{q})$ of degree $k \in \mathbb{Z}^\star$ satisfies the following conditions:

1. *there exists a non-zero $\boldsymbol{d} \in \mathbb{C}^n$ such that $V'(\boldsymbol{d}) = \boldsymbol{d}$, and*
2. *matrix $V''(\boldsymbol{d})$ is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_{n-1}, \lambda_n = k - 1$;*
3. *the system admits $2n - 1$ functionally independent first integrals $F_1 = H, F_2, \ldots, F_{2n-1}$ which are meromorphic in a connected neighbourhood of phase curve Γ_ε.*

Our Theorem

Theorem (continuation)

Then each (k, λ_i) *belongs to the list from Lemma K, and moreover*

- *if* $|k| > 2$*, then each pair* (k, λ_i) *for* $1 \leq i \leq n-1$*, belongs to items 3–9 of the table from Lemma K;*
- *if* $|k| \leq 2$*, then each pair* (k, λ_i)*, for* $1 \leq i \leq n-1$ *belongs to the following list*

case	k	λ
I.	-2	$1 - r^2$
II.	-1	1
III.	1	0
IV.	2	r^2

(4)

where $r \in \mathbb{Q}^\star$*;*

Separable potential

$$V = Aq_1^k + Bq_2^k$$

- Darboux points $\boldsymbol{d}_1 = (0, (\frac{1}{Bk})^{1/(k-2)})$ and $\boldsymbol{d}_2 = ((\frac{1}{Ak})^{1/(k-2)}, 0)$ for $k \neq 2$; for $\boldsymbol{d} = (1, 0)$ and $\boldsymbol{d} = (0, 1)$
- non-trivial eigenvalues $\lambda(\boldsymbol{d}_i) = 0$ for $k \neq 2$; for $k = 2$ $\lambda(\boldsymbol{d}_1) = B/A$ and $\lambda(\boldsymbol{d}_2) = A/B$
- by our theorem, if V is integrable, then either $k = -2$, or $k = 1$ or $k = 2$ and, in this last case, $A/B = r^2$ for $r \in \mathbb{Q}^\star$.

Radial potential

$$V = \alpha r^k, \quad r = \sqrt{q_1^2 + q_2^2}$$

- infintely many Darboux points
- non-trivial eigenvalue at each of them $\lambda(\boldsymbol{d}) = 1$. Thus, by our theorem, if V is superintegrable, then $k = -1$ or $k = 2$.

Three body problem

$$V = \frac{1}{k}\left[(q_1 - q_2)^k + (q_2 - q_3)^k + (q_3 - q_1)^k\right], \qquad k \in \mathbb{Z} \setminus \{0, 1\}$$

$$F_2 = p_1 + p_2 + p_3,$$

Lemma

Assume that $k \in \mathbb{Z} \setminus \{-2, 0, 1, 2, 4\}$. Then the potential V is not integrable by meromorphic first integrals in the Liouville sense.

- $k = 4$, one additional first integral F_3;
- $k = 2$ two additional first integrals F_3 and F_4;
- $k = -2$ three additional first integrals F_3, F_4 and F_5;

Lemma

Assume that $k \in \mathbb{Z} \setminus \{0, 1, -2\}$. Then the potential V is not maximally superintegrable by meromorphic first integrals.

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Lemma

Assume that $k \in \mathbb{Z} \setminus \{0, 1, -2\}$. Then the potential V is not maximally superintegrable by meromorphic first integrals.

Outline

Restriction of the problem

Problem

Whether a Hamiltonian system admits just one odditional first integral?

Lemma

If H admits an additional first integral meromorphic in a neighbourhood of Γ*, then* $\dim \mathcal{G}_{\mathrm{N}} < (n-1)(2n-1)$*, i.e.,* $\dim \mathcal{G}_{\mathrm{N}}$ *is not maximal.*

Certain Kolchin Theorem

$$w_i'' = r_i w_i, \qquad r_i \in \mathbb{C}(z), \quad i = 1, \dots m, \tag{P}$$

Theorem (Kolchin)

Assume then the differential Galois group G of the system (P) *has dimension smaller than* $3m$. *Then either the differential Galois group G_i for i-the equation has dimension smaller than 3, or there exist indices $1 \leq i < j \leq m$ and fundamental matrices W_i and W_j of solutions of i-th and j-th equations, such that*

$$W_i = \alpha A W_j \tag{5}$$

for a certain 2×2 matrix A with coefficients in $\mathbb{C}(z)$ and $\alpha^2 \in \mathbb{C}(z)$.

Our Lemma

Lemma

Assume that each equation in the product (P) *is Fuchsian and has singularities at* $S := \{z_1, \dots, z_l\} \in \mathbb{CP}^1$, *and differential Galois group* G *of the system* (P) *has dimension smaller than* $3m$. *If the second possibility the Kolchin Theorem occurs for* i*-th and* j*-th equations, then for all* $z_\star \in S$ *the local monodromy matrices* $M_i(z_\star)$, $M_j(z_\star)$ *around point* $z_\star$ *of the respective equations, satisfy either* $M_i(z_\star) = M_j(z_\star)$, *or* $M_i(z_\star) = -M_j(z_\star)$.

Our Theorem

Theorem

Assume that a homogeneous potential $V \in \mathbb{C}(\boldsymbol{q})$ of degree $k \in \mathbb{Z}^\star$ satisfies the following conditions:

1. *there exists a non-zero $\boldsymbol{d} \in \mathbb{C}^n$ such that $V'(\boldsymbol{d}) = \boldsymbol{d}$, and*
2. *matrix $V''(\boldsymbol{d})$ is semi-simple with eigenvalues $\lambda_1, \ldots, \lambda_n = k - 1$;*
3. *the system admits an additional meromorphic first integral F in a connected neighbourhood of the phase curve Γ_ε.*

Then either:

A1. *there exists $1 \leq r < n$ such that pair (k, λ_r) belongs to the list from Lemma K, or*

A2. *there exist $1 \leq i < j < n$ such that*

$$\frac{1}{2k}\sqrt{(k-2)^2 + 8k\lambda_i} = \frac{1}{2k}\sqrt{(k-2)^2 + 8k\lambda_j} + p, \qquad (6)$$

for some $p \in \mathbb{Z}$.

The second possibility

$$V = (q_1^2 + q_2^2)^2 + \frac{1}{2}\lambda q_3^2(q_1^2 + q_2^2) + \frac{1}{4}q_3^4, \quad \lambda \in \mathbb{C}.$$

- First integral $F = q_1 p_2 - q_2 p_1$.
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