The inception of Symplectic Geometry: the works of Lagrange and Poisson during the years 1808–1810

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Historical overview

The word "symplectic"

It seems that the word *symplectic* was used for the first time with its modern mathematical meaning by Hermann Weyl (1885–1955), in his book *Classical groups* [16]. It derives from a Greek word meaning *complex*, used by Weyl because the word *complex*, whose origin is Latin, was already in use in Mathematics with a different meaning.

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The concept of a symplectic structure

That concept appeared in Mathematics much earlier than the word *symplectic*, in the works of Joseph Louis Lagrange (1736–1813), first in his paper about the slow changes of the orbital elements of planets in the solar system, then in a following paper a little later, as a fundamental ingredient in the mathematical formulation of any problem in Mechanics.

Orbital elements of the planets (1)

First Kepler's law As a first (and very good) approximation, the orbit of each planet in the solar system is an ellipse, with the Sun at one of its foci : it is the first law discovered by *Johannes Kepler (1571–1630)*).

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Under Kepler's approximation, the knowledge of the *orbital elements* of a planet completely determines its position in space, for all times, past, present and future.

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- 2 for the determination of the plane which contains its orbit ; for example, once chosen a reference plane containing the Sun and a reference direction in that plane,
— the angle between that reference direction and the intersection of the orbit's plane with the reference plane,
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- a last one to determine the position of the planet on its orbit : its position at a particular time chosen for oirigin.

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J.-M. Souriau [14] has rigorously proven that the set of all possible motions of a given planet in the solar system, under Kepler's approximation, is indeed a smooth manifold of dimension 6 : the *manifold of motions* of the planet.

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By a method called *regularization of collisions*, we may avoid to exclude singular motions with collisions ; there is still a smooth manifold of motions, but it becomes non Hausdorff.

Beyond Kepler's approximation

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But in fact, even if we do not take into account the gravitational interaction between planets, the orbit of each planet would be an ellipse whose focus is the *center of mass* of the system planet-Sun, not the center of the Sun. That center of mass is different for each planet. Therefore the planets have two kinds of gravitational interactions between them : their direct mutual interactions, and the interaction that each of them exerts on all the others through its interaction with the Sun.

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It is to deal with that problem that Lagrange created the *method of variation of the constants*.

aplace, Lagrange and Poisson : chronolog

1773 : Laplace proves that there is no first order secular variation of the major axis of the elliptical orbits of the planets.

1776, 1781, 1782, . . . : Lagrange improves Laplace's result and considers the slow variations of other orbital elements.

20 June 1808 : Poisson introduces a new method in *Sur les inégalités séculaires des moyens mouvements des planètes*.

22 August 1808 : Lagrange considers again the same problem in *Mémoire sur la théorie des variations des éléments des planètes*.

13 March 1809 : Lagrange extends his method in *Mémoire sur la théorie générale de la variation des constantes arbitraires dans tous les problèmes de mécanique*.

agrange, Poisson and others : chronology (

16 October 1809 : Poisson introduces the *Poisson bracket* in *Sur la variation des constantes arbitraires dans les questions de mécanique*.

19 February 1810 : Lagrange presents his *Second mémoire sur la théorie de la variation des constantes arbitraires dans les problèmes de mécanique*. He recognizes Poisson's contribution, but claims that the main ideas already were in his previous paper.

15 January 1835 : Hamilton introduces what we call today *Hamilton's formalism* in his *Second essay on a general method in Dynamics*.

1837 (or maybe 1831 ?) : Cauchy gives a very clear presentation of Lagrange's method, using Hamilton's formalism, in his *Note sur la variation des constantes arbitraires dans les problèmes de mécanique*.

The method of varying constants

Lagrange had already used the idea of varying integration constants in his work on ordinary linear nonhomogeneous differential equations [7].

He used a similar idea to describe the planets as moving around the Sun on ellipses with slowly varying in time (instead of rigorously constant) orbital elements.

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He called his method *Méthode de variation des constantes* and presented it at the French Academy of Sciences on the 13rd of March 1809 [9].

Lagrange's paper of 1809 (1)

Lagrange considers a mechanical system with kinetic energy

$$T = T(r, s, u, \dots, r', s', u' \dots)\,,$$

where r, s, u, $\dots$ are independent real variables which describe the system's position in space. For a planet moving around the Sun, these variables are the three coordinates or the planet (in some reference frame). Let n be the number of these variables. In modern words, n is the dimension of the *configuration manifold*.

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Quantities r', s', u', $\dots$, are the derivatives of r, s, u, $\dots$, with respect to time t :

$$r' = \frac{dr}{dt}, \quad s' = \frac{ds}{dt}, \quad u' = \frac{du}{dt}, \quad , \dots$$

Lagrange's paper of 1809 (2)

As a first approximation, Lagrange assumes that the forces which act on t he system come from a potential V, which depends on r, s, u, $\ldots$, but not of the tume derivatives r', s', u', $\ldots$ For a planet's motion, V is the gravitational potential due to the Sun's attraction. The equations which govern the motion (established by Lagrange in his book [11]) are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial r'}\right) - \frac{\partial T}{\partial r} + \frac{\partial V}{\partial r} = 0\,,$$

and similar equations in which r and r' are replaced by s and s', u and u', $\ldots$

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and similar equations in which r and r' are replaced by s and s', u and u', $\ldots$

The general solution of this system of n second-order equations depends on the time t and on $2n$ integration constants. Lagrange denotes them a, b, c, f, g, $h, \ldots$

Lagrange's paper of 1809 (3)

That general solution may be written as

$$r = r(t, a, b, c, f, g, h, \ldots), \quad s = s(t, a, b, c, f, g, h, \ldots), \quad u = \ldots .$$

For a planet's motion, the $2n$ integration constants a, b, c, f, g, h, $\ldots$ are the *orbital elements* of the planet.

Lagrange's paper of 1809 (4)

As a better approximation, Lagrange assumes that the potential V does not fully describe the forces which act on the system, and should be replaced by $V - \Omega$, where Ω may depend on r, s, u, $\ldots$, and on the time t. For a planet's motion, Ω describes the gravitational interactions between the planet under consideration and all the other planets, which were considered as negligible in the first approximation. Ω depends on time, because the planets which are the source of these gravitational interaction are moving. The equations become

$$\frac{d}{dt}\left(\frac{\partial T}{\partial r'}\right) - \frac{\partial T}{\partial r} + \frac{\partial V}{\partial r} = \frac{\partial \Omega}{\partial r},$$

and similar equations in which r and r' are replaced by s and s', u and u', $\ldots$

Lagrange's paper of 1809 (5)

Lagrange writes the solution of this new system under the form

$$r = r\Big(t, a(t), b(t), c(t), f(t), g(t), h(t), \dots\Big)\,,$$

and similar expressions for s, u, $\dots$. The function

$$(t, a, b, c, f, g, h, \dots) \mapsto r(t, a, b, c, f, g, h \dots)$$

which appears in this expression, and the similar fuctions which appear in the expressions of s, u, $\dots$ are, of course, those previously found when solving the problem in its first approximation, with Ω replaced by 0. These functions are therefore considered as *known*.

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It only remains to find the $2n$ functions of time $t \mapsto a(t)$, $t \mapsto b(t)$, $\dots$. Of course, these functions will depend on time and on $2n$ arbitrary integration constants.

Lagrange's parentheses (1)

Lagrange obtains the differential equations which govern the time variations of these functions $a(t)$, $b(t)$, $\ldots$. The calculations by which he obtains these equations are at first very complicated, and he makes two successive improvements, first in an *Addition*, then in a *Supplément* to his initial paper. He finds a remarkable property : these equations become very simple when they are expressed in terms of quantities that he denotes by (a, b), (a, c), (a, f), (b, c), (b, f), $\ldots$. Today, these quantities are still in use and called *Lagrange's parentheses*.

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Lagrange's parentheses are functions of a, b, c, f, g, h, $\ldots$. They do not depend on time, nor on the additional forces which act on the system when Ω is taken into account.

Lagrange's parentheses (2)

J.-M. Souriau [15, 5] has shown that Lagrange's parentheses are the components of the *canonical symplectic* 2*-form* on the manifold of motions of the mechanical system, in the chart of that manifold whose local coordinates are a, b, c, f, g, h, $\ldots$. So Lagrange discovered the notion of a symplectic structure more than 100 years before that notion was so named by H. Weyl [16].

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We insist on the fact that Lagrange's parentheses are relative to the mechanical system with kinetic energy T and applied forces described by the potential V. The additional forces described by Ω play no part in Lagrange's parentheses : the consideration of these additional forces allowed the discovery of a structure in which they take no part !

Lagrange's parentheses (3)

At first, Lagrange obtained for the parentheses (a, b), (a, c), (b, c), $\ldots$ very complicated expressions. In the *Addition* to his paper (Section 26 of [9]), he obtains the much simpler texpressions which today are still in use :

$$(a, b) = \frac{\partial r}{\partial a}\frac{\partial p_r}{\partial b} - \frac{\partial r}{\partial b}\frac{\partial p_r}{\partial a} + \frac{\partial s}{\partial a}\frac{\partial p_s}{\partial b} - \frac{\partial s}{\partial b}\frac{\partial p_s}{\partial a} + \frac{\partial u}{\partial a}\frac{\partial p_u}{\partial b} - \frac{\partial u}{\partial b}\frac{\partial p_u}{\partial a} + \cdots ,$$

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and similar expressions for (a, c), (b, c), $\ldots$ We have set, as Hamilton [2,3] and Cauchy wil do 30 years later

$$p_r = \frac{\partial T}{\partial r'}, \quad p_s = \frac{\partial T}{\partial s'}, \quad p_u = \frac{\partial T}{\partial u'}.$$

Lagrange used the less convenient notations T', T'' and T''' instead of p_r, p_s and p_u.

Lagrange's parentheses (4)

We recall that r, s, u, $\ldots$ are local coordinates on the configuration manifold of the system, and r', s', u' their partial derivatives with respecto to time. The kinetic energy T, which depends on r, s, u, $\ldots$, r', s', u', $\ldots$, is a function defined on the tangent bundle to the configuration manifold, which is called the *manifold of kinematic states* of the system.

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$$(r, s, u, \dots, r', s', u', \dots) \mapsto (r, s, u, \dots, p_r, p_s, p_u, \dots),$$

called the *Legendre transformation*, is defined on the tangent bundle to the configuration manifold, and takes its values in the cotangent budle to that manifold, called the *phase space* of the system.

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called the *Legendre transformation*, is defined on the tangent bundle to the configuration manifold, and takes its values in the cotangent budle to that manifold, called the *phase space* of the system. When the kinetic energy is a positive definite quadratic form, that map is a diffeomorphism. This occurs very often, for example in the mechanical system considered by Lagrange, of the motion of a planet around the Sun.

Lagrange's parentheses (5)

Since the integration constants a, b, c, f, g, h, $\ldots$ make a system of local coordinates on the manifold of motions, they completely determine the motion of the system. We insist again that it is the system in his *first approximation*, with Ω replaced by 0. Therefore, for each time t, the instantaneous values of the quantities r, s, u, $\ldots$, r', s', u', $\ldots$, are determined as soon as a, b, c, f, g, h, $\ldots$ are given

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Conversely, the existence and uniqueness theorem for solutions of ordinary differential equations (implicitly considered as an evidence by Lagrange, at least for Kepler's problem whose solutions are explicitly known) shows that when the values of r, s, u, $\ldots$, r', s', u', $\ldots$ at any given time t are known, then the motion is determined, so a, b, c, f, g, h, $\ldots$ are known.

Lagrange's parentheses (6)

In short, for each time t, the map which associates to a motion of coordinates $(a,\ b,\ c,\ f,\ g,\ h,\ldots)$ the values at time t of $(r,\ s,\ u,\ldots,r',\ s',\ u',\ldots)$ is a diffeomorphism from the manifold of motions onto the manifold of kinematic states of the system.

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We compose that diffeomorphism with Legendre's transformation and we get, for each time t, a diffeomorphism from the manifold of motions onto the phase space

$$(a,\ b,\ c,\ f,\ g,\ h, \ldots) \mapsto \bigl(r(t),\ s(t),\ u(t), \ldots,\ p_r(t),\ p_s(t),\ p_u(t), \ldots\bigr)$$

(where $r(t)$, $s(t)$, $u(t)$, $p_r(t)$, $p_s(t)$, $p_u(t)$ are the values taken at time t by the corresponding quantities).

Lagrange's parentheses (7)

The partial derivatives which appear in the expression of Lagrange's parentheses

$$(a,b) = \frac{\partial r}{\partial a}\frac{\partial p_r}{\partial b} - \frac{\partial r}{\partial b}\frac{\partial p_r}{\partial a} + \frac{\partial s}{\partial a}\frac{\partial p_s}{\partial b} - \frac{\partial s}{\partial b}\frac{\partial p_s}{\partial a} + \frac{\partial u}{\partial a}\frac{\partial p_u}{\partial b} - \frac{\partial u}{\partial b}\frac{\partial p_u}{\partial a} + \cdots\,.$$

are the partial derivatives of the diffeomorphism

$$(a,\ b,\ c,\ f,\ g,\ h,\ldots) \mapsto (r(t),\ s(t),\ u(t),\ldots,\ p_r(t),\ p_s(t),\ p_u(t),\ldots)$$

where t is any value of the time, considered as fixed.

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$$(a,\ b,\ c,\ f,\ g,\ h,\ldots) \mapsto (r(t),\ s(t),\ u(t),\ldots,\ p_r(t),\ p_s(t),\ p_u(t),\ldots)$$

where t is any value of the time, considered as fixed.

Important remark Lagrange's parenthesis (a,b) has a meaning when a complete system of local coordinates $(a,\ b,\ c,\ f,\ g,\ h,\ldots)$ has been chosen on the manifold of motion : (a,b) depends not only of the functions a and b on that manifold : it also depends on all the other coordinates functions c, f, g, h, $\ldots$

Lagrange's parentheses (8)

Let us consider again the diffeomorphism

$$(a,\ b,\ c,\ f,\ g,\ h,\ldots) \mapsto (r(t),\ s(t),\ u(t),\ldots,\ p_r(t),\ p_s(t),\ p_u(t),\ldots)$$

where t is any value of the time, considered as fixed.

Lagrange's parentheses (8)

Let us consider again the diffeomorphism

$$(a,\ b,\ c,\ f,\ g,\ h, \ldots) \mapsto (r(t),\ s(t),\ u(t), \ldots,\ p_r(t),\ p_s(t),\ p_u(t), \ldots)$$

where t is any value of the time, considered as fixed.

Calculus of exterior differential forms, created by Élie Cartan at the beginning of the XX-th century, did not exist in Lagrange's times. Today, with this very efficient tool, it is very easy to check that Lagrange's parentheses are the components of the pull-back by that diffeomorphism, on the manifold of motions, of the canonical symplectic 2 form of the cotangent bundle to the configuration manifold.

Lagrange's parentheses (9)

$$
\begin{aligned}
&(a,b)\,da\wedge db+(a,c)\,da\wedge dc+\cdots+(b,c)\,db\wedge dc+\cdots\\
&\quad=\left(\frac{\partial r}{\partial a}da+\frac{\partial r}{\partial b}db+\cdots\right)\wedge\left(\frac{\partial p_r}{\partial a}da+\frac{\partial p_r}{\partial b}db+\cdots\right)\\
&\qquad+\left(\frac{\partial s}{\partial a}da+\frac{\partial s}{\partial b}db+\cdots\right)\wedge\left(\frac{\partial p_s}{\partial a}da+\frac{\partial p_s}{\partial b}db+\cdots\right)+\ldots\\
&\quad=dr\wedge dp_r+ds\wedge dp_s+du\wedge dp_u+\cdots\,.
\end{aligned}
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$$+\left(\frac{\partial s}{\partial a}da+\frac{\partial s}{\partial b}db+\cdots\right)\wedge\left(\frac{\partial p_s}{\partial a}da+\frac{\partial p_s}{\partial b}db+\cdots\right)+\dots$$
$$=dr\wedge dp_r+ds\wedge dp_s+du\wedge dp_u+\cdots\,.$$

That expresses a symplectic 2-form in Darboux coordinates.

Lagrange's parentheses (9)

$$
\begin{aligned}
&(a,b)\,da\wedge db+(a,c)\,da\wedge dc+\cdots+(b,c)\,db\wedge dc+\cdots\\
&\quad=\left(\frac{\partial r}{\partial a}da+\frac{\partial r}{\partial b}db+\cdots\right)\wedge\left(\frac{\partial p_r}{\partial a}da+\frac{\partial p_r}{\partial b}db+\cdots\right)\\
&\qquad+\left(\frac{\partial s}{\partial a}da+\frac{\partial s}{\partial b}db+\cdots\right)\wedge\left(\frac{\partial p_s}{\partial a}da+\frac{\partial p_s}{\partial b}db+\cdots\right)+\ldots\\
&\quad=dr\wedge dp_r+ds\wedge dp_s+du\wedge dp_u+\cdots\,.
\end{aligned}
$$

That expresses a symplectic 2-form in Darboux coordinates.

Lagrange has proven that although defined by means of a diffeomorphism which depends on time, his parentheses *do not depend directly on time* : they are functions on the manifold of motions. By proving that result, Lagrange has proven that *the canonical symplectic 2-form on phase space is invariant under the flow of the evolution vector field on that space*.

Formulae for the variation of constants

Lagrange proves that the derivatives with respect to time t of the "constants that are varied" a, b, $\ldots$, satisfy

$$\sum_{j=1}^{2n}(a_i, aj)\frac{da_j}{dt} = \frac{\partial\Omega}{\partial a_i}\,, \qquad 1 \leq i \leq 2n\,,$$

where, for short, I have written a_i, $1 \leq i \leq 2n$ instead of a, b, c, $\ldots$, and where I have taken into account the skew-symmetry $(a_j, a_i) = -(a_i, a_j)$.

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Lagrange indicates that by solving that linear system, one obtains something like

$$\frac{da_i}{dt} = \sum_{j=1}^{2n} L_{i\,j}\frac{\partial\Omega}{\partial a_j}, \quad 1 \leq i \leq 2n\,.$$

Formulae for the variation of constants (2)

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But Lagrange does not state their explicit expression.

That will be done by *Siméon Denis Poisson* (1781–1840) a few months later.

Poisson's paper of 1809

When he was a student at École Polytechnique, *Siméon Denis Poisson* (1781–1840) attended lectures by Lagrange. In a paper read at the French Academy of Sciences on the 16th of October 1809 [13], he adds a relatively important ingredient to Lagrange's *method of varying constants*. He introduces new quantities, defined on the manifold of motions, which he denotes by (a, b), (a, c), $\ldots$ These quantities *are not* Lagrange's parentheses. Today, they are called *Poisson brackets*. In his paper, Poisson uses also Lagrange's parentheses but he denotes them differently, by $[a, b]$ instead of (a, b), $[a, c]$ instead of (a, c), $\ldots$.

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We will keep Lagrange's notations (a, b), (a, c), $\ldots$ for Lagrange's parentheses and we will use $\{a, b\}$, $\{a, c\}$, $\ldots$ for Poisson brackets.

Poisson brackets (1)

The expression of Poisson brackets is

$$\{a, b\} = \frac{\partial a}{\partial p_r}\frac{\partial b}{\partial r} - \frac{\partial a}{\partial r}\frac{\partial b}{\partial p_r} + \frac{\partial a}{\partial p_s}\frac{\partial b}{\partial s} - \frac{\partial a}{\partial s}\frac{\partial b}{\partial p_s} + \frac{\partial a}{\partial p_u}\frac{\partial b}{\partial u} - \frac{\partial a}{\partial u}\frac{\partial b}{\partial p_u} + \cdots .$$

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Of course $\{a,c\}$, $\{b,c\}$, $\ldots$ are given by similar formulae. We observe that in these formulae appear the partial derivatives of the local coordinates a, b, c, $\ldots$ on the manifold of motions, considered as functions of the dynamical state of the system at time t, considered as fixed. The independent variables which describe that dynamical states are the values ; at time t, of the quantities r, p_r, s, p_s, u, p_u, $\ldots$.

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The above formula is the well known expression of the Poisson bracket of two functions a and b defined on a symplectic manifold, in Darboux coordinates.

Comparison of Poisson brackets with Lagrange's parentheses

$$\{a,b\} = \frac{\partial a}{\partial p_r}\frac{\partial b}{\partial r} - \frac{\partial a}{\partial r}\frac{\partial b}{\partial p_r} + \frac{\partial a}{\partial p_s}\frac{\partial b}{\partial s} - \frac{\partial a}{\partial s}\frac{\partial b}{\partial p_s} + \frac{\partial a}{\partial p_u}\frac{\partial b}{\partial u} - \frac{\partial a}{\partial u}\frac{\partial b}{\partial p_u} + \cdots .$$

$$(a,b) = \frac{\partial r}{\partial a}\frac{\partial p_r}{\partial b} - \frac{\partial r}{\partial b}\frac{\partial p_r}{\partial a} + \frac{\partial s}{\partial a}\frac{\partial p_s}{\partial b} - \frac{\partial s}{\partial b}\frac{\partial p_s}{\partial a} + \frac{\partial u}{\partial a}\frac{\partial p_u}{\partial b} - \frac{\partial u}{\partial b}\frac{\partial p_u}{\partial a} + \cdots .$$

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$$(a,b) = \frac{\partial r}{\partial a}\frac{\partial p_r}{\partial b} - \frac{\partial r}{\partial b}\frac{\partial p_r}{\partial a} + \frac{\partial s}{\partial a}\frac{\partial p_s}{\partial b} - \frac{\partial s}{\partial b}\frac{\partial p_s}{\partial a} + \frac{\partial u}{\partial a}\frac{\partial p_u}{\partial b} - \frac{\partial u}{\partial b}\frac{\partial p_u}{\partial a} + \cdots\,.$$

We see that these formulae involve the partial derivatives of two diffeomorphisms, each one being the inverse of the other one : the Poisson bracket involves the partial derivatives of coordinates a, b, $\dots$ on the manifold of motions with respect to coordinates r, s, r', s', $\dots$ on the pjhase space, while Lagrange's parentheses involve the partial derivatives of r, s, r', s', $\dots$ with respect to a, b, $\dots$

isson brackets vs. Lagrange's parentheses

As a conclusion :

- Lagrange's parentheses (a, b), (a, c), $\ldots$, are the components of the symplectic 2-form on the manifold of motions, in the chart of that manifold whose local coordinates are a, b, c, $\ldots$

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The matrix whose components are the Lagrange's parentheses (a, b), (a, c), $\ldots$, and the matrix whose components are the Poisson brackets $\{a, b\}$, $\{a, c\}$, $\ldots$, *are inverse of each other*. That property was clearly stated by Augustin Louis Cauchy (1789–1857) in a paper read at the Academy of Torino on 11th October 1831, 22 years later than Lagrange's and Poisson papers.

Poisson theorem

Important remark The Poisson bracket can be defined for any pair of smooth functions on the manifold of motions, and depends only on these two functions. Lagrange's parenthesis of two smooth functions has no meaning : Lagrange's parentheses *can be defined only for coordinates functions* : in other words, Lagrange's parenthesis (a, b) does not depend only on a and b : it depends on all the coordinate functions a, b, c, $\ldots$

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Poisson theorem It states that the Poisson bracket of two first integrals is a first integral, that means a function which remains constant on each trajectory of the system. Today, that result is presented as a consequence of *Jacobi identity*. That identity *was not known by Lagrange, nor by Poisson*, who considered the constancy of the Poisson bracket of two first integrals as due to the fact that it is a function defined on the manifold of motions.

Jacobi identity

Lagrange et Poisson noted the skew-symmetry of their parentheses and brackets, but said nothing about Jacobi identity for the Poisson bracket, nor about the relations between Lagrange's parentheses expressing that they are the components of a *closed* 2-form.

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Carl Gustav Jacob Jacobi (1804–1851) [6, 4] discovered that identity, understood its importance and proved that it is satisfied by Poisson's bracket as well as by the bracket of vector fields. That identity played an important part in the theory of Lie groups and Lie algebras developed by *Sophus Lie*.

$$\bigl\{f,\{g,h\}\bigr\}+\bigl\{g,\{h,f\}\bigr\}+\bigl\{h,\{f,g\}\bigr\}=0 \quad \text{for functions}\,,$$

$$\bigl[X,[Y,Z]\bigr]+\bigl[Y,[Z,X]\bigr]+\bigl[Z,[X,Y]\bigr]=0 \quad \text{for vector fields}\,.$$

Lagrange's paper of 1810

In this paper, Lagrange expresses more simply his previous results by using Poisson brackets. He writes the differential equations which govern the time variations of "constants" a, b, $\ldots$, under the form

$$\frac{da_i}{dt} = \sum_{j=1}^{2n} \{a_i, a_j\} \frac{\partial \Omega}{\partial a_j}, \quad 1 \leq i \leq 2n.$$

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$$\frac{da_i}{dt} = \sum_{j=1}^{2n} \{a_i, a_j\} \frac{\partial \Omega}{\partial a_j}\,, \qquad 1 \leq i \leq 2n\,.$$

I have denoted the constants by a_i, $1 \leq i \leq 2n$, which allows a more concise expression using the symbol $\sum_{i=1}^{2n}$.

Lagrange used longer expressions in which the constants were denoted by a, b, c, f, g, h, $\ldots$

Lagrange's paper of 1810 (2)

Let us observe that Lagrange could have written his equations under a simpler form

$$\frac{da_i}{dt} = \{a_i, \Omega\}\,, \quad 1 \leq i \leq 2n\,,$$

since Ω can be considered as a function defined on the product of the manifold of motions with the factor $\mathbb{R}$, for the time. Therefore the Poisson bracket $\{a_i, \Omega\}$ can be unambiguously defined :

$$\{a_i, \Omega\} = \sum_{j=1}^{2n} \{a_i, a_j\} \frac{\partial \Omega}{\partial a_j}\,.$$

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Lagrange did not use that simpler expression. Nor did Poisson in his paper of 1809. Lagrange and Poisson used the Poisson bracket only for coordinate functions a_i, not for more general functiuns such as Ω.

Cauchy's paper of 1837

This short paper (6 pages), published in the *Journal de Mathématiques pures et appliquées*, is extracted from a longer paper presented by Augustin-Louis Cauchy (1789-1857) at the Academy of Torino, on the 11th october 1831. Its title is almost the same as those of the papers by Lagrange and Poisson.

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Cauchy resolutely uses the Hamiltonian formalism. He explains very clearly the main results due to Lagrange and Poisson. However, he does not write Poisson brackets with the function Ω (which is denoted by R in his paper).

Cauchy proves (without using the word *matrix*) that the matrix whose coefficients are Lagrange's parentheses of the coordinates functions, and the matrix whose coefficients are the Poisson brackets of the same coordinates functions, are inverse of each other.

Variation of constants revisited (1)

I am now going to present, in modern language and with today's notations, the main results due to Lagrange and Poisson, about the method of varying constants. I will follow Cauchy's paper of 1837.

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Let (M, ω) be a $2n$-dimensional symplectic manifold, with a (maybe time-dependent) Hamiltonian function $Q : M \times \mathbb{R} \to \mathbb{R}$ (it is the notation used by Cauchy). Let M_0 be the manifold of motions of that Hamiltonian system and let $\Phi : \mathbb{R} \times M_0 \to M$, $(t, a) \mapsto \Phi(t, a)$ be the le "flow" of the Hamitonian vector field associated to Q. The easiest way of writing Hamilton's equation is the following : for each smooth function $g : M \to \mathbb{R}$

$$\frac{\partial\big(g \circ \Phi(t, a)\big)}{\partial t} = \{Q, g\}\big(\Phi(t, a)\big)\,.$$

Variation of constants revisited (2)

We assume now that the system's *true* Hamiltonian is $Q + R$ instead of Q, where R may be time dependent.

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The *method of varying constants*'aim is to transform the flow of the Hamiltonian vector field associated to Q into the flow of the Hamiltonian vector field associated to $Q + R$. More precisely, its aim is to find a map $\Psi : \mathbb{R} \times M_0 \to M_1$, $(t, b) \mapsto a = \Psi(t, b)$, where M_1 is the manifold of motions or the system with Hamiltonian $Q + R$, such that $(t, b) \mapsto \Phi\bigl(t, \Psi(t, b)\bigr)$ is the flow of the vector field with Hamiltonian $Q + R$.

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We must have, for any smooth function $g : M \to \mathbb{R}$,

$$\frac{d}{dt}\Bigl(g \circ \Phi\bigl(t, \Psi(t,b)\bigr)\Bigr) = \{Q+R, g\}\Bigl(\Phi\bigl(t, \Psi(t,b)\bigr)\Bigr)\,.$$

Variation of constants revisited (3)

For each value t_0 of the time t

$$\frac{d}{dt}\Bigl(g\circ\Phi\bigl(t,\Psi(t,b)\bigr)\Bigr)\Big|_{t=t_0} = \frac{d}{dt}\Bigl(g\circ\Phi\bigl(t,\Psi(t_0,b)\bigr)\Bigr)\Big|_{t=t_0} + \frac{d}{dt}\Bigl(g\circ\Phi\bigl(t_0,\Psi(t,b)\bigr)\Bigr)$$

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The first term of the right hand side is, since when t_0 is fixed, $\big(t,\Psi(t_0,b)\big)\mapsto\Phi\big(t,\Psi(t_0,b)\big)$ is the flow of the vector field with Hamiltonian Q,

$$\frac{d}{dt}\Big(g\circ\Phi\big(t,\Psi(t_0,b)\big)\Big)\Big|_{t=t_0} = \{Q,g\}\Big(\Phi\big(t_0,\Psi(t_0,b)\big)\Big)\,.$$

Variation of constants revisited (4)

Therefore the second term of the right hand side must be

$$\frac{d}{dt}\Big(g \circ \Phi\big(t_0, \Psi(t,b)\big)\Big)\Big|_{t=t_0} = \Big(\{Q+R, g\} - \{Q, g\}\Big)\Big(\Phi\big(t_0, \Psi(t_0,b)\big)\Big)$$

$$= \{R, g\}_M\Big(\Phi\big(t_0, \Psi(t_0,b)\big)\Big)$$

$$= \{R \circ \Phi_{t_0}, g \circ \Phi_{t_0}\}_{M_0}\big(\Psi(t_0,b)\big)\,.$$

The Poisson bracket of functions on M is denoted by $\{\ ,\ \}$ when there is no risk of confusion, and by $\{\ ,\ \}_M$ when we want to distinguish it from the Poisson bracket of functions defined on M_0, which is denoted by $\{\ ,\ \}_{M_O}$. For the last equality, we have used the fact that $\Phi_{t_0} : M_0 \to M$ is a Poisson map.

Variation of constants revisited (5)

The function $g_0 = g \circ \Phi_{t_0}$ can be any smooth function on M_0, so the last equality may be written as

$$\left\langle dg_0, \frac{\partial \Psi(t,b)}{\partial t} \right\rangle_{t=t_0} = \frac{d\Big(g_0\big(\Psi(t,b)\big)\Big)}{dt}\Big|_{t=t_0} = \{R\circ\Phi_{t_0}, g_0\}_{M_0}\Big(\Psi(t_0,b$$

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Now t_0 may take any value, so the last equation proves that for each $b \in M_1$ (the manifold of motions of the system with Hamiltonian $Q+R$), $t \mapsto \Psi(t,b)$ is an integal curve, drawn on the manifold M_0 of motions of the system with Hamiltonian Q, of the Hamiltonian system with the time-dependent Hamiltonian

$$(t,a) \mapsto R\bigl(t, \Phi(t,a)\bigr)\,, \quad (t,a) \in \mathbb{R} \times M_0\,.$$

It is the result discovered by Lagrange around 1808.

Thanks

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And I thank the persons who had the kindness and patience for listening to my talk.

Flow and manifold of motions

We consider the ordinary differential equation

$$\frac{d\varphi(t)}{dt} = X\bigl(t, \varphi(t)\bigr)$$

in which $X : \mathbb{R} \times M \to TM$ is a smooth, maybe time-dependent, vector field on a smooth manifold M.

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in which $X : \mathbb{R} \times M \to TM$ is a smooth, maybe time-dependent, vector field on a smooth manifold M.

The *flow* of that differential equation is the map, defined on an open subset of $\mathbb{R} \times \mathbb{R} \times M$, with values in M,

$$(t, t_0, x_0) \mapsto \Phi(t, t_0, x_0)$$

such that $t \mapsto \Phi(t, t_0, x_0)$ is the maximal solution of that differential equation which takes the value x_0 for $t = t_0$.

$$\frac{\partial\Phi(t, t_0, x_0)}{\partial t} = X\bigl(t, \Phi(t, t_0, x_0)\bigr)\,, \quad \Phi(t_0, t_0, x_0) = x_0\,.$$

Flow and manifold of motions (2)

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The *space of motions* of that differential equation is the set $\widehat{M}$ of all its maximal solutions $t \mapsto \varphi(t)$. That space is the quotient of $\mathbb{R} \times M$ by the equivalence relation

(t_2, x_2) and (t_1, x_1) are equivalent if (t_2, t_1, x_1) belongs to the open subset of $\mathbb{R} \times \mathbb{R} \times M$ on which Φ is defined and

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$$x_2 = \Phi(t_2, t_1, x_1)\,.$$

When Φ is defined on $\mathbb{R} \times \mathbb{R} \times M$, the space of motions $\widehat{M}$ is diffeomorphic to M. But *there is no canonical diffeomorphism* of $\widehat{M}$ onto M : when we choose a particular time $t_0 \in \mathbb{R}$, we have a diffeomorphism of $\widehat{M}$ onto M which associates with each motion $\varphi \in \widehat{M}$ the point $\varphi(t_0) \in M$. Of course *that diffeomorphism depends on* t_0.

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Flow and manifold of motions (3)

Generally speaking, the space of motions *has a smooth manifold structure, which may not be Hausdorff*.

The modified flow The value $\Phi(t, t_0, x_0)$ taken by the flow Φ at time t depends on t and on the *equivalence class* $a \in \widehat{M}$ of $(t_0, x_0) \in \mathbb{R} \times M$, rather than separately on t_0 and x_0. We see indeed that if (t_0, x_0) and (t_1, x_1) are equivalent, $x_1 = \Phi(t_1, t_0, x_0)$ and $\Phi(t, t_1, x_1) = \Phi\bigl(t, t_1, \Phi(t_1, t_0, x_0)\bigr) = \Phi(t, t_0, x_0)$. The *modified flow* of our differential equation is the map, defined on an open subset of $\mathbb{R} \times \widehat{M}$,

$$(t, a) \mapsto \widehat{\Phi}(t, a) = \Phi(t, t_0, x_0)\,,$$

where $(t_0, x_0) \in \mathbb{R} \times M$ is any element of the equivalence class $a \in \widehat{M}$. In 1809, Lagrange used that *point of view*.

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