Decoupling of second-order equations and quasi-bi-Hamiltonian systems

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Bedlewo   August 2008

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Generalities about 2nd-order equations

It is well known that a SODE field on TM

$$\Gamma = v^i \frac{\partial}{\partial q^i} + f^i(q,v) \frac{\partial}{\partial v^i}$$

comes with a canonically defined connection on $\tau : TM \to M$, determined by

$$X \in \mathcal{X}(M) \quad \mapsto \quad X^H = \frac{1}{2}\big(X^c + [X^V, \Gamma]\big)$$

[X^c: complete lift, X^V: vertical lift].

In coordinates:

$$X^H = X^i H_i, \quad H_i = \frac{\partial}{\partial q^i} - \Gamma^j_i \frac{\partial}{\partial v^j}, \quad \Gamma^i_j = -\frac{1}{2}\frac{\partial f^i}{\partial v^j}.$$

In turn, with the aid of the projection operators P_H and P_V of this non-linear connection on TM, one can construct a linear connection on the pullback bundle $\tau^*\tau : \ \tau^*TM \to TM$, said to be of Berwald type.

Essentially, this connection defines *vertical* and *horizontal covariant derivative operators* D^V_X and D^H_X on $\mathcal{X}(\tau)$, which in coordinates, are determined by the following action on functions F and basic vector fields (and then further extend by duality):

$$\mathrm{D}^V_X F = X^i\, V_i(F), \qquad \mathrm{D}^V_X \frac{\partial}{\partial q^i} = 0 \qquad \left(V_i := \frac{\partial}{\partial v^i}\right)$$

$$\mathrm{D}^H_X F = X^i\, H_i(F), \qquad \mathrm{D}^H_X \frac{\partial}{\partial q^i} = X^j V_i(\Gamma^k_j) \frac{\partial}{\partial q^k}$$

Of equal importance are

- the *dynamical covariant derivative* ∇ (degr. 0 derivation)
- a (1,1) tensor $\Phi \in V^1(\tau)$, called *Jacobi endomorphism*

which can implicitly be defined by $\quad \mathcal{L}_\Gamma X^H = (\nabla X)^H + \Phi(X)^V$.

$$\nabla F = \Gamma(F) \qquad \nabla \frac{\partial}{\partial q^i} = \Gamma^j_i \frac{\partial}{\partial q^j} \qquad \nabla dq^i = -\Gamma^i_j dq^j\,,$$

$$\Phi^i_j = -\frac{\partial f^i}{\partial q^j} - \Gamma^i_k \Gamma^k_j - \Gamma(\Gamma^i_j)\,.$$

Submersive equations and decoupling

SODEs are *submersive* (Kossowski and Thompson, 1991) if a number of the eqns decouple from the rest, i.e. in suitable coordinates (x^a, y^i), the system takes the form

$$\begin{aligned} \ddot{y}^i &= f^i(y), \\ \ddot{x}^a &= f^a(x, y). \end{aligned}$$

Such property is completely characterized intrinsically by:

existence of a distribution K along $\tau : TM \to M$, such that

$$\Phi(K) \subset K, \quad \nabla K \subset K, \quad \mathrm{D}^V_Z K \subset K \;\; \forall Z \in \mathcal{X}(\tau).$$

Indeed, those conditions have the following effect:

- D^V-invariance means that K is generated by basic vector fields,
- D^V- and ∇-invariance further implies D^H-invariance and as a result Frobenius integrability,
- if the x^a are coordinates on integral manifolds of K, ∇-invariance implies that the f^i do not depend on $\dot{x}^a$,
- finally Φ-invariance then implies $\partial f^i / \partial x^a = 0$ as well.

Assume now further that g is a Riemannian metric on M, satisfying

$$\nabla g = 0.$$

Let $K^\perp$ be the orthogonal complement of K: $\; g(K, K^\perp) = 0$.

It follows from $\nabla g = 0$ and $\mathrm{D}^V_X g = 0$ that

$$\nabla K^\perp \subset K^\perp, \quad \mathrm{D}^V_Z K^\perp \subset K^\perp \;\; \forall Z \in \mathcal{X}(\tau).$$

IMPORTANT SPECIAL CASE

If Γ represents a Lagrangian system on TM, the Hessian g of L has the property that $\Phi \lrcorner g$ is symmetric. If this g is Riemannian on M, the implication is that also $\Phi(K^\perp) \subset K^\perp$, i.e. $K^\perp$ satisfies all conditions of submersiveness as well (and there exist coordinates simultaneously adapted to K and $K^\perp$).

Conclusion: *A submersive Lagrangian system of mechanical type decouples into two separate systems.*

Cofactor systems

Lundmark, Wojciechowski and co-workers introduced *cofactor systems* as Newtonian type SODEs of the form

$$\ddot{q}^\alpha = -(A(q)^{-1})^{\alpha\beta}\frac{\partial W}{\partial q^\beta},$$

where the matrix A is of the form

$$A(q) = \mathrm{cof}\, G(q), \quad \text{with} \quad G^{\alpha\beta}(q) = a\, q^\alpha q^\beta + b^\alpha q^\beta + b^\beta q^\alpha + c^{\alpha\beta}.$$

These can be viewed as representing a class of non-conservative Lagrangian systems

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^\alpha}\right) - \frac{\partial T}{\partial q^\alpha} = Q_\alpha,$$

with a 'Euclidean kinetic energy' $T = \frac{1}{2}\sum(\dot{q}^\alpha)^2$ and non-conservative forces of some *quasi-potential* type.

The same authors later obtained remarkable results about a subclass of such systems which split into a so-called 'driving' and 'driven part'.

Driven cofactor systems

The basic assumption is one of submersiveness in the given coordinates, i.e. with $(q^\alpha) = (y^i, x^a)$, a given cofactor system $\ddot{q}^\alpha = f^\alpha$ is assumed to appear in the form:

$$\begin{aligned} \ddot{y}^i &= Q^i(y^j), \qquad i = 1, \ldots, m \\ \ddot{x}^a &= Q^a(y^i, x^b) \qquad a = 1, \ldots n. \end{aligned}$$

An extra hypothesis is: after solving the *driving eqns* for the y^i, the *driven system*

$$\ddot{x}^a = Q^a(y^i(t), x^b)$$

has a Lagrangian of mechanical type

$$T - V = \tfrac{1}{2} \sum (\dot{x}^a)^2 - V(y^i(t), x^a).$$

What is proved then is:

- the driving system is of cofactor type on $\mathbb{R}^m$,
- for any solution $y(t)$ of the driving system, the driven system has n (time-dependent) integrals,

- under some technical assumptions, primarily functional independence of eigenfunctions u^a of a suitably adapted reduced eigenvalue problem, there exists a canonical transformation $(x^a, p_a) \leftrightarrow (u^a, s_a)$ which has the effect that all the time-dependence in the transformed Hamiltonian is caught in an overall factor and the Hamilton-Jacobi eqn can be solved by separation of variables.

Cofactor and driven cofactor systems on Riemannian manifolds

Recall (Crampin and W.S. (2001)) the following characterization of a cofactor system on (the tangent bundle of) a Riemannian space (M, g).

Consider a non-conservative Lagrangian system determined by the 'Riemannian kinetic energy' $T = \frac{1}{2}\, g_{\alpha\beta}(q)\dot{q}^\alpha \dot{q}^\beta$ and non-conservative forces $Q^\alpha(q)$, given by some 1-form $\mu = Q_\alpha(q)dq^\alpha$ on M.

Definition: The pair (g, μ) determines a *cofactor system* if g admits a *special conformal Killing tensor* J and μ satisfies $D_J\mu = 0$.

A *special conformal Killing tensor* w.r.t. g is a (non-singular) type (1,1) tensor J on M, such that $J_{\alpha\beta} = g_{\alpha\gamma} J^\gamma_\beta$ is symmetric and (w.r.t. the Levi-Civita connection of g) has the property

$$J_{\alpha\beta|\gamma} = \tfrac{1}{2}(\sigma_\alpha g_{\beta\gamma} + \sigma_\beta g_{\alpha\gamma}), \qquad \text{which implies } \sigma = d\,\mathrm{tr}\, J. \qquad \text{(scKt)}$$

Secondly, D_J is a 'gauged' differential operator defined (for any 1-form ρ) by

$$D_J\rho = d_J\rho + d(\mathrm{tr}\, J) \wedge \rho \quad \text{or} \quad D_J\rho = (\det J)^{-1} d_J((\det J)\rho).$$

Note that the corresponding SODE field Γ on TM is of the form

$$\Gamma = \widetilde{\Gamma} + Q^\beta \frac{\partial}{\partial v^\beta}, \qquad Q^\beta = g^{\beta\alpha} Q_\alpha,$$

where $\widetilde{\Gamma}$ is the geodesic spray of g.

But $\widetilde{\Gamma}$ is Lagrangian, hence: $\widetilde{\nabla} g = 0$, $\quad \widetilde{\Phi} \lrcorner\, g$ is symmetric.

It follows that $\nabla g = 0$ as well, and we must insist on $d\mu \neq 0$ to avoid that Γ would be Lagrangian also and submersiveness would imply complete decoupling.

Assume now again that a distribution K makes Γ submersive and consider its complement $K^\perp$.

As before,

$$\nabla K^\perp \subset K^\perp, \quad \mathrm{D}^V_Z K^\perp \subset K^\perp \;\; \forall Z \in \mathcal{X}(\tau).$$

It follows from the submersiveness of $\widetilde{\Gamma}$ that $\mathrm{D}^H\mu(K, K^\perp) = 0$, and the condition to avoid splitting of Γ thus becomes: $\mathrm{D}^H\mu(K^\perp, K) \neq 0$.

In coordinates (y^i, x^a), simultaneously adapted to K and $K^\perp$

$$K = \mathrm{sp}\left\{\frac{\partial}{\partial x^a}\right\} \qquad K^\perp = \mathrm{sp}\left\{\frac{\partial}{\partial y^i}\right\},$$

the equations of motion will take the form

$$\begin{aligned}
\ddot{y}^i &= -\Gamma^i_{jk}(y)\dot{y}^j\dot{y}^k + Q^i(y), \\
\ddot{x}^a &= -\Gamma^a_{bc}(x)\dot{x}^b\dot{x}^c + Q^a(y,x).
\end{aligned}$$

Within this class of submersive non-conservative systems, two more assumptions are needed to capture and generalize the driven cofactor systems:

- one is the existence of a scKt J for the overall cofactor nature of the system,
- the other one is simply $d\mu(K,K)=0$, which in adapted coordinates means that the $Q_a(y,x)$ satisfy

$$\frac{\partial Q_a}{\partial x^b}-\frac{\partial Q_b}{\partial x^a}=0,$$

  and ensures that the 'driven' system is (parametrically) Hamiltonian.

Hence we arrive at the following coordinate-free characterization:

Definition: A *driven cofactor system* is a cofactor system (g,μ,J), determined by a Riemannian metric g, a basic 1-form μ and a scKt J on M, for which there exists a distribution K along τ, with the properties

$$\Phi(K)\subset K, \quad \nabla K\subset K, \quad \mathrm{D}^V_Z K\subset K$$

and

$$d\mu(K,K)=0, \quad \mathrm{D}^H\mu(K^\perp,K)\neq 0.$$

Further properties of cofactor systems

By now well known: if A is the cofactor tensor of a scKt J

$$A\,J = (\det J)I,$$

then A is a Killing tensor!

Also, $N_J = 0$, $D_J\mu = 0$ (in view of ${D_J}^2 = 0$) implies that there exists a function W on M, such that

$$A(\mu) = dW,$$

and we have the first integral

$$E = \tfrac{1}{2}A_{ij}(q)v^i v^j + W(q).$$

Denote by $\widetilde{J}$ the complete lift of J to T^*M:

$$\widetilde{J} = J^i_j \left(\frac{\partial}{\partial q^i} \otimes dq^j + \frac{\partial}{\partial p_j} \otimes dp_i \right) + p_k \left(\frac{\partial J^k_i}{\partial q^j} - \frac{\partial J^k_j}{\partial q^i} \right) \frac{\partial}{\partial p_i} \otimes dq^j.$$

$\widetilde{J}$ commutes with the standard Poisson map

$$P_0 : \mathcal{X}^*(T^*M) \to \mathcal{X}(T^*M),$$

and is the recursion operator of a Poisson-Nijenhuis structure on T^*M, which makes $P_J = \widetilde{J} \circ P_0$ and P_0 compatible.

Furthermore, if $\hat{\Gamma} \in \mathcal{X}(T^*M)$ denotes the image of Γ under the Legendre transform of the kinetic energy Lagrangian $L = \frac{1}{2} g_{ij} v^i v^j$, the cofactor system properties are necessary and sufficient for $\hat{\Gamma}$ to have a quasi-Hamiltonian representation:

$$F\,\hat{\Gamma} = P_J(dH)$$

with

$$H(= E) = \tfrac{1}{2} A^{ij} p_i p_j + W, \qquad F = \det J.$$

Finally (see e.g. Benenti (2005) or Marciniak and Blaszak (2008)), a time-scale transformation determined by

$$\frac{dt}{d\tilde{t}} = \det J,$$

will transform this representation, using results on geodesic equivalence of metrics, into a standard Hamiltonian system

$$\widetilde{\hat{\Gamma}} = P_0(d\widetilde{H}) \quad \text{with} \quad \widetilde{H}(q, \tilde{p}) = H(q, J\tilde{p}).$$

Cofactor pair systems

Suppose that a non-conservative system (g, μ) has two independent cofactor representations, so:

- J and K are two scKts for g
- $D_J\mu = D_K\mu = 0$.

Then, the scKt properties of J and K imply that $[J, K] = 0$ and hence that P_J and P_K are compatible.

Furthermore, $\forall a, b \in \mathbb{R}$ we have that $aJ + bK$ is also a scKt and the *gauged bi-differential operators* D_J and D_K behave as commuting derivations of degree 1, in the sense that

$$D_J{}^2 = 0, \quad D_K{}^2 = 0 \quad \text{and} \quad D_J D_K + D_K D_J = 0.$$

One can show then that there are n quadratic integrals, which are in involution w.r.t. both Poisson structures. Also, rescaling one or the other representation will lead to a quasi-bi-Hamiltonian system and Hamilton-Jacobi separability results of that theory become applicable.

And now back to the driven cofactor systems ...

With K and $K^{\perp}$ we have two complementary sub-bundles of TM (and corresponding sub-modules of $\mathcal{X}(\tau)$). Put

$$g_1 = g\big|_{K^{\perp}}, \qquad g_2 = g\big|_{K},$$

so that in adapted coordinates

$$g = g_1 + g_2 = g_{ij}(y)dy^i \otimes dy^j + g_{ab}(x)dx^a \otimes dx^b.$$

Consider further the projection operators

$$P_1 : \mathcal{X}(\tau) \to K^{\perp}, \qquad P_2 : \mathcal{X}(\tau) \to K,$$

and decompose the scKt J and the non-conservative forces μ accordingly:

$$J_1 = P_1 \circ J \circ P_1, \quad J_2 = P_2 \circ J \circ P_2, \quad J_{12} = P_1 \circ J \circ P_2, \quad J_{21} = P_2 \circ J \circ P_1$$

$$\mu_1 = P_1(\mu), \qquad \mu_2 = P_2(\mu).$$

Effect of the submersiveness on J

In adapted coordinates, we have

$$J_1 = J^i_j(y)\frac{\partial}{\partial y^i}\otimes dy^j, \qquad J_2 = J^a_b(x)\frac{\partial}{\partial x^a}\otimes dx^b,$$

$$J_{12} = J^i_a(y,x)\frac{\partial}{\partial y^i}\otimes dx^a, \qquad J_{21} = J^a_i(y,x)\frac{\partial}{\partial x^a}\otimes dy^i,$$

with

$$\frac{\partial J^a_i}{\partial y^j} = \frac{\partial J^a_j}{\partial y^i}, \qquad \frac{\partial J^i_a}{\partial x^b} = \frac{\partial J^i_b}{\partial x^a}.$$

In addition,

$$\begin{aligned} J_{ij|k} &= \tfrac{1}{2}(\sigma_{1i}g_{jk} + \sigma_{1j}g_{ik}), \qquad \sigma_1 = d\,\mathrm{tr}\,J_1, \\ J_{ab|c} &= \tfrac{1}{2}(\sigma_{2a}g_{bc} + \sigma_{2b}g_{ac}), \qquad \sigma_2 = d\,\mathrm{tr}\,J_2, \end{aligned}$$

meaning that J_1 and J_2 are scKts for g_1 and g_2 respectively.

A summary of results

The ‘driving system’, determined by (g_1, μ_1) is of cofactor type in its own right, with J_1 (assumed to be non-singular) as scKt, and thus has a quadratic integral which in adapted coordinates reads

$$E_1 = \tfrac{1}{2} A^1_{ij}(y)\dot{y}^i\dot{y}^j + W^1(y),$$

with $A^1 = \mathrm{cof}\, J_1$ and $A^1\mu_1 = dW^1$.

The projector P_2 is a degenerate scKt tensor for the complete system, so that we have a cofactor pair system in some sense.

The algorithm for generating integrals in involution of a cofactor pair system can be suitably adapted to this degenerate situation of (J, P_2): if $n = \dim K$, it produces $n+1$ integrals, one of which in fact is the E_1 of the driving system. The remaining n, say $H_{(1)}, \ldots, H_{(n)}$ will be first integrals of the ‘driven system’, along arbitrary solutions $y^i(t)$ of the driving system.

Since $\det J_1 \neq 0$, the block matrix structure of J can be decomposed as

$$\begin{pmatrix} J_1 & J_{12} \\ J_{21} & J_2 \end{pmatrix} = \begin{pmatrix} J_1 & 0 \\ J_{21} & 1 \end{pmatrix} \begin{pmatrix} 1 & {J_1}^{-1} J_{12} \\ 0 & J_2 - J_{21} {J_1}^{-1} J_{12} \end{pmatrix}$$

It follows that $\bar{J}_2 = J_2 - J_{21} {J_1}^{-1} J_{12}$ is non-singular and

$$\det J = (\det J_1)(\det \bar{J}_2).$$

Moreover, $\bar{J}_2$ is a scKt for the metric g_2 and gives rise to a cofactor representation of the driven system. Since, by assumption, the driven system also has a standard Hamiltonian representation, we seem to have the usual data for Hamilton-Jacobi separability.

However, the driven system, which comes to life along solutions of the driving system, is time-dependent!

The point is that, as in the Euclidean case, a cleverly chosen standard (time-dependent) canonical transformation will transform the original Hamiltonian into one of the form $H_{(1)}/\det J_1$ say, where all the time-dependence goes into the factor $(\det J_1)^{-1}$.

The fundamental technical assumption now is that the eqn

$$\det(J - \lambda P_2) = 0,$$

which is a polynomial of degree $n = \dim K$ due to degeneracy of P_2, has n functionally independent solutions $u^a(y, x)$.

One can show then that

$$\bar{J}_2(P_2(du^a)) = u^a \, P_2(du^a),$$

which reveals eigenfunctions and eigenforms of $\bar{J}_2$.

The idea is to use the u^a as new coordinates for the driven system. To discover the right canonical transformation $(x^a, p_a) \leftrightarrow (u^a, s_a)$, we first transform the original quasi-Hamiltonian representation of the full system via reparametrisation with factor $(\det J)$, but extract from that only the effect on the driven system, i.e.

$$p_\alpha = J^\beta_\alpha \check{p}_\beta \quad \Rightarrow \quad p_a = \bar{J}_{2a}^{\ b} \check{p}_b + \dots,$$

then we rescale the driven system with factor $(\det \bar{J}_2)^{-1}$

$$\check{p}_b = (\bar{J}_2^{-1})^c_b \, \tilde{p}_c \quad \Rightarrow \quad p_a = \tilde{p}_a + \dots,$$

then, in the variables $(x^a, \tilde{p}_a)$, consider the standard canonical transformation induced by the point transformation $x^a \to u^a(y(t), x)$, i.e.

$$s_a = \frac{\partial x^b}{\partial u^a}\tilde{p}_b = \frac{\partial x^b}{\partial u^a}(p_b + \ldots),$$

and finally rescale again with factor $(\det J_1)^{-1}$. Since

$$(\det J)/(\det \bar{J}_2) = \det J_1,$$

this means that we in the end come back to the original time-variable.

But in the original time, the driven system had a standard Hamiltonian description, hence the canonical tf will preserve this.

It turns out that the new Hamiltonian is

$$K = H_{(1)}(u^a, s_a, t)/(\det J_1)(t).$$

Since $H_{(1)}$ is a first integral of the driven system, it follows from K being a Hamiltonian that $H_{(1)}$ cannot have explicit time dependence.

Hamilton-Jacobi separability now follows.