Central Invariants of the Drinfeld – Sokolov Bihamiltonian Structures

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Based on joint works with Boris Dubrovin and Si-Qi Liu

1. Introduction

Classification of integrable systems of evolutionary PDEs

$$\frac{\partial w^i}{\partial t} = K^i(w; w_x, w_{xx}, \dots), \quad i = 1, \dots, n$$
$$w = (w^1, \dots, w^n) \in M^n.$$

Integrability: Hamiltonian system with a complete family of commuting Hamiltonians

Bihamiltonian evolutionary PDEs

$$K^i(w; w_x, w_{xx}, \dots) = \{w^i(x), H_1\}_1 = \{w^i(x), H_2\}_2$$

Consider bihamiltonian PDEs admitting a formal expansion w.r.t. a small parameter ϵ

$$w_t^i = A_j^i(w) w_x^j + \epsilon \left[B_j^i(w) w_{xx}^j + C_{jk}^i(w) w_x^j w_x^k \right] + \epsilon^2 \left[D_j^i(w) w_{xxx}^j + E_{j\,k}^i(w) w_x^j w_{xx}^k + F_{jkl}^i(w) w_x^j w_x^k w_x^l \right] + \dots,$$
$$i = 1, \dots, n$$

Classification w.r.t. certain extension of the group of local diffeomorphism of the manifold M, called the group of Miura-type transformations. We obtain the complete set of invariants of the bihamiltonian structure satisfying certain semisimplicity assumption.

The prototypical system: the Korteweg-de Vries

$$w_t = w w_x + \frac{\epsilon^2}{12} w_{xxx}, \quad w = w(x,t)$$

It has the first Hamiltonian structure (Faddeev & Zakharov; Gardner)

$$\{w(x), w(y)\}_1 = \delta'(x-y)$$

and the second one (Magri)

$$\{w(x), w(y)\}_2 = w(x)\,\delta'(x-y) + \frac{1}{2}\,w_x \delta(x-y) + \frac{\epsilon^2}{8}\delta'''(x-y)$$

Another example: The Camassa-Holm equation:

$$(q - \frac{\epsilon^2}{8} q_{xx})_t = q q_x - \frac{\epsilon^2}{12} q_x q_{xx} - \frac{\epsilon^2}{24} q q_{xxx}.$$

Set

$$w = q - \frac{\epsilon^2}{8} q_{xx}.$$

It have the Bihamiltonian structure

$$\{w(x), w(y)\}_1 = \delta'(x-y) - \frac{\epsilon^2}{8}\delta'''(x-y),$$
$$\{w(x), w(y)\}_2 = w(x)\delta'(x-y) + \frac{1}{2} w_x \delta(x-y)$$

The first part of these invariants: flat pencil of metrics

Defined on the manifold M, describes the bihamiltonian structure of the *hydrodynamic limit*

$$w^i_t = A^i_j(w) w^j_x, \quad i = 1, \dots, n$$

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The second part of the invariants: the central invariants

Comes from the deformation theory of the bihamiltonian structures of hydrodynamic type; it consists of n functions of one variable.

One of the motivations:

Relation of 2d TFT and Gromov-Witten theory with bihamiltonian integrable hierarchies

An important feature of such bihamiltonian structures:

For this class the flat pencil of metrics comes from a Frobenius structure on the manifold M and all the central invariants are constants equal to each other.

Problem: What are the flat pencils of metrics and the central invariants for the bihamiltonian hierarchies constructed by V. Drinfeld and V. Sokolov.

2. Definition of the Central invariants

Let $w = (w^1, \ldots, w^n) \in M$ be local coordinates on M. Denote by $\mathcal{B}$ the graded ring of polynomial functions on the jet bundle of M

$$\mathcal{B} = \varinjlim_{k} \mathcal{B}_k, \quad \mathcal{B}_k = \mathcal{C}^\infty(M)[w_x, w_{xx}, \ldots, w^{(k)}], \quad \deg \partial_x^k w^i = k.$$

The quotient space

$$\bar{\mathcal{B}} := \mathcal{B}[[\epsilon]]/\mathrm{Im}\,\partial_x$$

will be called the space of local functionals. Here

$$\partial_x = \sum_k w^{i,k+1} \frac{\partial}{\partial w^{i,k}}, \quad w^{i,k} := \frac{\partial^k w^i}{\partial x^k}.$$

Type of Poisson brackets:

For two local functionals

$$\bar{P} = \int P(w; w_x, \ldots; \epsilon)\, dx, \quad \bar{Q} = \int Q(w; w_x, \ldots; \epsilon)\, dx, \quad P, Q \in \mathcal{B}[[\epsilon]]$$

the Poisson bracket is a local functional of the form

$$\{\bar{P}, \bar{Q}\} = \int \frac{\delta \bar{P}}{\delta w^i(x)} \Pi^{ij} \frac{\delta \bar{Q}}{\delta w^j(x)}\, dx.$$

where

$$\Pi^{ij} = g^{ij}(w)\partial_x + \Gamma^{ij}_k(w) w^k_x + \sum_{k\geq 1} \epsilon^k \sum_{l=0}^{k+1} A^{ij}_{k,l}(w; w_x, \cdots, w^{(l)}) \partial_x^{k-l+1},$$

$$A^{ij}_{k,l} \in \mathcal{B}, \quad \deg A^{ij}_{k,l} = l. \quad \det(g^{ij}) \neq 0\,.$$

The Poisson bracket is also represented in the form

$$\{w^i(x), w^j(y)\} = \{w^i(x), w^j(y)\}^{[0]} + \sum_{k\geq 1} \epsilon^k \{w^i(x), w^j(y)\}^{[k]},$$

$$\{w^i(x), w^j(y)\}^{[k]} = \sum_{l=0}^{k+1} A^{ij}_{k,l}(w(x); w_x, \cdots, w^{(l)}(x))\delta^{(k-l+1)}(x-y)$$

Given a local functional

$$H = \int \sum_{k\geq 0} \epsilon^k P_k(w; w_x, w_{xx}, \ldots, w^{(k)})dx, \quad \deg P_k = k$$

we have a Hamiltonian system of evolutionary PDEs

$$w^i_t = \{w^i(x), H\} = \Pi^{ij} \frac{\delta H}{\delta w^j(x)}.$$

Poisson bracket of hydrodynamic type:

$$\{w^i(x), w^j(y)\}^{[0]} = g^{ij}(w(x))\delta'(x-y) + \Gamma^{ij}_k(w(x))w^k_x\delta(x-y).$$

Here $\det(g^{ij}) \neq 0$.

Anti-symmetry requires: g^{ij} defines a symmetric non-degenerate bilinear form on T^*M. We call it a metric.

The Jacobi identity implies: the curvature of the metric vanishes, and

$$\Gamma^{ij}_k = -g^{is}\Gamma^j_{sk}$$

(Dubrovin, Novikov 1983)

Miura-type transformations:

Definition.

A Miura-type transformation *is a change of variables of the form*

$$w^i \mapsto \tilde{w}^i(w; w_x, w_{xx}, \dots; \epsilon) = F_0^i(w) + \sum_{k\geq 1} \epsilon^k F_k^i(w; w_x, \cdots, w^{(k)})$$

where $F_k^i \in \mathcal{B}$ with $\deg F_k^i = k$*, and the map $w \mapsto F_0^i(w)$ is a diffeomorphism of M.*

All Miura-type transformations form a group $\mathcal{G}(M)$. It acts by automorphisms on the graded ring $\mathcal{B}[[\epsilon]]$. The action of the group $\mathcal{G}(M)$ on the Poisson brackets is given by the formula

$$\tilde{\Pi}^{kl} = L_i^k \Pi^{ij} L_j^{l\dagger}$$

Theorem. (Getzler; Degiovanni, Magri, Sciacca; Dubrovin, Z.)
Any Poisson bracket of the above form can be reduced to a Poisson bracket of hydrodynamic type given by its leading terms by a Miura-type transformation.

Thus a Poisson bracket of the above form is equivalent, w.r.t. the action of the group $\mathcal{G}(M)$, to

$$\{w^i(x), w^j(y)\} = \eta^{ij}\,\delta'(x-y), \quad \eta^{ij} = \text{constant}$$

The signature of the metric $g^{ij}(w)$ is the only local invariant of a *single* Poisson bracket.

Definition.

A pair of Poisson brackets $\{\ ,\ \}_1,\ \{\ ,\ \}_2$ *is called to be compatible if any of their linear combinations* $\{\ ,\ \}_1 + \lambda\{\ ,\ \}_2,\ \lambda \in \mathbb{R}$ *is also a Poisson bracket. A pair of compatible Poisson brackets forms a bihamiltonian structure.*

A system of evolutionary PDEs is called a bihamiltonian system if it can be represented as Hamiltonian systems w.r.t. to both Poisson bracket of a bihamiltonian structure

$$\frac{\partial w^i}{\partial t} = \{w^i(x), H_1\}_1 = \{w^i(x), H_2\}_2\,, \quad i = 1, \dots, n.$$

Flat pencil of metrics

The leading terms $\{\ ,\ \}_1^{[0]}$, $\{\ ,\ \}_2^{[0]}$ of a bihamiltonian structure is determined by a pair of flat metrics $g_1^{ij}(w)$, $g_2^{ij}(w)$ on M. They have the property that at any point $w \in M$ their arbitrary linear combination

$$a_1 g_1^{ij}(w) + a_2 g_2^{ij}(w)$$

has zero curvature, and the contravariant Christoffel coefficients for the above metric have the form of the same linear combination

$$a_1 \Gamma^{ij}_{k1} + a_2 \Gamma^{ij}_{k2}$$

Semisimplicity

Definition.

We say that a pair of metrics is strongly nondegenerate *if for any* $\lambda \in \mathbb{C}$ *the symmetric matrix* $(g_2^{ij}(w) - \lambda\, g_1^{ij}(w))$ *does not degenerate for generic* $w \in M$.

Definition.

A bihamiltonian structure is called semisimple if the associated pair of metrics (g_1^{ij}, g_2^{ij}) *is strongly nondegenerate, and it is semisimple at generic points of* M*, i.e., at a generic point* $w \in M$ *the roots* $\lambda = u^1(w), \dots, \lambda = u^n(w)$ *of the characteristic equation*

$$\det\left(g_2^{ij}(w) - \lambda\, g_1^{ij}(w)\right) = 0$$

are pairwise distinct.

Canonical coordinates:

The roots $\lambda = u^1(w), \ldots, \lambda = u^n(w)$ of the characteristic equation define a system of local coordinates (Ferapontov 2001). They are called the canonical coordinates of the bihamiltonian structure. In the canonical coordinates the two metrics are diagonal:

$$g_1^{ij}(u) = f^i(u)\delta_{ij}, \quad g_2^{ij}(u) = u^i f^i(u)\delta_{ij}, \quad i, j = 1, \cdots, n$$

for some functions $f^1(u), \ldots, f^n(u)$, $u = (u^1, \ldots, u^n) \in M$.

The central invariants

For a semisimple bihamiltonian structure

$$\{w^i(x), w^j(y)\}_a = \{w^i(x), w^j(y)\}_a^{[0]} + \sum_{k\geq 1} \epsilon^k \{w^i(x), w^j(y)\}_a^{[k]},$$

$$\{w^i(x), w^j(y)\}_a^{[k]} = \sum_{l=0}^{k+1} A^{ij}_{k,l;a}(w(x); w_x, \cdots, w^{(l)}(x))\delta^{(k-l+1)}(x-y)$$

Denote

$$\pi_a^{ij}(p; w) := \sum_{k=0}^{\infty} A^{ij}_{k,0;a}(w)\, p^k, \quad a = 1, 2.$$

The roots of the characteristic equation

$$\mathcal{R}(p, \lambda; w) := \det\left(\pi_2^{ij}(p; w) - \lambda\, \pi_1^{ij}(p; w)\right)$$

has the formal power series expansion in p

$$\lambda^i(p; w) = u^i(w) + \lambda_2^i(w)\, p^2 + \mathcal{O}\left(p^4\right), \quad i = 1, \ldots, n.$$

Denote

$$c_i(w) := \frac{1}{3} \frac{\lambda_2^i(w)}{f^i(w)}, \quad i = 1, \ldots, n$$

Theorem (Dubrovin, Liu, Z, 2006)

i) Each function $c_i(w)$ *defined above depends only on* $u^i(w)$*,*

$$c_i(w) = c_i\left(u^i(w)\right), \quad i = 1, \ldots, n.$$

ii) Two semisimple bihamiltonian structures with the same leading terms $\{\ ,\ \}_a^{[0]}$*,* $a = 1, 2$ *are equivalent* iff *they have the same set of central invariants* $c_i(u^i)$*,* $i = 1, \cdots, n$*.*

3. The Drinfeld - Sokolov bihamiltonian structures

Finite dimensional case:

Given a Poisson manifold $\mathcal{M}$, and a Poisson action of a simply connected Lie group G on $\mathcal{M}$. Let a family of Hamiltonians

$$H_1(x), \dots, H_N(x) \in \mathcal{C}^\infty(\mathcal{M})$$

generates the Poisson action.

The moment map

$$\mathcal{P} : \mathcal{M} \to \mathfrak{g}^*, \quad \mathcal{P}(x) = (H_1(x), \dots, H_N(x)) \in \mathfrak{g}^*$$

Given a Hamiltonian $H \in \mathcal{C}^\infty(\mathcal{M})$ invariant with respect to the action of the group G

$$\{H, H_i\} = 0, \quad i = 1, \dots, N$$

the goal of the reduction procedure is to reduce the order of the Hamiltonian system

$$\frac{dx}{dt} = \{x, H\}$$

i.e., to find a Poisson manifold $(\mathcal{M}^{\mathrm{red}}, \{\ ,\ \}_{\mathrm{red}})$ of a lower dimension and a Hamiltonian $H_{\mathrm{red}} \in \mathcal{C}^\infty(\mathcal{M}^{\mathrm{red}})$ such that problem of integration of the Hamiltonian system is reduced to the one for

$$\frac{dy}{dt} = \{y, H_{\mathrm{red}}\}_{\mathrm{red}}, \quad y \in \mathcal{M}^{\mathrm{red}}.$$

The reduced phase space

Consider a smooth common level surface of the Hamiltonians

$$\mathcal{M}_h := \mathcal{P}^{-1}(h)$$

where

$$h = (h_1, \ldots, h_N) \in \mathfrak{g}^*$$

is a regular value of the moment map. Denote $G_h \subset G$ the stabilizer of h with respect to the coadjoint action of G on $\mathfrak{g}^*$.

The reduced phase space

$$\mathcal{M}_h^{\mathrm{red}} := \mathcal{M}_h / G_h \,.$$

The reduced Poisson bracket

Assume that $G_h = G$

Functions on $\mathcal{M}_h^{\rm red}$ can be identified with G-invariant functions on $\mathcal{M}_h$. For any two G-invariant functions α, β on $\mathcal{M}_h$ denote $\hat{\alpha}$, $\hat{\beta}$ arbitrary extensions of these two functions on a neighborhood of $\mathcal{M}_h$. Then the Poisson bracket on the reduced phase space is given by

$$\{\alpha, \beta\}_{\rm red} := \{\hat{\alpha}, \hat{\beta}\}|_{\mathcal{M}_h}$$

Infinite dimensional case: the Drinfeld - Sokolov reduction

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, G the associated connected and simply connected Lie group. Fix a nondegenerate symmetric invariant bilinear form $\langle\ ,\ \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$. The central extension

$$0 \to \mathbb{C}\mathrm{k} \to \hat{\mathfrak{g}} \to L(\mathfrak{g}) \to 0$$

of the loop algebra $L(\mathfrak{g}) := C^\infty(S^1, \mathfrak{g})$ defined by the 2-cocycle

$$\omega(q,p) = -\int_{S^1} \langle q(x), p'(x)\rangle_{\mathfrak{g}} dx\,.$$

Denote by $\hat{\mathfrak{g}}^*$ be the space of linear functionals on $\hat{\mathfrak{g}}$ of the following form

$$\ell_{q(x)+a\mathrm{k}}[p(x)+b\mathrm{k}] = \int_{S^1} \langle q(x), p(x)\rangle_{\mathfrak{g}}\, dx + a\, b,$$

where $q(x), p(x) \in L(\mathfrak{g})$, $a, b \in \mathbb{C}$. We identify $\hat{\mathfrak{g}}^*$ with $\hat{\mathfrak{g}}$.

Let

$$\mathcal{M} = \{q(x) + \epsilon \mathrm{k} \,|\, q(x) \in L(\mathfrak{g})\} \subset \hat{\mathfrak{g}}^*$$

be the subspace of the linear functionals taking value ϵ at the central element k. The Lie Poisson bracket on $\hat{\mathfrak{g}}^*$ induces a Poisson bracket on $\mathcal{M}$

$$\{H_{a(x)}, H_{b(x)}\}[q] = \int_{S^1} \langle a(x), [b(x), \epsilon \frac{d}{dx} + q(x)]\rangle_{\mathfrak{g}}\, dx$$

The space $\mathcal{M}$ can be naturally identified with the following space of first order linear differential operators

$$\mathcal{M} = \left\{ \epsilon \frac{d}{dx} + q(x) \,\middle|\, q(x) \in L(\mathfrak{g}) \right\}$$

in such a way that the coadjoint action of

$$g = \exp(p(x) + b\mathrm{k}), \;\; p(x) \in L(\mathfrak{g})$$

on $\mathcal{M}$ is given by

$$\mathrm{Ad}^*_g : \;\; \epsilon \frac{d}{dx} + q(x) \mapsto \exp(\mathrm{ad}_{p(x)}) \left(\epsilon \frac{d}{dx} + q(x) \right).$$

we can regard it as an action of the loop group $L(G) := C^\infty(S^1, G)$ on $\mathcal{M}$.

We choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and denote by Φ the root system corresponding to $\mathfrak{h}$. Let $\Delta = (\alpha_1, \cdots, \alpha_n)$ be a base of Φ (where n is the rank of $\mathfrak{g}$), and Φ^+, Φ^- be the positive and negative root systems w.r.t. Δ, then we have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^- = \left(\bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha} \right).$$

Denote

$$\mathfrak{b} = \mathfrak{b}^+ = \mathfrak{n}^+ \oplus \mathfrak{h}, \ \ \mathfrak{b}^- = \mathfrak{n}^- \oplus \mathfrak{h}$$

the Borel subalgebras w.r.t. $\mathfrak{h}$, and

$$\mathfrak{n} = \mathfrak{n}^+ .$$

Let $N \subset G$ be the subgroup of the Lie group G associated with the Lie subalgebra $\mathfrak{n} \subset \mathfrak{g}$.

The coadjoint action defines a Poisson action of the loop group $L(N)$ on $\mathcal{M}$.

The moment map

$$\mathcal{P} : \mathcal{M} \to L(\mathfrak{n})^*$$

is given by

$$\mathcal{P}(q(x))(p(x)) = \int_{S^1} \langle p(x), q(x) \rangle_{\mathfrak{g}} dx,$$

where

$$q(x) \in L(\mathfrak{g}), \quad p(x) \in L(\mathfrak{n})$$

Now we choose a set of Weyl generators $X_i, H_i, Y_i\,(i = 1, \cdots, n)$ w.r.t. the Cartan decomposition,

$$X_i \in \mathfrak{g}_{\alpha_i},\ \ H_i \in \mathfrak{h},\ \ Y_i \in \mathfrak{g}_{-\alpha_i}.$$

Let

$$I = \sum_{i=1}^{n} Y_i \in \mathfrak{n}^-$$

be a *principal nilpotent element*. Denote

$$\mathcal{M}^I := \mathcal{P}^{-1}(I) = \epsilon \frac{d}{dx} + I + L(\mathfrak{b})$$

the level surface of $\mathcal{P}$ considering I as a constant map $S^1 \to \mathfrak{n}^-$.

We fix a subspace V of $\mathfrak{b}$ such that

$$\mathfrak{b} = V \oplus [I, \mathfrak{n}], \quad \dim V = \dim \mathfrak{b} - \dim \mathfrak{n} = n.$$

Proposition.

The Hamiltonian action of the loop group $L(N)$ on $\mathcal{M}^I$ is free, namely, each orbit contains a unique operator of the form

$$\epsilon \frac{d}{dx} + I + q^{\mathrm{can}}(x) \quad \textit{with} \quad q^{\mathrm{can}}(x) \in L(V) = C^\infty(S^1, V).$$

The reduced phase space

$$\mathcal{M}^I / L(N) \simeq \left\{ \epsilon \frac{d}{dx} + I + q^{\mathrm{can}}(x) \mid q^{\mathrm{can}}(x) \in L(V) \right\}$$

The action of the group $L(N)$ leaves $\mathcal{M}^I$ invariant. One can define a Poisson bracket on the reduced space

$$\tilde{\mathcal{M}} = \mathcal{M}^I / L(N)$$

as follows:

Identify the functionals on $\tilde{\mathcal{M}}$ with gauge invariant functionals F on $\mathcal{M}^I$, extend them arbitrary to functionals $\tilde{F}$ of the ambient space $\mathcal{M}$.

For any two extensions of gauge invariant functionals F_1, F_2 on $\mathcal{M}^I$, Define

$$\{F_1, F_2\} := \{\tilde{F}_1, \tilde{F}_2\}|_{\mathcal{M}^I}.$$

The bihamiltonian structure

Choose a base element α of the center of the nilpotent subalgebra $\mathfrak{n}$.

The Drinfeld - Sokolov bihamiltonian structure

$$\{\ ,\ \}_2 + \lambda\{\ ,\ \}_1$$

on the reduced space is obtained by the shift

$$q(x) \mapsto q(x) + \lambda\alpha$$

Fix a homogeneous basis $\gamma_1, \ldots, \gamma_n$ of $V = \oplus_{i=1}^n V_i$, we have

$$q^{\mathrm{can}}(x) = \sum_{i=1}^{n} w^i(x)\,\gamma_i \in L(V).$$

The bihamiltonian structures are expressed by differential polynomials of $w^1, \ldots, w^n$.

Example

The A_1 case

$$\{w(x), w(y)\}_1 = \delta'(x-y),$$
$$\{w(x), w(y)\}_2 = w(x)\delta'(x-y) + \frac{1}{2} w_x \delta(x-y) + \frac{\epsilon^2}{8}\delta''(x-y)$$

The bihamiltoian structure of the KdV hierarchy.

4. The associated Frobenius manifolds

Theorem. (Dubrovin, Liu, Z.)

The bihamiltonian structure of hydrodynamic type given by the leading terms of the Drinfeld - Sokolov bihamiltonian structure is equivalent to the one that is defined, in terms of the Frobenius manifold structure, on the loop space of the n-dimensional orbit space of the Weyl group of $\mathfrak{g}$.

Recall that the reduced phase space

$$\mathcal{M}^I/L(N) \simeq \left\{ \epsilon \frac{d}{dx} + I + q^{\mathrm{can}}(x) \mid q^{\mathrm{can}}(x) \in L(V) \right\}$$

where

$$\mathfrak{b} = V \oplus [I, \mathfrak{n}],$$

$q^{\mathrm{can}}(x)$ are gauge invariant differential polynomials of $q(x) \in L(\mathfrak{b})$

$$S^{-1}(x) \left(\epsilon \frac{d}{dx} + q + I \right) S(x) = \epsilon \frac{d}{dx} + q^{\mathrm{can}} + I$$

we have the isomorphism

$$\left\{ \begin{array}{l} \text{gauge invariant differential} \\ \text{polynomials } f(q; q_x, q_{xx}, \dots) \\ \text{on the space of differential} \\ \text{operators } \epsilon \frac{d}{dx} + q + I,\ q(x) \in \mathfrak{b} \end{array} \right\} \to \left\{ \begin{array}{l} \text{differential polynomials} \\ \text{on the affine algebraic} \\ \text{variety} \quad \mathfrak{h}/W \end{array} \right\}$$

The orbit space $M_{\mathfrak{g}} = \mathfrak{h}/W$ carries a flat pencil of metrics discovered in 1980 by K. Saito, T. Yano and J. Sekeguchi, this flat pencil of metrics defines a bihamiltonian structure on the loop space of $M_{\mathfrak{g}}$, and also a Frobenius manifold structure on $M_{\mathfrak{g}}$ by Dubrovin 1993.

The construction of Frobenius manifold structure on $M_{\mathfrak{g}}$

For the chosen basis of simple roots $\alpha_1, \ldots, \alpha_n \in \mathfrak{h}^*$ denote

$$G_{ab} = \langle \alpha_a^\vee, \alpha_b^\vee \rangle_{\mathfrak{g}}, \quad a,\, b = 1, \ldots, n$$

the Gram matrix of the invariant bilinear form. Let

$$\left(G^{ab}\right) = \left(G_{ab}\right)^{-1}$$

be the inverse matrix. It gives a (constant) bilinear form on the cotangent bundle $T^*\mathfrak{h}$. The projection of the bilinear form onto the quotient $\mathfrak{h}/W$ defines a bilinear form on $T^*M_{\mathfrak{g}}$ non-degenerate outside the locus $\Delta \subset M_{\mathfrak{g}}$ of singular orbits (the so-called *discriminant* of the Weyl group W).

The second flat metric on $M_{\mathfrak{g}}$

In order to represent this form in the coordinates let us choose a system of W-invariant homogeneous polynomials $y^1(\xi)$, ..., $y^n(\xi)$ generating the ring $\mathbb{C}[\mathfrak{h}^*]^W$. Here $\xi = \xi^a \alpha_a \in \mathfrak{h}$. The polynomial function

$$G^{ab} \frac{\partial y^i(\xi)}{\partial \xi^a} \frac{\partial y^j(\xi)}{\partial \xi^b}$$

is W-invariant for every $i, j = 1, \ldots, n$ and, thus, is a polynomial in y^1, ..., y^n. Denote $g_2^{ij}(y)$ these polynomials,

$$g_2^{ij}(y(\xi)) = G^{ab} \frac{\partial y^i(\xi)}{\partial \xi^a} \frac{\partial y^j(\xi)}{\partial \xi^b}.$$

The first flat metric

To define the first metric, let us assume that the invariant polynomial $y^1(\xi)$ has the maximal degree

$$\deg y^1(\xi) = h.$$

Here h is the Coxeter number of the Lie algebra $\mathfrak{g}$. Put

$$g_1^{ij}(y) := \frac{\partial g_2^{ij}(y)}{\partial y^1}.$$

The Frobenius manifold structure

Let $v^1(\xi), \ldots, v^n(\xi)$ be a system of flat coordinates for the first metric:

$$\eta^{ij} := (dv^i, dv^j)_1 = \text{const}.$$

Put

$$g^{ij}(v) := (dv^i, dv^j)_2.$$

Then there exists an element $F(v)$ of the degree $2h+2$ in the ring of W-invariant polynomials such that

$$\eta^{ik}\eta^{jl}\frac{\partial^2 F(v)}{\partial v^k \partial v^l} = \frac{h}{\deg v^i + \deg v^j - 2}\, g^{ij}(v).$$

The third derivatives

$$c_{ij}^k(v) := \eta^{kl} \frac{\partial^3 F(v)}{\partial v^l \partial v^i \partial v^j}$$

are the structure constants of the multiplication on the tangent space $T_v M_{\mathfrak{g}}$. This multiplication defines on $T_v M_{\mathfrak{g}}$ an associative commutative algebra with unity, and with a invariant bilinear form (A Frobenius algebra). The associativity condition is a nonlinear PDE for the function $F(v)$, called the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation.

Frobenius manifold is a coordinated free formulation of the WDVV equations.

A flat pencil of metrics on an arbitrary Frobenius manifold

A Frobenius manifold is equipped with a flat metric $\langle\ ,\ \rangle$, a product of tangent vectors $(a, b) \mapsto a \cdot b$, and an Euler vector field E. We put

$$(\ ,\ )_1 := \langle\ ,\ \rangle$$

and define the second metric on the cotangent bundle by

$$(\omega_1, \omega_2)_2 = i_E\, \omega_1 \cdot \omega_2$$

that must be valid for an arbitrary pair of 1-forms on the Frobenius manifold. In this formula the identification of tangent and cotangent spaces at every point is done by means of the *first* metric $(\ ,\ )_1$. By means of this identification one defines the product of 1-forms $\omega_1 \cdot \omega_2$ via the product of tangent vectors.

5. The central invariants of the Drinfeld - Sokolov bihamiltonian structure

Theorem. (Dubrovin, Liu, Z.)

For the Drinfeld - Sokolov bihamiltonian structure associated to a simple Lie algebra $\mathfrak{g}$ with an invariant bilinear form $\langle\ ,\ \rangle_{\mathfrak{g}}$, the central invariants are given by the following formulae up to a permutation:

$$c_i = \frac{1}{48}\langle \alpha_i^\vee, \alpha_i^\vee \rangle_{\mathfrak{g}}, \quad i = 1, \dots, n.$$

In particular, if we fix the invariant bilinear form $\langle\, ,\, \rangle_{\mathfrak{g}}$ as follows:

$$\langle a, b\rangle_{\mathfrak{g}} := \frac{1}{2h^{\vee}}\,\mathrm{tr}(\mathrm{ad}\, a \cdot \mathrm{ad}\, b),$$

where $h^{\vee}$ is the dual Coxeter number of $\mathfrak{g}$, then the above theorem shows that the central invariants for the Drinfeld - Sokolov bihamiltonian structures associated to the simple Lie algebras of A-D-E type are given by

$$c_1 = c_2 = \cdots = c_n = \frac{1}{24}$$

$\mathfrak{g}$	c_1	$\dots$	c_{n-1}	c_n
A_n	$\frac{1}{24}$	$\dots$	$\frac{1}{24}$	$\frac{1}{24}$
B_n	$\frac{1}{24}$	$\dots$	$\frac{1}{24}$	$\frac{1}{12}$
C_n	$\frac{1}{12}$	$\dots$	$\frac{1}{12}$	$\frac{1}{24}$
D_n	$\frac{1}{24}$	$\dots$	$\frac{1}{24}$	$\frac{1}{24}$
$E_n, \ n = 6, 7, 8$	$\frac{1}{24}$	$\dots$	$\frac{1}{24}$	$\frac{1}{24}$
$F_n, \ n = 4$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{12}$
$G_n, \ n = 2$	$\frac{1}{8}$			$\frac{1}{24}$

"Proof of the theorem for A-B-C-D cases"

For the A-B-C-D case, the Drinfeld - Sokolov bihamiltonian structure can be represented by a differential or pseudo-differential operator.

For the A_n case, the differential operator has the form

$$L = D^{n+1} + w_n(x)D^{n-1} + \cdots + w_2(x)D + w_1(x), \;\; D = \epsilon \frac{d}{dx}.$$

Define

$$\frac{\delta F}{\delta L} = \sum_{i=1}^{n} D^{-i} \frac{\delta F}{\delta w_i}.$$

and

$$\mathrm{Tr}\, A = \int \mathrm{res}\, A\, dx \in \bar{\mathcal{B}}, \quad \mathrm{res}\,(\sum_{i\le m} f_i D^i) = f_{-1}.$$

The Drinfeld – Sokolov bihamiltonian structure can be written as

$$\{F,G\}_\lambda = \{F,G\}_2 - \lambda\{F,G\}_1$$
$$= \frac{1}{\epsilon}\operatorname{Tr}\left((LY)_+LX - XL(YL)_+ + \frac{1}{n+1}X[L,g_Y]\right) - \lambda\frac{1}{\epsilon}\operatorname{Tr}\left([Y,X]L\right),$$

where $X = \frac{\delta F}{\delta L}$, $Y = \frac{\delta G}{\delta L}$, and the positive part of a pseudo-differential operator $Z = \sum z_i D^i$ is defined by

$$Z_+ = \sum_{i\geq 0} z_i D^i.$$

The function g_Y is defined by

$$g_Y = D^{-1}(\mathrm{res}[L, Y]).$$

Let

$$\lambda(x,p) = p^{n+1} + w_n(x)p^{n-1} + \cdots + w_2(x)p + w_1(x)$$

be the symbol of the Lax operator L. Then the dispersionless limit of the A_n Drinfeld – Sokolov bihamiltonian structure is given by the following formulae

$$\{\lambda(x,p),\lambda(y,q)\}_1 = \frac{\lambda'(p)-\lambda'(q)}{p-q}\,\delta'(x-y) + \left[\frac{\lambda_x(p)-\lambda_x(q)}{(p-q)^2} - \frac{\lambda'_x(q)}{p-q}\right]\delta(x-y),$$

$$\begin{aligned}&\{\lambda(x,p),\lambda(y,q)\}_2\\ &= \left(\frac{\lambda'(p)\lambda(q)-\lambda'(q)\lambda(p)}{p-q} + \frac{1}{n+1}\lambda'(p)\lambda'(q)\right)\delta'(x-y)\\ &\quad + \left[\frac{\lambda_x(p)\lambda(q)-\lambda_x(q)\lambda(p)}{(p-q)^2} + \frac{\lambda_x(q)\lambda'(p)-\lambda'_x(q)\lambda(p)}{p-q}\right.\\ &\quad \left. + \frac{1}{n+1}\lambda'(p)\lambda'_x(q)\right]\delta(x-y).\end{aligned}$$

For the $B - C - D$ cases,

$$B_n: \quad L = D^{2n+1} + \sum_{i=1}^{n} w_i(x)\, D^{2i-1} + \sum_{i=1}^{n} v_i(x)\, D^{2i-2}, \quad L + L^\dagger = 0$$

$$C_n: \quad L = D^{2n} + \sum_{i=1}^{n} w_i(x)\, D^{2i-2} + \sum_{i=2}^{n} v_i(x)\, D^{2i-3}, \qquad L = L^\dagger$$

$$D_n: \quad L = D^{2n-1} + \sum_{i=2}^{n} w_i(x)\, D^{2i-3} + \sum_{i=2}^{n} v_i(x)\, D^{2i-4} + \rho(x) D^{-1} \rho(x),$$
$$L + L^\dagger = 0.$$

Here $L^\dagger$ is the adjoint operator, the coefficients $v_i(x)$ are linear combinations of derivatives of $w_i(x)$ uniquely determined by the symmetry/antisymmetry conditions. We assume $w_1(x) = \rho^2(x)$ for the D_n case.

The variational derivative of a local functional w.r.t. L is now defined as

$$\frac{\delta F}{\delta L} = \frac{1}{2} \sum_{i=1}^{n} \left(D^{-2i+\nu} \frac{\delta F}{\delta w_i(x)} + \frac{\delta F}{\delta w_i(x)} D^{-2i+\nu} \right),$$

where $\nu = 0, 1, 2$ for the B_n, C_n and D_n cases respectively.

$$\{F, G\}_2 = \frac{1}{\epsilon} \operatorname{Tr} \left[(LY)_+ LX - XL(YL)_+ \right],$$

The first Poisson brackets are defined as the Lie derivatives of the second ones along the coordinate w_i, where $i = 1$ for B_n, C_n and $i = 2$ for D_n,

$$\{F, G\}_2(w_i, \cdots) - \lambda \{F, G\}_1(w_i, \cdots) = \{F, G\}_2(w_i - \lambda, \cdots).$$

"Proof of the theorem for the E_6, E_7, E_8, G_2, F_4 exceptional cases"
Dirac reduction formulation of the the DS reduction

[1] Boris Dubrovin, Si-Qi Liu, Youjin Zhang,
Frobenius Manifolds and Central Invariants for the Drinfeld - Sokolov Bihamiltonian Structures, *Adv. Math.* (2008), doi:10.1016/j.aim.2008.06.009

[2] Si-Qi Liu, Youjin Zhang,
On quasi-triviality and integrability of a class of scalar evolutionary PDEs, *J. Geom. Phys.* 57(2006),101-119.

[3] Boris Dubrovin, Si-Qi Liu, Youjin Zhang,
On Hamiltonian perturbations of hyperbolic systems of conservation laws I: quasi-triviality of bi-Hamiltonian perturbations,
Commun. Pure and Appl. Math. 59(2006), 559-615.

6. Conclusion

The Drinfeld - Sokolov bihamiltonian structures associated to the untwisted affine Lie algebras of $A - D - E$ type are topological deformations.

The A_n case: the Gelfand - Dickey hierarchy, corresponds to the A_n-topological minimal model
(a recent proof by C. Faber, S. Shadrin, D. Zvonkine, Tautological relations and the r-spin Witten conjecture, arXiv:math/0612510.)

The D-E case: A generalization of the intersection theory on the moduli space of r-spin curves was recently given to define 2d TFT corresponding to quasihomogenous hyper surface singularities.
(Fan Huijun, Javis Tyler, Ruan Yongbin)

Problem:

Starting from a semisimple FM to construct, by generalizing the DS construction, the topological deformation of the bihamiltonian structure of hydrodynamic type.

Thanks