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QUANTUM BI-HAMILTONIAN SYSTEMS

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We define quantum bi-Hamiltonian systems, by analogy with the classical case, as derivations in operator algebras which are inner derivations with respect to two compatible associative structures. We find such structures by means of the associative version of Nijenhuis tensors. Explicit examples, e.g. for the harmonic oscillator, are given.

1. Introduction

Bi-Hamiltonian systems at the classical level, as noticed by F. Magri,[1] play an important role in the discussion of complete integrability in the sense of Liouville.

At the quantum level, much earlier, E. P. Wigner[2] raised the question: *Do the equations of motion determine the quantum mechanical commutation relations?*

The way Wigner formulated his question was the following. Assuming the equations of motion

$$i\frac{\mathrm{d}}{\mathrm{d}t}\hat{q} = \frac{\hat{p}}{m}\,, \qquad i\frac{\mathrm{d}}{\mathrm{d}t}\hat{p} = -\frac{\widehat{\partial V}}{\partial q}\,, \tag{1}$$

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to find commutation relations such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{q} = -\frac{i}{\hbar}[\hat{q}, \hat{H}]\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\hat{p} = -\frac{i}{\hbar}[\hat{p}, \hat{H}]\,. \tag{2}$$

Wigner argued that equations of motion have a more immediate physical significance than the canonical commutation relations

$$[\hat{p}, \hat{q}] = -i\hbar\,. \tag{3}$$

The commutation relations we are searching for should define a "quantum Poisson bracket" in the terminology of Dirac.[3] Indeed, Dirac shows that if we look for a Lie algebra structure on the space of observables such that

$$[A, BC] = [A, B]C + B[A, C]\,, \tag{4}$$

then necessarily

$$[A, B] = \lambda(AB - BA)\,, \tag{5}$$

with λ being any complex number. To put it differently, according to Dirac, to look for alternative commutation relations (with the additional requirement (4)), it is equivalent to look for alternative products on the space of observables with the requirement that the equations of motion define a derivation with respect to the associative product.

Recently it has been shown,[4] in connection also with deformed oscillators, that one may obtain a large class of alternative associative products of the kind

$$A \circ_K B = AKB \tag{6}$$

for which the dynamics is a derivation any time K is an observable which is a constant of the motion. In particular, it has been applied to a precessing magnetic dipole.[5] It turns out that all these deformations are compatible among themselves in the sense we will explain later. This is rather unsatisfactory, because in considering the classical limit of these quantum cases we should be able to recover Poisson structures which are not necessarily compatible.

This paper is an attempt to put the search of alternative associative products in a more systematic setting.

2. Some Important Concepts in Cohomology of Algebras

Let $(\mathcal{A}, *)$ be an associative algebra and V be a $\mathcal{A}$-bimodule, respectively. In other words, V is a module that is the carrier space for a linear representation Ψ of $\mathcal{A}$ and a linear antirepresentation Ψ' of $\mathcal{A}$ that commute.

By a n-cochain we mean a n-linear mapping from $\mathcal{A} \times \cdots \times \mathcal{A}$ (n times) into V. We denote by $C^n(\mathcal{A}, V)$ the space of such n-cochains that can be regarded as an additive group. For every $n \in N$ we introduce the Hochschild[6]

coboundary operator, as defined by Eilenberg and MacLane, $\delta : C^n(\mathcal{A}, V) \to C^{n+1}(\mathcal{A}, V)$, by

$$(\delta\alpha)(a_1, \ldots, a_{n+1}) = a_1\alpha(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n}(-1)^i\alpha(a_1, \ldots, a_i * a_{i+1}, \ldots, a_{n+1}) + (-1)^{n+1}\alpha(a_1, \ldots, \ldots, a_n)a_{n+1}\,. \tag{7}$$

It is now easy to check that

$$\delta \circ \delta = 0\,.$$

The cohomology groups can be defined as follows: an n cochain $\alpha \in C^n(\mathcal{A}, V)$ is called an n-cocycle if $\delta\alpha = 0$, and an element of the form $\delta\beta$ where $\beta \in C^{n-1}(\mathcal{A}, V)$ is called an n-coboundary. These form a subgroup $B^n(\mathcal{A}, V)$ of the additive group $Z^n(\mathcal{A}, V)$ of n-cocycles. The cohomology group $H^n(\mathcal{A}, V)$ is defined as the quotient group $H^n(\mathcal{A}, V) = Z^n(\mathcal{A}, V)/B^n(\mathcal{A}, V)$.

For instance, when $n = 1$, we obtain

$$(\delta\alpha_1)(a_1, a_2) = a_1\alpha_1(a_2) - \alpha_1(a_1 * a_2) + \alpha_1(a_1)a_2\,,$$

and for $n = 2$,

$$(\delta\alpha_2)(a_1, a_2, a_3) = a_1\alpha_2(a_2, a_3) - \alpha_2(a_1 * a_2, a_3) + \alpha_2(a_1, a_2 * a_3) - \alpha_2(a_1, a_2)a_3\,.$$

The simplest example obtains when V is the additive group of $\mathcal{A}$, and then the $\mathcal{A}$-bimodule structure is given by left and right multiplication.

3. Compatible Associative Products and Associative Nijenhuis Tensors

By analogy with the classical case, where a bi-Hamiltonian system consists of two compatible Poisson brackets and a system which is Hamiltonian with respect to both brackets, by a *weak quantum bi-Hamiltonian system* we shall mean two Lie algebra structures on the space $\mathrm{Op}(\mathcal{H})$ of operators on a Hilbert space $\mathcal{H}$ (one of them will be usually the original one) which are compatible in the sense that the corresponding commutators are compatible Lie brackets (i.e. their sum is again a Lie bracket) and a derivation $D \in \mathrm{Der}(\mathrm{Op}(\mathcal{H}))$ which is an inner derivation with respect to both associative structures.[7]

Since we want the Leibniz rule

$$[A, B \circ C] = [A, B] \circ C + B \circ [A, C]\,, \tag{8}$$

in view of the Dirac's proof,[3] (pp. 85–86), that derivations of a sufficiently non-degenerate associative algebra are just adjoint operators, we would like to have a new bracket in the form of the commutator of a new associative structure. We will call such pairs of associative structures just *weak quantum bi-Hamiltonian* ones. A possible additional requirement is that both associative structures have the same unit **1**. Let us note that one can also consider a stronger version of compatibility of associative products "$\circ_1$" and "$\circ_2$" requiring that $\circ_1 + \lambda\circ_2$ is associative for all

$\lambda \in \mathbb{K}$, where $\mathbb{K}$ is the ground field (then the mean $(\circ_1 + \circ_2)/2$ is again associative with the same unit $\mathbf{1}$) and this is what we mean by a *quantum bi-Hamiltonian system*. We start with some pure algebraic observations.

Let $(\mathcal{A}, \cdot)$ be a unital associative algebra. A simple way to define a new associative product on $\mathcal{A}$ is to take an element $K \in \mathcal{A}$ and to define a new product by

$$A \circ_K B = AKB\,. \tag{9}$$

(We will usually skip the product symbol for the original associative structure.) Observe that the unit is not preserved unless $K = \mathbf{1}$ and that we have the homomorphism of the products

$$T_K(A \circ_K B) = T_K(A)T_K(B) \tag{10}$$

for T_K being the linear map

$$T_K : \mathcal{A} \to \mathcal{A}\,, \qquad T_K(A) = KA\,, \tag{11}$$

which is an isomorphism (nonunital, however) in case K is invertible.

This example can be generalized if we deform the associative structure by an associative analog of the Nijenhuis map (tensor), known better in the Lie algebra case.

Let $(\mathcal{A}, \mu)$ be an associative algebra over a field $\mathbb{K}$, with the product

$$\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}\,, \qquad (A, B) \mapsto AB \tag{12}$$

and let $N : \mathcal{A} \to \mathcal{A}$ be a linear map ($N \in \mathcal{A}^* \otimes \mathcal{A}$). If N is a derivation of the algebra $(\mathcal{A}, \mu)$, then $N(A)B + AN(B) - N(AB) = 0$. In any case, the map

$$\mu_N : (A, B) \mapsto A \circ_N B = N(A)B + AN(B) - N(AB)\,, \tag{13}$$

is a bilinear map and therefore it defines a new algebra structure $(\mathcal{A}, \mu_N)$. Using the terminology introduced in the preceding section, and considering the $\mathcal{A}$-bimodule structure in $\mathcal{A}$ as given by left and right multiplication, we can say that $A \circ_N B = \delta_\mu N(A, B)$ and therefore that N is a derivation of the original algebra if and only if N is a one-cocycle with respect to the Hochschild coboundary operator δ_μ associated with the product μ.

The obstruction for the linear map N to be a homomorphism of these products is measured by the μ-Nijenhuis torsion of N:

$$T_N(A, B) = N(A \circ_N B) - N(A)N(B)\,. \tag{14}$$

Definition 1. *We say that the linear map $N : \mathcal{A} \to \mathcal{A}$ is a μ-Nijenhuis tensor if the μ-Nijenhuis torsion of N vanishes, $T_N(A, B) = 0$, $\forall A,\, B \in \mathcal{A}$.*

In this case N is a homomorphism of the corresponding products:

$$N(A \circ_N B) = N(A)N(B)\,, \tag{15}$$

i.e.

$$N(N(A)B + AN(B) - N(AB)) - N(A)N(B) = 0\,.$$

Theorem 1. *The product μ_N defined by* (13) *is associative if and only if the μ-Nijenhuis torsion T_N of N is a two-Hochschild cocycle of the algebra $\mathcal{A}$, i.e.*

$$\delta_\mu T_N(A,B,C) := AT_N(B,C) - T_N(AB,C) + T_N(A,BC) - T_N(A,B)C = 0\,. \tag{16}$$

If this is the case, μ_N is an associative product compatible with μ, i.e. $\mu + \lambda\mu_N$ are associative for all $\lambda \in \mathbb{K}$. If μ is unital with the unit $\mathbf{1}$, then μ_N has the same unit providing that $N(\mathbf{1}) = \mathbf{1}$.

In particular, if N is a μ-Nijenhuis tensor, then μ_N is an associative product on $\mathcal{A}$ which is compatible with μ.

Proof. By direct computation,

$$\begin{aligned}(A \circ_N B) \circ_N C - A \circ_N (B \circ_N C) &= -AT_N(B,C) + T_N(AB,C) - T_N(A,BC) + T_N(A,B)C \\ &= -\delta_\mu T_N(A,B,C)\,.\end{aligned}$$

As for the compatibility, it suffices to prove

$$(AB) \circ_N C + (A \circ_N B)C = A(B \circ_N C) + A \circ_N (BC)\,, \tag{17}$$

which is straightforward:

$$\begin{aligned}(AB) \circ_N C + (A \circ_N B)C &= N(AB)C + ABN(C) - N(ABC) \\ &\quad + N(A)BC + AN(B)C - N(AB)C \\ &= N(A)BC + AN(B)C + ABN(C) - N(ABC) \\ &= A(B \circ_N C) + A \circ_N (BC)\,.\end{aligned}$$

□

The relation (17) means that the map μ_N, as seen as two-cochain in the algebra $(\mathcal{A}, \mu)$, is a two-cocycle because

$$\delta_{\mu_N}(A,B,C) = A(B \circ_N C) - (AB) \circ_N C + A \circ_N (BC) - (A \circ_N B)C\,.$$

Remark. Note that the compatibility condition (17) holds automatically, no matter if μ_N is associative or not. If we look for a new associative product $\circ$ which is compatible in the sense of (17), then this means that the new product is a Hochschild cocycle of the original associative algebra. If our algebra is, for instance, the algebra of $n \times n$ matrices, due to the Morita equivalence (cf. Ref. 8), its Hochschild cohomology are the same as the Hochschild cohomology of $\mathbb{K}$ (regarded as one-dimensional algebra over itself), thus vanish in dimensions > 0, so our product $\circ$ has to be a Hochschild coboundary, i.e. of the form $\circ_N$ for some N. This shows that we have not much freedom and, looking for compatible associative products, we must, in principle, work with Nijenhuis tensors.

The above observations can be reformulated in terms of the so called *Gerstenhaber bracket* $[\,,\,]_G$, which is a graded Lie bracket on the graded space of multilinear maps of $\mathcal{A}$ into $\mathcal{A}$ and which recognizes associative products (cf. Refs. 9 and 10), in full correspondence with the analogous theory for Nijenhuis tensors for Lie algebras and the Richardson–Nijenhuis bracket (cf. Ref. 11). In particular, $\mu_N = [\mu, N]_G$ and

$$2T_N(A,B) = [N,[\mu,N]_G]_G + [\mu,N^2]_G\,, \tag{18}$$

so that Theorem 1 is a direct consequence of the graded Jacobi identity for the Gerstenhaber bracket and the fact that $[\mu,\cdot]_G$ is proportional to the Hochschild coboundary operator δ_μ. (In particular, $[\mu,\mu]_G = 0$, so that $[\mu,[\mu,N]_G]_G = 0$, which is the compatibility condition (17).)

Now, we will show that a Nijenhuis tensor gives rise to a whole hierarchy of Nijenhuis tensors and associative structures, as has been already discovered by Saletan.[12] Putting N^k instead of N in the above, we can consider products μ_{N^k}.

Lemma 1. *If N is a μ-Nijenhuis tensor, then the products $\mu_{N^{k+r}}$ and μ_{N^k} are N^r-related, i.e.*

$$N^r(A \circ_{N^{k+r}} B) = N^r(A) \circ_{N^k} N^r(B) \tag{19}$$

for all k, $r = 0,1,2,\ldots$.

Proof. We will start with proving

$$N(A \circ_{N^{k+1}} B) = N(A) \circ_{N^k} N(B)\,. \tag{20}$$

Applying N^k to the Nijenhuis torsion

$$N(N(A)B) + N(AN(B)) - N^2(AB) - N(A)N(B)\,, \tag{21}$$

which vanish by assumption, we get

$$N^{k+2}(AB) - N^{k+1}(AN(B)) = N^{k+1}(N(A)B) - N^k(N(A)N(B))\,. \tag{22}$$

Using (22) inductively for $k := k-1$, we end up with

$$N^{k+2}(AB) - N^{k+1}(AN(B)) = N(N^{k+1}(A)B) - N^{k+1}(A)N(B)\,. \tag{23}$$

In a similar way, we get

$$N^{k+2}(AB) - N^{k+1}(N(A)B) = N(AN^{k+1}(B)) - N(A)N^{k+1}(B) \tag{24}$$

which, combined with (22), gives

$$N^{k+1}(AN(B)) - N^k(N(A)N(B)) = N(AN^{k+1}(B)) - N(A)N^{k+1}(B)\,. \tag{25}$$

Combining now (25) and (23), we find

$$\begin{aligned} &N^{k+2}(AB) - N^k(N(A)N(B)) \\ &\quad = N(N^{k+1}(A)B + AN^{k+1}(B)) - N^{k+1}(A)N(B) - N(A)N^{k+1}(B) \end{aligned}$$

which can be rewritten in the form

$$N^k(N(A))N(B) + N(A)N^k(N(B)) - N^k(N(A)N(B))$$
$$= N(N^{k+1}(A)B + AN^{k+1}(B) - N^{k+1}(AB))\,.$$

But the last one is exactly (20). Now, applying (20) inductively

$$N^r(A \circ_{N^{k+r}} B) = N^{r-1}N(A \circ_{N^{k+r}} B) = N^{r-1}(N(A) \circ_{N^{k+r-1}} N(B))\,, \tag{26}$$

we end up with (19). □

Theorem 2. *If N is a μ-Nijenhuis tensor, then*

$$(\mu_{N^i})_{N^k} = \mu_{N^{i+k}} \tag{27}$$

and N^r is a μ_{N^i}-Nijenhuis tensor, i.e.

$$N^r(A(\circ_{N^i})_{N^r}B) = N^r(A) \circ_{N^i} N^r(B) \tag{28}$$

for all $i, k = 0, 1, 2, \ldots$. In particular, all products μ_{N^k} are associative and compatible.

Proof. First, we show that

$$(\mu_{N^i})_N = \mu_{N^{i+1}}\,. \tag{29}$$

Indeed,

$$\begin{aligned}
A(\circ_{N^i})_N B &= N(A) \circ_{N^i} B + A \circ_{N^i} N(B) - N(A \circ_{N^i} B)\\
&= N^{i+1}(A)B + N(A)N^i(B) - N^i(N(A)B) + N^i(A)N(B)\\
&\quad + AN^{i+1}(B) - N^i(AN(B)) - N(A) \circ_{N^{i-1}} N(B)\\
&= N^{i+1}(A)B + AN^{i+1}(B) - N^{i+1}(AB) - N^i(N(A)B\\
&\quad + AN(B) - N(AB)) + N^{i-1}(N(A)N(B))\\
&= N^{i+1}(A)B + AN^{+1}(B) - N^{i+1}(AB)\\
&\quad - N^{i-1}(N(A \circ_N B) - N(A)N(B)) = A \circ_{N^{i+1}} B\,,
\end{aligned}$$

where we have used, according to Lemma, $N(A \circ_{N^i} B) = N(A) \circ_{N^{i-1}} N(B)$. Now, (29) together with (20) show that N is a μ_{N^i}-Nijenhuis tensor which produces a compatible associative product $(\mu_{N^i})_N = \mu_{N^{i+1}}$. Thus we can apply Lemma and (29) to μ_{N^i} instead of μ that proves the theorem. □

There is a way to obtain a new Nijenhuis tensor from two of them. Two Nijenhuis tensors N_1 and N_2 on $\mathcal{A}$ will be said to be *compatible* if $N_1 + N_2$ is again a Nijenhuis tensor.

Theorem 3. *Nijenhuis tensors N_1 and N_2 are compatible if and only if*

$$N_1(A \circ_{N_2} B) + N_2(A \circ_{N_1} B) = N_1(A)N_2(B) + N_2(A)N_1(B)\,. \tag{30}$$

If N_1 is compatible with $N_2, \ldots, N_k$, then it is compatible with any linear combination of them. If $N_1, \ldots, N_k$ are pairwise compatible, then any two linear combinations of them are compatible.

Proof. The first statement is a direct consequence of definitions if we only observe that $\circ_{N_1+N_2} = \circ_{N_1} + \circ_{N_2}$. The rest follows from the fact that (30) depends linearly on N_1 and N_2. □

Theorem 4. *If N is a Nijenhuis tensor, then all linear combinations of N^k, $k = 0, 1, 2, \ldots,$ are compatible.*

Proof. Indeed, for $k \geq r$,

$$\begin{aligned} &N^k(A \circ_{N^r} B) + N^r(A \circ_{N^k} B) \\ &\quad = N^{k-r}(N^r(A)N^r(B)) + N^r(A) \circ_{N^{k-r}} N^r(B) \\ &\quad = N^k(A)N^r(B) + N^r(A)N^k(B)\,, \end{aligned}$$

where we have used Theorem 2. Now, the assertion follows from Theorem 3. □

Remark. Let us observe that the product (6) can be obtained from the Nijenhuis tensor $N_K(A) = KA$. Indeed,

$$A \circ_{N_K} B = (KA)B + A(KB) - K(AB) = AKB\,. \tag{31}$$

The operator N_K is a Nijenhuis tensor, since

$$N_K(A \circ_{N_K} B) = K(AKB) = (KA)(KB) = N_K(A)N_K(B)\,. \tag{32}$$

In particular, the operators of multiplication by elements of the field $\mathbb{K}$ are Nijenhuis tensors. Other examples of Nijenhuis tensors can be constructed in the following way.

Theorem 5. *If $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ is a decomposition of an associative algebra $\mathcal{A}$ (nonunital in general) with the multiplication μ into two subalgebras ($\mathcal{A}$ with such a decomposition is called a twilled algebra) and P_1, P_2 denote the corresponding projections of $\mathcal{A}$ onto $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, then any linear combination $N = \lambda_1 P_1 + \lambda_2 P_2$ is a μ-Nijenhuis tensor.*

Proof. Since $\lambda_1 P_1 + \lambda_2 P_2 = (\lambda_1 - \lambda_2)P_1 + \lambda_2 I$, it is sufficient to show that P_1 is a μ-Nijenhuis tensor. Using the decomposition $A = A_1 + A_2$ etc. we have

$$A \circ_{P_1} B = A_1 B + A B_1 - (AB)_1\,, \tag{33}$$

so that $\mu_{P_1} = \mu$ on $\mathcal{A}_1$, $\mu_{P_1} = 0$ on $\mathcal{A}_2$, and

$$A_1 \circ_{P_1} B_2 = P_2(A_1 B_2)\,, \qquad A_2 \circ_{P_1} B_1 = P_2(A_2 B_1)\,. \tag{34}$$

Hence,

$$A \circ_{P_1} B = A_1 B_1 + P_2(A_1 B_2 + A_2 B_1) \tag{35}$$

and

$$P_1(A \circ_{P_1} B) = P_1(A_1 B_1 + P_2(A_1 B_2 + A_2 B_1)) = A_1 B_1 = P_1(A) P_1(B)\,, \tag{36}$$

so that P_1 is a Nijenhuis tensor. □

Example 1. Take $\mathcal{A} = M_2(\mathbb{K})$ — the algebra of 2×2-matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Take $\mathcal{A}_1$ to be the algebra of upper-triangular matrices $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and let $\mathcal{A}_2$ be the (commutative) algebra of strictly lower-triangular matrices $A = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$. Taking the Nijenhuis tensor P_1, we get new associative matrix multiplication in the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' & ab' + bd' \\ ca' + dc' & dd' \end{pmatrix}\,. \tag{37}$$

Note that the unit matrix I remains the unit for this new product and that inner derivations given by diagonal matrices are the same for both products.

Of course, we can use the complementary projection instead and get the product

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ' \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} bc' & 0 \\ 0 & cb' \end{pmatrix} \tag{38}$$

which is associative but not unital. Of course, this example admits an obvious generalization to algebras of matrices of any dimension. Note also that we can consider these products at the level of the operator algebra $\mathrm{Op}(\mathcal{H})$ over a Hilbert space $\mathcal{H}$ directly, viewing this algebra as algebra of infinite matrices, or using a decomposition of $\mathcal{H}$ into a direct sum of subspaces, so that we can write operators in a matrix form.

Remark. Observe that the product (35) is associative even if $\mathcal{A}_2$ is not a subalgebra but just a complementary subspace. Indeed,

$$\begin{aligned}(A \circ B) \circ C &= (A_1 B_1 + P_2(A_1 B_2 + A_2 B_1)) \circ C \\ &= A_1 B_1 C_1 + P_2(A_1 B_1 C_2 + P_2(A_1 B_2 + A_2 B_1) C_1) \\ &= A_1 B_1 C_1 + P_2(A_1 B_1 C_2 + A_1 B_2 C_1 + A_2 B_1 C_1) = A \circ (B \circ C)\,.\end{aligned}$$

However, this product is not of the form μ_{P_1} and it is, in general, not compatible with the original one.

Example 2. Again, for the matrix algebra $\mathcal{A} = M_2(\mathbb{K})$ take $\mathcal{A}_1 = span\langle I, C\rangle$, $\mathcal{A}_2 = span\langle A, B\rangle$, where

$$\begin{aligned} A &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ C &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{39}$$

Using the Nijenhuis tensor P_1, we get the product

$$\begin{aligned} &A \circ B = B \circ A = 0\,, && A \circ A = 0\,, && B \circ B = 0\,, \\ &A \circ C = B\,, && C \circ A = -B\,, \\ &B \circ C = -A\,, && C \circ B = A\,, && C \circ C = -I\,, \end{aligned}$$

and I remains the unit for this product. The inner derivation associated with C is the same for both products.

The product (35) is in fact a contraction of the original one, since

$$A \circ_{P_1} B = \lim_{h\to\infty} T_h^{-1}(T_h(A)T_h(B))\,, \tag{40}$$

where $T_h(A) = A_1 + hA_2$. Indeed,

$$\begin{aligned} T_h^{-1}(T_h(A)T_h(B)) &= T_h^{-1}(A_1B_1 + h(A_2B_1 + A_2B_1) + h^2A_2B_2) \\ &= A_1B_1 + P_2(A_2B_1 + A_1B_2) + hP_1(A_2B_1 + A_1B_2) \\ &\quad + hP_2(A_2B_2) + h^2P_1(A_2B_2) \end{aligned}$$

which tends to $A_1B_1 + P_2(A_2B_1 + A_1B_2)$ as $h \to 0$.

This can be generalized as follows. Using a decomposition of the algebra $\mathcal{A}$ into the direct sum $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, where $\mathcal{A}_1$ is assumed to be a subalgebra, we will write $A = A_1 + A_2$ for any element $A \in \mathcal{A}$ accordingly to this decomposition. Suppose that we have invertible linear operators N_1, N_2 acting, respectively, on $\mathcal{A}_1$ and $\mathcal{A}_2$. For any $h \in \mathbb{K}$ we define $T_h : \mathcal{A} \to \mathcal{A}$ by $T_h(A) = N_1(A_1) + hN_2(A_2)$ and put

$$A \circ_h B = T_h^{-1}(T_h(A)T_h(B))\,. \tag{41}$$

The product "$\circ_h$" is clearly associative and

$$\begin{aligned} A \circ_h B = {} & N_1^{-1}(N_1(A_1)N_1(B_1)) \\ & + N_2^{-1}((N_1(A_1)N_2(B_2) + N_2(A_2)N_1(B_1) + hN_2(A_2)N_2(B_2))_2) \\ & + hN_1^{-1}((N_1(A_1)N_2(B_2) + N_2(A_2)N_1(B_1) + hN_2(A_2)N_2(B_2))_1)\,. \end{aligned}$$

Passing formally with $h \to 0$, we get the contracted associative product

$$A \circ B = N_1^{-1}(N_1(A_1)N_1(B_1)) + N_2^{-1}((N_1(A_1)N_2(B_2) + N_2(A_2)N_1(B_1))_2)\,. \tag{42}$$

If we assume that there is an associative product $\circ_1$ on $\mathcal{A}_1$ such that $N_1(A_1 \circ_1 B_1) = N_1(A_1)N_1(B_1)$, then we can write the product (42) in the form

$$A \circ B = N_1(A_1) \circ_1 N_1(B_1) + N_2^{-1}((N_1(A_1)N_2(B_2) + N_2(A_2)N_1(B_1))_2)\,. \tag{43}$$

Now we can skip the assumption that N_1 is invertible. We can get even more, as one can check by direct calculations.

Theorem 6. *Let $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ be a decomposition of an associative algebra into subspaces such that $\mathcal{A}_1$ is a subalgebra. Let $\circ_1$ be an additional associative product on $\mathcal{A}_1$ and let N_1, $N_1' : \mathcal{A}_1 \to \mathcal{A}_1$ be homomorphisms of the product $\circ_1$ into the original one ($N_1(A_1 \circ_1 B_1) = N_1(A_1)N_1(B_1)$, etc.). Then, for any invertible linear map $N_2 : \mathcal{A}_2 \to \mathcal{A}_2$, the product*

$$A \circ B = A_1 \circ_1 B_1 + N_2^{-1}((N_1(A_1)N_2(B_2) + N_2(A_2)N_1'(B_1))_2) \tag{44}$$

is an associative product on $\mathcal{A}$.

We obtain a particular case of the above theorem if we start with a Nijenhuis tensor N_1 on the subalgebra $\mathcal{A}_1$ and we put $N_1' = N_1$ and $\circ_1 = \circ_{N_1}$.

Example 3. Let $\mathcal{A}$ be a matrix algebra, $\mathcal{A}_1$ be the subalgebra of diagonal matrices, and $\mathcal{A}_2$ be the complementary subspace of matrix with 0 on the diagonal. Denote by $\Delta(A)$ the diagonal part of the matrix A. We define $N_1 : \mathcal{A}_1 \to \mathcal{A}_1$ to be the multiplication by an invertible diagonal matrix K which is a Nijenhuis tensor. We have $A \circ_1 B = KAB$ for diagonal matrices A and B. Finally, putting $N_1' = N_1$ and $N_2 = I$ on $\mathcal{A}_2$, we get a new associative product (44)

$$\begin{aligned} A \circ B &= K\Delta(A)\Delta(B) + K\Delta(A)(B - \Delta(B)) + (A - \Delta(A))K\Delta(B) \\ &= K\Delta(A)B + AK\Delta(B) - K\Delta(A)\Delta(B)\,. \end{aligned} \tag{45}$$

W have used the fact that $\mathcal{A}_2$ is invariant with respect to the multiplication by diagonal matrices. Note also, that the above product is not $\mu_{K\Delta}$ since, in general, $\Delta(A)\Delta(B) \neq \Delta(AB)$.

To construct a Nijenhuis tensor N on $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ from N_1, we can use the following.

Theorem 7. *If N_1 is a Nijenhuis tensor on the subalgebra $\mathcal{A}_1$ in the decomposition $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, then $N(A) = N_1(A_1)$ is a Nijenhuis tensor on $\mathcal{A}$ if and only if*

$$\begin{aligned} N_1^2((A_2B_2)_1) &= 0\,, \\ N_1((N_1(A_1)B_2)_1 - N_1((A_1B_2)_1)) &= 0\,, \\ N_1((A_2N_1(B_1))_1 - N_1((A_2B_1)_1)) &= 0\,, \end{aligned} \tag{46}$$

far all $A_i, B_i \in \mathcal{A}_i$, $i = 1, 2$. In particular, this is the case for $\mathcal{A}_2$ being a (two-sided) ideal.

Proof. Since

$$\begin{aligned} A \circ_N B &= N_1(A_1)B + AN_1(B_1) - N_1((AB)_1) \\ &= A_1 \circ_{N_1} B_1 + N_1(A_1)B_2 + A_2N_1(B_1) - N_1((A_1B_2 + A_2B_1 + A_2B_2)_1)\,, \end{aligned}$$

we just rewrite the condition $N_1((A \circ_N B)_1) = N_1(A_1)N_1(B_1)$ using the fact that N_1 is a Nijenhuis tensor and that $A_i, B_i \in \mathcal{A}_i$ can be chosen independently. □

Example 4. Let $\mathcal{A}$ be the algebra of $n \times n$-matrices which are upper-triangular. Take $\mathcal{A}_1$ to be the commutative subalgebra of diagonal matrices and $\mathcal{A}_2$ to be the complementary subalgebra of strictly upper-triangular matrices. Put N_1 to be the multiplication by a diagonal matrix K from the left. Then N_1 is a Nijenhuis tensor on $\mathcal{A}_1$ which can be extended to the Nijenhuis tensor $N(A) = N_1(A_1)$ on $\mathcal{A}$. Indeed, in this case $\mathcal{A}_2$ is an ideal. The corresponding deformed product has the form

$$A \circ_N B = K\Delta(A)B + AK\Delta(B) - K\Delta(AB)\,, \tag{47}$$

where $\Delta(A)$ denotes the diagonal part of A.

Let us recall (cf. Ref. 10) that a Nijenhuis tensor for a Lie algebra $(\mathcal{L}, [\cdot,\cdot])$ is a linear mapping $N : \mathcal{L} \to \mathcal{L}$ such that $N([A,B]_N) = [N(A), N(B)]$, where

$$[A,B]_N = [N(A),B] + [A,N(B)] - N[A,B]\,. \tag{48}$$

It is well known (see e.g. Ref. 10) that $[\cdot,\cdot]_N$ is a compatible Lie bracket if N is a Nijenhuis tensor. The relation between Nijenhuis tensors in the associative and Lie algebra cases describes the following.

Theorem 8. *If N is a μ-Nijenhuis tensor for an associative algebra $(\mathcal{A}, \mu)$, then N is a Nijenhuis tensor for the Lie algebra $(\mathcal{A}, [\,,\,])$, where $[A,B] = AB - BA$ is the commutator, and*

$$[A,B]_N = A \circ_N B - B \circ_N A\,, \tag{49}$$

i.e. the deformed Lie bracket $[\cdot,\cdot]_N$ is the commutator of the deformed associative product $\circ_N$.

Proof. By definition,

$$\begin{aligned}
[A,B]_N &= [N(A),B] + [A,N(B)] - N[A,B] \\
&= N(A)B - BN(A) + AN(B) - N(B)A - N(AB - BA) \\
&= (N(A)B + AN(B) - N(AB)) - (N(B)A + BN(A) - N(BA)) \\
&= A \circ_N B - B \circ_N A.
\end{aligned}$$

Then,

$$\begin{aligned}
N([A,B]_N) &= N(A \circ_N B - B \circ_N A) \\
&= N(A)N(B) - N(B)N(A) = [N(A), N(B)]\,,
\end{aligned} \tag{50}$$

i.e. N is a Lie–Nijenhuis tensor. □

The above shows that we can apply the well–known theory of Nijenhuis tensors in the Lie algebra case for the "commutator part" of the associative Nijenhuis tensor to construct commuting elements etc. On the other hand, it is harder to

find associative Nijenhuis tensors, since vanishing of the Nijenhuis torsion in the Lie algebra case

$$N(A \circ_N B - B \circ_N A) = N(A)N(B) - N(B)N(B) \tag{51}$$

refers only to the skew-symmetrizations (commutators) of the corresponding products. Similarly, $[A, B]_N = A \circ_N B - B \circ_N A$ is a Lie bracket if and only if the total skew-symmetrization of the associator

$$Ass_N(A, B, C) = (A \circ_N B) \circ_N C - A \circ_N (B \circ_N C) \tag{52}$$

vanishes, which is weaker than just vanishing of the associator itself.

4. Final Examples

Example 5. Let now the algebra $\mathcal{A}$ be the algebra of infinite matrices concentrated about the diagonal, i.e. matrices which are null outside a diagonal strip. The algebra $\mathcal{A}$ represents then unbounded operators on a Hilbert space $\mathcal{H}$ with a common dense domain. We choose $\mathcal{A}_1$ to be a subalgebra of upper-triangular matrices and for $\mathcal{A}_2$ we take the supplementary algebra of strict lower-triangular matrices. Then, the mapping

$$N_\lambda(A) = (1 - \lambda)A_1 + \lambda A \tag{53}$$

is a Nijenhuis tensor on $\mathcal{A}$, in view of of Theorem 4, for every $\lambda \in \mathbb{C}$. Since the corresponding deformed associative products $\circ_\lambda$ give all the same result if one of factors is a diagonal matrix, the Hamiltonian H for the harmonic oscillator, $H(|e_n\rangle) = n|e_n\rangle$, describes the same motion for all deformed brackets. This time, however, $a^\dagger \circ_\lambda a = \lambda H$.

Example 6. Let us end up with a version of Example 3 for the algebra $\mathcal{A}$ of unbounded operators as above. Recall that our deformed product is

$$A \circ B = K\Delta(A)B + AK\Delta(B) - K\Delta(A)\Delta(B)\,, \tag{54}$$

where $\Delta(A)$ is the diagonal part of A. If A is diagonal, then $A \circ B = KAB$ and $B \circ A = BKA$, so that

$$[A, B]_\circ = [KA, B]\,. \tag{55}$$

Thus the dynamics described by the Heisenberg operator H as above is the same as the dynamics described by $K^{-1}H$ with respect to the new product. This time, however, the new product is not compatible with the standard one and we have $a^\dagger \circ a = 0$.

5. Conclusions

We have shown that in the Heisenberg picture alternative associative products are possible which allow to describe the same dynamics on the space of observables. We hope to consider the corresponding version on the phase space via the Wigner map, to compare these findings with those available at the classical level.

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