

CLASSIFICATION OF (1,1) TENSOR FIELDS AND BIHAMILTONIAN STRUCTURES

FRANCISCO JAVIER TURIEL

*Sección de Matemáticas, Facultad de Ciencias
A.P. 59, 29080 Málaga, Spain
E-mail: turiel@ccuma.sci.uma.es*

Abstract. Consider a (1,1) tensor field J , defined on a real or complex m -dimensional manifold M , whose Nijenhuis torsion vanishes. Suppose that for each point $p \in M$ there exist functions f_1, \dots, f_m , defined around p , such that $(df_1 \wedge \dots \wedge df_m)(p) \neq 0$ and $d(df_j(J(\cdot)))(p) = 0$, $j = 1, \dots, m$. Then there exists a dense open set such that we can find coordinates, around each of its points, on which J is written with affine coefficients. This result is obtained by associating to J a bihamiltonian structure on T^*M .

Introduction. Consider a (1,1) tensor field J , defined on a real or complex m -dimensional manifold M , whose Nijenhuis torsion vanishes. Suppose that for each point $p \in M$ there exist functions f_1, \dots, f_m , defined around p , such that $(df_1 \wedge \dots \wedge df_m)(p) \neq 0$ and $d(df_j \circ J)(p) = 0$, $j = 1, \dots, m$ [here $df \circ J$ means $df(J(\cdot))$]. In this paper we give a complete local classification of J on a dense open set that we call the regular open set. Moreover, near each regular point, i.e. each element of the regular open set, J is written with affine coefficients on a suitable coordinate system.

To express the condition about functions f_1, \dots, f_m , stated above, in a simple computational way we introduce the invariant P_J (see section 1). This invariant only depends on the 1-jet of J at each point, and $P_J(p) = 0$ iff functions f_1, \dots, f_m as before exist. When J defines a G-structure, the first-order structure function being zero implies $P_J = 0$ and $N_J = 0$ (this last property is well known). Besides all points of M are regular; therefore this work generalizes the main result of [5]. On the other hand N_J and P_J both together can be considered as a generalization of the first-order structure function.

This kind of tensor fields appear in a natural way in Differential Geometry. For example, on the base space of a bilagrangian fibration (see [1]) there exists a tensor field J ,

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with $N_J = 0$, such that if (x_1, \dots, x_m) are action coordinates then each $dx_j \circ J$ is closed; so $P_J = 0$. From a wider viewpoint, when $N_J = 0$, we can study the equation:

$$(1) \quad d(df \circ J) = 0;$$

i.e. the existence of conservation laws for J . Our classification shows that the existence, close to p , of m functionally independent solutions to equation (1) is equivalent to $P_J = 0$ near p .

Partial answers to the foregoing question may be found in [2], [6] and [7]. In [4], by using eigenvalues and eigenspaces, J. Grifone and M. Mehdi give an elegant necessary and sufficient condition for the existence of enough local solutions to equations (1) when J is real analytic. With the Grifone-Mehdi condition all points are regular and a calculation shows that it implies $P_J = 0$. Therefore the Grifone-Mehdi result follows from ours.

Finally, let us sketch the way for classifying J . As $N_J = 0$ we can construct a bihamiltonian structure on T^*M and from it a $(1, 1)$ tensor field J^* , prolongation of J to T^*M (see [8]). The main result of [9] gives us the local model of J^* on a dense open set and now a J^* -invariant cross section of T^*M allows us to find a model of J . This cross section exists because $P_J = 0$ implies that the behaviour of J^* does not change along each fiber of T^*M .

In a forthcoming paper we will study some cases where $P_J \neq 0$.

1. The first step. Consider a $(1, 1)$ tensor field J on a real or complex manifold M of dimension m . We recall that the Nijenhuis torsion of J is the $(1, 2)$ tensor field given by the formula

$$N_J(X, Y) = [JX, JY] + J^2[X, Y] - J[X, JY] - J[JX, Y].$$

If τ is a 1-form $\tau \circ J$ will mean the 1-form defined by $(\tau \circ J)(X) = \tau(JX)$.

For each $p \in M$ let $F(2, J)(p)$ be the vector subspace of all the 2-forms β_σ defined by $\beta_\sigma(v, w) = \sigma(Jv, w) - \sigma(v, Jw)$ where $v, w \in T_pM$ and σ is a symmetric bilinear form on T_pM . Observe that $F(2, J^k)(p) \subset F(2, J)(p)$ for each $k \in \mathbb{N}$. Set

$$F_J(p) = \frac{\Lambda^2 T_p^* M}{F(2, J)(p)}.$$

Given $\alpha \in T_p^*M$ and a function f defined around p such that $df(p) = \alpha$, the class of $d(df \circ J)(p)$ on $F_J(p)$ only depends on α . That defines a linear map $P_J(p) : T_p^*M \rightarrow F_J(p)$ or, from a global viewpoint, $P_J : T^*M \rightarrow F_J$ where F_J is the disjoint union of all $F_J(p)$.

Note that $P_J(p) = 0$ if and only if there exist functions f_1, \dots, f_m , defined around p , such that $(df_1 \wedge \dots \wedge df_m)(p) \neq 0$ and $d(df_j \circ J)(p) = 0$, $j = 1, \dots, m$. When the characteristic polynomial of $J(p)$ equals its minimal polynomial, i.e. when T_pM is cyclic, then $F(2, J)(p) = \Lambda^2 T_p^*M$ and automatically $P_J(p) = 0$. If $J^2 = -\text{Id}$ a straightforward calculation shows that $N_J = 0$ implies $P_J = 0$. However J can be semisimple, $N_J = 0$ and $P_J \neq 0$; e.g. on \mathbb{R}^m , $m \geq 2$, $J = e^{x_1} \text{Id}$.

Let $\mathbb{K}_N[t]$ be the polynomial algebra in one variable over the ring of differentiable functions on a manifold N . Here differentiable means C^∞ if N is a real manifold ($\mathbb{K} = \mathbb{R}$) and holomorphic in the complex case ($\mathbb{K} = \mathbb{C}$). A polynomial $\varphi \in \mathbb{K}_N[t]$ is called *irreducible* if it is irreducible at each point of N . We shall say that $\varphi, \rho \in \mathbb{K}_N[t]$ are

relatively prime if they are at each point. Consider an endomorphism field H of a vector bundle $\pi : V \rightarrow N$, i.e. a cross section of $V \otimes V^*$. We will say that H has *constant algebraic type* if there exist relatively prime irreducible polynomials $\varphi_1, \dots, \varphi_\ell \in \mathbb{K}_N[t]$ and natural numbers $a_{ij}, i = 1, \dots, r_j, j = 1, \dots, \ell$, such that for each $p \in N$ the family $\{\varphi_j^{a_{ij}}(p)\}, i = 1, \dots, r_j, j = 1, \dots, \ell$, is the family of elementary divisors of $H(p)$.

Suppose that J defines a G -structure, i.e. J has constant algebraic type on M and $\varphi_1, \dots, \varphi_\ell \in \mathbb{K}[t]$. If its first-order structure function vanishes then $P_J = 0$. Indeed, around each point $p \in M$ there exists a linear connection ∇ , whose torsion at p vanishes, such that $\nabla J = 0$. Let f_1, \dots, f_m be normal coordinates with origin p ; then $d(df_j \circ J)(p) = 0$ and $P_J(p) = 0$. Conversely $N_J = 0$ and $P_J = 0$ imply that the first-order structure function equals zero. In a word, *the invariants N_J and P_J can be seen as a generalization of the first-order structure function to the case where J does not define a G -structure.*

Henceforth we shall suppose $N_J = 0$. Set $g_k = \text{trace}(J^k)$ and $E = \bigcap_{j=1}^m \text{Ker } dg_j$. It is well known that $(k + 1)dg_k \circ J = kdg_{k+1}$ and $JE \subset E$ (see [9]).

We say that a point $p \in M$ is *regular* if there exists an open neighbourhood A of p such that:

- (1) J has constant algebraic type on A ,
- (2) E , restricted to A , is a vector subbundle of TA .
- (3) The restriction of J to E has constant algebraic type on A .

The set of all regular points is a dense open set of M which we shall call the *regular open set*. Our local classification of J only refers to the regular open set.

Now suppose that on an open neighbourhood of a regular point p the characteristic polynomial φ of J is the product $\varphi_1 \cdot \varphi_2$ of two monic relatively prime polynomials φ_1 and φ_2 . Then around p the structure (M, J) decomposes into a product of two similar structures $(M_1, J_1) \times (M_2, J_2)$, where φ_1 is the characteristic polynomial of J_1 (more exactly φ_1 is the pull-back of the characteristic polynomial of J_1) and φ_2 that of J_2 (see [3] and [9]). Moreover $N_{J_1} = 0, N_{J_2} = 0$, and p_1 and p_2 are regular points where $p = (p_1, p_2)$. On the other hand $P_{J_1} = 0$ and $P_{J_2} = 0$ if $P_J = 0$.

This splitting property reduces the classification to the case where the characteristic polynomial φ of J is a power of an irreducible one. Therefore we have only two possibilities: $\varphi = (t + f)^m$, or $\varphi = (t^2 + ft + g)^n$ where $m = 2n$ and M is a real manifold.

2. The case $\varphi = (t + f)^m$. In this section, by associating to J a bihamiltonian structure on T^*M , we prove the following result:

THEOREM 1. *Consider a (1,1) tensor field J such that $N_J = 0$ and $P_J = 0$. Suppose that its characteristic polynomial is $(t + f)^m$. Then around each regular point p there exist coordinates $((x_i^j), y)$ with origin p , i.e. $p \equiv 0$, such that:*

- (a) $i = 1, \dots, r_j$ and $r_1 \geq r_2 \geq \dots \geq r_\ell$. Moreover we also consider the case with no coordinates (x_i^j) , i.e. $\ell = 0$, and the case with coordinates (x_i^j) only.

(b) $J = (y + a) \text{Id} + H + Y \otimes dy$ where

$$H = \sum_{j=1}^{\ell} \left(\sum_{i=1}^{r_j-1} \frac{\partial}{\partial x_{i+1}^j} \otimes dx_i^j \right) \quad \text{and} \quad Y = \frac{\partial}{\partial x_1^1} + \sum_{j=1}^{\ell} \left(\sum_{i=2}^{r_j} (1-i)x_i^j \frac{\partial}{\partial x_i^j} \right).$$

Remark. In the first special case $m = 1$ and $J = (y + a) \text{Id}$; in the second one $m = r_1 + \dots + r_{\ell}$ and $J = a \text{Id} + \sum_{j=1}^{\ell} (\sum_{i=1}^{r_j-1} \partial/\partial x_{i+1}^j \otimes dx_i^j)$. The elementary divisors of J determine its model completely. If there is no coordinate y , i.e. if J defines a G-structure, they are: $\{(t - a)^{r_j}\}$, $j = 1, \dots, \ell$. Otherwise they are: $(t - (y + a))^{r_1+1}$; $\{(t - (y + a))^{r_j}\}$, $j = 2, \dots, \ell$.

Let $c_J : T^*M \rightarrow T^*M$ be the morphism of T^*M defined by $c_J(\tau) = \tau \circ J$ and let ω be the Liouville symplectic form of T^*M . Set $\omega_1 = (c_J)^*\omega$ where c_J is regarded as a differentiable map. Consider the (1, 1) tensor field J^* , on T^*M , defined by $\omega_1(X, Y) = \omega(J^*X, Y)$. Then $N_{J^*} = 0$, because $N_J = 0$, and $\{\omega, \omega_1\}$ is a bihamiltonian structure (see [8]). If (x_1, \dots, x_m) are coordinates on M , $(x_1, \dots, x_m, z_1, \dots, z_m)$ the associated coordinates on T^*M , and $J = \sum_{i,j=1}^m f_{ij} \partial/\partial x_i \otimes dx_j$ then

$$J^* = \sum_{i,j=1}^m f_{ij} \left(\frac{\partial}{\partial x_i} \otimes dx_j + \frac{\partial}{\partial z_j} \otimes dz_i \right) + \sum_{i,j,k=1}^m z_i \left(\frac{\partial f_{ij}}{\partial x_k} - \frac{\partial f_{ik}}{\partial x_j} \right) \frac{\partial}{\partial z_j} \otimes dx_k.$$

Hence $\pi_* \circ J^* = J \circ \pi_*$.

Throughout the rest of this section J is as in theorem 1. By the local expression of J^* given above, its characteristic polynomial is $\varphi^* = (t + f \circ \pi)^{2m}$. Since $P_J = 0$, around each regular point $p \in M$ there exist coordinates (x_1, \dots, x_m) such that $d(dx_i \circ J)(p) = 0$, $i = 1, \dots, m$. Even more if $df(p) \neq 0$ [regularity implies $df(p) = 0$ iff f is constant near p] we can suppose $f = x_1$ because $g_1 = -mf$ and $dg_1 \circ J = \frac{dg_2}{2}$. But $dx_i \circ J = \sum_{j=1}^m f_{ij} dx_j$, then $\frac{\partial f_{ij}}{\partial x_k}(p) = \frac{\partial f_{ik}}{\partial x_j}(p)$ and

$$J^*(p, z) = \sum_{i,j=1}^m f_{ij}(p) \left(\frac{\partial}{\partial x_i} \otimes dx_j + \frac{\partial}{\partial z_j} \otimes dz_i \right) (p, z).$$

Therefore the elementary divisors of $J(p)$ and $(J|_E)(p)$ determine those of $J^*(p, z)$ and $(J^*|_{E^*})(p, z)$ completely, and the pull-back of the regular open set of J is included in the regular open set of J^* . This is the role of the assumption $P_J = 0$ while $N_J = 0$ assures us that $\{\omega, \omega_1\}$ is bihamiltonian.

The zero cross section allows us to consider M as a submanifold of T^*M . Take a regular point $p \in M$ such that $df(p) = 0$, i.e. f constant near p . By theorem 3 of [9] there exist coordinates (y_1, \dots, y_{2m}) on an open neighbourhood A of p , with origin this point, on which ω and ω_1 are written with constant coefficients and J^* as well. By rearranging coordinates (y_1, \dots, y_{2m}) if necessary, we can suppose that $\{\frac{\partial}{\partial y_1}(p), \dots, \frac{\partial}{\partial y_m}(p)\}$ spans $T_p M$ and $\{\frac{\partial}{\partial y_{m+1}}(p), \dots, \frac{\partial}{\partial y_{2m}}(p)\}$ spans the vertical subspace $\text{Ker } \pi_*(p)$ at p . Both subspaces are J^* -invariant as the local expression of J^* shows. Set $A_0 = \{y \in A : y_{m+1} = \dots = y_{2m} = 0\}$. As $\text{rank}((\pi|_{A_0})(p)) = m$ we can choose an open neighbourhood B of p on A_0 such that $\pi(B)$ is open and $\pi : B \rightarrow \pi(B)$ a diffeomorphism.

By construction $J^*(TA_0) \subset TA_0$. Let J' be the restriction of J^* to A_0 . The tensor

field J' is written with constant coefficients on A_0 . Moreover $(\pi_{|A_0})_* \circ J' = J \circ (\pi_{|A_0})_*$ since $\pi_* \circ J^* = J \circ \pi_*$. Then J is written with constant coefficients on $\pi(B)$, which proves theorem 1 when $df(p) = 0$.

The proof of the other case is basically the same but we have to rearrange coordinates in a more sophisticated way. Let V be a real or complex vector space of dimension $2n$. Consider $\alpha, \alpha_1 \in \Lambda^2 V^*$ such that $\alpha^n \neq 0$. Let \tilde{J} be the endomorphism of V given by $\alpha_1(v, w) = \alpha(\tilde{J}v, w)$. Suppose \tilde{J} nilpotent (see proposition 1 of [9] for the model of $\{\alpha, \alpha_1\}$). An n -dimensional vector subspace W of V is called bilagrangian if $\alpha(v, w) = \alpha_1(v, w) = 0$ for all $v, w \in W$; in other words W is lagrangian for α and $JW \subset W$. When W is bilagrangian and there exists another bilagrangian subspace W' such that $V = W \oplus W'$ we shall say that W is superlagrangian. A bilagrangian subspace W is superlagrangian if and only if the elementary divisors of $J|_W$ are half those of J ; i.e. if $\{t^{r_j}\}, j = 1, \dots, \ell$, are the elementary divisors of $J|_W$ then $\{t^{r_j}, t^{r_j}\}, j = 1, \dots, \ell$, are those of J .

LEMMA 1. Consider a basis $\{e_i^j\}, i = 1, \dots, 2r_j, j = 1, \dots, \ell$, of V such that

$$\alpha = \sum_{j=1}^{\ell} \left(\sum_{k=1}^{r_j} e_{2k-1}^{*j} \wedge e_{2k}^{*j} \right) \quad \text{and} \quad \alpha_1 = \sum_{j=1}^{\ell} \left(\sum_{k=1}^{r_j-1} e_{2k-1}^{*j} \wedge e_{2k+2}^{*j} \right).$$

Let W be the vector subspace spanned by $\{e_{2k-1}^j\}, k = 1, \dots, r_j, j = 1, \dots, \ell$. Then for each superlagrangian subspace W' of V there exists $T \in GL(V)$ such that $T^*\alpha = \alpha, T^*\alpha_1 = \alpha_1$ and $W \cap TW' = \{0\}$. Moreover if $e_{2r_1-1}^1 \notin W'$ we can choose T in such a way that $Te_1^1 = e_1^1$.

Now take a regular point $p \in M$. Suppose $df(p) \neq 0$. By theorem 3 of [9] there exist coordinates $(x, y) = ((x_i^j), y_1, y_2), i = 1, \dots, 2r_j$ and $r_1 \geq r_2 \geq \dots \geq r_\ell$, with origin p , such that

$$\omega = \sum_{j=1}^{\ell} \left(\sum_{k=1}^{r_j} dx_{2k-1}^j \wedge dx_{2k}^j \right) + dy_1 \wedge dy_2$$

and $\omega_1 = (y_2 + a)\omega + \tau + \alpha \wedge dy_2$ where

$$\tau = \sum_{j=1}^{\ell} \left(\sum_{k=1}^{r_j-1} dx_{2k-1}^j \wedge dx_{2k+2}^j \right)$$

and

$$\alpha = dx_2^1 + \sum_{j=1}^{\ell} \left(\sum_{k=1}^{r_j} [(k + 1/2)x_{2k}^j dx_{2k-1}^j + (k - 1/2)x_{2k-1}^j dx_{2k}^j] \right).$$

Hence $J^* = (y_2 + a)\text{Id} + H^* + \frac{\partial}{\partial y_1} \otimes \alpha - Z \otimes dy_2$ where

$$H^* = \sum_{j=1}^{\ell} \left(\sum_{k=1}^{r_j-1} \frac{\partial}{\partial x_{2k+1}^j} \otimes dx_{2k-1}^j + \sum_{k=2}^{r_j} \frac{\partial}{\partial x_{2k-2}^j} \otimes dx_{2k}^j \right)$$

and

$$Z = \frac{\partial}{\partial x_1^1} + \sum_{j=1}^{\ell} \left(\sum_{k=1}^{r_j} \left[(k - 1/2)x_{2k-1}^j \frac{\partial}{\partial x_{2k-1}^j} - (k + 1/2)x_{2k}^j \frac{\partial}{\partial x_{2k}^j} \right] \right).$$

LEMMA 2. *The vector $\frac{\partial}{\partial x_{2r_1-1}^1}(p)$ does not belong to the vertical subspace $\text{Ker } \pi_*(p)$.*

PROOF. By the local expression of J^* in the coordinates $(x_1, \dots, x_m, z_1, \dots, z_m)$ given at the beginning of this section, $\text{Ker } \pi_*(p)$ and $T_p M$ are $J^*(p)$ -invariant, and $J|_{\text{Ker } \pi_*(p)}$ and $J|_{T_p M}$ have the same elementary divisors. As $p \equiv 0$ in coordinates (x, y) , the elementary divisors of $J^*(p)$ are $(t - a)^{r_1+1}$; $(t - a)^{r_1+1}$; $\{(t - a)^{r_j}, (t - a)^{r_j}\}$, $j = 2, \dots, \ell$. Therefore there exists $v \in T_p M$ spanning a cyclic subspace U of dimension $r_1 + 1$ such that $U \cap \text{Ker } \pi_*(p) = \{0\}$.

Moreover $v = a \frac{\partial}{\partial y_2}(p) + b \frac{\partial}{\partial x_{2r_1}^1}(p) + v_1$ where $(J^*(p) - a \text{Id})^{r_1} v_1 = 0$.

By construction

$$(J^*(p) - a \text{Id})^{r_1} v = a \frac{\partial}{\partial x_{2r_1-1}^1}(p) + b \frac{\partial}{\partial y_1}(p)$$

does not belong to $\text{Ker } \pi_*(p)$. As $\omega(\partial/\partial y_1, \cdot) = dy_2 = -d(f \circ \pi)$ and $\omega = \sum_{j=1}^m dz_j \wedge dx_j$ in coordinates $(x_1, \dots, x_m, z_1, \dots, z_m)$ of T^*M , the vector $\frac{\partial}{\partial y_1}(p)$ belongs to $\text{Ker } \pi_*(p)$. So $\frac{\partial}{\partial x_{2r_1-1}^1}(p) \notin \text{Ker } \pi_*(p)$. ■

Set $\omega' = \sum_{j=1}^{\ell} (\sum_{k=1}^{r_j} dx_{2k-1}^j \wedge dx_{2k}^j)$.

LEMMA 3. *The vector subspace $(\text{Ker } \pi_* \cap \text{Ker } dy_1 \cap \text{Ker } dy_2)(p)$, regarded as a subspace of $T_0 \mathbb{K}^{2m-2}$, is superlagrangian with respect to $\{\omega'(0), \tau(0)\}$.*

PROOF. As $f \circ \pi = -(y_2 + a)$, $\text{Ker } \pi_*(p) \subset \text{Ker } dy_2(p) = \text{Ker } d(f \circ \pi)(p)$. Now note that $((J^* - a \text{Id})^{r_1} \text{Ker } \pi_*)(p)$ is a 1-dimensional subspace of $\text{Ker } \pi_*(p) \cap \mathbb{K}\{\frac{\partial}{\partial x_{2r_1-1}^1}(p), \frac{\partial}{\partial y_1}(p)\}$ (here $\mathbb{K}\{v_1, \dots, v_s\}$ is the space spanned by $\{v_1, \dots, v_s\}$). So $((J^* - a \text{Id})^{r_1} \text{Ker } \pi_*)(p) = \mathbb{K}\{\frac{\partial}{\partial y_1}(p)\}$ since $\frac{\partial}{\partial x_{2r_1-1}^1}(p) \notin \text{Ker } \pi_*(p)$.

On the other hand $T_0 \mathbb{K}^{2m-2}$ can be seen as the quotient space $\text{Ker } dy_2(p) / \mathbb{K}\{\frac{\partial}{\partial y_1}(p)\}$, which identifies $(\text{Ker } \pi_* \cap \text{Ker } dy_1 \cap \text{Ker } dy_2)(p)$ with $\text{Ker } \pi_*(p) / \mathbb{K}\{\frac{\partial}{\partial y_1}(p)\}$, and $(H^* + a \text{Id})(0)$ as the endomorphism induced by $J^*|_{\text{Ker } dy_2(p)}$. Therefore the elementary divisors of $H^*|_{(\text{Ker } \pi_* \cap \text{Ker } dy_1 \cap \text{Ker } dy_2)(p)}$ are $\{t^{r_j}\}$, $j = 1, \dots, \ell$. ■

LEMMA 4. *Let $\{e_i^j\}$, $i = 1, \dots, 2r_j$, $j = 1, \dots, \ell$, be the canonical basis of $\mathbb{K}^{2m-2} = \mathbb{K}^{2r_1} \times \dots \times \mathbb{K}^{2r_\ell}$. Set $\alpha = \sum_{j=1}^{\ell} (\sum_{k=1}^{r_j} e_{2k-1}^{*j} \wedge e_{2k}^{*j})$ and $\alpha_1 = \sum_{j=1}^{\ell} (\sum_{k=1}^{r_j-1} e_{2k-1}^{*j} \wedge e_{2k+2}^{*j})$. Given $T \in GL(\mathbb{K}^{2m-2})$ if $T e_1^1 = e_1^1$; $T^* \alpha = \alpha$ and $T^* \alpha_1 = \alpha_1$, there exists a germ of diffeomorphism $\tilde{G} : (\mathbb{K}^{2m}, 0) \rightarrow (\mathbb{K}^{2m}, 0)$ such that $\tilde{G}(x, y) = (G(x), y)$; $\tilde{G}^* \omega = \omega$; $\tilde{G}^* \omega_1 = \omega_1$ and $G_*(0) = T$.*

PROOF. We will adapt to our case the proof of proposition 3 of [9]. Consider the map $G_T : \mathbb{K}^{2m} \rightarrow \mathbb{K}^{2m}$ given by $G_T(x, y) = (Tx, y)$. Then $G_T^* \omega = \omega$ and $G_T^* \omega_1 = \omega_1 + dg \wedge dy_2$ where g is a quadratic function such that $d(dg \circ H^*) = 0$. Indeed G_T preserves $dx_2^1(0) = \omega(\frac{\partial}{\partial x_1^1}, \cdot)(0)$ and H^* , and $d(\alpha \circ H^*) = -2\tau$.

Let D and \mathbb{L} be the exterior derivative and the Lie derivative *with respect to the variables x only*. We begin searching for a vector field $X_t = \sum_{j=1}^{\ell} (\sum_{i=1}^{2r_j} \varphi_i^j(x, t) \frac{\partial}{\partial x_i^j})$, defined on an open neighbourhood of the compact $\{0\} \times [0, 1] \subset \mathbb{K}^{2m-2} \times \mathbb{K}$, such that:

- (1) $\mathbb{L}_{X_t} \omega' = \mathbb{L}_{X_t} \tau = 0$.
- (2) $\mathbb{L}_{X_t} (\alpha + tDg) = Dg$ (remark that $dg = Dg$).
- (3) For each $i = 1, \dots, 2r_j$ and $j = 1, \dots, \ell$, φ_i^j and $D\varphi_i^j$ vanish on $\{0\} \times [0, 1]$.

Consider the vector field Z_t given by $\omega'(Z_t, \cdot) = \alpha + tDg$. Take a function $f(x, t)$, defined around $\{0\} \times [0, 1]$, such that:

- (I) $Z_t f = -f - g$.
- (II) $D(Df \circ H^*) = 0$.
- (III) For all $i = 1, \dots, 2r_j$, $j = 1, \dots, \ell$, $k = 1, \dots, 2r_s$ and $s = 1, \dots, \ell$, the partial derivatives $\partial f / \partial x_i^j$ and $\partial^2 f / \partial x_k^s \partial x_i^j$ vanish on $\{0\} \times [0, 1]$.

Let X_t the vector field defined by $\omega'(X_t, \cdot) = Df$. Then X_t satisfies conditions (1), (2) and (3). By proposition 1.A (see the appendix) this kind of functions exists because g is quadratic, $D(Dg \circ H^*) = 0$, $Z_t(0) = \partial / \partial x_1^1$, and $\mathbb{L}_{Z_t} H^* = -H^*$ since $\mathbb{L}_{Z_t} \omega' = D(\alpha + tDg) = -\omega'$ and $\mathbb{L}_{Z_t} \tau = D(\alpha \circ H^* + tDg \circ H^*) = -2\tau$.

By integrating the vector field $-X_t$ we obtain a germ of diffeomorphism $F : (\mathbb{K}^{2m-2}, 0) \rightarrow (\mathbb{K}^{2m-2}, 0)$ such that $F^* \omega' = \omega'$; $F^* \tau = \tau$; $F^*(\alpha + Dg) = \alpha$ and $F_*(0) = \text{Id}$. Now set $\tilde{G} = \tilde{F} \circ G_T$ where $\tilde{F}(x, y) = (F(x), y)$. ■

Let W be the subspace of $T_p T^* M$ spanned by $\{\frac{\partial}{\partial x_{2k-1}^j}(p)\}$, $k = 1, \dots, r_j$, $j = 1, \dots, \ell$. By lemmas 1, 2, 3 and 4 we can suppose, without loss of generality, $W \cap (\text{Ker } \pi_* \cap dy_1 \cap dy_2)(p) = \{0\}$, which implies $(W \oplus \mathbb{K}\{\frac{\partial}{\partial y_2}(p)\}) \cap \text{Ker } \pi_*(p) = \{0\}$. Indeed $\dim(\text{Ker } \pi_* \cap dy_1 \cap dy_2)(p) = m - 1$ (lemma 3) and $\frac{\partial}{\partial y_1}(p) \in \text{Ker } \pi_*(p)$ (lemma 2, proof); then $\text{Ker } \pi_*(p) = \mathbb{K}\{\frac{\partial}{\partial y_1}(p)\} \oplus (\text{Ker } \pi_* \cap dy_1 \cap dy_2)(p)$.

Set $A_0 = \{(x, y) \in A : x_{2k}^j = y_1 = 0, k = 1, \dots, r_j, j = 1, \dots, \ell\}$ where A is the domain of coordinates (x, y) . Then $J^*(TA_0) \subset TA_0$ and $T_p A_0 \oplus \text{Ker } \pi_*(p) = T_p T^* M$. Finally, by reasoning as in the case $df(p) = 0$ we can state:

PROPOSITION 1. *Under the assumptions of theorem 1, if $df(p) \neq 0$ then there exist coordinates $((x_i^j), y)$ as in this theorem such that $J = (y + a) \text{Id} + H + Y \otimes dy$ where*

$$H = \sum_{j=1}^{\ell} \left(\sum_{i=1}^{r_j-1} \frac{\partial}{\partial x_{i+1}^j} \otimes dx_i^j \right) \quad \text{and} \quad Y = \frac{\partial}{\partial x_1^1} + \sum_{j=1}^{\ell} \left(\sum_{i=1}^{r_j} (1/2 - i) x_i^j \frac{\partial}{\partial x_i^j} \right).$$

When $df(p) \neq 0$, proposition 1 shows that the local model of J only depends on its elementary divisors.

LEMMA 5. *Consider on $\mathbb{K}^m = \mathbb{K}^{r_1} \times \dots \times \mathbb{K}^{r_\ell} \times \mathbb{K}$, with $r_1 \geq \dots \geq r_\ell$ if $\ell > 0$, coordinates $((x_i^j), y)$. Let \mathbb{L} be the Lie derivative with respect to variables (x_i^j) only. Set $J = (y + a) \text{Id} + H + Y \otimes dy$ where Y is a vector field defined around the origin such that $dy(Y) = 0$ and $H = \sum_{j=1}^{\ell} (\sum_{i=1}^{r_j-1} \frac{\partial}{\partial x_{i+1}^j} \otimes dx_i^j)$. If $\mathbb{L}_Y H = H$ and $H^{r_1-1} Y(0) \neq 0$, then $N_J = 0$ and close to the origin $P_J = 0$ and J has constant algebraic type.*

The elementary divisors of J , near the origin, are the same both for proposition 1 and lemma 5: $(t - (y + a))^{r_1+1}$; $\{(t - (y + a))^{r_j}\}$, $j = 2, \dots, \ell$. So their models are equivalent. We finish the proof of theorem 1 by taking

$$Y = \frac{\partial}{\partial x_1^1} + \sum_{j=1}^{\ell} \left(\sum_{i=2}^{r_j} (1-i)x_i^j \frac{\partial}{\partial x_i^j} \right).$$

The model announced by the author in a lecture at the Banach Center is obtained by setting

$$Y = \frac{\partial}{\partial x_1^1} - \sum_{j=1}^{\ell} \left(\sum_{i=1}^{r_j} ix_i^j \frac{\partial}{\partial x_i^j} \right).$$

Another interesting model is given by taking

$$Y = \frac{\partial}{\partial x_1^1} + \sum_{j=1}^{\ell} \left(\sum_{i=1}^{r_j} (r_j + 1 - i)x_i^j \frac{\partial}{\partial x_i^j} \right).$$

For this model the forms $dy \circ J = (y + a)dy$ and $dx_{r_j}^j \circ J = (y + a)dx_{r_j}^j + x_{r_j}^j dy + dx_{r_j-1}^j$ are closed. As $N_J = 0$ all the forms $dx_{r_j}^j \circ J^k$ are closed too. Therefore if the characteristic polynomial of J is $(t + f)^m$, for each regular point p and for all $\lambda_0 \in T_p^*M$ there exists a closed 1-form λ , defined near p , such that $\lambda(p) = \lambda_0$ and $d(\lambda \circ J) = 0$; usually λ is called a *conservation law*. In other words, the equation $d(df \circ J) = 0$ has enough local solutions on the regular open set.

3. The case $\varphi = (t^2 + ft + g)^n$. Since our problem is local we can suppose M connected and all of its points regular. Set $J_0 = 2(4g - f^2)^{-\frac{1}{2}}J + f(4g - f^2)^{-\frac{1}{2}}\text{Id}$ which makes sense because $f^2 - 4g < 0$. By construction J_0 defines a G-structure and $(J_0^2 + \text{Id})^n = 0$. Let H be the semisimple part of J_0 . Then H is a complex structure, J a holomorphic tensor field and $(t + h)^n$ its complex characteristic polynomial, where $h = \frac{1}{2}(f - i(4g - f^2)^{\frac{1}{2}})$ is holomorphic.

Indeed, consider $\{\omega, \omega_1\}$ and J^* on T^*M as in section 2. Now the characteristic polynomial of J^* is $\varphi^* = (t^2 + (f \circ \pi)t + (g \circ \pi))^{2n}$. Let A be the regular open set of J^* . Set $J_0^* = 2((4g - f^2)^{-\frac{1}{2}} \circ \pi)J^* + ((f(4g - f^2)^{-\frac{1}{2}}) \circ \pi)\text{Id}$. On each connected component of A the tensor field J_0^* defines a G-structure; moreover $((J_0^*)^2 + \text{Id})^{2n} = 0$. Let H^* be the semisimple part of J . In section 6 of [9] we showed that H^* is a complex structure, J^* holomorphic and $(t + h^*)^{2n}$ its complex characteristic polynomial, where $h^* = \frac{1}{2}(f \circ \pi - i(4g - f^2)^{\frac{1}{2}} \circ \pi)$ is a holomorphic function. On the other hand $\pi_* \circ J_0^* = J_0 \circ \pi_*$ and $\pi_* \circ H^* = H \circ \pi_*$ because $\pi_* \circ J^* = J \circ \pi_*$. So holomorphy holds on $\pi(A)$, and on M as well since A is dense on T^*M and $\pi(A)$ on M .

The complex regular set of J is M (see section 6 of [9] again).

Suppose $P_J = 0$. Let $f = f_1 + if_2$ a holomorphic function. Then $d(df \circ J) = d(df_1 \circ J) + i(d(df_2 \circ J))$ is a holomorphic 2-form, so $d(df_1 \circ J)(HX, Y) = d(df_1 \circ J)(X, HY)$ and $d(df_2 \circ J)(X, Y) = -d(df_1 \circ J)(HX, Y)$. As $P_J(p) = 0$ from the real viewpoint, there exists a real symmetric bilinear form σ on T_pM such that $d(df_1 \circ J)(p)(v, w) = \sigma(J(p)v, w) - \sigma(v, J(p)w)$. Set $\sigma_1(v, w) = \frac{1}{2}(\sigma(v, w) - \sigma(H(p)v, H(p)w))$ and $\tilde{\sigma}(v, w) = \sigma_1(v, w) - i\sigma_1(H(p)v, w)$. As J and H commute $\tilde{\sigma}$ is a complex symmetric bilinear form

and $d(df \circ J)(p)(v, w) = \bar{\sigma}(J(p)v, w) - \bar{\sigma}(v, J(p)w)$. In other words $P_J = 0$ from the complex viewpoint. So to find a model of J , regard M as a complex manifold of dimension n and apply theorem 1. Then forget the complex structure and regard J as a real tensor field.

THEOREM 2. *Suppose $N_J = 0$ and $P_J = 0$. Then the local model of J around each regular point is a finite product of models chosen among:*

- (a) *For a complex manifold, those of theorem 1.*
- (b) *For a real manifold, those of theorem 1 and those obtained considering the complex models of that theorem from the real viewpoint.*

The local model of J is completely determined by its elementary divisors.

Remark. Suppose $N_J = 0$. Let p be a regular point. By theorem 2 there exist enough solutions to the equation $d(df \circ J) = 0$, i.e. conservation laws, near p iff P_J vanishes around this point. Nevertheless the existence of this kind of functions does not imply $N_J = 0$; e.g. on \mathbb{K}^2 consider $J = e^{x_2} \text{Id} + \partial/\partial x_2 \otimes dx_1$; $f_1 = x_1 - e^{x_2}$ and $f_2 = x_2$.

Appendix. Consider an open set A of \mathbb{K}^n endowed with a nilpotent constant coefficient (1, 1) tensor field H . Let B be a differentiable manifold (the parameter space). Elements of $A \times B$ will be denoted by (x, y) while by D , $D^{(2)}$ and L we mean the exterior derivative, the second-order differential and the Lie derivative, all of them with respect to the variables (x_1, \dots, x_n) only. Let Z be a vector field on A depending on the parameter $y \in B$. We say that Z is generic at a point (x, y) if the dimension of the cyclic subspace spanned by $Z(x, y)$ equals the degree of the minimal polynomial of H .

PROPOSITION 1.A. *Suppose given $p \in A$, a compact set $K \subset B$, a scalar $a \in \mathbb{K}$ and a function $g : A \times B \rightarrow \mathbb{K}$, such that: (1) $L_Z H = cH$ where $c \in \mathbb{K}$; (2) Z is generic on $\{p\} \times K$; (3) $D(Dg \circ H) = 0$, $g(\{p\} \times B) = 0$ and $Dg(\{p\} \times B) = 0$.*

Then there exist an open neighbourhood U of p , an open set $V \supset K$ and a function $f : U \times V \rightarrow \mathbb{K}$ such that: (I) $Zf = af + g$; (II) $D(Df \circ H) = 0$; (III) $Df(\{p\} \times V) = 0$ and $D^{(2)}f(\{p\} \times V) = 0$. Moreover if $Dg(\text{Ker } H^r) = 0$ we can choose f in such a way that $Df(\text{Ker } H^r) = 0$.

The proof of this result is essentially that of proposition 1.A of [9]. Before lemma 2.A no change is needed at all. This last result should be replaced with:

LEMMA 2'.A. *Consider a function $h_1 : A \times B \rightarrow \mathbb{K}$. Suppose $Dh_1(\text{Ker } H) = 0$ and $D(Dh_1 \circ H) = 0$. Then there exist an open neighbourhood U of p and a function $h : U \times B \rightarrow \mathbb{K}$ such that: (1) $Dh \circ H = Dh_1$; (2) $h(\{p\} \times B) = 0$; (3) $Dh(p, y) = 0$ for all $y \in B$ such that $Dh_1(p, y) = 0$; $D^{(2)}h(p, y) = 0$ for each $y \in B$ such that $Dh_1(p, y) = 0$ and $D^{(2)}h_1(p, y) = 0$.*

Proof. There exist a vector subbundle E of TA and a morphism $\rho : TA \rightarrow TA$ such that $TA = E \oplus \text{Ker } H$ and $(\rho \circ H)|_E = \text{Id}$. Set $\alpha = Dh_1 \circ \rho$. Obviously $\alpha \circ H = Dh_1$. Let C be the set of all $y \in B$ such that $Dh_1(p, y) = 0$ and $D^{(2)}h_1(p, y) = 0$. Suppose $\alpha = \sum_{j=1}^n g_j dx_j$. Then $g_j(\{p\} \times C) = 0$ and $Dg_j(\{p\} \times C) = 0$, $j = 1, \dots, n$.

By rearranging coordinates (x_1, \dots, x_n) we can suppose the foliation $\text{Ker } H$ given by $dx_1 = \dots = dx_k = 0$. From lemma 1.A, $D\alpha(\text{Im } H, \text{Im } H) = 0$ so $D\alpha = \sum_{j=1}^k (\sum_{i=1}^n f_{ij} dx_i) \wedge dx_j$ where each f_{ij} equals zero on $\{p\} \times C$.

Let $U = \prod_{i=1}^n U_i$ be an open neighbourhood of p , product of intervals ($\mathbb{K} = \mathbb{R}$) or disks ($\mathbb{K} = \mathbb{C}$). As $D\alpha$ is closed, there exist functions $\tilde{f}_j : U \times B \rightarrow \mathbb{K}$ such that $\partial \tilde{f}_j / \partial x_i = f_{ij}$ and $\tilde{f}_j(U_1 \times \dots \times U_k \times \{(p_{k+1}, \dots, p_n)\} \times B) = 0$, $i = k+1, \dots, n$, $j = 1, \dots, k$, where $p = (p_1, \dots, p_n)$. Therefore $\tilde{f}_j(\{p\} \times B) = 0$ and $D\tilde{f}_j(\{p\} \times C) = 0$.

Set $\beta = D\alpha - D(\sum_{j=1}^k \tilde{f}_j dx_j) = \sum_{i,\ell=1}^k e_{i\ell} dx_i \wedge dx_\ell$. As $D\beta = 0$, the functions $e_{i\ell}$ do not depend on (x_{k+1}, \dots, x_n) . By construction $e_{i\ell}(\{p\} \times C) = 0$.

Now we can find functions $e_2, \dots, e_k : U \times B \rightarrow \mathbb{K}$, which do not depend on (x_{k+1}, \dots, x_n) , such that $\partial e_j / \partial x_1 = e_{1j}$ and $e_j(\{p_1\} \times U_2 \times \dots \times U_n \times B) = 0$, $j = 2, \dots, k$. So $e_j(\{p\} \times B) = 0$ and $De_j(\{p\} \times C) = 0$. Set $\beta' = \sum_{j=2}^k e_j dx_j$. Then $\beta_1 = \beta - D\beta'$ is closed and $\beta_1(\{p\} \times C) = 0$. Moreover β_1 only involves the variables (x_2, \dots, x_k) and differentials dx_2, \dots, dx_k . By induction we construct $\tilde{\beta} = \sum_{j=1}^k \tilde{e}_j dx_j$ such that $D\tilde{\beta} = \beta$, $\tilde{e}_j(\{p\} \times B) = 0$ and $D\tilde{e}_j(\{p\} \times C) = 0$, $j = 1, \dots, k$.

Set $\alpha_1 = \sum_{j=1}^k f_j dx_j$ where $f_j = \tilde{f}_j + \tilde{e}_j$. Again $f_j(\{p\} \times B) = 0$ and $Df_j(\{p\} \times C) = 0$, $j = 1, \dots, k$. By construction $\alpha_1 \circ H = 0$ and $D(\alpha - \alpha_1) = 0$. Therefore there exists a function $h : U \times B \rightarrow \mathbb{K}$ such that $h(\{p\} \times B) = 0$ and $Dh = \alpha - \alpha_1$. Now $Dh \circ H = \alpha \circ H = Dh_1$ and $Dh(p, y) = \alpha(p, y) = (Dh_1 \circ \rho)(p, y)$, which proves (1), (2) and (3). Finally, note that $Dh = \sum_{j=1}^k (g_j - f_j) dx_j + \sum_{j=k+1}^n g_j dx_j$ so $D^{(2)}h(\{p\} \times C) = 0$. ■

Beyond this point both propositions have the same proof (lemma 2'. A assures us that $Dg_0(\{p\} \times B) = 0$).

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