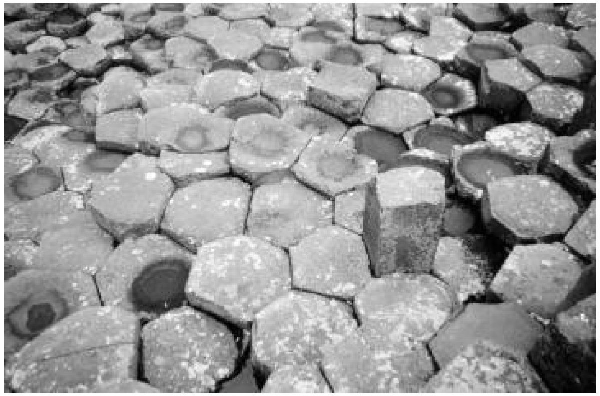
# Computer Modeling of Physical Phenomena


Lecture 4 - Growth and patterns

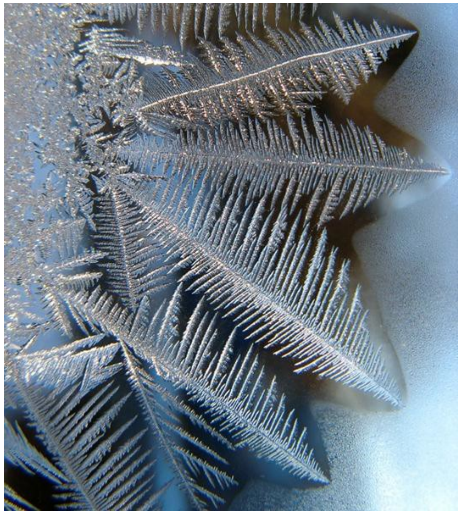
# Large variety of patterns - both in inaminate…



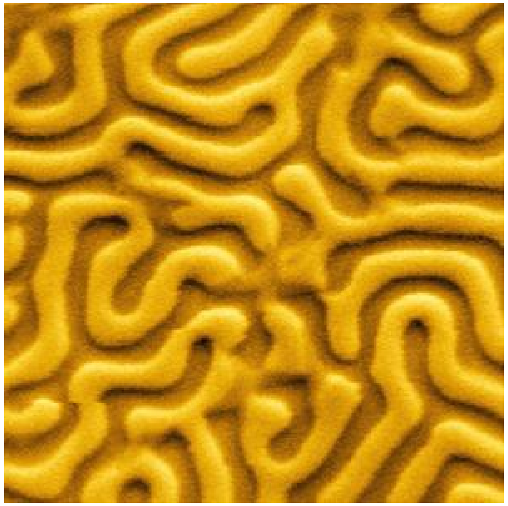
Giant's Causeway

dunes and sand ripples

river networks



ice, snowflakes



magnetic domains

# …and animate nature

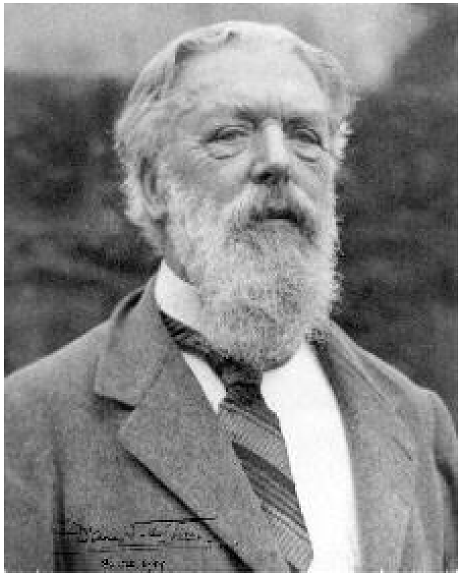
leaf

fern

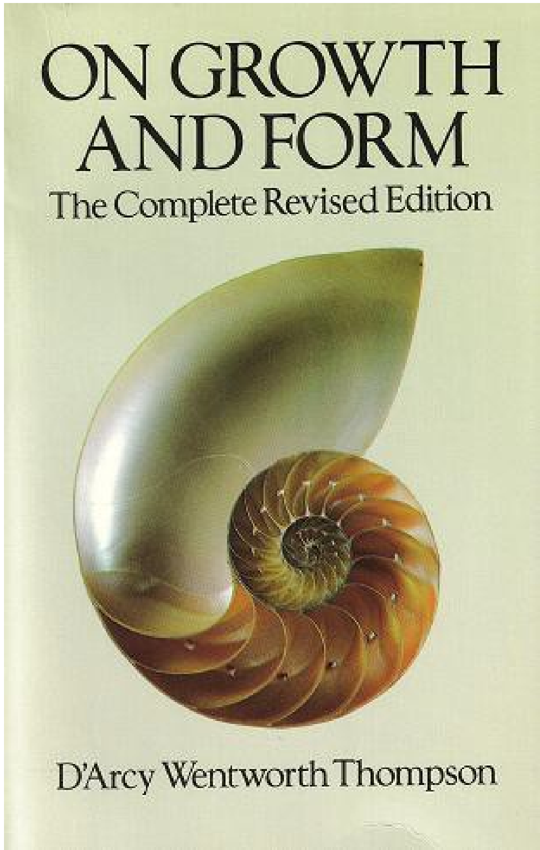
Roman cauliflower

honeycomb

# Thompson’s manifesto



Sir D'Arcy Wentworth Thompson 1860-1948



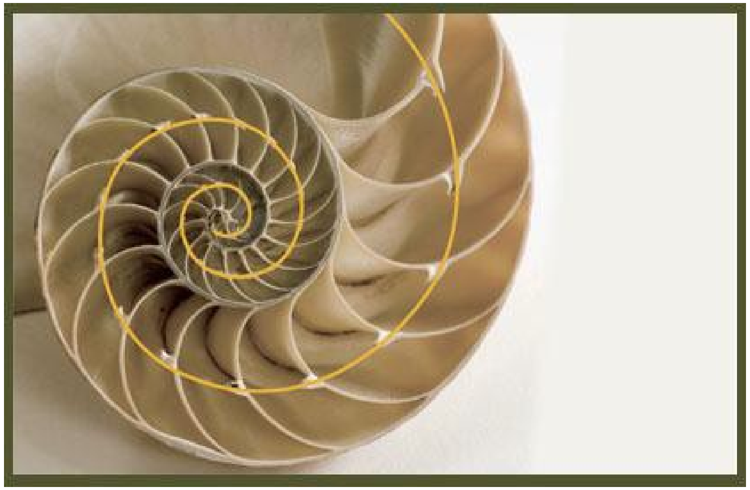

„On growth and form”
(1917, 1942)

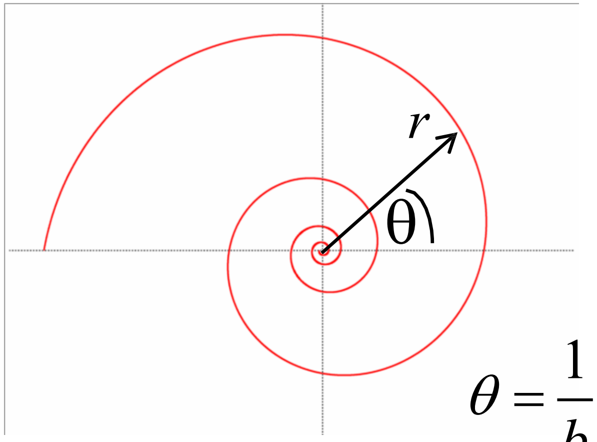
The waves of the sea, the little ripples on the shore, the sweeping curve of the sandy bay between the headlands, the outline of the hills, the shape of the clouds, all these are so many riddles of form, so many problems of morphology, and all of them the physicist can more or less easily read and adequately solve...

# First example: mouflon/nautilus

the sprial shape appears when the growth velocities of internal and external surface are different

# Logarithmic spiral






$$\theta = \frac{1}{b} \log\ r$$

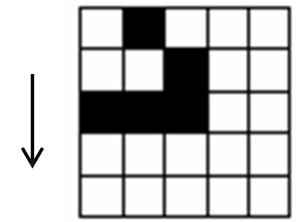
Invariant to scaling

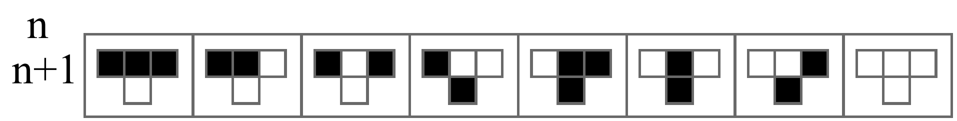
$$r \rightarrow e^{2\pi b} r$$

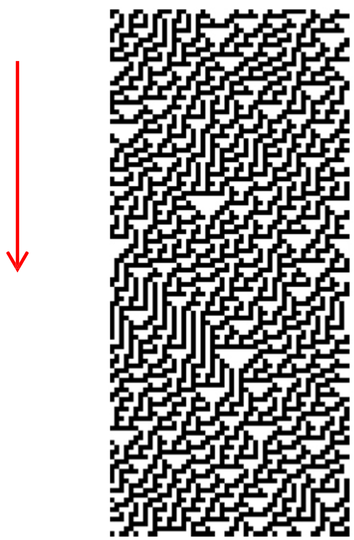
# Second example: textile cone

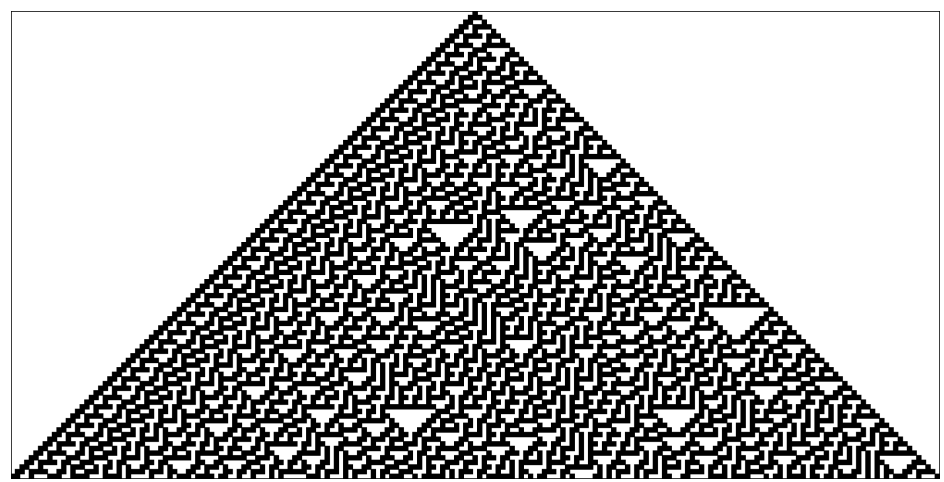
# Textile cone: a model

- shell of a cone grows gradually – row after row
- cells of a growing row are affected by chemical sygnals from the previous row
- complex patterns on a shell might be a result of relatively simple rules



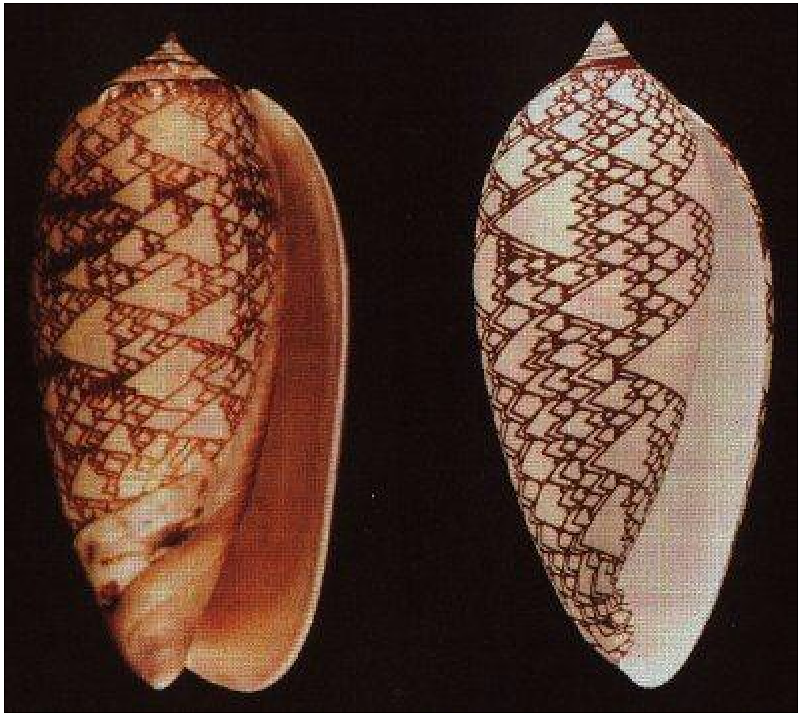




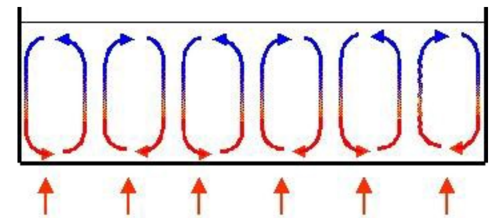
cellular automata

# Cone vs cone

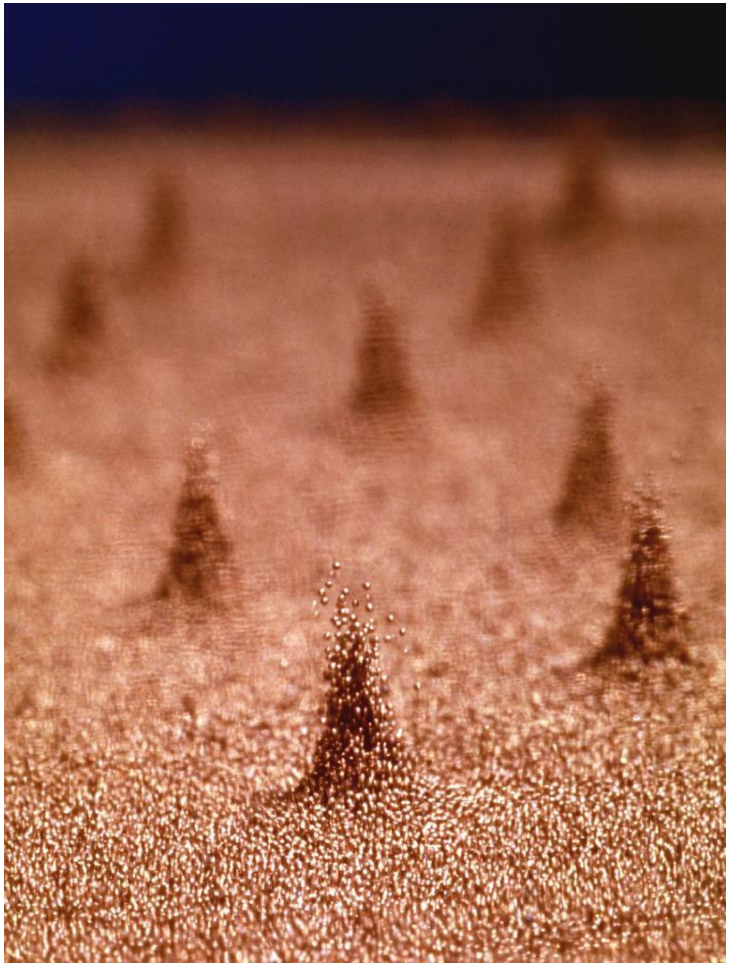


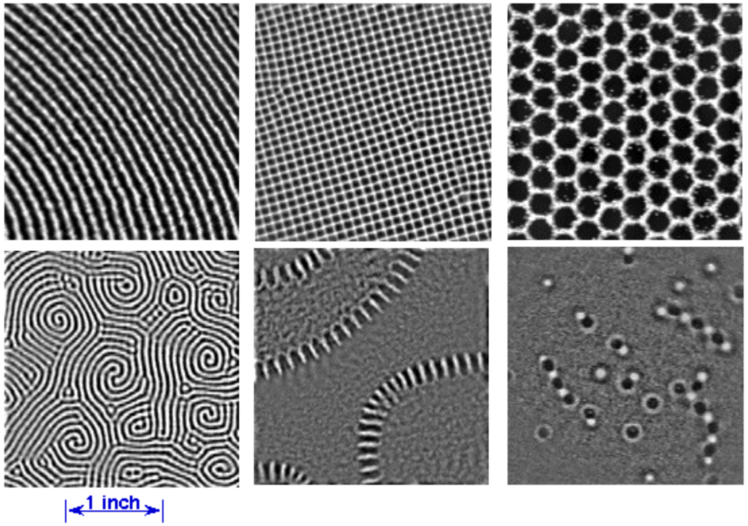
natural simulated

# Dynamic structures: Rayleigh–Bénard cells

# Oscillons

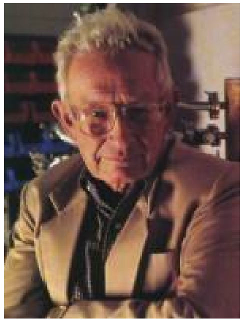





Umbahovar, Melo, Swinney, 1996

# More is different

- Complex systems are made up of a large number of entities that by interacting locally with each other give rise to qualitatively new global properties or appearance of ordered structures
- These novel, emergent properties – arising on each level of complexity - are not a mere summation of properties of parts of the system
- "Cell is not a tiger, just as a single gold atom is not yellow and gleaming"



P.W. Anderson (Nobel, 1977)

„More is different", ***Science***, 4 Aug. 1972 Vol. 177

# Pattern formation is favored by

- self-activation (short-ranged positive feedback loop): small perturbation of the uniform state will have a tendency to grow in time
- screening (long-ranged inhibition): the appearance of a structure in a given point decreases the probability of its formation in its neighbourhood

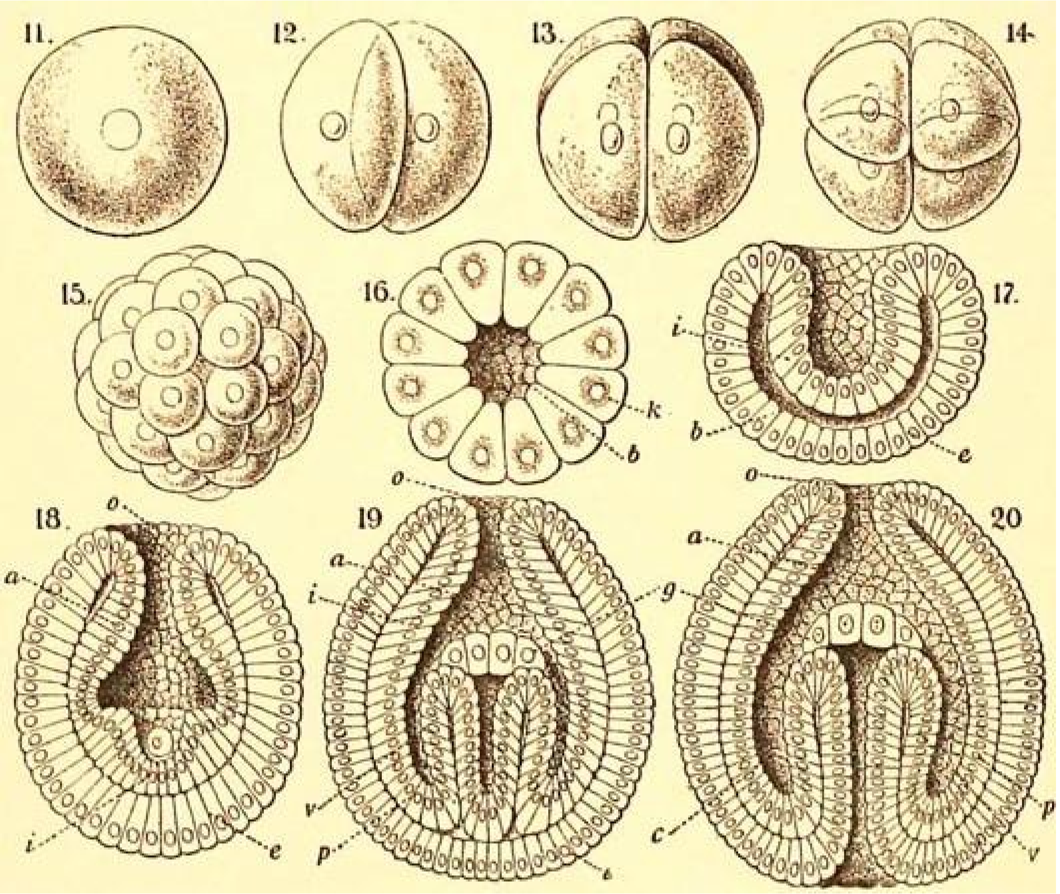
# How the leopard gets its spots?

# Alan Turing (1912-1954)


father of modern computing

The Chemical Basis of Morphogenesis, Phil. Trans. Royal Soc. London B 237, 37-72, 1952

# Embriogenesis



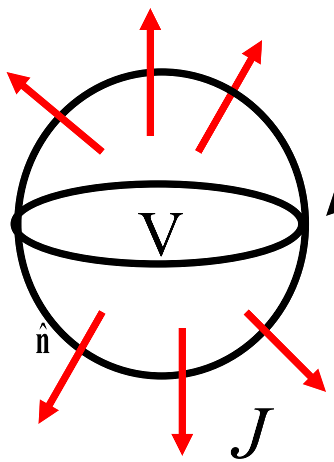

How is the initial isotropy of an embryo broken?

# Turing's hypothesis

- morphogenesis is controlled by the concentration of certain chemicals: morphogens

Why is the concentration of morphogens inhomogeneous?

# Reaction-diffusion equation




volume V

mass balance:

reaction source term

$$\frac{\partial N}{\partial t} = -\oint_S \vec{J} \cdot \vec{n}\, dS + \int_V f(c) dV$$

transport through the boundaries

diffusive current:

$$J = -D \nabla c$$

but

$$N = \int_V c\, dV$$

$$\frac{\partial N}{\partial t} = -\oint_S \vec{J} \cdot \vec{n}\, dS + \int_V f(c) dV$$

(Gauss law)

# Reaction-diffusion equation(2)

$$\int_V \frac{\partial c}{\partial t} dV = \int_V \text{-div}\,\vec{J}\ dV + \int_V f(c)\ dV$$

⇩

$$\frac{\partial c}{\partial t} = -\text{div}\,\vec{J} + f(c)$$

$$J = -D\nabla c$$

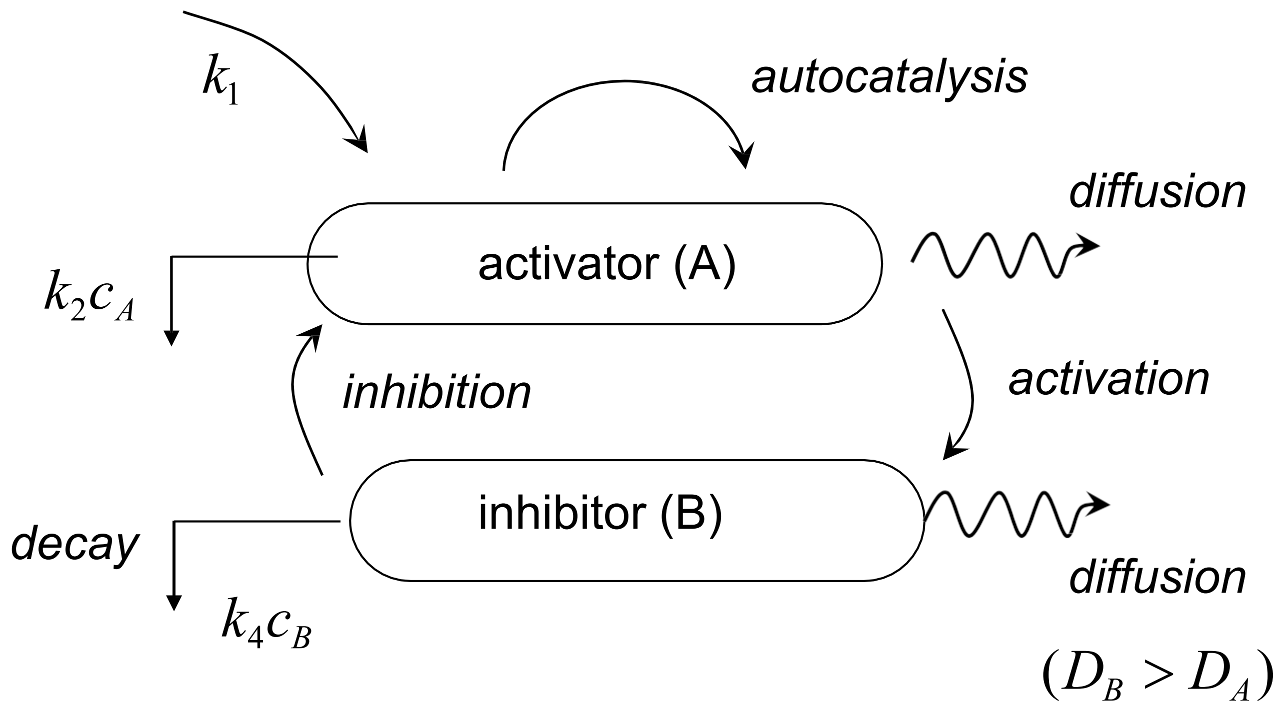
⇩

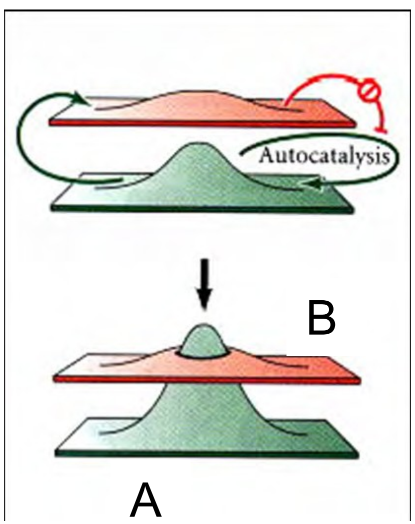
$$\frac{\partial c}{\partial t} = D\nabla^2 c + f(c)$$

# Turing's model

$$\frac{\partial c_A}{\partial t} = D_A \nabla^2 c_A + k_1 - k_2 c_A + k_3 \frac{c_A^2}{c_B}$$

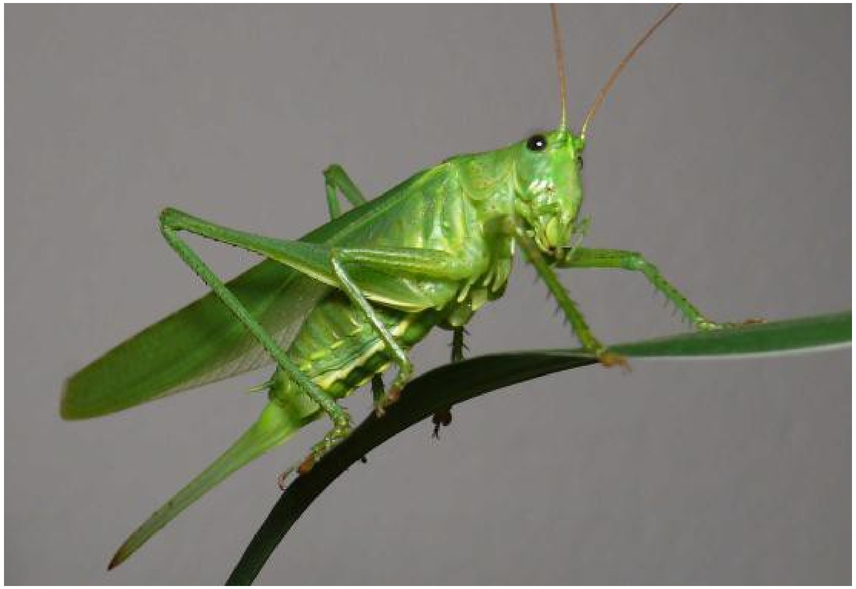
$$\frac{\partial c_B}{\partial t} = D_B \nabla^2 c_B - k_4 c_B + k_5 c_A^2$$

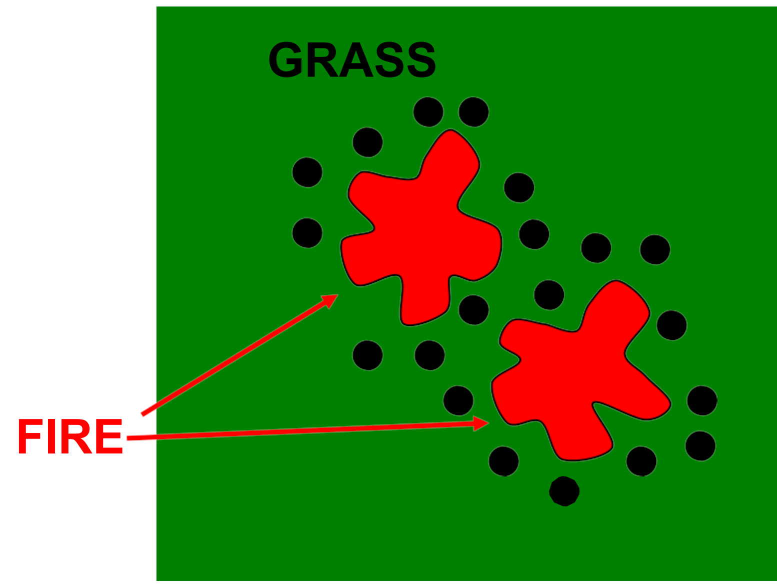


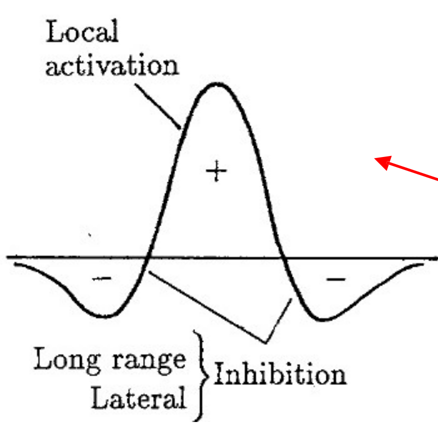




Turing, 1952
Gierer and Meinhardt, 1972

# Sweating grasshoppers



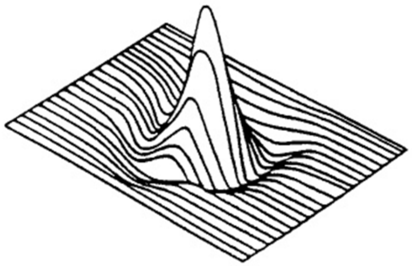


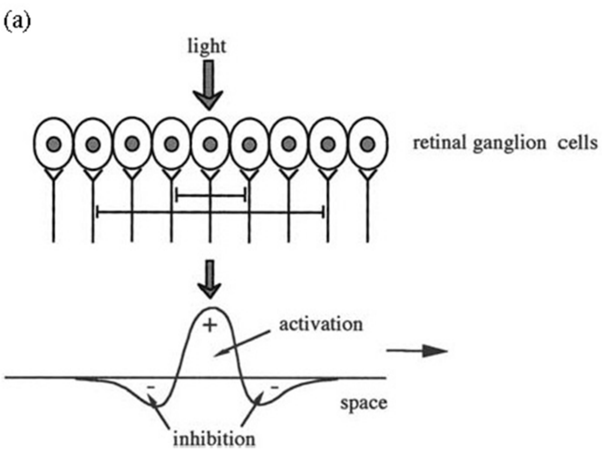



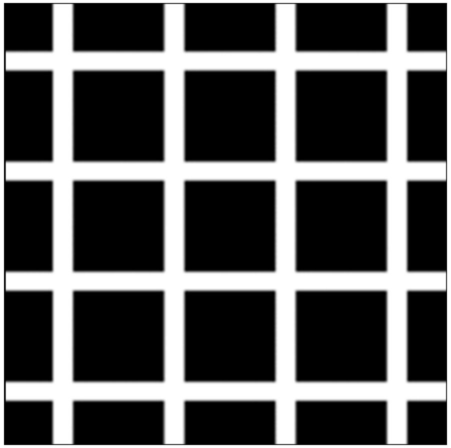

lateral inhibition

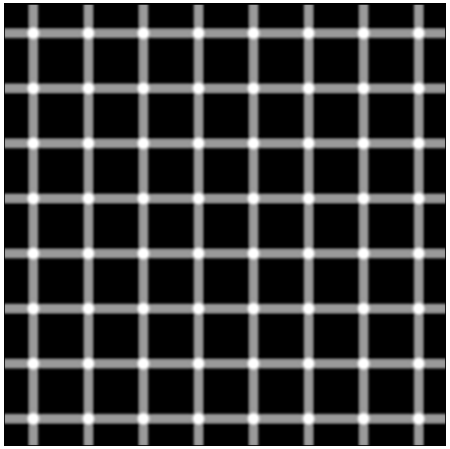
J.D. Murray („Mathematical Biology”)

# Lateral inhibition and Herman illusion







# Turing model and animal coat formation

- Skin and hair pigments, such as malanin are produced by specialized skin cells called melanocytes
- Turing model assumes that - already at the embryo stage - melanocytes are activated by a high concentration of an activator (A) and turned off by an inhibitor (B)

# Dimensionless variables

Scalings:

$$\hat{t} = k_2 t, \qquad \hat{l} = \sqrt{\frac{k_2}{D_A}}\, l, \qquad u = \frac{k_2}{k_1} c_{A,} \qquad v = \frac{k_2^2 k_4}{k_1^2 k_5} c_{B,}$$

$$a = \frac{k_3 k_4}{k_1 k_5}, \qquad d = \frac{D_B}{D_A}, \qquad \mu = \frac{k_4}{k_2}$$

lead to

$$\frac{\partial u}{\partial t} = \nabla^2 u + 1 - u + a \frac{u^2}{v}$$

$$\frac{\partial v}{\partial t} = d \nabla^2 v + \mu \left( u^2 - v \right)$$

# Uniform base state

$$\frac{\partial u}{\partial t} = \nabla^2 u + 1 - u + a\frac{u^2}{v} = 0$$

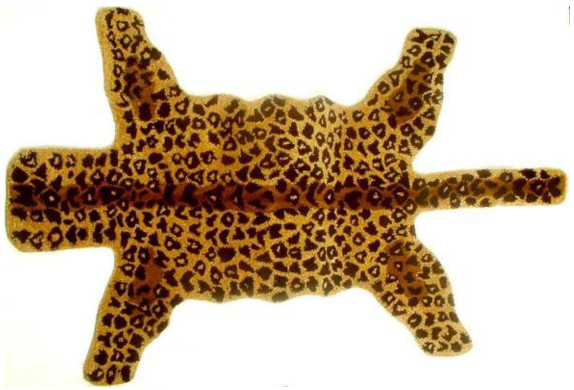
$$\frac{\partial v}{\partial t} = d\nabla^2 v + \mu\left(u^2 - v\right) = 0$$

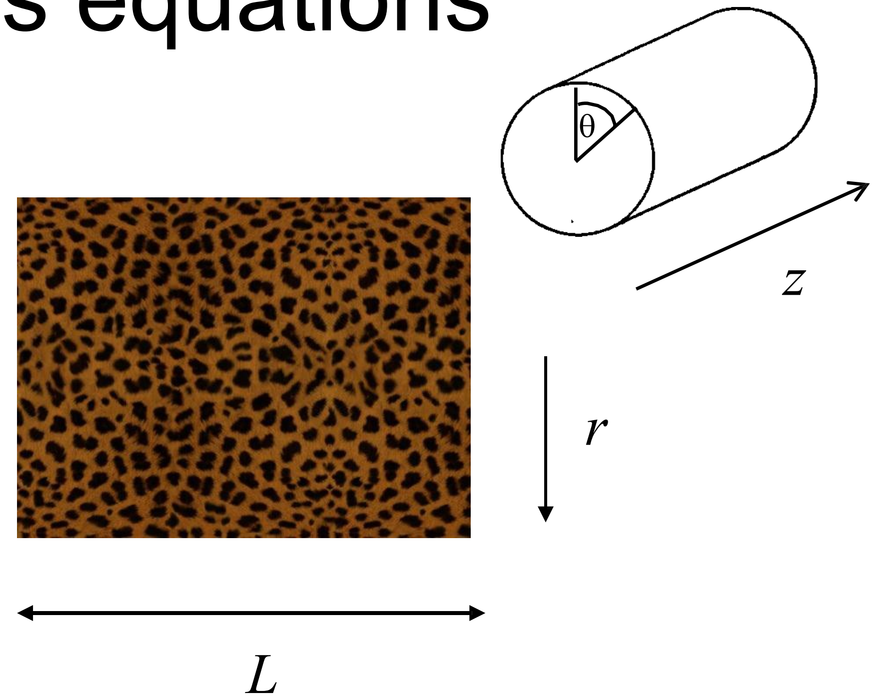
$$u_0 = 1 + a$$

$$v_0 = (1 + a)^2$$

# Solving Turing's equations

over the skin area:

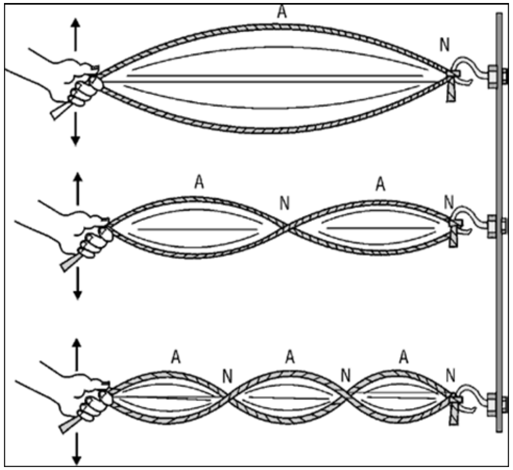





linear stability analysis – expansion around the uniform state

$$C - C_0 = \sum_{n,m} a_{n,m} \exp[\lambda(k^2)t] \cos\frac{n\pi z}{l} \cos m\theta$$

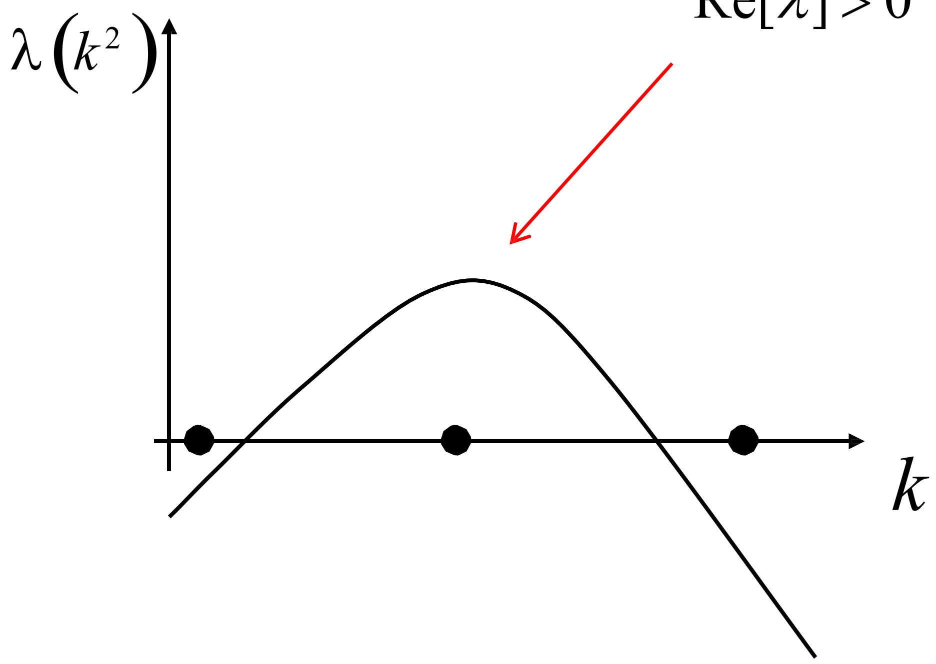
$$k^2 = \frac{\pi^2 n^2}{L^2} + \frac{m^2}{r^2}$$



# Stability of the solutions

$$C - C_0 = \sum_{n,m} a_{n,m} \exp[\lambda(k^2)t] \cos\frac{n\pi z}{l} \cos m\theta$$

$\mathrm{Re}[\lambda] > 0$ - instability



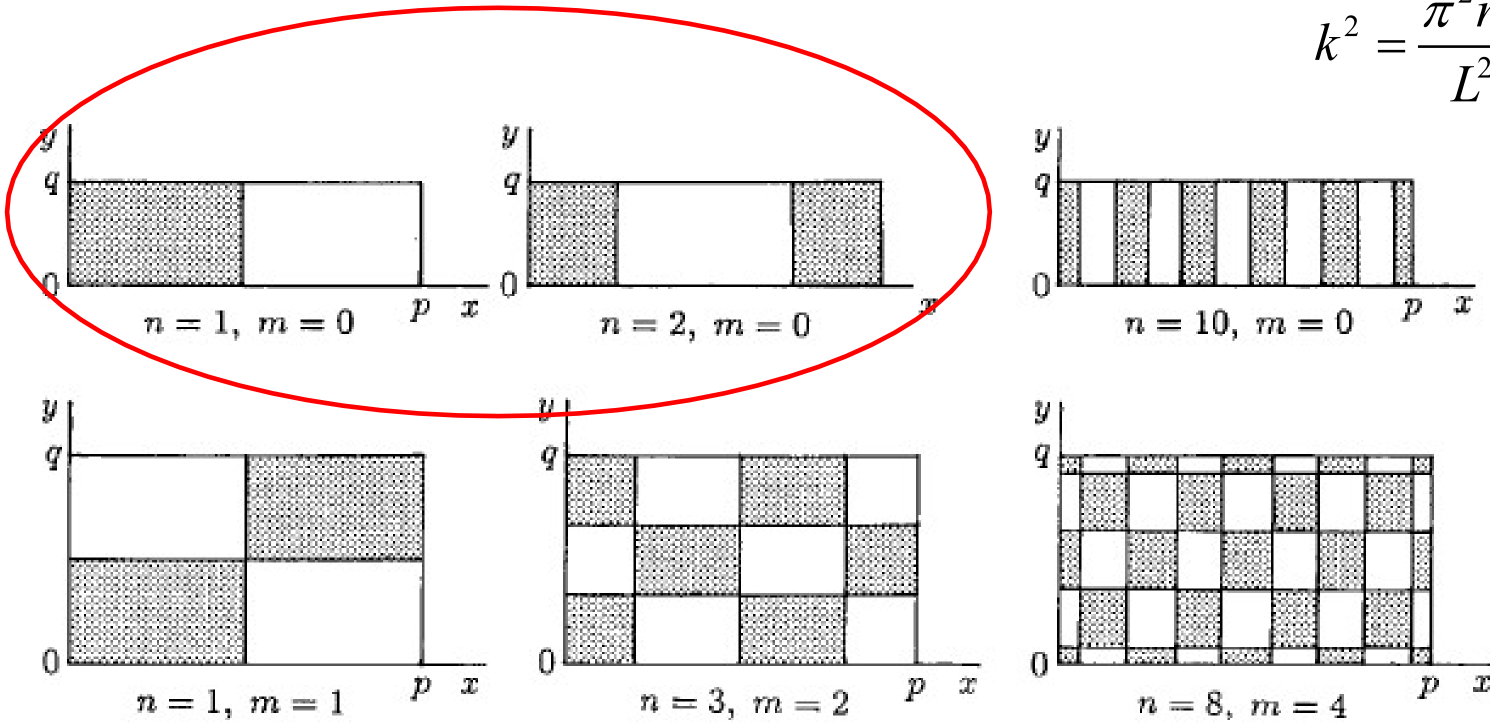

$$k^2 = \frac{\pi^2 n^2}{L^2} + \frac{m^2}{r^2}$$

$n = 0, 1, 2 \ldots \quad m = 0, 1, 2 \ldots$

for small skin dimensions (*r,L*) not a single *k>0* is found in an instability region – the solution remains uniform

# Solution – the lowest modes

$$k^2 = \frac{\pi^2 n^2}{L^2} + \frac{m^2}{r^2}$$




usually *r*<*L*, thus the lowest modes which become unstable are inhomogeneous along the body (z axis) and homogeneous along the angular coordinate ($\theta$)

# The lowest modes

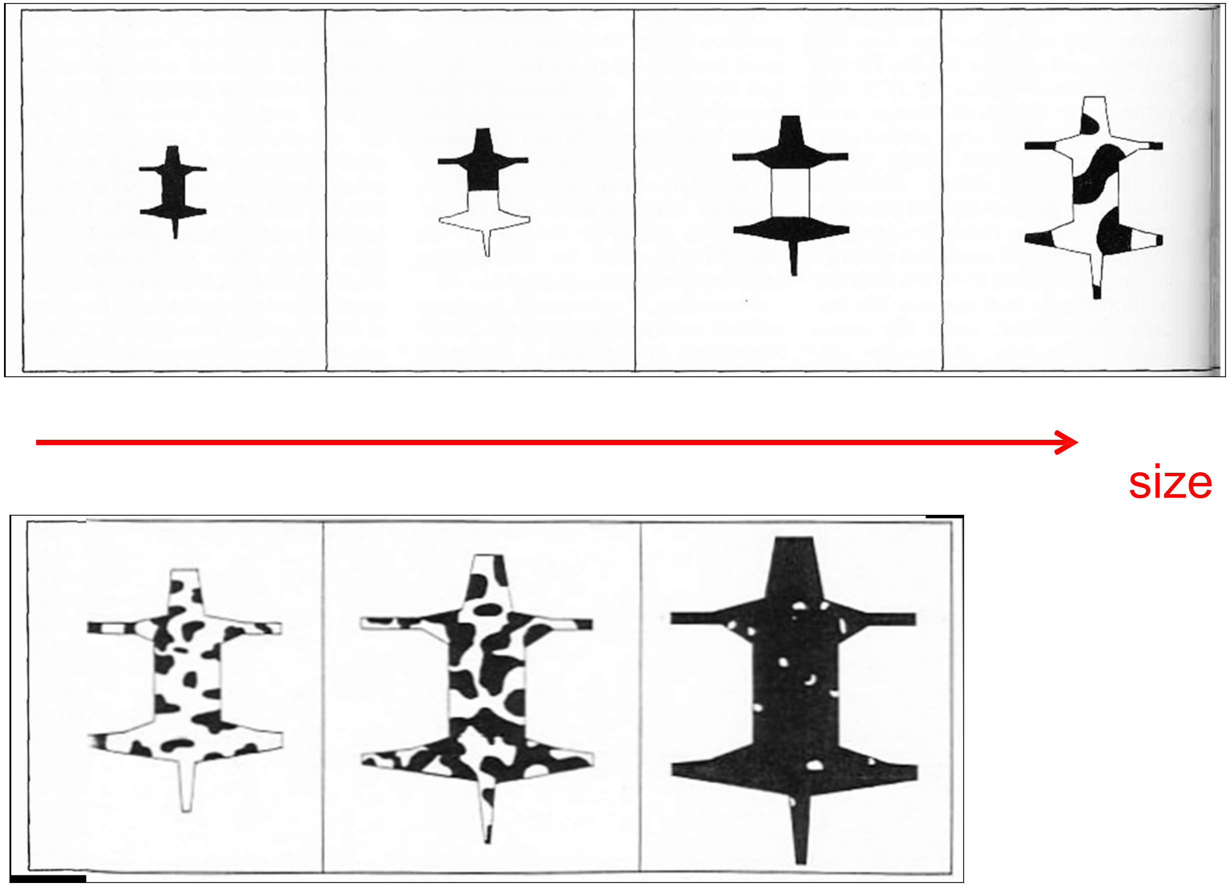
$$\cos\frac{\pi z}{L}$$

$$\cos\frac{2\pi z}{L}$$

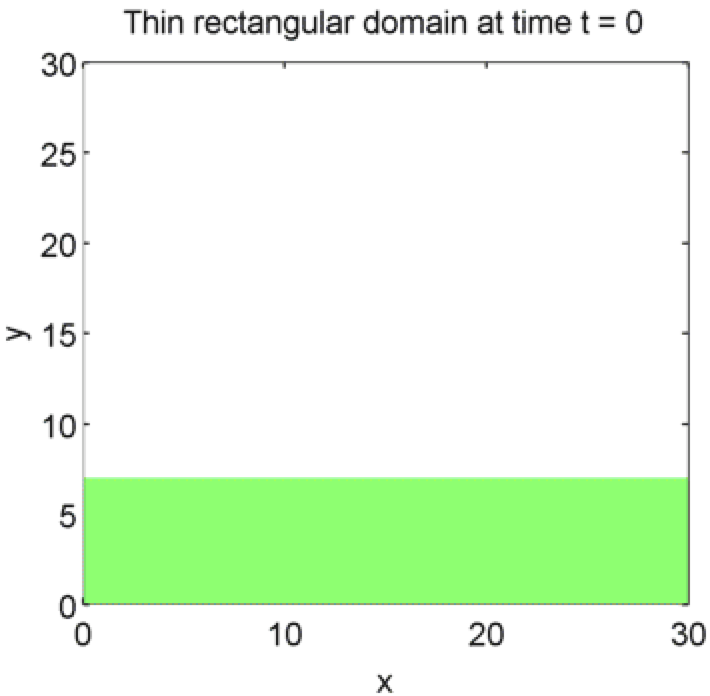
Valais goat

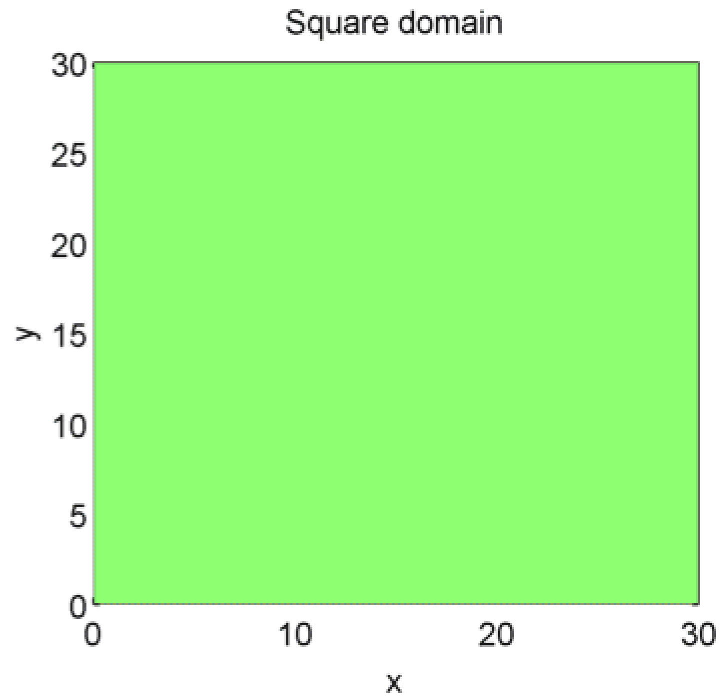
Galloway cattle

# ...and higher modes



# Modes in different geometries - simulation



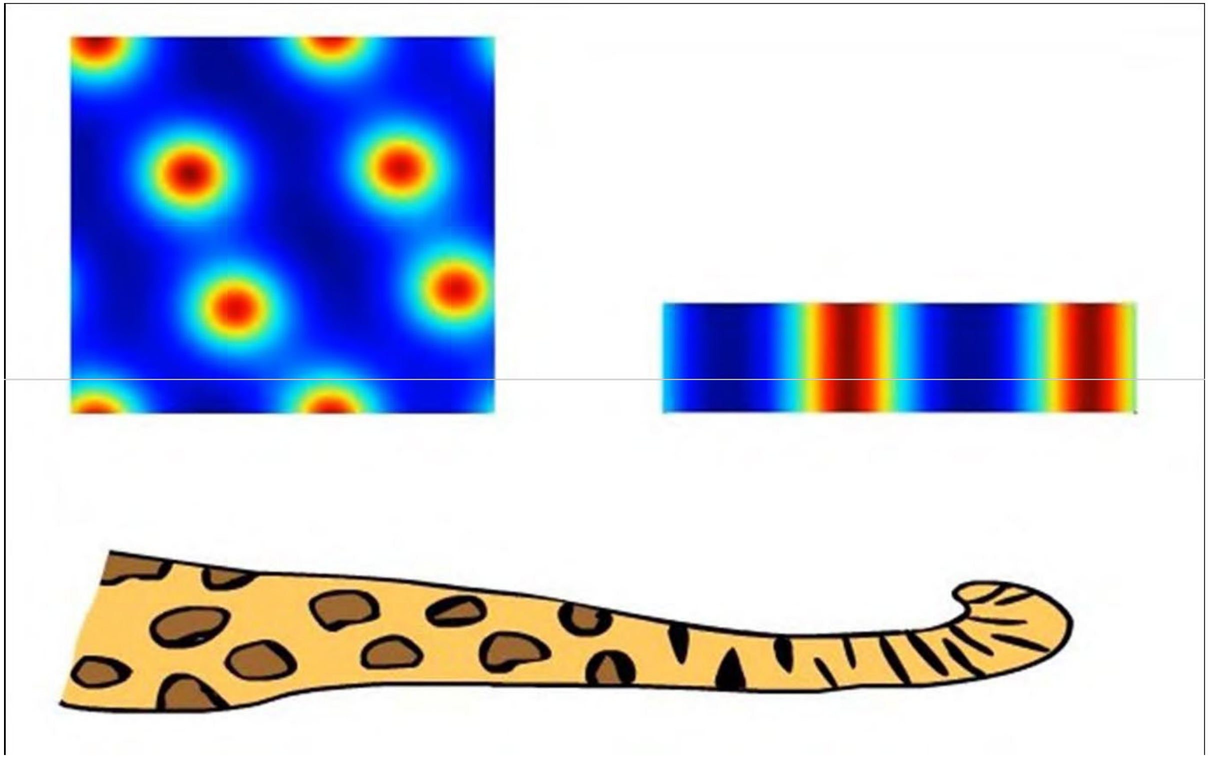
# Tail theorem

There are no striped animals with spotted tails, although there are spotted animals with striped tails

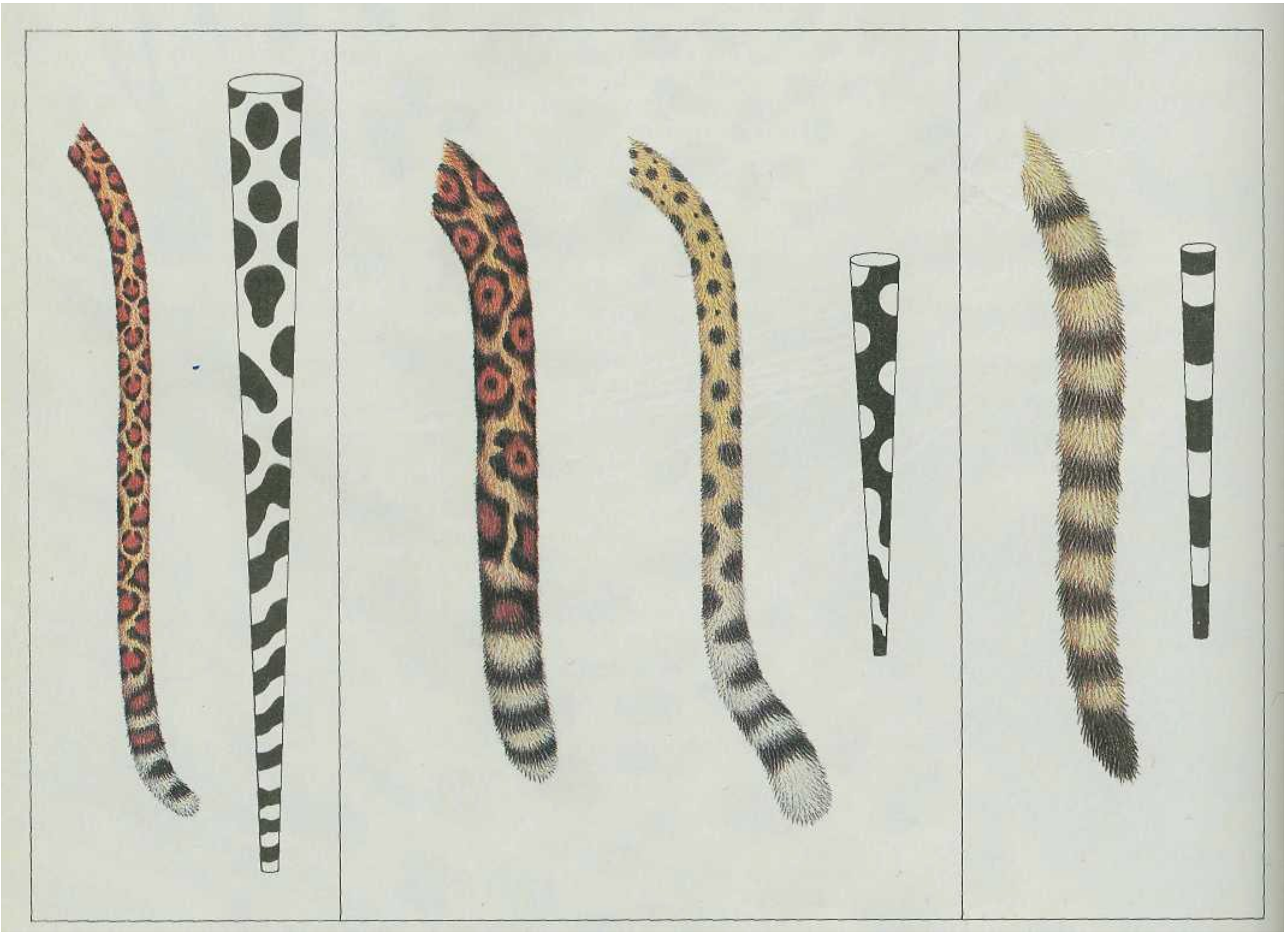
genetta

gepard

# Tail theorem



# Tail theorem

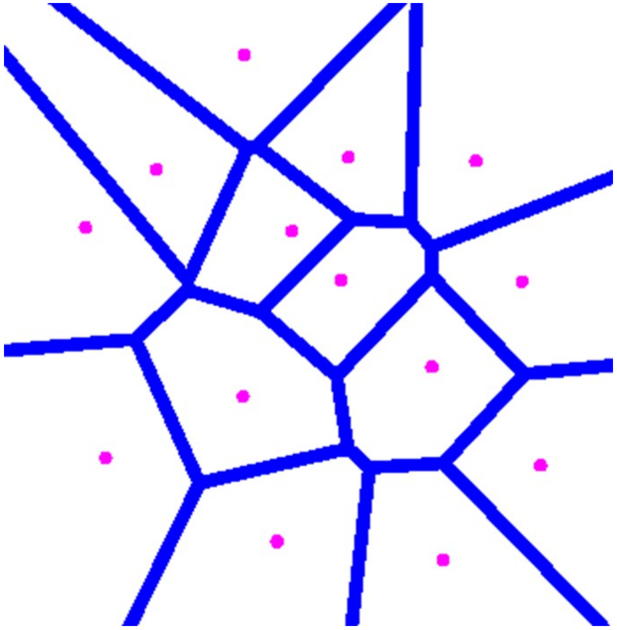


leopard jaguar gepard genetta

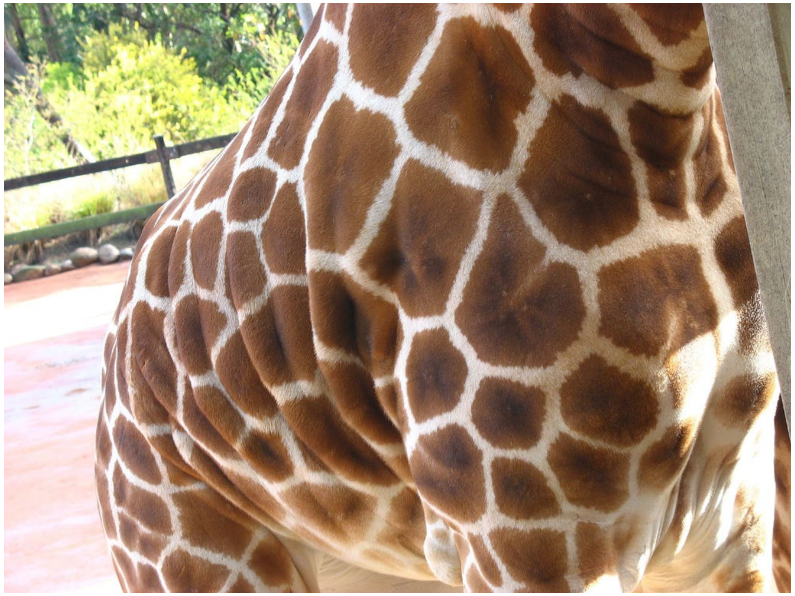
# Tesselating girrafe

## Dirichlet tessellation / Voronoi cells

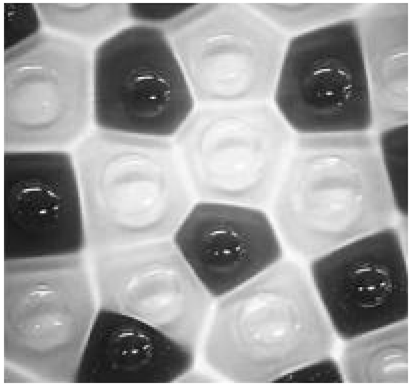
Given n points on the plane (the so-called Voronoi centers), the Voronoi cell V(s) corresponding to a given center A is the set of points whose distance from A is less than the distance from all other centers.



# Voronoi cells in nature



giraffe



Petri dish

frying pan

# Giraffe problem

$$\frac{\partial a}{\partial t} = D_a \nabla^2 c_a + \rho_a \left[ \frac{a^2 s}{1+\kappa_a a^2} - a \right]$$

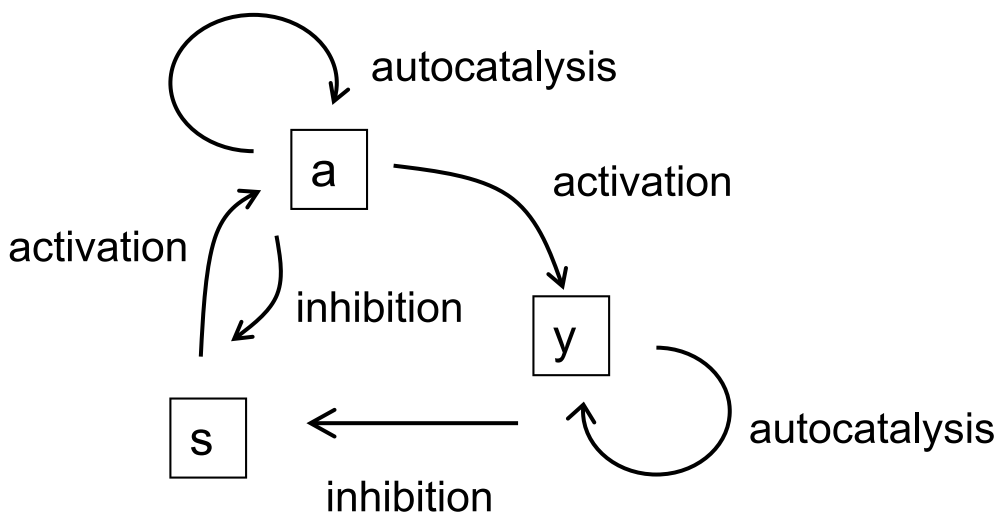
activator (morphogen)

$$\frac{\partial s}{\partial t} = D_s \nabla^2 c_s + \frac{\sigma_s}{1+\kappa_s y} - \frac{\rho_a a^2 s}{1+\kappa_a a^2} - \mu_s s$$

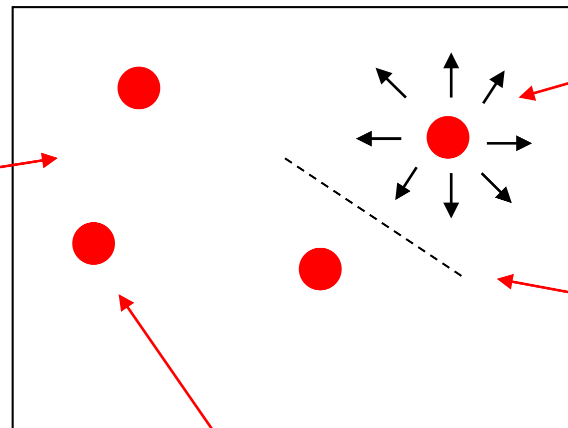
substrate (needed for activator production)

$$\frac{\partial y}{\partial t} = \rho_y \frac{y^2}{1+\kappa_a y^2} - \mu_y y + \sigma_y a$$

melanocyte activity
(y=1 – melanine production)
(y=0 – no production)



# Giraffe: mechanism



propagating chemical wave of substance *a* - follows the direction of high substrate concentration

initial state:
y=0, a=0, s=$s_0$

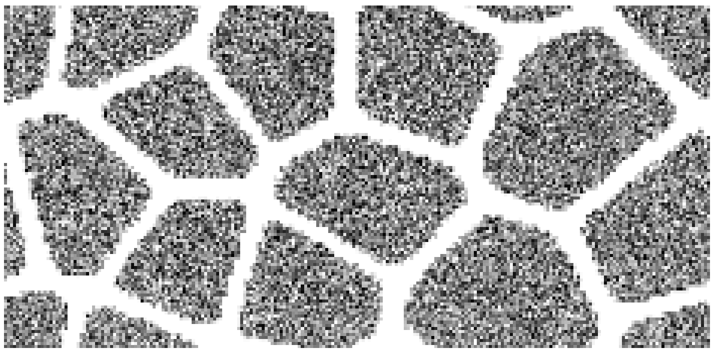
'annihilation line' formed when two waves meet (low concentration of *a* due to substrate depletion)

random spots where $a=a_0$; high concentration of *a* triggers activation of melanocytes

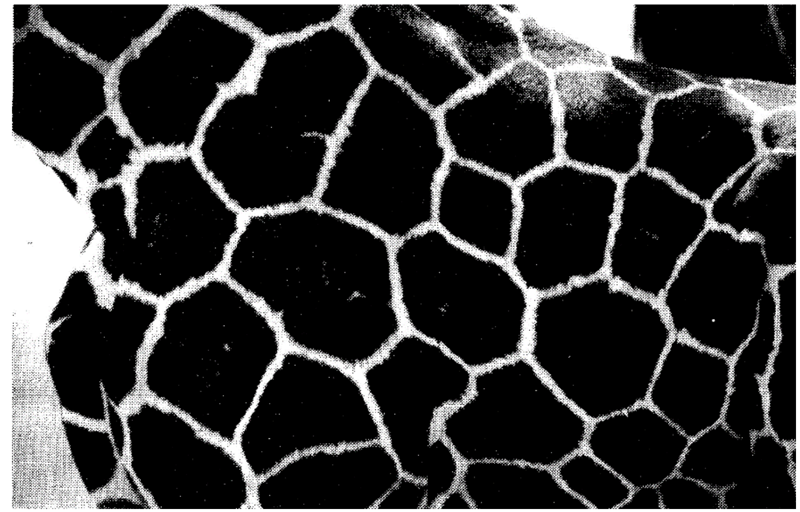
Moreover:

- after activation, *y* remains non-zero even when *a* disappears
- high *y* value reduces *s* concentration and stops morphogenesis

# Giraffe-results

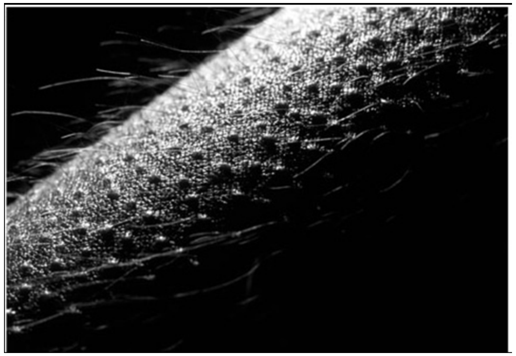


concentration of y

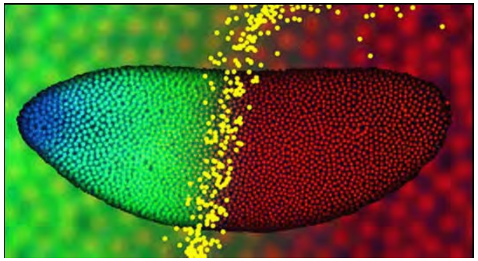


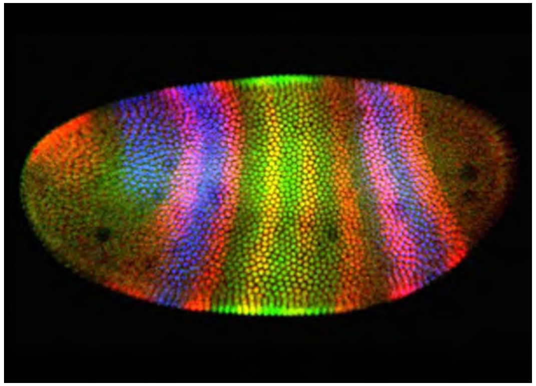
# Experimental confirmation?

In several cases, it has been possible to identify the morphogens responsible for a given morphogenetic process, e.g. the distribution of hair stalks on mouse skin is controlled by an activator (Wnt) and an inhibitor (Dkk)



Morphogens have also been identified in the case of fruit fly embryogenesis; there, however, the process proceeds in a much more controlled manner than that proposed by Turing





# Literature


James D. Murray, *Mathematical Biology*, Springer (1993)

A.J. Koch i H. Meinhardt, *Biological pattern formation: from basic mechanisms to complex structures,* Rev. Mod. Phys., **66**, 1481 (1994)

P. Ball, *Nature's Patterns: a Tapestry in Three Parts*, Oxford, (2011)