Instability growth rate for small permeability contrasts

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We begin with a dimensionless form of Eqs. (1)-(3), using the downstream scaling of length:
\[
\xi = \frac{x'}{l_d}, \quad \eta = \frac{y}{l_d}, \quad \tau = \frac{\gamma_0 v_0 t}{l_d},
\]
and dimensionless velocity, concentration, and porosity fields:
\[
\hat{v} = \frac{v}{v_0}, \quad \hat{c} = \frac{c}{c_{in}}, \quad \hat{\phi} = \frac{\phi - \phi_0}{\phi_{max} - \phi_0}.
\]

The pressure can be eliminated by replacing Darcy’s equation with the compatibility equation (6):
\[
\hat{v}_\xi \partial_\xi \hat{c} + \hat{v}_\eta \partial_\eta \hat{c} = \frac{1 + P e^{-1}}{\partial_\xi} \hat{c},
\]
\[
\partial_\tau \hat{\phi} - \partial_\xi \hat{\phi} = \frac{1 + P e^{-1}}{\partial_\xi} \hat{c},
\]
\[
\hat{v}_\xi \hat{\phi}_\xi + \hat{v}_\eta \hat{\phi}_\eta = 0,
\]
\[
\partial_\eta \hat{v}_\xi - \alpha \hat{\phi}_\eta \hat{\phi} = \partial_\xi \hat{v}_\eta - \alpha \hat{v}_\eta \hat{\phi}_\xi,
\]
where the (time-dependent) terms in \(\gamma\) have been dropped, corresponding to the typical limiting case of small acid capacity.

Substituting perturbations of the form of Eq. (11) leads to coupled equations for the one-dimensional fields \(\delta \hat{\phi}(\xi), \delta \hat{c}(\xi),\) and \(\delta \hat{v}(\xi)\):
\[
(\partial_\xi \hat{c}_b) \delta \hat{v} = (\partial_\xi - \hat{\omega}) \delta \hat{\phi} + [P e^{-1} (\partial_\xi^2 \hat{u}^2 - \hat{u}^2)] \delta \hat{c},
\]
\[
(1 + P e^{-1}) \delta \hat{c} = (-\partial_\xi + \hat{\omega}) \delta \hat{\phi},
\]
\[
\alpha \hat{u}^2 \delta \hat{\phi} = (-\partial_\xi^2 + \alpha (\partial_\xi \hat{\phi} b) \partial_\xi + \hat{u}^2) \delta \hat{v}.
\]

For the purposes of numerical solution it is convenient to combine these equations into a single fifth-order equation for \(\delta \hat{\phi},\)
\[
\{ [\partial_\xi^2 + \alpha e^{-\xi} \partial_\xi - \hat{u}^2] e^\xi [P e^{-1} (\partial_\xi^2 \hat{u}^2 - \hat{u}^2 - 1) - \partial_\xi - 1] [\partial_\xi - \hat{\omega}] + \alpha \hat{u}^2 \} \delta \hat{\phi} = 0.
\]
The equations for the upstream perturbations (without the reaction terms and with constant porosity) are:

\[(\partial_\xi \delta \hat{c}) \delta \hat{v} = \left[ Pe^{-1}(\partial_\xi^2 - \hat{u}^2) - \partial_\xi \right] \delta \hat{c}; \quad (11)\]

\[0 = (-\partial_\xi^2 + \hat{u}^2) \delta \hat{v}. \quad (12)\]

The upstream and downstream perturbations are connected by continuity conditions in \(\hat{\phi}, \hat{v}, \hat{\phi}_n, \hat{c},\) and \(\partial_\xi \hat{c}\) at the front \(\xi = \xi_f\), where perturbation in the front position also grows exponentially \(\xi_f = \xi_0 \cos(\hat{u} \eta)e^{\hat{\omega} \tau}\).

In the limit that the permeability contrast \(\alpha\) is small, we can make a regular perturbation expansion around \(\alpha = 0\). Expanding \(\hat{\omega}, \delta \hat{\phi}, \delta \hat{c},\) and \(\delta \hat{v}\) in powers of \(\alpha\),

\[\hat{\omega} = \hat{\omega}_0 + \alpha \hat{\omega}_1 + \ldots, \quad (13)\]

the zeroth order solution for \(\delta \hat{\phi}\) is

\[(\delta \hat{\phi})_0 = A_0 e^{-(\hat{u}+1)\xi} + B_0 e^{\lambda \xi}, \quad (14)\]

with

\[\lambda = \frac{1}{2} \left( Pe - \sqrt{(Pe + 2)^2 + 4\hat{u}^2} \right). \quad (15)\]

Applying the continuity conditions at the front we find \((\delta \hat{\phi})_0 = \xi_0 e^{\lambda \xi}\) and

\[w_0 = \beta = \frac{1}{2} \left( Pe - \sqrt{Pe^2 + 4\hat{u}^2} \right). \quad (16)\]

The first-order (in \(\alpha\)) solution can be found in a similar fashion; after applying the continuity conditions at the front, we obtain an explicit expression for the first-order growth rate, which is always positive:

\[\hat{\omega}_1 = \frac{c_1 + c_2 \beta - c_3 \lambda - c_4 \beta \lambda}{2(1 + Pe)[1 + Pe(1 + \hat{u})][(1 + 2\hat{u} + Pe(1 + \hat{u})]}, \quad (17)\]

where the coefficients are polynomials in \(\hat{u}\) and \(Pe\):

\[c_1 = \hat{u}\{1 + 2\hat{u} + Pe(1 + \hat{u})[3 + \hat{u} + Pe(3 + \hat{u} + Pe)]\}; \quad (18)\]

\[c_2 = 1 + 2\hat{u} + Pe\{3 + 4\hat{u} + \hat{u}^2 + Pe[2 + 3\hat{u} + Pe(1 + \hat{u})]\}; \quad (19)\]

\[c_3 = Pe \hat{u}^2; \quad (20)\]

\[c_4 = Pe[\hat{u} + Pe(1 + \hat{u})]. \quad (21)\]