ZADANIE 1

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \qquad \Bigg\{ \; P^2 = \begin{bmatrix} \frac{3}{8} & \frac{5}{16} & \frac{5}{16} \\ \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{5}{16} & \frac{5}{16} \end{bmatrix}$$

P-STWO PRZEJŚCIA $i \to j$ W n KROKACH $= [P^n]_{(j,i)}$

OBLICZAM P^n

$$\det(P - \lambda \mathbb{1}) = \det \begin{bmatrix} \frac{1}{2}-\lambda & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2}-\lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4}-\lambda \end{bmatrix} =$$

$$= \left(\tfrac{1}{2}-\lambda\right)^2\left(\tfrac{1}{4}-\lambda\right) + \tfrac{1}{16} - \tfrac{1}{8}\left(\tfrac{1}{2}-\lambda\right) - \tfrac{1}{8}\left(\tfrac{1}{2}-\lambda\right)$$

$$= \tfrac{1}{16}\left((1-2\lambda)^2(1-4\lambda) + 1 - 2(1-2\lambda)\right)$$

$$= \tfrac{1}{16}\left((1-4\lambda+4\lambda^2)(1-4\lambda) \underline{-1+4\lambda}\right)$$

$$= \tfrac{1}{16}(1-4\lambda)\left(1-4\lambda+4\lambda^2-1\right) = \tfrac{1}{16}(1-4\lambda)4\lambda(1-\lambda)$$

WARTOŚCI WŁASNE $P: 0, \frac{1}{4}, 1$

$P^n = f(P)$, GDZIE $f(x) = x^n$

$$f(x) = \varphi(x)\, w_p(x) + \underbrace{ax^2+bx+c}_{\text{RESZTA}} \qquad w_p \text{ – WIEL. CHAR. MACIERZY } P$$

$x = 0, \frac{1}{4}, 1$:

$$\begin{cases} 0 = c \\ \left(\frac{1}{4}\right)^n = \frac{a}{16} + \frac{b}{4} + c \\ 1 = a+b+c \end{cases} \nearrow \quad \begin{cases} c = 0,\ b = 1-a \\ \left(\frac{1}{4}\right)^n = \frac{a}{16} + \frac{1-a}{4} \end{cases}$$

$$\left(\tfrac{1}{4}\right)^{n-2} = a + 4 - 4a = 4 - 3a$$

$$3a = 4 - \left(\tfrac{1}{4}\right)^{n-2}$$

$$a = \tfrac{1}{3}\left(4 - \left(\tfrac{1}{4}\right)^{n-2}\right), \quad b = 1-a = \tfrac{1}{3}\left(\left(\tfrac{1}{4}\right)^{n-2} - 1\right)$$

$$P^n = \tfrac{1}{3}\left(4 - \left(\tfrac{1}{4}\right)^{n-2}\right)P^2 + \underbrace{\tfrac{1}{3}\left(\left(\tfrac{1}{4}\right)^{n-2} - 1\right)P}$$ TA MACIERZ MA WYRAZ $(2,1) = 0$

$$[P^n]_{(2,1)} = \tfrac{1}{3}\left(1 - \left(\tfrac{1}{4}\right)^{n-1}\right), \quad \text{BO } [P^2]_{(2,1)} = \tfrac{1}{4}.$$

ZADANIE 2

$$A := \begin{bmatrix} 1 & 2+i \\ 2-i & 5 \end{bmatrix}$$

$$\det(A - \lambda \mathbb{1}) = \det \begin{bmatrix} 1-\lambda & 2+i \\ 2-i & 5-\lambda \end{bmatrix} = (1-\lambda)(5-\lambda) - 5 = \lambda^2 - 6\lambda + 5 - 5 = \lambda(\lambda - 6)$$

WARTOŚCI WŁASNE A: $0, 6$

PODPRZESTRZENIE WŁASNE:

- $$\ker(A - 6\mathbb{1}) = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \,\middle|\, \begin{bmatrix} -5 & 2+i \\ 2-i & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} =$$

$$= \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \,\middle|\, b = (2-i)a \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2-i \end{bmatrix} \right\}$$

$$e_1 := \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$$ ← TEN WEKTOR MA DŁUGOŚĆ 1:

$$\|e_1\|^2 = \frac{1}{6} \begin{bmatrix} 1 & 2+i \end{bmatrix} \begin{bmatrix} 1 \\ 2-i \end{bmatrix} =$$

$$= \frac{1}{6}(1+5) = 1$$

- $$\ker A = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \,\middle|\, \begin{bmatrix} 1 & 2+i \\ 2-i & 5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \,\middle|\, a = -(2+i)b \right\} = \operatorname{span}\left\{ \begin{bmatrix} -(2+i) \\ 1 \end{bmatrix} \right\}$$

$$e_2 := \frac{1}{\sqrt{6}} \begin{bmatrix} -(2+i) \\ 1 \end{bmatrix}$$ ← TAK JAK POPRZEDNIO $\|e_2\| = 1$

RZUTY ORTOGONALNE NA PODPRZESTRZENIE WŁASNE:

$$P_1 = |e_1)(e_1| = \frac{1}{6} \begin{bmatrix} 1 \\ 2-i \end{bmatrix} \begin{bmatrix} 1 & 2+i \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & 2+i \\ 2-i & 5 \end{bmatrix}$$

$$P_2 = |e_2)(e_2| = \frac{1}{6} \begin{bmatrix} -(2+i) \\ 1 \end{bmatrix} \begin{bmatrix} -(2-i) & 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & -(2+i) \\ -(2-i) & 1 \end{bmatrix}$$

ZADANIE 3

P-Ń ZDARZEŃ ELEMENTARNYCH Ω MA 36 PUNKTÓW, KAŻDY O P-STWIE $\frac{1}{36}$.

WYPISUJĘ WARTOŚCI ZMIENNEJ X:

(1,1) 1	(1,2) 2	(1,3) 3	(1,4) 4	(1,5) 5	(1,6) 6
(2,1) 2	(2,2) 4	(2,3) 6	(2,4) 8	(2,5) 10	(2,6) 12
(3,1) 3	(3,2) 6	(3,3) 9	(3,4) 12	(3,5) 15	(3,6) 18
(4,1) 4	(4,2) 8	(4,3) 12	(4,4) 16	(4,5) 20	(4,6) 24
(5,1) 5	(5,2) 10	(5,3) 15	(5,4) 20	(5,5) 25	(5,6) 30
(6,1) 6	(6,2) 12	(6,3) 18	(6,4) 24	(6,5) 30	(6,6) 36

$\{1,2,3,4,5,6,8,9,10,12,15,16,18,20,24,25,30,36\}$

ROZKŁAD: $P(X=1)=\frac{1}{36}$, $P(X=2)=\frac{2}{36}$, $P(X=3)=\frac{2}{36}$, $P(X=4)=\frac{3}{36}$, $P(X=5)=\frac{2}{36}$, $P(X=6)=\frac{4}{36}$

$P(X=8)=\frac{2}{36}$, $P(X=9)=\frac{1}{36}$, $P(X=10)=\frac{2}{36}$, $P(X=12)=\frac{4}{36}$, $P(X=15)=\frac{2}{36}$, $P(X=16)=\frac{1}{36}$

$P(X=18)=\frac{2}{36}$, $P(X=20)=\frac{2}{36}$, $P(X=24)=\frac{2}{36}$, $P(X=25)=\frac{1}{36}$, $P(X=30)=\frac{2}{36}$, $P(X=36)=\frac{1}{36}$

WARTOŚĆ OCZEKIWANA

$$E(X)=\sum_{k=1}^{36} k\,P(X=k)$$

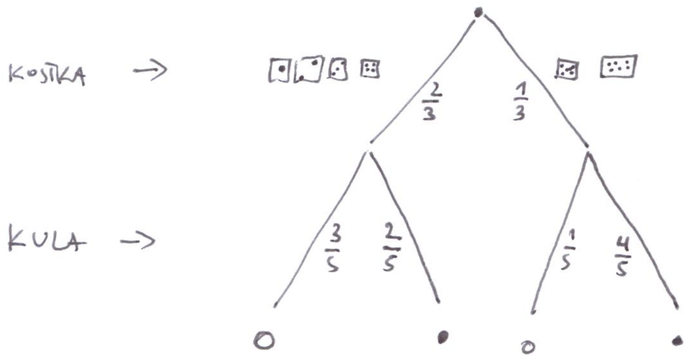
$\leftarrow$ DLA k NIEBĘDĄCYCH WARTOŚCIAMI ZMIENNEJ X MAMY $P(X=k)=0$

$$=\frac{1}{36}\big(1\cdot 1+2\cdot 2+3\cdot 2+4\cdot 3+5\cdot 2+6\cdot 4$$
$$+8\cdot 2+9\cdot 1+10\cdot 2+12\cdot 4+15\cdot 2+16\cdot 1$$
$$+18\cdot 2+20\cdot 2+24\cdot 2+25\cdot 1+30\cdot 2+36\cdot 1\big)$$

$$=\frac{1}{36}\cdot 441=12+\frac{1}{4}$$

ZADANIE 4

RZUTY KOSTKĄ I LOSOWANIA KUL SĄ NIEZALEŻNE, WIĘC ODPOWIEDNIE P-STWA SIĘ MNOŻĄ:



$$P(\circ) = \frac{2}{3}\cdot\frac{3}{5} + \frac{1}{3}\cdot\frac{1}{5} = \frac{7}{15}.$$

ZADANIE 5

MAMY 12 ZDARZEŃ ELEMENTARNYCH. WYPISUJĘ ICH P-STWA W TABELI:

	⚀	⚁	⚂	⚃	⚄	⚅
○	$\frac{1}{6}\cdot\frac{3}{5}=\frac{1}{10}$	$\frac{1}{6}\cdot\frac{3}{5}=\frac{1}{10}$	$\frac{1}{6}\cdot\frac{3}{5}=\frac{1}{10}$	$\frac{1}{6}\cdot\frac{3}{5}=\frac{1}{10}$	$\frac{1}{6}\cdot\frac{1}{5}=\frac{1}{30}$	$\frac{1}{6}\cdot\frac{1}{5}=\frac{1}{30}$
●	$\frac{1}{6}\cdot\frac{2}{5}=\frac{1}{15}$	$\frac{1}{6}\cdot\frac{2}{5}=\frac{1}{15}$	$\frac{1}{6}\cdot\frac{2}{5}=\frac{1}{15}$	$\frac{1}{6}\cdot\frac{2}{5}=\frac{1}{15}$	$\frac{1}{6}\cdot\frac{4}{5}=\frac{2}{15}$	$\frac{1}{6}\cdot\frac{4}{5}=\frac{2}{15}$

ZMIENNA Y MA WARTOŚCI:

	⚀	⚁	⚂	⚃	⚄	⚅
○	1	1	1	1	1	1
●	0	1	0	1	0	1

$$D^2(Y) = E(Y^2) - E(Y)^2 = \{ Y = Y^2 \} = E(Y) - E(Y)^2$$

$$E(Y) = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{30} + \frac{1}{30} + \frac{1}{15} + \frac{1}{15} + \frac{2}{15} =$$

$$= \frac{1}{30}(3+3+3+3+1+1+2+2+4) = \frac{22}{30} = \frac{11}{15}$$

$$D^2(Y) = \frac{11}{15} - \left(\frac{11}{15}\right)^2 = \frac{11\cdot 15 - 11\cdot 11}{(15)^2} = 11\cdot\frac{4}{225} = \frac{44}{225}$$