Equivariant cyclic homology and quantum groups

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Bornological vector spaces

A **bornology** on a (complex) vector space V is a collection $\mathfrak{S}(V)$ of subsets of V satisfying some conditions. A linear map $f : V \to W$ between bornological vector spaces is bounded if $f(S) \in \mathfrak{S}(W)$ for all $S \in \mathfrak{S}(V)$.

An important example of a bornology is given by the **bounded subsets in a locally convex vector space** V.

The category of (complete) bornological vector spaces and bounded linear maps is equipped with a natural tensor product $\hat{\otimes}$ such that

$$\mathrm{Hom}(U \hat{\otimes} V, W) \cong \mathrm{Hom}(U, \mathrm{Hom}(V, W))$$

holds in general.

Throughout we work in the **category of complete bornological vector spaces**.

Noncommutative equivariant differential forms

Let G be a finite group. A **G-algebra** is a bornological algebra with an action of G by bounded algebra automorphisms.

Let A be a G-algebra. We define the **equivariant differential forms** over A by

$$\Omega^n_G(A) = \mathcal{O}_G \hat{\otimes} \Omega^n(A)$$

where $\mathcal{O}_G$ is the algebra of functions on G and

$$\Omega^n(A) = A^+ \hat{\otimes} A^{\hat{\otimes} n} \cong A^{\hat{\otimes} n+1} \oplus A^{\hat{\otimes} n}$$

for $n \geq 1$ and $\Omega^0(A) = A$. Here A^+ is the unitarization of A.

The equivariant Hochschild operator $b : \Omega^n_G(A) \to \Omega^{n-1}_G(A)$ is defined by

$$b(f(s) \otimes x_0 dx_1 \cdots dx_n) = \sum_{j=0}^{n-1} (-1)^j f(s) \otimes x_0 dx_1 \cdots d(x_j x_{j+1}) \cdots dx_n$$
$$+ (-1)^n f(s) \otimes (s^{-1} \cdot x_n) x_0 dx_1 \cdots dx_{n-1}.$$

The operator $B : \Omega_G^n(A) \longrightarrow \Omega_G^{n+1}(A)$ is given by

$$B(f(s) \otimes x_0 dx_1 \cdots dx_n) =$$
$$\sum_{i=0}^{n} (-1)^{ni} f(s) \otimes s^{-1} \cdot (dx_{n+1-i} \cdots dx_n) dx_0 \cdots dx_{n-i}.$$

Moreover we define the operator $T : \Omega_G^n(A) \longrightarrow \Omega_G^n(A)$ by

$$T(f(s) \otimes \omega) = f(s) \otimes s^{-1} \cdot \omega.$$

1 Lemma *We have the relations*

$$b^2 = 0, \qquad B^2 = 0$$

and

$$bB + Bb = 1 - T.$$

The operator $1 - T$ is in general NOT zero. Hence equivariant differential forms do not satisfy the axioms of a mixed complex.

Covariant modules

Let again A be a G-algebra. There is a G-action on $\Omega^n_G(A)$ defined by

$$t \cdot (f(s) \otimes \omega) = f(t^{-1}st) \otimes t \cdot \omega$$

and an $\mathcal{O}_G$-module structure given by

$$g \cdot (f \otimes \omega) = gf \otimes \omega.$$

These actions combine to give $\Omega^n_G(A)$ the structure of a covariant module in the following sense.

2 Definition *A G-covariant module is a vector space M which is both an $\mathcal{O}_G$-module and a G-module such that*

$$t \cdot (f \cdot m) = (t \cdot f) \cdot (t \cdot m)$$

for all $t \in G, f \in \mathcal{O}_G$ and $m \in M$. Here $(t \cdot f)(s) = f(t^{-1}st)$ for $f \in \mathcal{O}_G$.

A linear map $\phi : M \to N$ between covariant modules is called covariant if it is $\mathcal{O}_G$-linear and equivariant. We write $\mathfrak{Hom}_G(M, N)$ for the space of covariant maps.

The operators b and B on $\Omega_G(A)$ are covariant.

Equivariant cyclic homology

Let A be a G-algebra. We equip $\Omega_G(A)$ with the usual even/odd-grading and the differential $\partial = b + B$.

3 Definition *The bivariant equivariant periodic cyclic homology of the G-algebras A and B is*

$$HP^G_*(A, B) = H_*\mathfrak{Hom}_G(\theta\Omega_G(A)), \theta\Omega_G(B)))).$$

Here the differential in $\mathfrak{Hom}_G(\theta\Omega_G(A)), \theta\Omega_G(B)))$ is defined by the usual formula

$$\partial(\phi) = \phi\partial_A - (-1)^{|\phi|}\partial_B\phi$$

for a homogenous element ϕ.

We have to check that this definition makes sense since

$$\partial^2 = (b + B)^2 = Bb + bB = 1 - T \neq 0$$

on the individual complexes!

4 Lemma *For every covariant map*

$$\phi \in \mathfrak{Hom}_G(\theta\Omega_G(A)), \theta\Omega_G(B)))$$

we have $T\phi = \phi T$.

Since $\partial_A^2 = 1 - T$ and similarly $\partial_B^2 = 1 - T$ we compute

$$\begin{aligned} \partial^2(\phi) &= \phi\,\partial_A^2 + (-1)^{|\phi|}(-1)^{|\phi|-1}\partial_B^2\,\phi \\ &= \phi(1-T) - (1-T)\phi = T\phi - \phi T = 0 \end{aligned}$$

Hence the Hom-complex is indeed a complex.

Group actions on manifolds

Let G be a **finite group** acting on a compact smooth manifold X. The **Brylinski space** associated to this action is

$$\hat{X} = \{(s, x) \in G \times X |\; s \cdot x = x\}$$

equipped with the G-action

$$t \cdot (s, x) = (tst^{-1}, t \cdot x).$$

The manifold $\hat{X}$ is the **disjoint union of the fixed point sets** X^s **for elements** $s \in G$.

The space of (ordinary) differential forms $\mathcal{A}(\hat{X})$ on $\hat{X}$ is a covariant module by considering the induced G-action and the $\mathcal{O}_G$-module structure

$$(f \cdot \sigma)(s, x) = f(s)\sigma(s, x)$$

for $\sigma \in \mathcal{A}(\hat{X})$. The de Rham cohomology of $\hat{X}$ becomes a covariant module as well.

5 Theorem *Let G be a finite group acting on compact smooth manifolds X and Y. Then there is a natural isomorphism*

$$HP^G_*(C^\infty(X), C^\infty(Y)) = \mathfrak{Hom}_G(H^*_{dR}(\hat{X}), H^*_{dR}(\hat{Y})).$$

The equivariant Chern character

6 Proposition *Let the finite group G act on compact smooth manifolds X and Y. Then there is a natural isomorphism*

$KK^G_*(C(X), C(Y)) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathfrak{Hom}_G(K^*(\hat{X}) \otimes_{\mathbb{Z}} \mathbb{C}, K^*(\hat{Y}) \otimes_{\mathbb{Z}} \mathbb{C}).$

The **equivariant Chern character**

$$\mathrm{ch}^G_* : KK^G_*(C(X), C(Y)) \to HP^G_*(C^\infty(X), C^\infty(Y))$$

is obtained by applying the classical Chern character

$$\mathrm{ch}_* : K^*(\hat{X}) \to H^*_{dR}(\hat{X})$$

and similarly for $\hat{Y}$.

Bornological quantum groups

Bornological quantum groups are a **generalization of algebraic quantum groups** in the sense of van Daele. They are obtained by **allowing for nontrivial bornologies**.

A basic example is **the algebra $C_c^\infty(G)$ of smooth functions with compact support on a Lie group G.**

In this case the **comultiplication**

$$\Delta : C_c^\infty(G) \to M(C_c^\infty(G) \hat{\otimes} C_c^\infty(G)) = C^\infty(G \times G)$$

is given by $\Delta(f)(r, s) = f(rs)$.

The **counit**

$$\epsilon : C_c^\infty(G) \to \mathbb{C}$$

is given by $\epsilon(f) = f(e)$ where e is the unit element of G.

The **antipode**

$$S : C_c^\infty(G) \to C_c^\infty(G)$$

is obtained by setting $S(f)(t) = f(t^{-1})$.

A left invariant **integral** $\phi : C_c^\infty(G) \to \mathbb{C}$ is given by integration

$$\phi(f) = \int_G f(t)dt$$

with respect to a left Haar measure.

A second example is the dual quantum group to $C_c^\infty(G)$. Explicitly, it can be described as the **smooth group algebra** $\mathcal{D}(G)$ of G. We let $\mathcal{D}(G) = C_c^\infty(G)$ be equipped with the convolution product

$$(f * g)(t) = \int_G f(s)g(s^{-1}t)ds.$$

The **comultiplication**

$$\Delta : \mathcal{D}(G) \to M(\mathcal{D}(G)\hat{\otimes}\mathcal{D}(G)) = \mathcal{E}'(G \times G)$$

is given by $\Delta(f)(h) = \int_G f(r)h(r,r)dr$.

The **counit**

$$\epsilon : \mathcal{D}(G) \to \mathbb{C}$$

is given by $\epsilon(f) = \int_G f(s)ds$.

The **antipode**

$$S : \mathcal{D}(G) \to \mathcal{D}(G)$$

is obtained by setting $S(f)(t) = \delta(t)f(t^{-1})$ where δ is the modular function for G.

A (left) invariant **integral** $\phi : \mathcal{D}(G) \to \mathbb{C}$ is given by evaluation

$$\phi(f) = f(e)$$

at the identity.

Equivariant differential forms

Let H be a bornological quantum group and let $\hat{H}$ be the dual quantum group.

A **smooth H-module** is an H-module V such that the module action $H \hat{\otimes} V \to V$ is a bornological quotient map.

An **H-algebra** is a smooth H-module equipped an algebra structure $A \hat{\otimes} A \to A$ which is equivariant.

If A is an H-algebra we obtain a left action of H on **the space $\Omega_H(A) = H \hat{\otimes} \Omega^n(A)$ of equivariant differential forms** by setting

$$r \cdot (t \otimes \omega) = r_{(3)} t S(r_{(1)}) \otimes r_{(2)} \cdot \omega$$

for $r, t \in H$ and $\omega \in \Omega^n(A)$. Moreover there is a left action of $\hat{H}$ on this space given by

$$f \cdot (t \otimes \omega) = f(t_{(2)}) t_{(1)} \otimes \omega.$$

The equivariant Hochschild operator $b : \Omega_H^n(A) \to \Omega_H^{n-1}(A)$ is defined by

$$b(t \otimes x_0 dx_1 \cdots dx_n) = \sum_{j=0}^{n-1} (-1)^j t \otimes x_0 dx_1 \cdots d(x_j x_{j+1}) \cdots dx_n$$
$$+ (-1)^n t_{(2)} \otimes (S^{-1}(t_{(1)}) \cdot x_n) x_0 dx_1 \cdots dx_{n-1}.$$

Next we define $B : \Omega_H^n(A) \to \Omega_H^{n-1}(A)$ by

$$B(t \otimes x_0 dx_1 \cdots dx_n) =$$
$$\sum_{i=0}^{n} (-1)^{ni} t_{(2)} \otimes S^{-1}(t_{(1)}) \cdot (dx_{n+1-i} \cdots dx_n) dx_0 \cdots dx_{n-i}$$

Furthermore we define the operator $T : \Omega_H^n(A) \to \Omega_H^n(A)$ by

$$T(t \otimes \omega) = t_{(2)} \otimes S^{-1}(t_{(1)}) \cdot \omega = S^{-1}(t_{(2)}) \cdot (t_{(1)} \otimes \omega).$$

One checks that b, B and T are compatible with the actions of H and $\hat{H}$.

7 Lemma *We have the relations*

$$b^2 = 0, \qquad B^2 = 0$$

and

$$bB + Bb = 1 - T$$

on $\Omega_H(A)$.

Covariant modules and Yetter-Drinfeld modules

The actions of H and $\hat{H}$ on $\Omega^n_H(A)$ combine to define the structure of a **covariant module** in the following sense.

8 Definition *Let H be a bornological quantum group. An H-covariant module is a smooth H-module which is also a smooth $\hat{H}$-module such that*

$$t \cdot (f \cdot m) = f_{(1)}(S^{-1}(t_{(3)}))f_{(3)}(S^2(t_{(1)}))\, f_{(2)} \cdot (t_{(2)} \cdot m)$$

for all $t \in H, f \in \hat{H}$ and $m \in M$.

Using the regular actions of H on $\hat{H}$ given by

$$t \rightharpoonup f = f_{(1)}\, f_{(2)}(t), \qquad f \leftharpoonup t = f_{(1)}(t) f_{(2)}$$

one may rewrite the above condition as

$$t \cdot (f \cdot m) = (S^2(t_{(1)}) \rightharpoonup f \leftharpoonup S^{-1}(t_{(3)})) \cdot (t_{(2)} \cdot m).$$

The notion of covariant modules is related to the definition of **Yetter-Drinfeld modules**.

9 Definition *Let H be a bornological quantum group. An H-Yetter-Drinfeld module is a smooth H-module which is also a smooth $\hat{H}$-module such that*

$$t \cdot (f \cdot m) = f_{(1)}(S^{-1}(t_{(3)}))f_{(3)}(t_{(1)})f_{(2)} \cdot (t_{(2)} \cdot m)$$

for all $t \in H, f \in \hat{H}$ and $m \in M$.

This may also be written as

$$t \cdot (f \cdot m) = (t_{(1)} \rightharpoonup f \leftharpoonup S^{-1}(t_{(3)})) \cdot (t_{(2)} \cdot m).$$

Important observation:

Every H-Yetter-Drinfeld module is also an $\hat{H}$-Yetter-Drinfeld module. That is, the compatibility conditions for the actions of H and $\hat{H}$ are the same in both cases. **This is not true for covariant modules.**

Modular pairs

10 Definition *Let H be a bornological quantum group. A* **modular pair** *$\tau = (\sigma, \delta)$ for H consists of a group-like element $\sigma \in M(H)$ and a character $\delta : H \to \mathbb{C}$ satisfying the conditions*

a) $(\phi \hat{\otimes} \mathrm{id})\Delta(f) = \sigma^{-2}\phi(f)$

b) $\phi(fg) = \phi(g\delta \rightharpoonup (\sigma f \sigma^{-1}) \leftharpoonup \delta)$

c) $S^2(f) = \delta^{-1} \rightharpoonup (\sigma f \sigma^{-1}) \leftharpoonup \delta$

d) $\delta(\sigma) = 1$

where ϕ is a left invariant Haar measure on H.

11 Proposition *Let $\tau = (\sigma, \delta)$ be a modular pair for the bornological quantum group H. Then $\hat{\tau} = (\delta, \sigma)$ is a modular pair for the dual quantum group $\hat{H}$.*

Condition a) in the definition says that σ^2 is the **modular function of H**. It follows in particular that σ^2 is **uniquely determined** by H. The same holds true for δ^2.

The quantum group $C_c^\infty(G)$ of **smooth functions on a Lie group** G is equipped with a natural modular pair $\tau = (\delta^{\frac{1}{2}}, \epsilon)$ where $\delta^{\frac{1}{2}} \in M(C_c^\infty(G)) = C^\infty(G)$ is the square root of the modular function of G.

The algebra of regular functions $\mathcal{O}(G)$ on a **compact quantum group** G has a canonical modular pair $(1, \delta)$ where δ is the modular character f_1. Every **discrete quantum group** is equipped with a modular pair as well.

12 Proposition *Let (σ, δ) be a modular pair for H. Then the category of H-covariant modules is naturally isomorphic to the category of H-Yetter-Drinfeld modules.*

Proof. Let M be an H-covariant module. Define new actions of H and $\hat{H}$ on M by

$$t \bullet m = \sigma^{-1}(t_{(1)}) t_{(2)} \cdot m, \qquad f \bullet m = \delta(f_{(2)}) f_{(1)} \cdot m$$

for $t \in \hat{H}$ and $f \in H$. Then M becomes a Yetter-Drinfeld module with these actions.

Equivariant cyclic homology

Let H be a bornological quantum group and let A be an H-algebra. The **crossed product** $A \rtimes H$ is $A \hat{\otimes} H$ with multiplication

$$(a \otimes r)(b \otimes t) = ar_{(1)} \cdot b \otimes r_{(2)} t$$

The algebra $A \rtimes H$ carries an action of $\hat{H}$ given by

$$f \cdot (a \otimes r) = f(r_{(2)})\, a \otimes r_{(1)}.$$

In particular, one may form the crossed product $A \rtimes H \rtimes \hat{H}$ which is again an H-algebra.

13 Definition *The bivariant equivariant periodic cyclic homology of the H-algebras A and B is*

$$HP^H_*(A, B) = H_* \mathfrak{Hom}_H(\theta\Omega_H(A \rtimes H \rtimes \hat{H}), \theta\Omega_H(B \rtimes H \rtimes \hat{H}))).$$

In the case $H = \mathbb{C}G$ for a finite group G one has

$$HP^{\mathbb{C}G}_*(A, B) = HP^G_*(A, B).$$

Baaj-Skandalis duality

14 Theorem *Let A be an H-algebra. There exists a chain map*

$$D_H : \theta\Omega_{\hat{H}}(A \rtimes H) \to \theta\Omega_H(A)$$

which is also a map of Yetter-Drinfeld modules. If A is of the form $A = B \rtimes H \rtimes \hat{H}$ for some H-algebra B this map is a homotopy equivalence.

As a corollary one obtains the following result.

15 Theorem *Let H be a bornological quantum group. Then there exists a natural isomorphism*

$$J_H : HP^H_*(A, B) \to HP^{\hat{H}}_*(A \rtimes H, B \rtimes H)$$

for all H-algebras A and B. This isomorphism is compatible with the composition product.

Proof. There is a natural isomorphism

$$HP^H_*(A, B) \cong HP^H_*(A, B \rtimes H \rtimes \hat{H}).$$

Using this one defines the map J_H by

$$J_H(\phi) = D_H \cdot \phi \cdot D_{\hat{H}}.$$

Some Applications

16 Theorem *Let H be the group algebra of a compact quantum group. Then there is a natural isomorphism*

$$HP^H_*(\mathbb{C}, A) \cong HP_*(A \rtimes H)$$

for all H-algebras A.

17 Theorem *Let H be the group algebra of a discrete quantum group. Then there is a natural isomorphism*

$$HP^H_*(A, \mathbb{C}) \cong HP^*(A \rtimes H)$$

for all H-algebras A.

The assembly map in cyclic homology

Let G be a Lie group with finitely many connected components. Then $\underline{E}G$ can be chosen as G/K where K is a maximal compact subgroup of G.

The Mishenko line bundle is a class $[\mathcal{L}]$ in $K_0(C_c^\infty(G/K) \rtimes G)$. Let $\mathrm{ch}(\mathcal{L})$ in $HP_0(\mathbb{C}, C_c^\infty(G/K))$ be the Chern character of $[\mathcal{L}]$ and define

$$HP_*^{top}(G; B) = HP_*^G(C_c^\infty(G/K), B)$$

For any G-algebra B we define the **assembly map**

$$\mu_B : HP_*^{top}(G; B) \to HP_*(B \rtimes G)$$

by

$$\begin{tikzcd}
HP_*^G(C_c^\infty(G/K), B) \arrow[d, "J_{\mathcal{D}(G)}"] \\
HP_*^{C_c^\infty(G)}(C_c^\infty(G/K) \rtimes G, B \rtimes G) \arrow[d, "forget"] \\
HP_*(C_c^\infty(G/K) \rtimes G, B \rtimes G) \arrow[d, "\mathrm{ch}(\mathcal{L})\cdot -"] \\
HP_*(B \rtimes G)
\end{tikzcd}$$