Transgression of the Chern character for families and cyclic cohomology

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0 Introduction

The aim of this note is to establish a link between the Chern character in K-homology [3] and the version of the standard Chern character and of its transgression developed by D. Quillen [5] and J.-M. Bismut [2]. We show that the Chern character in K-homology, which associates to every finitely summable Fredholm module over an involutive algebra a cyclic cohomology class, is equal via the Loday-Quillen isomorphism to the transgressed Chern character of the family of conjugates of the Fredholm operator by the unitary group of the algebra.

Here is a brief look at the content. Section 1 recalls the definition of the Chern character of a finitely summable unbounded Fredholm module. Section 2 introduces the appropriate space of vector potentials, acted upon by the unitary groups as a gauge transformation group. In Section 3 we use the superconnection formalism to produce a transgressed Chern character for the family of translates of the given invertible unbounded Fredholm operator. The Loday-Quillen isomorphism, whose cohomological version is recalled in Section 4, provides the means for converting the restriction to unitary orbits of the transgressed Chern character into a cyclic cocycle. This is explained in Section 5 where we also identify the cyclic cohomology class of this cocycle with the Chren character of the given Fredholm module.

1 Chern character for unbounded p-summable Fredholm modules

In this section we recall some basic notions concerning the Chern character in K-homology [3]. Given a Hilbert space h we let $L(h)$ denote the involutive algebra of bounded operators in h and, for $p \in [1, \infty[$, we let $L^p(h)$ be the two-sided ideal of p-summable compact operators on h.

Let A be a unital involutive algebra over $\mathbb{C}$ and let $p \in [1, \infty[$. An *odd* p-summable unbounded Fredholm module over A is given by:

(i) a Hilbert space h together with an involutive homomorphism of A in $L(h)$;

(ii) a selfadjoint invertible unbounded operator D on h such that $D^{-1} \in L^p(h)$ and $[D, a] \in L(h)$ for any $a \in A$.

An even p-summable unbounded Fredholm module over A is given by a pair (h, D) satisfying (i) and (ii) together with a $\mathbb{Z}/2$ grading of h, *i.e.* a selfadjoint involution $\varepsilon \in L(h)$, such that:

(iii) $\varepsilon\, a = a\, \varepsilon$, $\forall\, a \in A$ and $\varepsilon\, D = -D\, \varepsilon$.

Departing slightly from the conventions adopted in [3], we shall omit the powers of $2\pi i$ in the definition of the Chern character, thus using the following definitions.

For any odd integer $n > p-1$, the n-dimensional character of an odd p-summable unbounded Fredholm module (h, D) over A is the n-cyclic cocycle on A given by the equality:

$$\tau_n(a^0, a^1, \ldots, a^n) = (-1)^{\frac{n-1}{2}} \, \frac{n}{2}\left(\frac{n}{2}-1\right)\ldots\frac{1}{2}\,\mathrm{Trace}\left(P[P,a^0][P,a^1]\ldots[P,a^n]\right),$$

where

$$P = D\,|D|^{-1} = \frac{2}{\pi}\int_0^\infty D(D^2+\mu^2)^{-1}\,d\mu$$

is the phase of D. For any even integer $n > p-1$, the n-dimensional character of an even (h, D) is the n-cyclic cocycle on A given by:

$$\tau_n(a^0, a^1, \ldots, a^n) = \left(\frac{n}{2}\right)!\;\frac{1}{2}\,\mathrm{Trace}\left(\varepsilon P[P,a^0][P,a^1]\ldots[P,a^n]\right).$$

One has the equality $S\tau_n = \tau_{n+2}$, in the cyclic cohomology group $H_\lambda^{n+2}(A)$, for any $n > p-1$. (*Cf.* [3] , Ch. 1.)

For $q \in \mathbb{N}$, let $h_q = h \otimes \mathbb{C}^q$, $D_q = D \otimes 1$. The pair (h_q, D_q) in the odd case, resp. $(h_q, D_q, \varepsilon_q = \varepsilon \otimes 1)$ in the even case, yields a p-summable Fredholm module over the algebra $M_q(A) = A \otimes M_q(\mathbb{C})$, whose characters τ_n^q are equal to $\tau_n \,\#\, \mathrm{Trace}$, with the obvious notation [3].

Once (h, D) is fixed, we shall regard A as an involutive normed algebra, with the norm

$$\|a\| = \|a\|_\infty + \|[D,a]\|_\infty \quad (= \text{norm of } \begin{bmatrix} a & [D,a] \\ 0 & a \end{bmatrix}),$$

where $\| \;\|_\infty$ denotes the operator norm.

We conclude by mentioning an important example of such a Fredholm module. Let M be a closed spin manifold, S the bundle of spinors over M and D the Dirac operator on $h = L^2(M, S)$. By forming the Dirac Hamiltonian D_m with mass $m \neq 0$ (see [3], Ch. 1, §6) one obtains an unbounded Fredholm module (h_m, D_m) over $A = C^\infty(M)$ which is p-summable for each $p > \dim M$.

2 Vector potentials and the action of the gauge group

Let (h, D) be a fixed p-summable unbounded Fredholm module over A. We define a subspace V of $L(h)$ by setting:

$$V = \{A \in L(h)\,;\; A = A^*\,,\; A = \sum_{i=1}^r a_i[D, b_i]\,,\; a_i, b_i \in A\}\,.$$

If $c = \sum_{i=1}^r a_i \otimes b_i \in A \otimes A$ satisfies $\sum_{i=1}^r a_i\, b_i = 0$ and $\sum_{i=1}^r a_i \otimes b_i = \sum_{i=1}^r b_i^* \otimes a_i^*$, then $\rho(c) = \sum_{i=1}^r a_i[D, b_i] = \sum_{i=1}^r a_i\, Db_i$ belongs to V. Conversely, every element of V is of the form $\rho(c)$ with c as above.

Let U be the unitary group of A, *i.e.* $U = \{u \in A\,;\; u^*u = uu^* = 1\}$. We let U act on V as follows:

$$\gamma_u(A) = \rho(u \otimes u^* - 1 \otimes 1) + uAu^* = u\,[D, u^*] + uAu^*\,,$$

for $u \in U$ and $A \in V$. We also let $D_A = D + A$, $\forall\, A \in V$.

Lemma 1.

(i) *For any $A \in V$ the operator D_A is selfadjoint and has finite dimensional kernel.*

(ii) *If D_A is injective then it is invertible with $D_A^{-1} \in L^p(h)$.*

(iii) *γ is an affine action of U on V and $D_{\gamma_u(A)} = u\, D_A\, u^*$, for any $u \in U$ and $A \in V$.*

Proof.

(i) D_A has the same domain as D and it is selfadjoint (*cf.* [6]). One has $\mathrm{Ker}\, D_A = \mathrm{Ker}\,(1 + D^{-1}A)$; the latter is finite dimensional since $D^{-1}A$ is a compact operator.

(ii) Assume D_A injective. Then $\mathrm{Ker}\,(1{+}D^{-1}A) = \mathrm{Ker}\,(D^{-1}D_A) = 0$ and $\mathrm{Ker}\,(1{+}AD^{-1}) = \mathrm{Ker}\,(D_A\, D^{-1}) = 0$. Thus $1 + D^{-1}A$ is a Fredholm operator which is injective and has dense range, since $(1+D^{-1}A)^* = 1 + AD^{-1}$. Therefore $1 + D^{-1}A$ is invertible, hence $D_A^{-1} = (1 + D^{-1}A)^{-1}D^{-1}$ belongs to $L^p(h)$.

(iii) One has $D + \gamma_u(A) = D + u\,[D, u^*] + uAu^* = u\,(D + A)\,u^*$. □

We let $V^{-1} = \{A \in V\,;\ D_A \text{ is injective}\}$. It is an open subset of V, once we endow $V \subset L(h)$ with the norm topology. In fact, $A \in V^{-1}$ iff $\det_n(1 + D^{-1}A) \neq 0$, where n is an integer larger than p and $\det_n$ is the modified determinant (see [7]).

3 Transgression of the Chern character

In this section we shall exploit the formalism of superconnections and the corresponding version of Chern character and its transgression developed in [5] and [2]. We let A and (h, D) be as in the previous section and assume that we are in the even case, *i.e.* a $\mathbb{Z}/2$ grading ε of h, satisfying (iii) of Section 1, is also given. We shall view V as an infinite dimensional affine manifold modeled on the normed vector space V.

Let $L_a^r(V, \mathbb{C})$ denote the Banach space of norm continuous r-multilinear alternating maps from the normed vector space V to $\mathbb{C}$. We then let $\Omega^r(V)$ be the space of smooth maps of V in $L_a^r(V, \mathbb{C})$ and $(\Omega(V), d)$ be the de Rham algebra of V. We endow the trivial bundle $H = h \times V$ over V with the trivial $\mathbb{Z}_2$ graded connection ∇, given for any $X \in T_A(V) \cong V$ by the equality:

$$(\nabla_X\, \xi)(A) = \varepsilon(\mathbf{X}\,\xi)(a)\,, \qquad \forall\, \xi \in \Gamma(H) \cong C^\infty(V, h)\,,$$

where $\mathbf{X}$ is the translation invariant vector field determined by $X \in V$, *i.e.*

$$(\mathbf{X}\,\xi)(A) = \left.\frac{d}{dt}\right|_{t=0} \xi(A + tX)\,, \qquad \forall\, \xi \in \Gamma(H)\,.$$

For each integer $r \geq 0$, we let $L_a^r(V, h)$ denote the Banach space of norm continuous r-multilinear alternating maps from the normed vector space V to h. We then let $\Omega^r(V, H)$ be the space of smooth maps from V to $L_a^r(V, h)$. $\Omega(V, H) = \underset{r \geq 0}{\oplus}\, \Omega^r(V, H)$ will be viewed as a right module over the graded algebra $\Omega(V)$. We extend ∇ to $\Omega(V, H)$ by the formula

$$(\nabla\omega)(\mathbf{X}^0, \ldots, \mathbf{X}^r) = \varepsilon \sum_{i=0}^{r} (-1)^i\, \mathbf{X}^i\, \omega(\mathbf{X}^0, \ldots, \hat{\mathbf{X}}^i, \ldots, \mathbf{X}^r)\,, \tag{1}$$

where $\omega \in \Omega^r(V, H)$ and $X^0, \ldots, X^r \in V$.

Consider now the (unbounded) bundle endomorphism D of H given at each point $A \in V$ by the unbounded selfadjoint operator $D_A = D + A$, acting on the fiber $H_A = h$, and extend it in the obvious way to $\Omega(V, H)$.

Lemma 2. *The covariant derivative $D' = \nabla D + D\nabla$ of D is the endomorphism of $\Omega(V,H)$ given by the formula:*

$$(D'\omega)_A(X^0,\ldots,X^r) = \varepsilon \sum_{i=0}^{r} (-1)^i \, X^i \, \omega_A(X^0,\ldots,\hat{X}^i,\ldots,X^r)\,,$$

for $\omega \in \Omega^r(V,H)$, where each $X^j \in T_A(V) \cong V$ is viewed as an element of $L(h)$.

Proof. Let $X \in V = T_A(V)$ and $\omega \in \Omega^r(V,H)$. Then

$$\begin{aligned} X D\,\omega(\mathbf{Y}^1,\ldots,\mathbf{Y}^r) &= \left(\frac{d}{dt} D_{A+tX}\right)\Big|_{t=0} \omega_A(Y^1,\ldots,Y^r) + D_A\, X\omega(\mathbf{Y}^1,\ldots,\mathbf{Y}^r) \\ &= X\,\omega_A(Y^1,\ldots,Y^r) + D_A\, X\omega(\mathbf{Y}^1,\ldots,\mathbf{Y}^r)\,, \end{aligned}$$

i.e.

$$\varepsilon X D\,\omega(\mathbf{Y}^1,\ldots,\mathbf{Y}^r) + D_A\,\varepsilon X\omega(\mathbf{Y}^1,\ldots,\mathbf{Y}^r) = X\omega_A(Y^1,\ldots,Y^r)\,.$$

The statement now follows from formula (1).

Since $\nabla^2 = 0$, one has $(\nabla + zD)^2 = zD' + z^2D^2$, for any $z \in \mathbb{C}$. As in [5], $\exp -(\nabla + zD)^2$ is an endomorphism of the right module $\Omega(V,H)$ over the $\mathbb{Z}/2$ graded algebra $\Omega(V)$. From the expansional formula (*cf.* [1]) it follows that its component of degree n is:

$$\begin{aligned} \Omega_z^{(n)} &= z^n \int_{0\le s_1\le\ldots\le s_n\le 1} \exp(-s_1\, z^2\, D^2)D' \, \exp((s_1 - s_2)\, z^2\, D^2)D' \\ &\qquad \ldots \exp((s_{n-1} - s_n)\, z^2\, D^2)D' \, \exp((s_n - 1)\, z^2\, D^2)\, ds_1 \ldots ds_n\,, \end{aligned} \tag{2}$$

which is bounded for $\mathrm{Re}(z^2) > 0$.

The Hölder inequality [7] shows that if $0 \le s_1 \le \ldots \le s_n \le 1$ and $T_1,\ldots,T_n \in L(h)$, then:

$$\|\exp(-s_1\, z^2\, D_A^2)\, T_1 \exp((s_1 - s_2)\, z^2\, D_A^2)\, T_2 \ldots \exp((s_n - 1)\, z^2\, D_A^2)\|_1 \le \|\exp(-z^2\, D_A^2)\|_1\, \|T_1\| \ldots \|T_n\|\,, \tag{3}$$

where $A \in V$ and $\|\ \|_1$ denotes the L^1 norm. When $\mathrm{Re}(z^2) > 0$, $\exp(-z^2\, D_A^2) \in L^1(h)$ so that the right hand side is finite. By applying Lemma 2 and integrating over the standard n-simplex, one obtains for $X^1,\ldots,X^n \in V$ and $\mathrm{Re}(z^2) > 0$:

$$\|\Omega_{z,A}^{(n)}\,(X^1,\ldots,X^n)\|_1 \le |z|^n\, \|\exp(-z^2\, D_A^2)\|_1\, \|X^1\| \ldots \|X^n\|\,, \tag{4}$$

where $\Omega_{z,A}^{(n)}$ is the restriction of $\Omega_z^{(n)}$ to the fiber over $A \in V$.

Let now $\omega_z = \mathrm{Trace}\,(\varepsilon \exp -(\nabla + zD)^2) \in \Omega(V)$, *i.e.*

$$\omega_{z,A}^{(n)}\,(X^1,\ldots,X^n) = \mathrm{Trace}\,(\varepsilon\, \Omega_{z,A}^{(n)}(X^1,\ldots,X^n))\,, \quad \forall\, A \in V\,,\ X^1,\ldots,X^n \in V\,. \tag{5}$$

Proposition 1.

(a) *For any $z \in \mathbb{C}$ with $\mathrm{Re}(z^2) > 0$, ω_z is a closed U-invariant differential form on V, whose odd degree components are equal to 0.*

(b) *If $\mathrm{Re}(z^2) \to \infty$, then $\omega_{z,A}^{(n)} \to 0$ pointwise, for each $A \in V^{-1}$.*

(c) *If $n \ge p$ and $0 < \mathrm{Re}(z^2) \to 0$, then $\Omega_{z,A}^{(n)} \to 0$ pointwise, for each $A \in V$.*

Proof.

(a) The fact that Ω_z is closed and all its odd degree components vanish follows as in [5]. The U-invariance of ω_z follows from Lemma 1 (iii).

(b) When $\mathrm{Re}(z^2) \to \infty$, the L^1 norm of $z^n \exp -(z^2 D_A)^2$ tends to 0 if D_A is invertible. Thus, the inequality (4) implies that $\omega_{z,A}^{(n)}(X^1, \ldots, X^n) \to 0$.

(c) Again by (4), it is enough to prove that

$$t^n \,\| \exp(-t^2 D_A^2)\|_1 \to 0 \quad \text{when} \quad 0 < t \to 0\,.$$

By Lemma 1 (i), (ii), we can assume that $T = D_A^2$ is invertible and that $T^{-1} \in L^{p/2}(h)$. Let then μ be the spectral measure of T, *i.e.* $\int f d\mu = \mathrm{Trace}\,(f(T))$ for any positive function f. Thus $\int \lambda^{-p/2}\, d\mu(\lambda) < \infty$. One has

$$t^n \,\| \exp(-t^2\, T)\|_1 = t^n \int \exp(-t^2\,\lambda)\, d\mu(\lambda) = \int t^n\, \lambda^{p/2} \exp(-t^2\,\lambda)(\lambda^{-p/2}\, d\mu(\lambda))\,.$$

The result now follows from the dominated convergence theorem, since

$$t^n\, \lambda^{p/2} \exp(-t^2\,\lambda) \leq C_p\, t^{n-p}\,, \quad \text{with} \quad C_p = \left(\frac{p}{2e}\right)^{p/2}\,.$$

□

It is clear that $\omega_{z,A}(X^1, \ldots, X^n)$ is an analytic function of z for $\mathrm{Re}(z^2) > 0$; therefore $\partial_z\, \omega_{z,A}(X^1, \ldots, X^n)$ makes sense.

Lemma 3. *Let $A \in V^{-1}$. Then for any $s \in [0,1]$ one has:*

$$s\, \|D_A \exp(-st^2\, D_A^2)\|_{\frac{1}{s}} \leq C_p\, t^{-ps-1} (\mathrm{Trace}\, |D_A|^{-p})^s\,,$$

where $C_p = \sup\limits_{s\in[0,1]} \left(s\left(\frac{1}{2e}\left(p + \frac{1}{s}\right)\right)^{\frac{1}{2}+\frac{ps}{2}} \right)$.

Proof. As before, let μ be the spectral measure of D_A^2. One has

$$\|D_A \exp(-st^2\, D_A^2)\|_{1/s} = \left(\int \lambda^{\frac{1}{2s}} \exp(-\lambda\, t^2)\, d\mu(\lambda) \right)^s\,.$$

Since $\int \lambda^{\frac{1}{2s}} \exp(-\lambda\, t^2)\, d\mu(\lambda) = \int \lambda^{\frac{1}{2s}}\, \lambda^{\frac{p}{2}} \exp(-\lambda\, t^2)\, \lambda^{-\frac{p}{2}}\, d\mu(\lambda)$ and $\lambda^{\frac{1}{2s}}\, \lambda^{\frac{p}{2}} \exp(-\lambda\, t^2) \leq \left(\frac{1}{et^2}\left(\frac{1}{2s} + \frac{p}{2}\right)\right)^{\frac{1}{2s}+\frac{p}{2}}$ the conclusion follows from the fact that $\sup\limits_{s\in[0,1]} \left(s\left(\frac{1}{2e}\left(p + \frac{1}{s}\right)\right)^{\frac{1}{2}+\frac{ps}{2}} \right) < \infty$. □

Now, as in [5], one has:

$$\partial_t\, \omega_t = d\theta_t \tag{6}$$

where θ_t is the coefficient of dt in the following differential form on $V \times\,]0, \infty[$:

$$\begin{aligned}
&\mathrm{Trace}\,(\varepsilon \exp -(\varepsilon\, dt\, D + tD' + t^2\, D^2)) = \\
&\sum_{n=1}^{\infty} t^{n-1} \int_{0\leq s_1 \leq \ldots \leq s_{n-1} \leq 1} \mathrm{Trace}\,(\varepsilon \exp(-s_1(\varepsilon\, dt\, D + t^2\, D^2))\, D' \ldots \\
&\exp((s_{n-2} - s_{n-1})(\varepsilon\, dt\, D + t^2\, D^2))\, D' \exp(s_{n-1} - 1)(\varepsilon\, dt\, D + t^2\, D^2))\, ds_1 \ldots ds_{n-1}\,.
\end{aligned}$$

The finiteness of the latter follows from Lemma 3, by noting that

$$\exp(-s(\varepsilon\, dt\, D + t^2\, D^2)) = \exp(-st^2\, D^2)(1 - s\,\varepsilon\, dt\, D)$$

and applying the Hölder inequality. In fact, applying Lemma 2 and the Hölder inequality one obtains the estimate:

$$\|\theta_{t,A}^{(n-1)}\| = \sup_{\|X^j\|\le 1} |\theta_{t,A}(X^1,\ldots,X^{n-1})| \le n t^{n-1} \sup_{s\in[0,1]} (s\, \|D_A \exp(-st^2\, D_A^2)\|_{\frac{1}{s}}\, \|\exp(-t^2\, D_A^2)\|_1^{1-s})\,, \tag{7}$$

which together with Lemma 3 implies:

$$\|\theta_{t,A}^{(n-1)}\| \le n C_p\, t^{n-p-2} \sup_{s\in[0,1]} ((\mathrm{Trace}\, |D_A|^{-p})^s\, \|t^p \exp(-t^2\, D_A^2)\|_1^{1-s})\,.$$

Since, as shown in the proof of Proposition 1 (c), $\|t^p \exp(-t^2\, D_A^2)\|_1 \to 0$ when $t \to 0$, it follows that:

$$\|\theta_{t,A}^{(n-1)}\| = o\,(t^{-n+p+2})\,, \quad \text{as} \quad 0 < t \to 0\,, \quad \text{for each} \quad A \in V^{-1}\,. \tag{8}$$

Theorem 1. Let $m > p$ be an odd integer. Then, for each $A \in V^{-1}$, the integral

$$\theta_A^{(m)} = \int_0^\infty \theta_{t,A}\, dt \tag{9}$$

converges and defines a closed U-invariant differential form of degree m on V^{-1}. Furthermore, one has

$$\begin{aligned}\theta_A^{(m)}(X^1,\ldots,X^m) \;&=\; -\Gamma\left(\frac{m+1}{2}\right)^{-1} \sum_\sigma \mathrm{sgn}(\sigma) \int_0^\infty \mathrm{Trace}\,(\varepsilon\, D_A (D_A^2+\mu^2)^{-1}\, X^{\sigma(1)}\, (D_A^2+\mu^2)^{-1} \ldots \\ &\quad \ldots X^{\sigma(m)} (D_A^2+\mu^2)^{-1})\, \mu^m\, d\mu\end{aligned} \tag{10}$$

where $X^1,\ldots,X^m \in V$, σ runs over all permutations of $\{1,\ldots,m\}$ and $\mathrm{sgn}(\sigma) =$ signature of σ.

Proof. The estimate (8) ensures the convergence of the integral (9) on $[0,1]$. To prove convergence on $[1,\infty[$, we note that if $s \in \left[0,\frac{1}{2}\right]$, then $s\,\|D_A \exp(-t^2\, D_A^2)\|_{\frac{1}{s}}$ stays bounded (*cf.* Lemma 3) while $\|\exp -t^2\, D_A^2\|_1^{1-s}$ decreases exponentially with $t \to \infty$, since D_A is invertible. On the other hand, if $s \in \left[\frac{1}{2},1\right]$, then $\|D_A \exp(-st^2\, D_A^2)\|_{\frac{1}{s}}$ decreases exponentially with $t \to \infty$, uniformly in s. Thus, the inequality (7) implies that $\|\theta_{t,A}^{(m)}\|$ decreases exponentially as $t \to \infty$, which ensures the convergence of (9) on $[0,\infty[$, as well as the finiteness of $\|\theta_A^{(m)}\| = \sup_{\|X^j\|\le 1} |\theta_A^{(m)}(X^1,\ldots,X^m)|$.

The fact that $\theta_A^{(m)}$ is closed follows from the equality (6) and Proposition 1 (b), (c), while the invariance under the action of U is obvious by construction.

Let now $X^1,\ldots,X^m \in V$. According to the definition of $\theta^{(m)}$ one has:

$$\begin{aligned}\theta_A^{(m)}(X^1,\ldots,X^m) \;&=\; \sum_\sigma \mathrm{sgn}(\sigma) \int_0^\infty \int_{0\le s_1\le\ldots\le s_m\le 0} t^m\, \mathrm{Trace}\,(\varepsilon \exp -s_1\, t^2\, D_A^2)(1 - s_1\, dt\, D_A)\, X^{\sigma(s)} \ldots \\ &\quad \exp((s_{m-1}-s_m)\, t^2\, D_A^2)(1 + (s_{m-1}-s_m)\, dt\, D_A)\, X^{\sigma(m)} \\ &\quad \exp((s_{m-1}-1)\, t^2\, D_A^2)(1 + (s_{m-1}-1)\, dt\, D_A)\, ds_1 \ldots ds_m\,.\end{aligned}$$

Performing the change of variables

$$u_1 = s_1\, t^2\,,\ u_2 = (s_2 - s_1)\, t^2, \ldots, u_m = (s_m - s_{m-1})\, t^2\,,\ u_{m+1} = (1 - s_m)\, t^2\,,$$

with

$$ds_1 \dots ds_m \, t^m \, dt = \frac{1}{2} \left(\sum_{i=1}^{m+1} u_i \right)^{-\frac{m+1}{2}} du_1 \dots du_m \, du_{m+1}$$

and using the identity

$$\left(\sum_{i=1}^{m+1} u_i \right)^{-\frac{m+1}{2}} = \Gamma \left(\frac{m+1}{2} \right)^{-1} \int_0^\infty \nu^{\frac{m+1}{2}} \exp \left(-\nu \sum_{i=1}^{m+1} u_i \right) \frac{d\nu}{\nu} \, ,$$

one obtains

$$\begin{aligned} \theta_A^{(m)}(X^1, \dots, X^m) \quad = \quad & -\frac{1}{2} \Gamma \left(\frac{m+1}{2} \right)^{-1} \sum_\sigma \mathrm{sgn}(\sigma) \int_0^\infty \mathrm{Trace} \, (\varepsilon \, D_A (D_A^2 + \nu)^{-1} \, X^{\sigma(1)} \, \nu (D_A^2 + \nu)^{-1} \dots \\ & \dots X^{\sigma(m)} (D_A^2 + \nu)^{-1}) \, \nu^{\frac{m+1}{2}} \, \frac{d\nu}{\nu} \, . \end{aligned}$$

Finally, changing ν to μ^2 gives formula (10). □

The transgressed form $\theta^{(m)}$ on V^{-1} given by Theorem 1 will be denoted $\mathrm{Tch}^{(m)} \, (H, D)$.

Remark 1. Let $q \in \mathbb{N}$, replace A by $M_q(A) = A \otimes M_q(\mathbb{C})$, (h, D) by (h_q, D_q). Then V is replaced by $V_q = M_q(V)$, and for any $A \in V^{-1}$, $X^j \in V$, $a^j \in M_q(\mathbb{C})$ one has:

$$\begin{aligned} & \theta_{A \otimes 1}(X^1 \otimes a^1, \dots, X^m \otimes a^m) = -\Gamma \left(\frac{m+1}{2} \right)^{-1} \sum \mathrm{Sgn}(\sigma) \\ & \left(\int_0^\infty \mathrm{Tr}_s \left(\frac{D_A}{D_A^2 + \mu^2} \, X^{\sigma(1)} \dots \frac{1}{D_A^2 + \mu^2} \, X^{\sigma(m)} \, \frac{1}{D_A^2 + \mu^2} \right) \mu^m \, d\mu \right) \mathrm{Trace} \, (a^{\sigma(1)} \dots a^{\sigma(m)}) \, . \end{aligned}$$

Remark 2. The case of odd Fredholm modules can be reduced to the even case for $\mathbb{Z}/2$ graded algebras as in [3], Section 7, it yields the following formula for the transgressed Chern character:

$$\theta_A(X^1, \dots, X^m) = \frac{1}{\Gamma \left(\frac{m+1}{2} \right)} \sum_\sigma \mathrm{Sgn}(\sigma) \int_0^\infty \mathrm{Tr} \left(\frac{D_A}{D_A^2 + \mu^2} \, X^{\sigma(1)} \dots \frac{1}{D_A^2 + \mu^2} \, X^{\sigma(m)} \, \frac{1}{D_A^2 + \mu^2} \right) \mu^m \, d\mu \, . \tag{11}$$

Remark 3. The Hölder inequality shows that the integrals (10) and (11) make sense even for $m = p$ and also that the norm of θ_A is finite.

Remark 4. The above construction still works under the following weaker hypothesis on (h, D): there exist $p_1, p_2 \geq 1$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ such that: (a) $D^{-1} \in L^{p_1}(h)$; (b) $[D, a] \in L^{p_2}(h)$ for any $a \in L(h)$.

4 The Loday-Quillen isomorphism in cohomology

Let A be an involutive unital Banach algebra over $\mathbb{C}$. Following [4], [8] we shall describe the natural isomorphism between the continuous cyclic cohomology of A and the primitive part of the continuous Lie algebra cohomology of the unitary group of the algebra $M_\infty(A) = \varinjlim M_q(A)$. We let U_∞ be the inductive limit of the unitary groups U_q of $M_q(A)$. It contains as a subgroup the inductive limit U_∞ of the unitary groups of $M_q(\mathbb{C})$. We identify the complexified Lie algebra of U_∞ with $M_\infty(A)$ and endow it with the inductive limit topology. A continuous m-multilinear alternating form α on $M_\infty(A)$ will be called: 1) *invariant* if it is invariant under the adjoint action of U_∞; 2) *primitive* if for any $p, q \in \mathbb{N}$ one has $\Delta^*\alpha = i_1^*\alpha + i_2^*\alpha$ where Δ, i_1 and i_2 are the Lie algebra homomorphisms of $M_p(A) \times M_q(A)$ to $M_{p+q}(A)$ given by: $i_1(a,b) = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, $i_2(a,b) = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$, $\Delta(a,b) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ for $a \in M_p(A)$, $b \in M_q(A)$. We let $(\Lambda^*(A)^U, d)$ be the cochain complex of all such invariant primitive continuous forms. By construction Λ_∞^* is the projective limit of the complexes Λ_q^* of left invariant continuous forms on U_q satisfying 1) and 2).

Let, for each n, $C_\lambda^n(A)$ be the space of continuous cyclic $(n+1)$ linear forms on A and let $b : C_\lambda^n(A) \to C^{n+1}(A)$ be the Hochschild coboundary [3]. The following result follows from [4] (*cf.* also [8]).

Proposition 2. *The map* $\Phi : C_\lambda^* \to \Lambda_\infty^{*+1}$ *given by* $\Phi(\tau)(X^0, \dots, X^m) = (-1)^m \sum \operatorname{Sgn}(\sigma)(\tau \# \operatorname{Tr})$ $(X^{\sigma(0)}, X^{\sigma(1)}, \dots, X^{\sigma(m)})$ *is an isomorphism of complexes.*

Remark 5. Note that (*cf.* [4]) one recovers τ from $\alpha = \Phi(\tau)$ by the equality $\tau(a^0, \dots, a^m) = (-1)^m$ $\alpha(a^0 \otimes e_{01}, a^1 \otimes e_{12}, \dots, a^m \otimes e_{m0})$ where $e_{ij} \in M_\infty(\mathbb{C})$ are the obvious matrix units.

5 Restriction of the transgressed Chern character to unitary orbits

Let A, h, D and ε be as in Section 2. We consider A as a normed $*$ algebra with the norm $\|a\| = \|a\|_\infty + \|[D,a]\|_\infty$, $\forall\, a \in A$. For each $q \in \mathbb{N}$, let $(h_q, D_q, \varepsilon_q)$ be the corresponding Fredholm module over $M_q(A)$ (*cf.* Section 1 and Section 2, Remark 1). Fix $A \in V^{-1}$, and let $A_q = A \otimes 1 \in V_q$. By Remark 1, for $m > p$, the component of degree m of $\operatorname{Tch}(H_q, D_q)$ is a U_q invariant closed differential form on V_q^{-1} and thus its restriction to the U_q orbit of $A_q = A \otimes 1$ yields a left invariant form on U_q. The tangent map at the origin of U_q to the map $u \to \gamma_u(A_q) \in V_q^{-1}$, is given by $X = -X^* \in M_q(A) \to [X, D_{A_q}] \in V_q$. Thus, by Remark 1, the restriction θ_q to the U_q orbit of $A_q = A \otimes 1$, is given by:

$$\theta_q(x^1 \otimes a^1, \dots, x^m \otimes a^m) = -\Gamma\left(\frac{m+1}{2}\right)^{-1} \sum \operatorname{Sgn}(\sigma)$$
$$\left(\int_0^\infty \operatorname{Tr}_s\left(\frac{D_A}{D_A^2+\mu^2}[D_A, x^{\sigma(1)}]\frac{1}{D_A^2+\mu^2}\dots[D_A, x^{\sigma(m)}]\frac{1}{D_A^2+\mu^2}\right)\mu^m\,d\mu\right)\operatorname{Trace}(a^{\sigma(1)}\dots a^{\sigma(m)}).$$

This formula makes it obvious that θ_q satisfies conditions 1 and 2 and that the restriction of $\theta_{q'}$, $q' \geq q$ to $M_q(A)$ is θ_q. Thus the family (θ_q) defines an invariant primitive m-form on U_∞, which will be denoted $\theta_\infty = \operatorname{Tch}(h, D_A)$. Its continuity, *i.e.* the continuity of each θ_q is insured by Remark 3 and the continuity of the map from A to $L(h)$ given by $x \to [D_A, x]$. According to Propositon 2 there exists a unique cyclic cocycle $\tau \in Z_\lambda^{m-1}(A)$ such that $\theta_\infty = \Phi(\tau)$. Moreover, by Remark 5, τ is given by

$$\tau(x^1, \dots, x^m) = \Gamma\left(\frac{m+1}{2}\right)^{-1} \sum_\lambda \int_0^\infty \operatorname{Tr}_s\left(\frac{D_A}{D_A^2+\mu^2}[D_A, x^{\lambda(1)}]\frac{D_A}{D_A^2+\mu^2}\dots[D_A, x^{\lambda(m)}]\frac{1}{D_A^2+\mu^2}\right)\mu^m\,d\mu$$

where λ runs through all cyclic permutation of $\{1, \dots, m\}$. We can summarize the above discussion as follows.

Theorem 2. *Let* (h, D, ε) *be an unbounded even* p*-summable Fredholm module over* A *and* $n > p - 1$ *an even integer.*

1) *The following equality defines an* n*-cyclic cocycle on* A*:*

$$\tau_D(x^0, \dots, x^n) = \Gamma\left(\frac{n}{2}+1\right)^{-1} \sum_\lambda \int_0^\infty \operatorname{Tr}_s\left(\frac{D}{D^2+\mu^2}[D, x^{\lambda(0)}]\frac{1}{D^2+\mu^2}\dots[D, x^{\lambda(n)}]\frac{1}{D^2+\mu^2}\right)\mu^{n+1}\,d\mu$$

where λ *runs through all cyclic permutations of* $\{0, 1, \dots, n\}$*.*

2) *For any* $A \in V^{-1}$ *the restriction* θ_∞ *to the* U_∞ *orbit of* $A \otimes 1$ *in* V_∞ *of the transgressed Chern character* $\mathrm{Tch}\,(H,D)^{(n+1)}$ *is the primitive invariant form* $\Phi(\tau_{D_A})$.

The explicit formula for τ as an integral does not make it obvious that it is a cyclic cocycle; this property follows from Proposition 2. For $n = 0$ one has:

$$\tau(a) = \int_0^\infty \mathrm{Tr}_s \left(\frac{D}{(D^2+\mu^2)^2}\,[D,A] \right) \mu\, d\mu = \mathrm{Tr}_s(D^{-1}\,[D,a])\,.$$

A similar discussion holds in the odd case, yielding:

Theorem 3. *Let* (h,D) *be an unbounded odd* p*-summable Fredholm module over* A *and* $n > p-1$ *an odd integer.*

1) *The following equality defines an* n*-cyclic cocycle on* A*:*

$$\begin{aligned}\tau_D(x^0,\dots,x^n) \;=\;& \Gamma\left(\frac{n}{2}+1\right)^{-1} \sum_\lambda \mathrm{sgn}(\lambda) \int_0^\infty \mathrm{Tr} \\ & \left(\frac{D}{D^2+\mu^2}\,[D,x^{\lambda(0)}]\,\frac{1}{D^2+\mu^2} \cdots [D,x^{\lambda(n)}]\,\frac{1}{D^2+\mu^2} \right) \mu^{n+1}\, d\mu\,,\end{aligned}$$

where λ *runs through all cyclic permutations of* $\{0,1,\dots,n\}$.

2) *For any* $A \in V^{-1}$ *the restriction* θ_∞ *to the* U_∞ *orbit of* $A \otimes 1$ *in* V_∞ *of the transgressed Chern character* $\mathrm{Tch}\,(H,D)^{(n+1)}$ *is the primitive invaraiant form* $\Phi(\tau_{D_A})$.

For $n = 1$ one can show that the cocycle τ_D is given by:

$$\tau_D(a^0,a^1) = \frac{\pi}{8}\,\mathrm{Tr}\,(P\,[P,a^0]\,[P,a^1])$$

where $P = D\,|D|^{-1}$ is the phase of D.

We shall now connect in full generality, the cocycle τ_D with the Chern character in K homology (*cf.* Section 1) of (h,D).

Lemma 4. *For* $\alpha \in [0,1]$, *let* $D_\alpha = D\,|D|^{-\alpha}$. *If* $p' > p$, *then* $[D_\alpha,a] \in L^{p'/\alpha}$ *for any* $a \in A$ *and* $\|[D_\alpha,a]\|_{p'/\alpha} \le C''\,\|[D,a]\|_\infty(1+\|D^{-1}\|_p^2)$, *where* C'' *is a finite constant depending only on* p *and* p'.

Proof. One has $D_\alpha = \frac{\sin\left(\frac{\pi}{2}\,\alpha\right)}{\frac{\pi}{2}} \int_0^\infty \frac{D}{D^2+\mu^2}\,\mu^{1-\alpha}\,d\mu$. Thus, with the notation $\delta(X) = [X,a]$, where $a \in A$ is fixed, one has:

$$\delta(D_\alpha) = \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} \int_0^\infty (\delta(D)(D^2+\mu^2)^{-1} - D(D^2+\mu^2)^{-1}(\delta(D)\,D + D\delta(D))(D^2+\mu^2)^{-1})\,\mu^{1-\alpha}\,d\mu\,.$$

One has

$$\begin{aligned}\|D^2(D^2+\mu^2)^{-1}\,\delta(D)(D^2+\mu^2)^{-1}\|_{p'/\alpha} \;\le\;& \|\delta(D)\|_\infty\,\|(D^2+\mu^2)^{-1}\|_{p'/\alpha} \\ & \|D(D^2+\mu^2)^{-1}\,\delta(D)\,D(D^2+\mu^2)^{-1}\|_{p'/\alpha} \\ \le\;& \|\delta(D)\|_\infty\,\|D(D^2+\mu^2)^{-1}\|_{2p'/\alpha}^2 \\ =\;& \|\delta(D)\|_\infty\,\|D^2(D^2+\mu^2)^{-2}\|_{p'/\alpha} \\ \le\;& \|\delta(D)\|_\infty\,\|(D^2+\mu^2)^{-1}\|_{p'/\alpha}\,.\end{aligned}$$

Thus

$$\|\delta(D_\alpha)\|_{p'/\alpha} \leq 3\, \frac{\sin\left(\frac{\pi}{2}\,\alpha\right)}{\frac{\pi}{2}}\, \|\delta(D)\|_\infty \int_0^\infty \|(D^2+\mu^2)^{-1}\|_{p'/\alpha}\, \mu^{1-\alpha}\, d\mu\,.$$

Let ρ be the spectral measure of D; one has $(\|D^{-1}\|_p)^p = \int \lambda^{-p}\, d\rho(\lambda)$ and

$$(\|(D^2+\mu^2)^{-1}\|_{p'/\alpha})^{p'/\alpha} = \int (\lambda^2+\mu^2)^{-p'/\alpha}\, \lambda^p\, \lambda^{-p}\, d\rho(\lambda) \leq (\|D^{-1}\|_p)^p \sup_\lambda \lambda^p(\lambda^2+\mu^2)^{-p'/\alpha}\,.$$

The latter is reached for $\lambda = \left(\frac{p}{2\,\frac{p'}{\alpha}-p}\right)^{1/2} \mu$ and is equal to $c_\alpha\, \mu^{p-\frac{2p'}{\alpha}}$ where $c_\alpha = p^{p/2}\left(2\,\frac{p'}{\alpha}\right)^{-p'/\alpha}$ $\left(2\,\frac{p'}{\alpha}-p\right)^{p'/\alpha-p/2}$. One has $(c_\alpha)^{\alpha/p'} = \left(1-\frac{\alpha p}{2p'}\right) p^{\frac{\alpha p}{2p'}} \left(2\,\frac{p'}{\alpha}-p\right)^{-\frac{\alpha p}{2p'}}$ which is bounded for $\alpha \in [0,1]$ by a constant C' depending only on p, p'. Thus

$$\|(D^2+\mu^2)^{-1}\|_{p'/\alpha} \leq \|D^{-1}\|_p^{p\alpha/p'}\, C'\, \mu^{-2+\frac{\alpha p}{p'}}\,,$$

whence

$$\int_1^\infty \|(D^2+\mu^2)^{-1}\|_{p'/\alpha}\, \mu^{1-\alpha}\, d\mu \leq \|D^{-1}\|_p^{p\alpha/p'}\, C' \frac{1}{\left(1-\frac{p}{p'}\right)}\, \frac{1}{\alpha}\,.$$

On the other hand for $\mu \in [0,1]$ we can use the inequality $\|(D^2+\mu^2)^{-1}\|_{p'/\alpha} \leq \|D^{-2}\|_{p'/\alpha} \leq (\|D^{-1}\|_p)^2$. Thus

$$\int_0^\infty \|D^2+\mu^2\|_{p'/\alpha}\, \mu^{1-\alpha}\, d\mu \leq C''\, \frac{1}{\alpha}\, (1+\|D^{-1}\|_p^2)$$

and

$$\|[D_\alpha, a]\|_{p'/\alpha} \leq 3\, \frac{\sin\left(\frac{\pi}{2}\,\alpha\right)}{\left(\frac{\pi}{2}\,\alpha\right)}\, \|[D,a]\|_\infty\, C''(1+\|D^{-1}\|_p^2)\,.$$

□

Theorem 4. *Let (h,D) (resp. (h,D,ε)) be an unbounded odd (resp. even) p-summable Fredholm module. Let n be an odd (resp. even) integer $n > p-1$. For any $A \in V^{-1}$, let $\mathrm{Ch}_n(h,D_A) \in H^n_\lambda(A)$ be the n dimensional Chern character of (h,D_A)* (cf. Section 1).

1) *One has $\tau_{D_A} = (n!)^{-1}\,\mathrm{Ch}_n(h,D_A)$ in $H^n_\lambda(A)$.*
2) *The restriction of* $\mathrm{Tch}^{(n+1)}(H,D)$ *to the U_α orbit of $A\otimes 1$ in U_∞ is cohomologous to* $(n!)^{-1}\,\Phi(\mathrm{Ch}_n(h,D_A))$.
3) *For any $n > p-1$, $\tau^{n+2}_{D_A} = ((n+2)(n+1))^{-1}\, S\, \tau^n_{D_A}$ where S is the periodicity operator in cyclic cohomology* [3].

Proof. We can assume that $A = 0$. Let $p' \in]p, n+1[$. Consider the family $D_\alpha = D\,|D|^{-\alpha}$ for $\alpha \in [0,1]$. By Lemma 4, $[D_\alpha, a]$ belongs to $L^{p'/\alpha}$ for any $a \in A$. Let V^α be the subspace of $L^{p'/\alpha}$ formed of selfadjoint operators of the form $\sum a_i[D_\alpha, b_i]$. Remark 4 allows to apply the above construction for every α, yielding the following family of cyclic cocycles on A:

$$\tau_\alpha(a^0,\ldots,a^n) = \Gamma\left(\frac{n}{2}+1\right)^{-1} \sum \mathrm{Sgn}(\lambda) \int_0^\infty \mathrm{Tr}\left(\frac{D_\alpha}{D_\alpha^2+\mu^2}\,[D_\alpha,a^0]\ldots[D_\alpha,a^{\lambda(n)}]\,\frac{1}{D_\alpha^2+\mu^2}\right) \mu^{n+1}\, d\mu\,.$$

For $\alpha = 1$, one has $D_1 = P = \mathrm{Phase}\ D$ and $D_1^2 = 1$ so that τ_1 is equal to $(n!)^{-1}\,\mathrm{Ch}_n(h,D)$. Applying the construction of Tch on the total space of the bundle $(U^\alpha_\infty)_{\alpha\in[0,1]}$ on $[0,1]$ yields, by restriction to U_∞, a closed invariant primitive form $\tilde\theta$ on $U_\infty \times [0,1]$, such that $\tilde\theta \mid U_\infty \times \{0\} = \Phi(\tau_{D_0})$ and $\tilde\theta \mid U_\infty \times \{1\} = \Phi(\tau_{D_1})$. Together with Proposition 2, this completes the proof of 1). The statement 2) alos follows from Proposition 2, while 3) is Theorem 1 in Section 4 of [3].

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