TRANSVERSE GEOMETRY AND MODULAR FORMS

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Survey of joint work with

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1 Hopf Algebras in Transverse Geometry

The 'space of leaves' of a foliation $(V, \mathcal{F})$ can be described in terms of (M, Γ), with M = *complete transversal* and Γ = *holonomy pseudogroup*. The noncommutative 'transverse coordinates' are, by definition

$$\mathcal{A}_M^\Gamma := C_c^\infty(FM) \rtimes \Gamma, \quad FM = \text{frame bundle}$$

consisting of finite sums of monomials of the form

$$\sum f\, U_\phi^*\,, \quad f \in C_c^\infty(FM)\,,\, \phi \in \Gamma\,,$$

with the product

$$f\, U_\phi^* \,\cdot g\, U_\psi^* \,=\, (f \,\cdot g|\phi)\, U_{\psi\,\phi}^*\,.$$

If M is equipped with a *flat affine connection* (ω^i_j) with $\{X_k\}$, $\{Y^j_i\}$ the horizontal and vertical vector fields, one has an action of $\mathfrak{a}(n,\mathbb{R}) = \mathfrak{gl}(n,\mathbb{R}) \rtimes \mathbb{R}^n$,

$$[Y^j_i, Y^\ell_k] = \delta^j_k Y^\ell_i - \delta^\ell_i Y^j_k\,,\ [Y^j_i, X_k] = \delta^j_k X_i\,,\ [X_k, X_\ell] = 0.$$

Each $Z \in \mathfrak{a}(n,\mathbb{R})$ acts on $\mathcal{A}^\Gamma_M$ as a linear transformation: $\quad Z(fU^*_\psi) = Z(f)U^*_\psi\,.$

Define $\mathcal{H}_n$:= the algebra of linear transformations of $\mathcal{A}^\Gamma_M$ generated by $\{\, X_k,\, Y^j_i,\, \delta^i_{jk,\ell_1\ldots,\ell_r}\,\}$, where

$$\delta^i_{jk,\ell_1\ldots,\ell_r}(f\,U^*_\varphi) \;=\; \gamma^i_{jk,\ell_1\ldots\ell_r}\cdot f\,U^*_\varphi\,, \quad \text{with}$$

$$\gamma^i_{jk} = \langle \widetilde{\varphi}^*\omega^i_j,\, X_k\rangle \quad , \quad \gamma^i_{jk,\ell_1\ldots\ell_r} = X_{\ell_r}\cdots X_{\ell_1}(\gamma^i_{jk})\,.$$

The generators satisfy, when acting on $\mathcal{A}_M^\Gamma$:

$$\begin{aligned}
Y_i^j(a^1a^2) &= Y_i^j(a^1)a^2 + a^1Y_i^j(a^2)\,, \\
X_k(a^1a^2) &= X_k(a^1)a^2 + a^1X_k(a^2) + \delta_{jk}^i(a^1)Y_i^j(a^2), \\
\delta_{jk}^i(a^1a^2) &= \delta_{jk}^i(a^1)\,a^2 + a^1\,\delta_{jk}^i(a^2)\,.
\end{aligned}$$

By multiplicativity

$$h(a^1a^2) \;=\; \sum\; h_{(1)}(a^1)\,h_{(2)}(a^2)\,,$$

giving a *coproduct*

$$\Delta\, h = \sum\; h_{(1)} \otimes h_{(2)}\,, \qquad \forall\, h \in \mathcal{H}\,,$$

compatible with the algebra structure, and satisfying all the Hopf algebra axioms. The Hopf algebra $\mathcal{H}_n$ can also be described as a *bicrossproduct* of two Hopf algebras, one co-commutative and the other commutative.

2 The Hopf Algebra $\mathcal{H}_1$

As *algebra* = enveloping algebra of the Lie algebra with basis $\{X, Y, \delta_n\,; n \geq 1\}$ and brackets

$$[Y, X] = X\ ,\ [Y, \delta_n] = n\,\delta_n\ , [X, \delta_n] = \delta_{n+1},\ [\delta_k, \delta_\ell] = 0\,;$$

as *coalgebra*, $\quad \Delta\, Y = Y \otimes 1 + 1 \otimes Y\ ,$

$$\Delta\, X = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y$$

$$\Delta\, \delta_1 = \delta_1 \otimes 1 + 1 \otimes \delta_1$$

$$\text{plus} \quad \Delta(h^1\, h^2) = \Delta h^1 \cdot \Delta h^2\,;$$

antipode : $\quad S(Y) = -Y\ , \qquad S(X) = -X + \delta_1 Y\ ,$

$$S(\delta_1) = -\delta_1\ , \quad \text{plus} \quad S(h^1\, h^2) = S(h^2)\, S(h^1)\,;$$

counit $\quad \varepsilon(h) = \text{const. term of} \quad h \in \mathcal{H}_1\ .$

A discrete group $\Gamma \subset \mathrm{Diff}^+(M^1)$ acts on $J^1_+(M^1) \simeq M \times \mathbb{R}^+$ by $\varphi(y, y_1) = (\varphi(y),\, \varphi'(y) \cdot y_1)$ and $\mathcal{H}_1$ acts on $\mathcal{A}_\Gamma = C_c^\infty(J^1_+(M^1)) \rtimes \Gamma$ by

$$\begin{aligned} Y(fU_\varphi^*) &= y_1 \frac{\partial f}{\partial y_1} U_\varphi^*, \\ X(fU_\varphi^*) &= y_1 \frac{\partial f}{\partial y} U_\varphi^*, \\ \delta_n(fU_\varphi^*) &= y_1^n \frac{d^n}{dy^n} \left(\log \frac{d\varphi}{dy} \right) fU_\varphi^* . \end{aligned}$$

The trace

$$\tau(fU_\varphi^*) = \begin{cases} \int_{J^1_+(M^1)} f(y, y_1) \frac{dy \wedge dy_1}{y_1^2} & \text{if } \varphi = 1\,, \\ 0 & \text{if } \varphi \neq 1\,, \end{cases}$$

satisfies $\tau(h(a)) = \delta(h)\, \tau(a)\,, \qquad \forall \quad h \in \mathcal{H}_1\,,$ where

$\delta \in \mathcal{H}_1^* : \qquad \delta(Y) = 1, \quad \delta(X) = 0 \quad \text{and} \quad \delta(\delta_n) = 0.$

The *twisted antipode* $\widetilde{S} = \delta * S$ satisfies $\widetilde{S}^2 = \mathrm{Id}$.

3 Hopf Cyclic Cohomology

- *Modular pair* a *character* $\delta : \mathcal{H} \to \mathbb{C}$, a *group-like element* $\sigma \in \mathcal{H}$, $\Delta(\sigma) = \sigma \otimes \sigma$, $\varepsilon(\sigma) = 1$, such that $\delta(\sigma) = 1$. With $\widetilde{S}(h) := \sum_{(h)} \delta(h_{(1)})\, S(h_{(2)})$, $h \in \mathcal{H}$, we require $\widetilde{S}^2 = Ad(\sigma)$.

- *Cylic object* $\mathcal{H}^\natural_{Hopf} = \{C^q_{Hopf}(\mathcal{H})\}_{q\geq 0}$ $(\sigma = 1)$:

$$\partial_0(h^1 \otimes \ldots \otimes h^{n-1}) = 1 \otimes h^1 \otimes \ldots \otimes h^{n-1},$$

$$\partial_j(h^1 \otimes \ldots \otimes h^{n-1}) = h^1 \otimes \ldots \otimes \Delta h^j \otimes \ldots \otimes h^{n-1},$$

$$\partial_n(h^1 \otimes \ldots \otimes h^{n-1}) = h^1 \otimes \ldots \otimes h^{n-1} \otimes \sigma \;;$$

$$\sigma_i(h^1 \otimes \ldots \otimes h^{n+1}) \;=\; h^1 \otimes \ldots \otimes \varepsilon(h^{i+1}) \otimes \ldots \otimes h^{n+1};$$

$$\tau_n(h^1 \otimes \ldots \otimes h^n) \;=\; (\Delta^{n-1}\widetilde{S}(h^1)) \cdot h^2 \otimes \ldots \otimes h^n \otimes \sigma\,.$$

• An action of $\mathcal{H}$ on an algebra $\mathcal{A}$ with invariant trace

induces $\quad \chi^{\Gamma}_{Hopf} : HC^*_{\text{Hopf}}(\mathcal{H}) \to HC^*(\mathcal{A})\,,$ via the

cochain map

$$\chi_\tau(h^1 \otimes \ldots \otimes h^n)(a^0, \ldots, a^n) \,=\, \tau(a^0\, h^1(a^1) \ldots h^n(a^n))\,.$$

Theorem 1. *There is a canonical isomorphism*

$$\kappa^*_n \,:\, H^*(\mathfrak{a}_n, SO_n) \xrightarrow{\simeq} PHC^*_{\text{Hopf}}(\mathcal{H}_{FM}, SO(n))\,,$$

such that $\quad \chi^{\Gamma}_{Hopf} \circ \kappa^*_n \,=\, \chi^{\Gamma}_{GF}\,, \quad$ *where*

$$\chi^{\Gamma}_{GF} : H^*(\mathfrak{a}_n, SO_n) \longrightarrow HC^*(\mathcal{A}^{\Gamma}_{PM}), \quad PM = FM/SO(n)$$

is the cyclic version of the Chern-Weil map.

4 Hopf Cyclic Classes

• The generating class $[\delta_1] \in PHC^{\text{odd}}_{\text{Hopf}}(\mathcal{H}_1)$

$$b(\delta_1) \;=\; 1 \otimes \delta_1 - \Delta\delta_1 + \delta_1 \otimes 1 = 0\,,$$

$$\tau_1(\delta_1) \;=\; \widetilde{S}(\delta_1) = S(\delta_1) \;= -\delta_1\,;$$

$[\delta_1]$ gives the *Godbillon-Vey class* of foliations.

• The cocycle $\delta_2' := \delta_2 - \frac{1}{2}\delta_1^2 \in \mathcal{H}_1$ acts on $\mathcal{A}_\Gamma$ by

$\delta_2'(fU_\varphi^*) = y_1^2\,\{\varphi(y)\,;y\}\,fU_\varphi^*$ where

$$\{y\,;x\} \;:=\; \frac{d^2}{dx^2}\left(\log\frac{dy}{dx}\right) - \frac{1}{2}\left(\frac{d}{dx}\left(\log\frac{dy}{dx}\right)\right)^2.$$

representing the *Schwarzian* $[\delta_2'] \in HC^1_{\text{Hopf}}(\mathcal{H}_1)\,.$

• The generating class $[T] \in PHC^{\text{even}}_{\text{Hopf}}(\mathcal{H}_1)$

$$T \;:=\; X \otimes Y - Y \otimes X - \delta_1\, Y \otimes Y$$

gives the *transverse fundamental class* of foliations.

5 Modular Forms and Hecke Operators

Modular form of weight $k =$ holomorphic function f on

$\mathfrak{H} = \{z \in \mathbb{C}, \mathrm{Im}(z) > 0\}$ satisfying

$$f|_k\,\gamma \;=\; f\,, \quad \forall\,\gamma \in \mathrm{SL}(2,\mathbb{Z})\,, \quad \text{where}$$

$$f|_k\,\alpha\,(z) \;=\; \det(\alpha)^{k/2}\,(cz+d)^{-k} f(\alpha\cdot z)\,,$$

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G^+(\mathbb{R}) := \mathrm{GL}^+(2,\mathbb{R}), \quad \alpha\cdot z = \frac{az+b}{cz+d}$$

plus holomorphicity at ∞, that is f_∞ is holomorphic at

$q = 0$; f is a *cusp form* if $f_\infty(0) = 0$.

More generally, one can replace $\Gamma(1)$ by

$$\Gamma(N) = \left\{ \gamma \in \mathrm{SL}(2,\mathbb{Z})\,; \quad \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}.$$

One obtains algebras of modular forms of *all levels*

$$\mathcal{M} := \varinjlim_{N\to\infty} \mathcal{M}(\Gamma(N)) \, , \qquad \mathcal{M}_k := \varinjlim_{N\to\infty} \mathcal{M}_k(\Gamma(N)) \, .$$

The group $G^+(\mathbb{Q}) := \mathrm{GL}^+(2,\mathbb{Q})$ acts 'sideways' on the tower defining the projective limit $\mathfrak{H}_{\mathbb{A}}$.

One gets a 'transverse space' with 'coordinates' = the crossed-product algebra $\mathcal{A} \equiv \mathcal{A}_{G^+(\mathbb{Q})} := \mathcal{M} \rtimes G^+(\mathbb{Q})$, consisting of finite sums of symbols of the form

$$\sum f \, U_\gamma^* \, , \qquad \text{with} \quad f \in \mathcal{M} \, , \quad \gamma \in G^+(\mathbb{Q}) \, ,$$

and with the product given by the rule

$$f \, U_\alpha^* \, \cdot g \, U_\beta^* = \, (f \, \cdot g|\alpha) \, U_{\beta\,\alpha}^* \, .$$

6 Action of the Hopf Algebra $\mathcal{H}_1$

The action of X is given by the 'Ramanujan operator'

$$X := \frac{1}{2\pi i}\frac{d}{dz} - \frac{1}{12\pi i}\,\frac{d}{dz}(\log\Delta)\cdot Y$$
$$= \frac{1}{2\pi i}\,\frac{d}{dz} - \frac{1}{2\pi i}\,\frac{d}{dz}(\log\eta^4)\cdot Y\,,$$

where $Y(f) = \frac{k}{2}\cdot f\,,\quad \forall\, f\in\mathcal{M}_k$ and

$$\Delta(z) \;=\; (2\pi)^{12}\,\eta^{24}(z) \;=\; (2\pi)^{12}\,q\,\prod_{n=1}^{\infty}(1-q^n)^{24}\,.$$

One has $[Y,X] \;=\; X$. While Y is invariant under the

action of $\gamma\in G^+(\mathbb{Q})$, one has

$$\big(X(f|_k\,\gamma^{-1})\big)\,|_{k+2}\,\gamma \;=\; X(f) \;-\; \mu_\gamma\cdot Y(f)\,,$$

where $\mu_\gamma(z) = \frac{1}{12\pi i}\,\frac{d}{dz}\log\frac{\Delta|\gamma}{\Delta} = \frac{1}{2\pi i}\,\frac{d}{dz}\log\frac{\eta^4|\gamma}{\eta^4}$

is an *Eisenstein series* of weight 2.

Theorem 2. *There is a unique Hopf action of* $\mathcal{H}_1$ *on the algebra* $\mathcal{A}_{G^+(\mathbb{Q})}$ *, such that*

$$X(f\,U_\gamma^*) \;=\; X(f)\,U_\gamma^*\,, \qquad Y(f\,U_\gamma^*) = Y(f)\,U_\gamma^*\,,$$

$$\delta_n(f\,U_\gamma^*) \;=\; X^{n-1}(\mu_\gamma)\cdot f\,U_\gamma^*\,, \quad n \geq 1\,.$$

The Schwarzian cocycle acts by the derivation

$$\delta_2'(f\,U_\gamma^*) = \left(X(\mu_\gamma) - \frac{1}{2}\,\mu_\gamma^2\right)\cdot f\,U_\gamma^*$$

which is inner

$$\delta_2'(a) \;=\; \Omega\cdot a - a\cdot\Omega\,, \qquad a \in \mathcal{A}_{G^+(\mathbb{Q})}$$

implemented by the weight 4 *modular form*

$$\Omega := -\frac{E_4}{72}\,, \qquad E_4(q) := 1 + 240\,\sum_1^\infty n^3 \frac{q^n}{1-q^n}$$

7 Transverse Cocycle and Euler Class

A partial analogue of the Godbillon-Vey cocycle is

$$\mathrm{GV}\,(a,b) \;=\; P(a\cdot\delta_1(b))\,, \qquad a,\,b\in\mathcal{A}_{G^+(\mathbb{Q})}\,,$$

where P is the projection onto the weight 2 component, promoted as usual to $P:\mathcal{A}_{G^+(\mathbb{Q})}\;\to\;\mathcal{M}_2\,.$

For $a,b\,\in\mathcal{A}_{G^+(\mathbb{Q})}$ and $h\in\mathcal{H}_1$, one has

$$P\,(h(a)\cdot b) = P\,(a\cdot\widetilde{S}(h)\,(b))\,,$$

but there is no invariant trace. Moreover, GV reduces to the *transverse* group 1-cocycle

$$\mathbf{E}(\gamma) \;:=\; \mathrm{GV}\,(U^*_{\gamma^{-1}},U^*_{\gamma}) \;=\; \mu_\gamma|\gamma^{-1}\,.$$

A formal analogue of Godbillon-Vey cocycle can be obtained by cup product with the *period cocycle*:

$$\Psi(\gamma)\,(\omega) \equiv \langle \omega, \Psi(\gamma) \rangle \ := \ \int_{z_0}^{\gamma \cdot z_0} \omega \, , \qquad \gamma \in G^+(\mathbb{Q});$$

$[\Psi] \in H^1(G^+(\mathbb{Q}), (\Omega^1)^*)$ is independent of $z_0 \in \mathfrak{H}$. The coupling of the two 1-cocycles gives a 2-cocycle

$$\tau(\gamma_1, \gamma_2) \ := \ \int_{z_0}^{\gamma_2 \cdot z_0} \mu_{\gamma_1}(z)\, dz \, .$$

Theorem 3. *The* 2*-cocycle* $\operatorname{Re} \tau$ *represents the Euler class* $e \in H^2\,(\mathrm{SL}(2, \mathbb{Q}),\, \mathbb{R})$ *. Its imaginary part* $\operatorname{Im} \tau$ *is a coboundary.*

Replacing periods by *modular symbols* leads to

$$\theta(\gamma_1,\gamma_2) \;=\; \int_{z_0}^{\gamma_2\cdot z_0} \mu_{\gamma_1}(z)dz \;-\; z_0\cdot \mathbf{a}_0(\mu_{\gamma_1}|\gamma_2 - \mu_{\gamma_1}) \\ +\; \int_{z_0}^{i\infty} (\widetilde{\mu_{\gamma_1}|\gamma_2} - \widetilde{\mu_{\gamma_1}})(z)\,dz$$

whose real part of θ is rational

$$\rho \;:=\mathrm{Re}\;\theta\;\in Z^2\,(G^+(\mathbb{Q}),\,\mathbb{Q})$$

and has an interesting arithmetic expression.

Specifically, let

$$\gamma_1 \;=\; \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad \gamma_2 \;=\; \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \;\in\; M_2^+(\mathbb{Z})\,,$$

represent *modulo scalar matrices* elements in $\mathrm{PSL}(2,\mathbb{Q})$.

Theorem 4. *The cocycle* $\rho \in Z^2(\mathrm{PSL}(2,\mathbb{Q}),\mathbb{Q})$ *can be explicitly expressed as follows:*

$$\text{if } c_2 = 0\,, \quad \rho(\gamma_1,\gamma_2) = \left(\sum_{\mathbf{x}\cdot\check{\gamma}_1=\mathbf{0}} \mathbf{B}_2(x_1) - \mathbf{B}_2(0)\right)\frac{b_2}{d_2}\,;$$

$$\begin{aligned}\text{if } c_2 > 0\,, \quad \rho(\gamma_1,\gamma_2) \,=& \left(\sum_{\mathbf{x}\cdot\check{\gamma}_1=\mathbf{0}} \mathbf{B}_2(x_1) - \mathbf{B}_2(0)\right)\frac{a_2}{c_2} \\ &+ \left(\sum_{\mathbf{x}\cdot\check{\gamma}_2\check{\gamma}_1=\mathbf{0}} \mathbf{B}_2(x_1) - \sum_{\mathbf{x}\cdot\check{\gamma}_2=\mathbf{0}} \mathbf{B}_2(x_1)\right)\frac{d_2}{c_2} \\ &-2\sum_{\mathbf{x}\cdot\check{\gamma}_1=\mathbf{0}}\sum_{j=0}^{c_2-1}\mathbf{B}_1\left(\frac{x_1+j}{c_2}\right)\mathbf{B}_1\left(a_2\frac{x_1+j}{c_2}+x_2\right) \\ &+2\sum_{j=0}^{c_2-1}\mathbf{B}_1\left(\frac{j}{c_2}\right)\mathbf{B}_1\left(\frac{a_2 j}{c_2}\right)\end{aligned}$$

where $\mathbf{x} \in \mathbb{Q}^2/\mathbb{Z}^2\,, \check{\gamma} := \det(\gamma)\cdot\gamma^{-1}$ *and* $\forall\, x \in \mathbb{Q}\,,$

$$\mathbf{B}_1(x) := x-[x]-\frac{1}{2}\,, \quad \mathbf{B}_2(x) := (x-[x])^2-(x-[x])+\frac{1}{6}\,.$$

Corollary 5. *One has the following 'splitting formula': $\forall\,\gamma_1,\gamma_2 \in \mathrm{PSL}(2,\mathbb{Q})$,*

$$\frac{1}{2}\rho(\gamma_1,\gamma_2) - \varpi(\gamma_1,\gamma_2) \;=\; \Phi(\gamma_1\gamma_2) - \Phi(\gamma_1) - \Phi(\gamma_2)\,,$$

where ϖ is defined by means of the Hilbert symbol

$$\varpi(\gamma_1,\gamma_2) \;=\; (x(\gamma_1), x(\gamma_2)) \;-\; (-x(\gamma_1)x(\gamma_2), x(\gamma_1\gamma_2))$$

and Φ is given for $\gamma_1 \in \mathrm{SL}(2,\mathbb{Z})$ by the Dedekind-Rademacher function

$$\Phi(\gamma_1) \;=\; \frac{a_1 + d_1}{12\,c_1} \;-\; \sum_{j=0}^{c_1-1} \mathbf{B}_1\left(\frac{j}{c_1}\right)\mathbf{B}_1\left(\frac{a_1\,j}{c_1}\right)$$

and for γ_2 in the Borel subgroup of $\mathrm{SL}(2,\mathbb{Q})$ by

$$\Phi\begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix} \;=\; \frac{1}{12}\frac{b_2}{d_2}\,.$$

8 Perturbed $\mathcal{H}_1$-actions on $\mathcal{A}_{G^+(\mathbb{Q})}$

• A 1-*cocycle* $\gamma \in Z^1(\mathcal{H}_1, \mathcal{A})$ is an invertible element of the convolution algebra of *linear* maps $\mathrm{Hom}(\mathcal{H}, \mathcal{A})$, such that $\gamma(h\,h') = \sum \gamma(h_{(1)})\, h_{(2)}(\gamma(h'))$, $h, h' \in \mathcal{H}$.

The γ-perturbed action of $\mathcal{H}_1$ on $\mathcal{A}$ is

$$h \cdot_\gamma a := \sum \gamma(h_{(1)})\, h_{(2)}(a)\, \gamma^{-1}(h_{(3)})\,.$$

Theorem 6. *For any* $\nu \in \mathcal{M}_2$ *there is a unique* 1-*cocycle* $\gamma_\nu \in Z^1(\mathcal{H}_1, \mathcal{A}_{G^+(\mathbb{Q})})$ *such that* $\gamma(X) = 0 = \gamma(Y)$ *and* $\gamma(\delta_1) = \nu$. *Then* $X \mapsto X + \nu Y$ *and the Schwarzian acts as* $\delta_2' \cdot_\nu a = [X(\nu) + \dfrac{\nu^2}{2} + \omega,\ a]$.

There is no choice of ν *for which* $\delta_2' \cdot_\nu \equiv 0$.

Remark: *Comp.* [Zagier]'s gauge equivalence.

9 Rankin-Cohen Deformations of $\mathcal{A}_{G^+(\mathbb{Q})}$

• The nth *Rankin-Cohen bracket* of two modular forms

$f \in \mathcal{M}_k\,,\ g \in \mathcal{M}_\ell$ is $\quad RC_n(f,g) \in \mathcal{M}_{k+\ell+2n}$

$$:= \sum_{r+s=n} (-1)^r \binom{n+k-1}{s} \binom{n+\ell-1}{r} f^{(r)} g^{(s)}\,.$$

• For a graded commutative algebra $\mathcal{A} = \sum_{n\geq 0} \mathcal{A}_n$ with a derivation X of degree 2, the *standard Rankin-Cohen brackets*, cf. [Zagier], are: $\forall a \in \quad RC_n(a,b)$

$$:= \sum_{r+s=n} (-1)^r \binom{n+k-1}{s} \binom{n+\ell-1}{r} X^r(a) X^s(b)\,,$$

$\forall\, a \in \mathcal{A}_k\,,\ b \in \mathcal{A}_\ell\,.$

Theorem 7. *There is a unique $G^+(\mathbb{Q})$-covariant extension of the RC-brackets from modular forms to $\mathcal{A}_{G^+(\mathbb{Q})}$.*

Using [Zagier] and [Cohen-Manin-Zagier], one obtains:

Theorem 8. *The following $*$-products define associative deformations of the algebra $\mathcal{A}_{G^+(\mathbb{Q})}$:*

$$a *_t b := \sum_{n\geq 0} t^n RC_n(a,b)\,, \quad a,b \in \mathcal{A}_{G^+(\mathbb{Q})}\,,$$

and more generally, $\quad \forall \kappa \in \mathbb{C}$

$$a *_t^\kappa b \,:=\, \sum_{n\geq 0} t^n \, RC_n\left(\mathbf{t}_n^\kappa(Y\otimes 1, 1\otimes Y)(a\otimes b)\right)\,,$$

$$\mathbf{t}_n^\kappa(\alpha,\beta) := (-\frac{1}{4})^n \sum_{j\geq 0} \binom{n}{2j} \frac{\binom{-\frac{1}{2}}{j}\binom{\kappa-\frac{3}{2}}{j}\binom{\frac{1}{2}-\kappa}{j}}{\binom{-\alpha-\frac{1}{2}}{j}\binom{-\beta-\frac{1}{2}}{j}\binom{n+\alpha+\beta-\frac{3}{2}}{j}}\,.$$

10 Rankin-Cohen Deformations of $\mathcal{H}_1$

• *Deformation* of a Hopf algebra $\mathcal{H}$:

$$\begin{aligned} m_t &= m + tm_1 + t^2 m_2 + ... \quad \text{and} \\ \Delta_t &= \Delta + t\Delta_1 + t^2\Delta_2 + ... \,. \end{aligned}$$

• *Twisting element*: $F \in \mathcal{H}[[t]] \otimes_{\mathbb{C}[[t]]} \mathcal{H}[[t]]$ invertible,

$$\begin{aligned} (\Delta \otimes \mathrm{Id})(F) \cdot F \otimes 1 &= (\mathrm{Id} \otimes \Delta)(F) \cdot 1 \otimes F \quad \text{and} \\ (\varepsilon \otimes \mathrm{Id})(F) &= 1 \otimes 1 = (\mathrm{Id} \otimes \varepsilon)(F)\,. \end{aligned}$$

$\Delta_t(h) = F^{-1}\Delta(h)F$ gives deformation $\mathcal{H}[[t]]^F$.

• If $\mathcal{H}$ acts on an algebra $\mathcal{A}$ and $F \in \mathcal{H}[[t]] \otimes_{\mathbb{C}[[t]]} \mathcal{H}[[t]]$

is twisting, then $\mu_t^F(a \otimes b) = \mu\left(F(a \otimes b)\right), \quad a, b \in \mathcal{A}$

gives an *associative deformation* of $\mathcal{A}$.

• *Projective reduction*: $\mathcal{H}_{\rm pr} := \mathcal{H}_1/\big(\delta_2'\big)$, so that

$$\delta_n = \frac{(n-1)!}{2^{n-1}}\delta_1^n .$$

By 'extension of scalars' one 'thickens' $\mathcal{H}_{\rm pr}$ to $\widetilde{\mathcal{H}}_{\rm pr}$, defined over a free commutative algebra $\mathcal{P} = \mathbb{C}[Z_0, Z_1, \ldots]$, to which all actions in Theorem 6 extend canonically. This uniquely determines *universal brackets*

$$\widetilde{RC}_n \in \widetilde{\mathcal{H}}_{\rm pr} \otimes_{\mathcal{P}} \widetilde{\mathcal{H}}_{\rm pr} .$$

Theorem 9. *The element*

$$\widetilde{F}^{RC} := \sum_{n\geq 0} t^n \, \widetilde{RC}_n \in \left(\widetilde{\mathcal{H}}_{\rm pr} \otimes_{\mathcal{P}} \widetilde{\mathcal{H}}_{\rm pr}\right) [[t]]$$

defines a twist over $\mathcal{P}$.

By specializing $\widetilde{F}^{RC}$ at $Z_k = 0$, one obtains:

Theorem 10. $F^{RC} := \sum_{n\geq 0} t^n \, RC_n$, *where* RC_n :=

$$\sum_{k=0}^{n} \frac{S(X)^k}{k!} (2Y+k)_{n-k} \otimes \frac{X^{n-k}}{(n-k)!} (2Y+n-k)_k \, ,$$

$$Z_k = Z\,(Z+1)....(Z+k-1) \qquad \textit{and}$$

$$S(X)^k \; = \; \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{\delta_1^j}{2^j} X^{k-j} (2Y+k-j)_j \, ,$$

defines a twisting of $\mathcal{H}_{\mathrm{pr}}$ *along the direction*

$$F_1^{RC} = -2T = S(X) \otimes 2Y \; + \; 2Y \otimes X \, .$$

Remark: This can be viewed as analogue of Drinfeld's theorem for universal enveloping algebras $\mathfrak{A}(\mathfrak{g})$, asserting that such a twist F exists (and satisfies QYBE) if and only if its direction $F_1 - F_1^{21}$ satisfies CYBE.

11 RC-deformations of Projective Structures

An algebra $\mathcal{A}$ on which $\mathcal{H}_1$ acts such that

$$\delta_2'(a) = \Omega \cdot a - a \cdot \Omega \quad \text{and} \quad \delta_k(\Omega) = 0\,, \forall k \in \mathbb{N}$$

can be viewed as a *noncommutative analogue* of a classical *projective structure*, with Ω playing the role of the *quadratic differential.* Since such an $\mathcal{A}$ can be turned into an $\widetilde{\mathcal{H}}_{\mathrm{pr}}$-module algebra, by Theorem 9 one obtains a surprisingly general deformation result for algebras:

Theorem 11. *Let $\mathcal{A}$ be an algebra endowed with a projective structure. Then*

$$a *_t b := \sum_{n \geq 0} t^n \widetilde{RC}_n(a, b)\,,$$

defines an associative deformation of $\mathcal{A}$.