E-THEORETIC DUALITY FOR HIGHER RANK GRAPH ALGEBRAS

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Abstract. We prove that there is a Poincaré type duality in E-theory between higher rank graph algebras associated with a higher rank graph and its opposite correspondent. We obtain an r-duality, that is the fundamental classes are in E^r. The basic tools are a higher rank Fock space and higher rank Toeplitz algebra which has a more interesting ideal structure than in the rank 1 case. The K-homology fundamental class is given by an r-fold exact sequence whereas the K-theory fundamental class is given by a homomorphism. The E-theoretic products are essentially pull-backs so that the computation is done at the level of exact sequences.

1. Introduction

According to Kasparov (cf. [Co] and references in [Ka]) two C^*-algebras A and B are K-dual if there exist two classes $\delta \in KK^r(\mathbb{C}, A \otimes B)$, the K-theory fundamental class, and $\Delta \in KK^r(A \otimes B, \mathbb{C})$, the K-homology fundamental class, such that

$$\delta \otimes_B \sigma^{12}(\Delta) = 1_A \text{ and } \sigma_{12}(\delta) \otimes_A \Delta = (-1)^r 1_B.$$

Here σ^{12} and σ_{12} are the flip homomorphism on the first respectively the second argument of the KK-bifunctor and the tensor product is the minimal one. More precisely this describes an r-duality. (We have adopted the $(-1)^r$-sign from [Em].) Such duality classes define canonical isomorphisms between K-theory and K-homology of A and B. In the sequel we will omit the flips as is customary in most other references.

In [KaPu] a 1-duality between Cuntz-Krieger algebras O_A and O_{A^T}, where A^T is the transpose of A was exhibited. In fact a slightly weaker result was proved asserting only that the Kasparov product with certain classes implement isomorphisms between K-theory and K-homology of O_A and O_{A^T}. However an easy homotopy argument (untwist) shows that the classes defined in [KaPu] are indeed duality classes in the above sense.

In this paper we generalize this result to a class of higher rank graph algebras which were introduced in [KuPa]. This class of algebras is in turn a more conceptual and graphic approach to higher rank Cuntz-Krieger algebras which were introduced in [RS] in connection with discrete groups acting on buildings. Tensor products of Cuntz-Krieger algebras are amongst the simplest examples of such algebras. Let O_Λ and $O_{\Lambda^{op}}$ be the higher rank graph algebras defined by the higher rank graph Λ and its opposite correspondent Λ^{op}. In the case of ordinary Cuntz-Krieger algebras O_A the K-theory class in $KK^1(\mathbb{C}, O_A \otimes O_{A^T})$ was defined in [KaPu] by a morphism from $\mathcal{S}$ to $O_A \otimes O_{A^T}$. The K-homology class

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in $KK^1(O_A \otimes O_{A^T}, \mathbb{C}) \cong Ext(O_A \otimes O_{A^T})$ is defined using a short exact sequence which comes from a Fock space type construction.

To get an idea of what the duality classes should look like in the higher rank case consider first the particular case when O_Λ is a tensor product of Cuntz-Krieger algebras $O_{A_1} \otimes \ldots \otimes O_{A_r}$. In that case $O_{\Lambda^{op}} = O_{A_1^T} \otimes \ldots \otimes O_{A_r^T}$ and we have 1-dualities between O_{A_i} and $O_{A_i^T}$. We can define a K-theory class $\delta \in KK^r(\mathbb{C}, O_A \otimes O_{A^T})$ simply as the tensor product of the r classes $\delta_i \in KK^1(\mathbb{C}, O_{A_i} \otimes O_{A_i^T})$. A K-homology fundamental class is given by a suitable r-fold Kasparov intersection product of the $\Delta_i \in KK^1(O_{A_i} \otimes O_{A_i^T}, \mathbb{C})$. As usual for K-homology, it is much more complicated to determine because it amounts to computing the intersection product of the corresponding extensions.

In [Ze] another description of KK-theory was given in which the cycles are possibly longer exact sequences and the intersection product is simply the Yonneda product (splice), similar to the Yonneda product from homological algebra. For example the K-homology class for $\mathbb{R}^r$ or $\mathbb{T}^r$, both of which we shall denote by $\mathcal{T}^{\otimes r}$, can be expressed by a splice of r copies of the Toeplitz extension suitably tensored. For $r = 2$ we obtain the sequence

$$0 \to \mathcal{K} \otimes \mathcal{K} \to \mathcal{T} \otimes \mathcal{K} \to C(\mathbb{T}) \otimes \mathcal{T} \to C(\mathbb{T}) \otimes C(\mathbb{T}) \to 0$$

and for $r \geq 2$ the r-fold exact sequence

$$0 \to \mathcal{T}_0 \to \mathcal{T}_1 \to \cdots \to \mathcal{T}_r \to C(\mathbb{T}^r) \to 0$$

represents a class in $KK^r(C(\mathbb{T}^r), \mathbb{C})$. One is therefore lead to the idea of representing the K-homology fundamental class (Dirac class) by an r-fold exact sequence. Looking at the rank 1 case, we can find a candidate for a Fock space type construction: we define a higher rank Fock space as $\mathsf{F}_M = l^2(\overline{W})$ where $\overline{W}$ is the set of morphisms of non-zero shape together with a vacuum word on which there are left and right creations. The left and right creation operators both taken together generate the higher rank version of the Toeplitz algebras. In the case of ordinary Cuntz-Krieger algebras we get a slightly modified version of the ordinary A-Fock space which gives a modified Toeplitz algebra. This is mainly due to the way the rank 1 algebras are presented in the setting of graph algebras. The ideal structure of the higher rank Toeplitz algebras is more complicated than in the rank 1 case. To obtain the defining relations for O_Λ and $O_{\Lambda^{op}}$ we have to divide out by an ideal which strictly includes the ideal of compact operators. It is precisely this property which is used to define an r-fold exact sequence of the form

$$0 \longrightarrow \mathcal{K} \xrightarrow{i_0} \mathcal{E}_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{r-1}} \mathcal{E}_r \xrightarrow{i_r} O_\Lambda \otimes O_{\Lambda^{op}} \longrightarrow 0 \quad (*)$$

in section 3 as a candidate for the K-homology class.

We need to pay attention to an important technical issue. $(*)$ is given by splicing the r short exact sequences $0 \to \text{Ker } i_k \to \mathcal{E}_k \to \text{Im } i_k \to 0$. [Ze] calls longer extensions completely semisplit if all these 1-fold extensions admit completely positive liftings. Only under this condition it is guaranteed that it defines an element $\Delta \in KK^r(O_\Lambda \otimes O_{\Lambda^{op}}, \mathbb{C})$. We have not been able to prove the complete semisplitness condition in full generality and leave this to future work, although it seems very plausible and is certainly met in the case of tensor products.

However, using E-theory this difficulty can be circumvented. Each of the short exact sequences $0 \to \mathrm{Ker}\ i_k \to \mathcal{E}_k \to \mathrm{Im}\ i_k \to 0$ defines asymptotic morphisms from $\mathcal{S}(\mathrm{Im}\ i_k)$ to $\mathrm{Ker}\ i_k$. We associate to our long exact sequence the product of the corresponding E-theory classes and prove the E-theoretic version of our duality. The result is valid in KK-theory as well since the higher rank graph algebras are nuclear. Our proof of the duality is a generalization of the arguments in [KaPu] to r-fold extensions. Two important ingredients appearing in [KaPu] are still available: the gauge action and the untwist. In general the equivalence relations appearing in the definition of Ext-groups are quite complicated but in our particular situation what makes arguments work in E-theory is a strong isomorphism between sequences obtained after untwisting.

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2. The higher rank Toeplitz algebra

In this section we shall introduce most of our notation which is based on [KuPa]. Let Λ be a higher rank graph of rank r. By A we denote the set of objects of Λ, by σ the shape functor on Λ and Λ^{op} with j-th component $\sigma(w)_j$; o and t are the origin and terminal maps (also called range and source). Define the set

$$W(\Lambda) = \{w \in \Lambda \mid \sigma(w) \neq 0\}$$

of morphisms of non-zero shape. We shall write W instead of $W(\Lambda)$ if it is clear which Λ we mean. The set of all morphisms including shape 0 is denoted by $\dot{W} = \dot{W}(\Lambda)$. In [KuPa] a higher rank graph algebra O_Λ is defined as the universal C^*-algebra generated by partial isometries s_u indexed by $u \in \dot{W}$ satisfying

(i) $\{s_a = p_a \mid a \in A\}$ are pairwise orthogonal projections;
(ii) $s_{uv} = s_u s_v$ for all $u, v \in \dot{W}$ such that $t(u) = o(v)$;
(iii) $s_u^* s_u = s_{t(u)}$ for all $u \in \dot{W}$;
(iv) $s_a = \sum_{\sigma(u)=n, o(u)=a} s_u s_u^*$ for all $n \in \mathbb{N}^r$.

We denote by Λ^{op} the opposite category which is a higher rank graph in the obvious way. A possible source of confusion is the composition of u and v in Λ^{op} which is vu. This could be avoided by denoting by $\overline{u}$ the element u whenever we refer to it as a morphism in Λ^{op}. However, the choice will always be clear from the context because we use the multiplication mainly for indices of partial isometries of the algebras O_Λ or $O_{\Lambda^{op}}$.

Some conditions on Λ must be be assumed if duality is to be true. One reason is the existence of some higher rank graphs whose C^*-algebras are AF. For example in [KuPa] (ch.5) the skew product of a graph with its shape gives an AF-algebra. For Cuntz algebras it is Morita equivalent to the UHF algebra $U(n)$. If there was a dual B for $U(n)$ we would have $K_i(B) = K^1(U(n))$ where i is 0 or 1. But $K^1(U(n))$ is uncountable so that B cannot be separable. Usually we have to assume our algebras to be separable, at least in one argument of the bivariant theory. In order to avoid infinite sums in a C^*-algebra, we assume that the set of objects A is finite. We shall need several times the uniqueness of O_Λ so we assume that Λ is aperiodic (see Def.4.3 and Thm.4.6 of [KuPa]). Thus O_Λ and $O_{\Lambda^{op}}$ are nuclear and under mild further assumptions simple purely infinite ([KuPa]).

We define the Fock space associated with Λ by $\mathsf{F}_\Lambda = \mathsf{F} = \mathbb{C}\Omega \oplus l^2(W)$. Thinking of Ω as a word we may write $\mathsf{F} = \ell^2(\overline{W})$, where $\overline{W} = W \cup \{\Omega\}$. For any $u \in W$ we define left and right creations L_u and R_u by $L_u\Omega = R_u\Omega = \delta_u$ and

$$L_u\delta_w = \begin{cases} \delta_{uw} & \text{if } t(u) = o(w) \\ 0 & \text{otherwise,} \end{cases}$$

$$R_u\delta_w = \begin{cases} \delta_{wu} & \text{if } t(w) = o(u) \\ 0 & \text{otherwise.} \end{cases}$$

The closed linear span of a set S in a normed linear space is denoted by $[S]$; the projection onto a closed subspace $\mathcal{L}$ of a Hilbert space $\mathcal{H}$ by $P_{\mathcal{L}}$. For any $j \in \{1, \dots, r\}$ and $a \in A$ we define

$$P_a = P_{[\Omega, \delta_u | o(u)=a]}\ ,\ P_a^j = P_{[\Omega, \delta_u | o(u)=a \text{ and } \sigma(u)_j = 0]}\ ,$$

$$Q_a = P_{[\Omega, \delta_u | t(u)=a]}\ ,\ Q_a^j = P_{[\Omega, \delta_u | t(u)=a \text{ and } \sigma(u)_j = 0]}\ ,$$

$$P^j = P_{[\Omega, \delta_u | \sigma(u)_j = 0]}.$$

Define $L_\Omega = 1 = R_\Omega$ and for $a \in A$ let L_a be the projection P_a and R_a the projection Q_a. It is easy to check that for any $w \in W$, $j \in \{1, \dots, r\}$ and $a \in A$ we have

$$L_w^* L_w = P_{t(w)},\ P_a = \sum_{o(u)=a; \sigma(u)=e_j} L_u L_u^* + P_a^j,$$

$$R_w^* R_w = Q_{o(w)},\quad Q_a = \sum_{t(u)=a; \sigma(u)=e_j} R_u R_u^* + Q_a^j.$$

Note also that

$$1 - \sum_{\sigma(u)=e_j} L_u L_u^* = P^j = 1 - \sum_{\sigma(u)=e_j} R_u R_u^*$$

and for all $k \in \mathbb{Z}_+^r \setminus \{0\}$ we have

$$\sum_{\sigma(u)=k} L_u L_u^* = \sum_{\sigma(u)=k} R_u R_u^* = P_{[\delta_w | \sigma(w) \geq k]}.$$

Let $\mathcal{E} = C^*(L_u, R_u \mid u \in W)$ be the (two-sided) Toeplitz algebra. Denote by $J_j = \langle P^j \rangle$ the closed two-sided ideals generated by P^j in $\mathcal{E}$. In the rank 1 case there is only the ideal J_1 of compact operators on the Fock space. But for $r > 1$ the ideals J_k strictly contain the compact operators and are different ideals.

Lemma 2.1. *We have* $\bigcap_{j=1}^r J_j = \mathcal{K}(\mathsf{F})$.

Proof. It is easy to see that $\mathcal{K}(\mathsf{F}) \subseteq J_1 \cdots J_r$ since $P_a^1 \dots P_a^r = P_\Omega$ is a rank one projection. For the opposite inclusion we prove that if x_k is a finite product of L_u, R_u, L_u^*, R_u^*, the element $x_1 P^1 x_2 P^2 \dots x_r P^r x_{r+1}$ is a finite rank operator. To this end observe that the shape of any element of the form $x_j P^j x_{j+1} \delta_w$ is bounded in the direction j by a constant depending only on x_j. This is true because the shape of $P^j x_{j+1} \delta_w$ is 0 in the direction j. It is clear that the shape of $x_1 P^1 x_2 P^2 \dots x_r P^r x_{r+1} \delta_w$ is bounded in all directions by a constant depending only on $x_1, \dots, x_r$, hence it is a finite rank operator. It follows that $J_1 \cdots J_r \subseteq \mathcal{K}(\mathsf{F})$. ☐

For a nonempty subset $S \subseteq \{1, \ldots, r\}$ let us define

$$J^S = \bigcap_{j \in S} J_j \quad \text{and} \quad J_S = \sum_{j\in S} J_j.$$

So the smallest ideal in $\mathcal{E}$ is $J^{\{1,\ldots,r\}} = \mathcal{K}(\mathsf{F})$ whereas $J_1 + J_2 + \ldots + J_r = J_{\{1,\ldots,r\}}$ is the largest. Indeed the quotient by the latter is $O_\Lambda \otimes O_{\Lambda^{op}}$ as is shown in the next Theorem. Especially later we will use the abreviation $\{k, k+1, \ldots, l\} = \overline{k,l}$.

Theorem 2.2. *Denoting the generators of O_Λ, $O_{\Lambda^{op}}$ by s_u, t_u there is an isomorphism $\mathcal{E}/J_{\{1,\ldots,r\}} \longrightarrow O_\Lambda \otimes O_{\Lambda^{op}}$ given by $\widehat{L_u} \mapsto s_u \otimes 1$ and $\widehat{R_u} \mapsto 1 \otimes t_u$.*

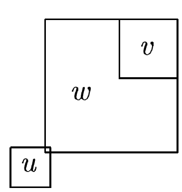
Proof. $\mathcal{E}/J_{\{1,\ldots,r\}}$ is generated by $\widehat{L_u}$ and $\widehat{R_u}$. We check first that the quotient is not trivial. It is enough to show that $\widehat{P_a} \neq 0$ because $L_u^* L_u = P_{t(u)}$ and $R_u^* R_u = Q_{o(u)}$. From the proof of the previous Lemma we know that if $x, y \in \mathcal{E}$ are fixed in the $*$-algebra generated by all L_u and R_u we can find w with shape big enough in the direction j such that $\|(P_a - yP^j x)\delta_w\| = \|P_a \delta_w\| = 1$. Thus $\operatorname{dist}(P_a, J_j) = 1$ for all j and so $\|\widehat{P_a}\| = 1$.

Next we check that the partial isometries $\{\widehat{L_u} \mid u, v \in W\}$ and $\{\widehat{R_u} \mid u, v \in W\}$ satisfy the defining relations for O_Λ respectively $O_{\Lambda^{op}}$. Clearly $\widehat{P_a}\widehat{P_b} = \widehat{P_\Omega} = 0$ for $a \neq b$ since $P_\Omega \in \mathcal{K} \subseteq J_1 + \ldots + J_r = J_{\{1,\ldots,r\}}$. Also $\widehat{L_u}\widehat{L_v} = \widehat{L_{uv}}$ for $u, v \in \dot{W}$ and $\widehat{L_u}^* \widehat{L_u} = \widehat{P_{t(u)}}$ are verified because this is already true in $\mathcal{E}$. Since $P_a^j \in J_{\{1,\ldots,r\}}$ for all j it follows that

$$\widehat{P_a} = \sum_{\substack{o(u)=a \\ \sigma(u)=e_j}} \widehat{L_u L_u^*} + \widehat{P_a^j} = \sum_{\substack{o(u)=a \\ \sigma(u)=e_j}} \widehat{L_u L_u^*}.$$

This shows that the $\widehat{L_u}$ are generators for O_Λ (cf. [KuPa], Remark 1.6.(iii)). Similarly the $\widehat{R_u}$ generate $O_{\Lambda^{op}}$.

Finally we show that $C^*(\widehat{L_u})$ commutes with $C^*(\widehat{R_v})$. It is clear from the definition that $L_u R_v = R_v L_u$ so it is enough to show that $\widehat{L_u R_v^*} = \widehat{R_v^* L_u}$. We show that $L_u R_v^* - R_v^* L_u = (L_u R_v^* - R_v^* L_u)X$ with X a projection in $J_{\{1,\ldots,r\}}$. It is easily seen that $L_u R_v^* \delta_w = R_v^* L_u \delta_w$ whenever $\sigma(w) \geq \sigma(v)$. This may be illustrated for $r = 2$ by the following diagram.



Now let

$$X_j = P^j + \sum_{\sigma(u)=e_j} L_u P^j L_u^* + \sum_{\sigma(u)=2e_j} L_u P^j L_u^* + \cdots + \sum_{\sigma(u)=le_j} L_u P^j L_u^*.$$

where $l = \sigma(v)_j$. X_j is the projection onto words w with $0 \leq \sigma(w)_j \leq \sigma(v)_j$. $X_1, \ldots, X_r$ is a commuting family of projections in $J_{\{1,\ldots,r\}}$ hence the projection X onto the range of $X_1 + \cdots + X_r$ is also in $J_{\{1,\ldots,r\}}$ and this X is as required. $\square$

3. The long exact sequence

In this section we shall construct an r-fold exact sequence. As we have already pointed out, this is inspired by the particular case of tensor products of ordinary Cuntz-Krieger algebras. For example if $O_\Lambda = O_A \otimes O_B$ then the K-homology class is given by a 2-fold exact sequence after tensoring and splicing (see [KaPu] for notations):

$$0 \to \mathcal{K}(\mathcal{F}_A)\otimes\mathcal{K}(\mathcal{F}_B) \to \mathcal{E}_A\otimes\mathcal{K}(\mathcal{F}_B) \to O_A\otimes O_{A^T}\otimes\mathcal{E}_B \to O_A\otimes O_{A^T}\otimes O_B\otimes O_{B^T} \to 0$$

If J_1 and J_2 are the ideals generated in $\mathcal{E}_A \otimes \mathcal{E}_B$ by $P_\Omega \otimes 1$ respectively $1 \otimes P_\Omega$ we see that

$$\begin{aligned}
\mathcal{K}(\mathcal{F}_A) \otimes \mathcal{K}(\mathcal{F}_B) &= J_1 \cap J_2 = J_1 J_2 \\
\mathcal{E}_A \otimes \mathcal{K}(\mathcal{F}_B) &= J_2 \\
O_A \otimes O_{A^T} \otimes \mathcal{E}_B &= \mathcal{E}_A \otimes \mathcal{E}_B \big/ J_1 \\
O_A \otimes O_{A^T} \otimes O_B \otimes O_{B^T} &= \mathcal{E}_A \otimes \mathcal{E}_B \big/ J_1 + J_2.
\end{aligned}$$

From this class of examples we are lead to define

$$\begin{aligned}
\mathcal{E}_0 &= J_1 \cap \ldots \cap J_r = J^{\overline{1,r}} = \mathcal{K}(\mathsf{F}) \\
\mathcal{E}_1 &= J_2 \cap \ldots \cap J_r = J^{\overline{2,r}} \\
\mathcal{E}_k &= J_{k+1} \cap \ldots \cap J_r \big/ (J_1 + \ldots + J_{k-1}) \cap J_{k+1} \cap \ldots \cap J_r = J^{\overline{k+1,r}} \big/ J_{\overline{1,k-1}} \cap J^{\overline{k+1,r}},
\end{aligned}$$

for $k \in \{2, \ldots r-1\}$ and

$$\begin{aligned}
\mathcal{E}_r &= \mathcal{E} \big/ J_1 + \ldots + J_{r-1} = \mathcal{E} \big/ J^{\overline{1,r-1}} \\
\mathcal{E}_{r+1} &= \mathcal{E} \big/ J_1 + \ldots + J_r = \mathcal{E} \big/ J^{\overline{1,r}}.
\end{aligned}$$

$\mathcal{E}_0 \subseteq \mathcal{E}_1$ so we can define $i_0 : \mathcal{E}_0 \hookrightarrow \mathcal{E}_1$.

$\mathcal{E}_1 \subseteq J_3 \cap \ldots \cap J_r$ so there is a map $i_1 : \mathcal{E}_1 \to \mathcal{E}_2$ given by the inclusion composed with the quotient map.

For $k \in \{2, \ldots, r-2\}$ we have

$$J^{\overline{k+1,r}} = J_{k+1} \cap \ldots \cap J_r \subseteq J_{k+2} \cap \ldots \cap J_r = J^{\overline{k+2,r}}$$

and

$$J_{\overline{1,k-1}} \cap J^{\overline{k+1,r}} \subseteq J_{\overline{1,k}} \cap J^{\overline{k+2,r}}$$

so that we can again define a map $i_k : \mathcal{E}_k \to \mathcal{E}_{k+1}$. Writing $\mathcal{E}_{r-1} = J_r \big/ J_{\overline{1,r-2}} = J_1 + \cdots + J_{r-2} + J_r \big/ J_1 + \cdots + J_{r-2}$ there is also a natural homomorphism $i_{r-1} : \mathcal{E}_{r-1} \to \mathcal{E}_r$ since $J_1 + \cdots + J_{r-2} + J_r \subseteq \mathcal{E}$ and $J_1 + \cdots + J_{r-2} \subseteq J_1 + \cdots + J_{r-1}$. Finally $i_r : \mathcal{E}_r \to \mathcal{E}_{r+1}$ is defined since $J_1 + \cdots + J_{r-1} \subseteq J_1 + \cdots + J_r$.

Proposition 3.1. *The r-fold sequence*

$$0 \longrightarrow \mathcal{E}_0 \xrightarrow{i_0} \mathcal{E}_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{r-1}} \mathcal{E}_r \xrightarrow{i_r} \mathcal{E}_{r+1} \longrightarrow 0$$

is exact and such that $\mathcal{E}_0 = \mathcal{K}(\mathsf{F})$ *and* $\mathcal{E}_{r+1} = O_\Lambda \otimes O_{\Lambda^{op}}$.

Proof. $\mathcal{E}_0 = \text{Im } i_0 = \text{Ker } i_1 = \mathcal{E}_1 \cap J_1 \cap \ldots \cap J_r = J_1 \cap \ldots \cap J_r$. For any $k \in \overline{1, r-1}$ we have

$$\text{Im } i_k = J^{\overline{k+1,r}} \Big/ J_{\overline{1,k}} \cap J^{\overline{k+2,r}} \subset \mathcal{E}_{k+1},$$

and if $k < r-2$

$$\begin{aligned} \text{Ker } i_{k+1} &= J^{\overline{k+1,r}} \cap J_{\overline{1,k}} \cap J^{\overline{k+2,r}} \Big/ J_{\overline{1,k-1}} \cap J^{\overline{k+1,r}} \\ &= J_{\overline{1,k}} \cap J^{\overline{k+1,r}} \Big/ J_{\overline{1,k-1}} \cap J^{\overline{k+1,r}} \\ &= J^{\overline{k,r}} \Big/ J_{\overline{1,k-1}} \cap J^{\overline{k+1,r}} \subseteq \mathcal{E}_k. \end{aligned}$$

Finally i_r is clearly surjective,

$$\mathcal{E}_{r-1} = J_r \Big/ J_{\overline{1,r-2}} \cap J_r \cong J_{\overline{1,r-2}} + J_r \Big/ J_{\overline{1,r-2}}$$

so that $\text{Im } i_{r-1} = J_{\overline{1,r}} \Big/ J_{\overline{1,r-1}} = \text{Ker } i_r$ and $\text{Ker } i_{r-1} = J_{r-1} \cap J_r \Big/ J_{\overline{1,r-2}} \cap J_r = \text{Im } i_{r-2}$. This shows the exactness of the long sequence. From Lemma 2.1 we have $\mathcal{E}_0 = \mathcal{K}(\mathsf{F})$ and from Theorem 2.2 $\mathcal{E}_{r+1} = O_\Lambda \otimes O_{\Lambda^{op}}$. □

By [Ze] if this sequence is completely semisplit it defines an element $\Delta \in KK^r(O_\Lambda \otimes O_{\Lambda^{op}}, \mathbb{C}) = K^r(O_\Lambda \otimes O_{\Lambda^{op}})$. We have not been able to prove semisplitness. However, we can associate an element in $E^r(O_\Lambda \otimes O_{\Lambda^{op}}, \mathbb{C})$ by considering it as a splice or r short exact sequences

$$0 \to \text{Ker } i_k \to \mathcal{E}_k \to \text{Im } i_k \to 0.$$

Each sequence gives rise to a class in $E(\mathcal{S}\text{Im } i_k, \text{Ker } i_k) = E^1(\text{Im } i_k, \text{Ker } i_k)$ so we can define $\Delta \in E^r$ as the product of them.

Remark 3.2. (i) If $\mathcal{E}$ is nuclear then the sequences $0 \to \text{Ker } i_k \to \mathcal{E}_k \to \text{Im } i_k \to 0$ consist of nuclear C^*-algebras and so the long sequence is completely semisplit. In this case we can work with Zekri's *Ext*-theory. The advantage of E-theory is that we do not need the semisplitness condition. Is $\mathcal{E}$ nuclear in general? We have not been able to prove it.

(ii) The above describes a general procedure for obtaining an $r+2$-term exact sequence from an r-tuple of ideals $J_1, \ldots, J_r$ in a C^*-algebra B. If B is nuclear this defines an element in $KK^r\left(B \big/ J_{\overline{1,r}}, \mathbb{C}\right)$. Is there a converse of this process i.e. given $x \in KK^r(A, \mathbb{C})$ can we find a Toeplitz algebra B with suitable ideals such that x is given by the above sequence?

4. The K-theory class

We shall define $\delta \in E^r(\mathbb{C}, O_\Lambda \otimes O_{\Lambda^{op}}) = KK(\mathcal{S}^{\otimes r}, O_\Lambda \otimes O_{\Lambda^{op}})$ by restricting a morphism $w : C(\mathbb{T}^r) \to O_\Lambda \otimes O_{\Lambda^{op}}$ to $\mathcal{S}^{\otimes r}$. We shall use the same letter for w and its restriction. To define w it suffices to find commuting normal partial isometries $w_1, \ldots, w_r$ in $O_\Lambda \otimes O_{\Lambda^{op}}$ because then we can define $w(z_k) = w_k$ and its restriction to $\mathcal{S}^{\otimes r}$ is given by $w(z_k - 1) = w_k - 1$. Inspired by [KaPu] we define

$$w_k = \sum_{\sigma(u) = e_k} s_u^* \otimes t_u.$$

Note that in the rank one case this element is different from that in [KaPu] because the Fock space and the Toeplitz algebra are slightly different in our setting being defined in terms of the edge matrix (see [KPRR] for the graph approach to Cuntz-Krieger algebras).

Proposition 4.1. *For all i, j we have*

(i) $w_i w_j = w_j w_i$,
(ii) $w_i^* w_j = w_j w_i^*$.

Proof. (i)

$$\begin{aligned} w_i w_j &= \sum_{\substack{\sigma(v)=e_j \\ \sigma(u)=e_i}} s_u^* s_v^* \otimes t_u t_v \\ &= \sum_{\substack{\sigma(v)=e_j \\ \sigma(u)=e_i}} (s_v s_u)^* \otimes t_u t_v \\ &= \sum_{\substack{\sigma(v)=e_j \\ \sigma(u)=e_i}} s_{vu}^* \otimes t_{vu} \\ &= \sum_{\sigma(w)=e_j+e_i} s_w^* \otimes t_w. \end{aligned}$$

Note that we used our convention about multiplication in either Λ or Λ^{op} in passing from the above second equality to the third one. Similarly

$$w_j w_i = \sum_{\sigma(w)=e_i+e_j} s_w^* \otimes t_w.$$

(ii) If $i = j$ we have

$$\begin{aligned} w_j w_j^* &= \sum_{\sigma(u)=\sigma(v)=e_j} s_u^* s_v \otimes t_u t_v^* \\ &= \sum_{\sigma(u)=e_j} s_{t(u)} \otimes t_u t_u^* \\ &= \sum_a s_a \otimes \Big(\sum_{\sigma(u)=e_j; t(u)=a} t_u t_u^* \Big) = \sum_{a \in A} s_a \otimes t_a, \end{aligned}$$

where for the second equality we used $s_u^* s_v = s_u^* s_u s_u^* s_v s_v^* s_v = \delta_{u,v} s_{t(u)}$. Similarly

$$w_j^* w_j = \sum_{a \in A} s_a \otimes t_a.$$

If $i \neq j$ we have for any $u, v \in W$ such that $\sigma(u) = e_i, \sigma(v) = e_j, o(u) = o(v)$

$$\begin{aligned} s_v^* s_u &= \sum_{\sigma(x)=e_i;\ \sigma(y)=e_j} s_x s_x^* s_v^* s_u s_y s_y^* \\ &= \sum_{\sigma(x)=e_i;\ \sigma(y)=e_j} s_x s_{vx}^* s_{uy} s_y^* \\ &= \sum_{\substack{\sigma(x)=e_i;\ \sigma(y)=e_j \\ vx=uy}} s_x s_y^* \end{aligned}$$

and hence

$$\begin{aligned} w_i^* w_j &= \sum_{\substack{\sigma(v)=e_j \\ \sigma(u)=e_i}} s_u s_v^* \otimes t_u^* t_v \\ &= \sum_{\substack{\sigma(v)=e_j \\ \sigma(u)=e_i}} s_u s_v^* \otimes \Big(\sum_{\substack{\sigma(x)=e_j;\ \sigma(y)=e_i \\ xu=yv}} t_x t_y^* \Big) \\ &= \sum_{\substack{\sigma(y)=\sigma(u)=e_i \\ \sigma(x)=\sigma(v)=e_j \\ xu=yv}} s_u s_v^* \otimes t_x t_y^*. \end{aligned}$$

Similarly

$$\begin{aligned} w_j w_i^* &= \sum_{\sigma(v)=e_j;\ \sigma(u)=e_i} s_v^* s_u \otimes t_v t_u^* \\ &= \sum_{\substack{\sigma(u)=\sigma(x)=e_i \\ \sigma(v)=\sigma(y)=e_j \\ vx=uy}} s_x s_y^* \otimes t_v t_u^* \\ &= \sum_{\substack{\sigma(y)=\sigma(u)=e_i \\ \sigma(x)=\sigma(v)=e_j \\ xu=yv}} s_u s_v^* \otimes t_x t_y^*. \end{aligned}$$

□

5. The untwist

In this short section we describe an easy homotopy argument (the untwist) which also works in the rank 1 case and simplifies some of the constructions in [KaPu]. Let $\Theta : C(\mathbb{T}^r) \otimes O_\Lambda \longrightarrow C(\mathbb{T}^r) \otimes O_\Lambda$, $\Theta(f)(z) = \gamma_z(f(z))$, where γ is the gauge action on O_Λ. (Its restriction to $\mathcal{S}^{\otimes r} \otimes O_\Lambda$ will also be denoted by Θ.)

Theorem 5.1. *The restriction of Θ to $\mathcal{S}^{\otimes r} \otimes O_\Lambda$ is homotopic to the identity, so that $[\Theta]$ and $[id]$ (or 1) give rise to the same element in $E^0(\mathcal{S}^{\otimes r} \otimes O_\Lambda, \mathcal{S}^{\otimes r} \otimes O_\Lambda)$.*

Proof. We identify $\mathcal{S} = C_0(\mathbb{R})$ with $\{f \in C(\mathbb{T}) \mid f(1) = 0\} \subseteq C(\mathbb{T})$, using the Cayley transformation $\mathbb{R} \ni t \longmapsto \frac{t+i}{t-i} \in \mathbb{T}$. Θ is then given by

$$\Theta(f)(t) = \gamma_{z_t}(f(t)) \quad \text{where} \quad t \in \mathbb{R}^r$$

and $z_t = \left(\frac{t_j - i}{t_j + i}\right)_{j=\overline{1,r}} \in \mathbb{T}^r$. For any $s \in [0,1]$ let

$$\Theta_s(f)(t) = \gamma_{z_t^s}(f(t)) \text{ with } z_t^s = \left(\frac{t_j s - i}{t_j s + i} e^{i(1-s)\pi}\right)_{j=\overline{1,r}} \in \mathbb{T}^r.$$

This family is pointwise continuous that is for any $f \in C_0(\mathbb{R}^r, O_\Lambda)$ the family $s \mapsto \Theta_s(f)$ is continuous on $[0,1]$. Indeed, this is more or less clear whenever $f \in C_c(\mathbb{R}^r, O_\Lambda)$ i.e. has compact support and since $\|\Theta_s\| \leq 1$ for all $s \in [0,1]$ and $C_c(\mathbb{R}^r, O_M)$ is dense in $C_0(\mathbb{R}^r, O_\Lambda)$ it is true for all $f \in C_0(\mathbb{R}^r, O_\Lambda)$. Now it is clear that $\Theta_0 = id$ and $\Theta_1 = \Theta$ defining the desired homotopy between id and Θ. □

6. Computation of the product

The computation will be done mainly with exact sequences. We need the following Lemma ([GHT], Prop 5.8). It is in turn an E-theoretic version of Lemma 1.5 of [Sk]. Using the associativity of composition in E-theory, it is still true if we use longer horizontal sequences.

Lemma 6.1. *Suppose we have the following commutative diagram*

$$\begin{array}{ccccccccc} 0 \to & B_1 & \to & D_1 & \to & A_1 & \to 0 \\ & \downarrow \phi & & \downarrow \phi & & \downarrow \psi & \\ 0 \to & B_2 & \to & D_2 & \to & A_2 & \to 0 \end{array}$$

the horizontal sequences giving rise to E-theory classes $\Delta_1 \in E(\mathcal{S}A_1, B_1)$ and $\Delta_2 \in E(\mathcal{S}A_2, B_2)$. Then we have $\psi^*(\Delta_2) = \phi_*(\Delta_1) \in E(\mathcal{S}A_1, B_2)$

As in [KaPu] let $\tau^D : E^i(A,B) \to E^i(D \otimes A, D \otimes B)$ and $\tau_D : E^i(A,B) \to E^i(A \otimes D, B \otimes D)$ denote the standard maps for C^*-algebras A, B, D. We shall compute $\delta \otimes_{O_{\Lambda^{op}}} \Delta = \tau_{O_\Lambda}(\delta) \otimes \tau^{O_\Lambda}(\Delta)$ in $E^0(\mathcal{S}^{\otimes r} \otimes O_\Lambda, \mathcal{S}^{\otimes r} \otimes O_\Lambda)$ by determining $[\Theta] \otimes \tau_{O_\Lambda}(\delta) \otimes \tau^{O_\Lambda}(\Delta) = [\tau_{O_\Lambda}(\delta) \circ \Theta] \otimes \tau^{O_\Lambda}(\Delta)$ which is equal to $\delta \otimes_{O_{\Lambda^{op}}} \Delta$ because $[\Theta] = [id]$. Since $\tau^{O_\Lambda}(\delta) \circ \Theta$ is a morphism, the previous Lemma shows that this product is a pull-back given by the top row of the following commutative diagram:

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & O_\Lambda \otimes \mathcal{E}_0 & \longrightarrow & \mathcal{E}'_1 & \longrightarrow & \cdots & \longrightarrow & \mathcal{E}'_r & \longrightarrow & C(\mathbb{T}^r) \otimes O_\Lambda & \longrightarrow & 0 \\ & & \Big\| & & & & & & & & \downarrow \Theta & & \\ & & & & & & & & & & C(\mathbb{T}^r) \otimes O_\Lambda & & \\ & & & & & & & & & & \downarrow \delta \otimes id & & \\ 0 & \longrightarrow & O_\Lambda \otimes \mathcal{E}_0 & \longrightarrow & O_\Lambda \otimes \mathcal{E}_1 & \longrightarrow & \cdots & \longrightarrow & O_\Lambda \otimes \mathcal{E}_r & \longrightarrow & O_\Lambda \otimes O_{\Lambda^{op}} \otimes O_\Lambda & \longrightarrow & 0 \end{array}$$

More precisely, $\delta \otimes_{O_{\Lambda^{op}}} \Delta$ corresponds to a further pull-back of the top row ending with $\mathcal{S} \otimes O_\Lambda$.

Thinking of the r-fold exact sequence as r splices of 1-fold exact sequences this pull-back is constructed inductively. Only $\mathcal{E}'_r$ is changed.

The next task is to show that this sequence and $\tau^{O_\Lambda}(\mathcal{T}^{\otimes r})$ give rise to the same E-theory class in $E^r(\mathcal{S}^{\otimes r} \otimes O_\Lambda, O_{\Lambda^{op}})$. Since $\mathcal{T}^{\otimes r} \in E^r(\mathcal{S}^r, \mathbb{C})$ corresponds to $1_\mathbb{C} \in KK^0(\mathbb{C}, \mathbb{C})$ via Bott periodicity and $\tau^{O_\Lambda}(1_\mathbb{C}) = 1_{O_\Lambda}$ this indeed proves $\delta \otimes_{O_{\Lambda^{op}}} \Delta = 1_{O_\Lambda}$.

Once we have that we can show that $\delta \otimes_{O_\Lambda} \Delta = (-1)^r 1_{O_{\Lambda^{op}}}$ analogously, by exchanging the roles of Λ and Λ^{op}. The sign $(-1)^r$ appears since $w_k = \sum_{\sigma(u)=e_k} s_u^* \otimes t_u$ has to be replaced by $\bar{w}_k = \sum_{\sigma(u)=e_k} t_u^* \otimes s_u$ when swaping O_Λ and $O_{\Lambda^{op}}$. The corresponding K-theory class is $(-1)^r \delta$.

The idea is to find a subsequence of the previous pull-back isomorphic to $\tau^{O_\Lambda}(\mathcal{T}^{\otimes r})$ (in the strongest possible way) such that the vertical inclusion at the left end is an equivalence and at the right end is equality (Prop.6.6). The previous lemma then finishes the proof.

Let

$$W_j = \sum_{\sigma(u)=e_j} s_u^* \otimes R_u.$$

The W_j are commuting partial isometries with $W_j^* W_j \geq W_j W_j^* \neq W_j^* W_j$. More precisely:

Proposition 6.2. *One has*

(i) $W_j W_i = W_i W_j$;
(ii) $W_i^* W_j = W_j W_i^*$ *for* $i \neq j$;
(iii) $W_j(1 \otimes L_u) = (1 \otimes L_u) W_j$;
(iv) $W_j^*(1 \otimes L_u) = (1 \otimes L_u) W_j^*$ *for* $\sigma(u) = e_i, i \neq j$;
(v) $W_j^* W_j - W_j W_j^* = \sum_{a \in A} s_a \otimes Q_a^j$.

Proof. (i) The relations

$$W_j W_i = \sum_{\sigma(w)=e_i+e_j} s_w^* \otimes R_w = W_i W_j$$

follow exactly as in Proposition 4.1 (i) replacing t_w by R_w.

(ii) We have

$$W_i^* W_j = \sum_{\sigma(u)=e_i;\sigma(v)=e_j} s_u s_v^* \otimes R_u^* R_v = \sum_{\sigma(u)=e_i;\sigma(v)=e_j} s_u s_v^* \otimes R_u^* R_v$$

and so as in 4.1 (ii)

$$\begin{aligned} W_j W_i^* &= \sum_{\sigma(u)=e_i;\sigma(v)=e_j} s_v^* s_u \otimes R_v R_u^* \\ &= \sum_{\substack{\sigma(v)=\sigma(y)=e_j \\ \sigma(u)=\sigma(x)=e_i}} s_x s_{vx}^* s_{uy} s_y^* \otimes R_v R_u^* \\ &= \sum_{\substack{\sigma(x)=\sigma(u)=e_i \\ \sigma(y)=\sigma(v)=e_j \\ vx=uy}} s_x s_y^* \otimes R_v R_u^* \end{aligned}$$

for the case $i \neq j$.

Therefore it is enough to show that $R_x^* R_y = \sum_{vx=uy} R_v R_u^*$, where the sum is taken subject to $\sigma(u) = \sigma(x) = e_i, \sigma(v) = \sigma(y) = e_j$. We note here that in the following calculation the product is taken in Λ not in Λ^{op}. If $w \in W$ is such that $R_x^* \delta_{wy} = R_x^* R_y \delta_w \neq 0$ then $wy = w_1 x$, in particular $\sigma(w)_i > 0$ since $i \neq j$ and

$(\sigma(w)+\sigma(y))_i \geq \sigma(x)_i$. We have $w = w'u$ for some u with $\sigma(u) = \sigma(x) = e_i$. Similarly $w_1 = w_1'v$ with $\sigma(v) = \sigma(y) = e_j$. We have $uy = vx$ and $w' = w_1'$ since $w'uy = w_1'vx$ and $\sigma(uy) = \sigma(vx)$.

Thus we obtain

$$R_x^* R_y \delta_w = \delta_{w_1} = \delta_{w_1'v} = R_v R_u^* \delta_{w_1'u} = R_v R_u^* \delta_w,$$

hence

$$\begin{aligned} R_x^* R_y \delta_w &= R_x^* R_y \delta_{w'u} = R_x^* \delta_{w'uy} \\ &= R_x^* \delta_{w'vx} = \delta_{w'v} \\ &= R_v R_u^* \delta_w. \end{aligned}$$

Conversely, if $R_v R_u^* \delta_w \neq 0$ with $vx = uy$ then $w = w'u$ and

$$\begin{aligned} R_x^* R_y \delta_w &= R_x^* \delta_{w'uy} = R_x^* \delta_{w'vx} \\ &= R_x^* \delta_{w'vx} = \delta_{w'v} \\ &= R_v R_u^* \delta_w. \end{aligned}$$

(iii) This is clear from the commutation relations $L_u R_v = R_v L_u$.

(iv) It suffices to show that $R_v^* L_u = L_u R_v^*$ whenever $\sigma(u) = e_i$, $\sigma(v) = e_j$ and $i \neq j$. This is true since if $R_v^* L_u \delta_w \neq 0$ then $uw = w'v$ whence $\sigma(uw) = \sigma(w'v)$ that is $\sigma(u) + \sigma(w) = \sigma(w') + \sigma(v)$. Since $\sigma(u) = e_i \neq e_j = \sigma(v)$ we must have $\sigma(w)_j > 0$ that is $w = w''x$ with $\sigma(x) = e_j = \sigma(v)$. From $uw = w'v$ we deduce $x = v$ and $L_u R_v^* \delta_w = \delta_{uw''} = R_v^* L_u \delta_{w''v} = R_v^* L_u \delta_w$. Conversely, if $L_u R_v^* \delta_w \neq 0$ then $w = w'v$ and $L_u R_v^* \delta_w = \delta_{uw'} = R_v^* L_u \delta_{w'v} = R_v^* L_u \delta_w$.

(v) We compute both products.

$$\begin{aligned} W_j^* W_j &= \sum_{\sigma(u)=\sigma(v)=e_j} s_u s_v^* \otimes R_u^* R_v \\ &= \sum_{\sigma(u)=e_j} s_u s_u^* \otimes R_u^* R_u \\ &= \sum_{\sigma(u)=e_j} s_u s_u^* \otimes Q_{o(u)} \\ &= \sum_{a \in A} s_a \otimes Q_a \end{aligned}$$

and on the other hand

$$\begin{aligned} W_j W_j^* &= \sum_{\sigma(u)=e_j} s_{t(u)} \otimes R_u R_u^* \\ &= \sum_{\sigma(u)=e_j} s_{t(u)} \otimes (R_u^* R_u - Q_{t(u)}^j) \\ &= \sum_{a \in A} s_a \otimes (Q_a - Q_a^j). \end{aligned}$$

□

Proposition 6.3. *For any i, j and $u \in W$ with $\sigma(u) = e_i$ we have the following commutation relations:*

(i) $[W_j, W_i^*(1 \otimes L_u)] = 0$;
(ii) $[W_j^*, W_i^*(1 \otimes L_u)] = 0$.

Proof. In case $i \neq j$ (i) and (ii) follow from the relations (i)-(iv) of the previous Proposition. So we only have to consider the case where $i = j$. From (v) of the previous Proposition we have

$$W_j W_j^*(1 \otimes L_u) = (W_j^* W_j - \sum_{a \in A} s_a \otimes Q_a^j)(1 \otimes L_u).$$

Since $Q_a^j L_u = 0$ for $\sigma(u) = e_j$ it follows that

$$W_j W_j^*(1 \otimes L_u) = W_j^* W_j (1 \otimes L_u) = W_j^*(1 \otimes L_u) W_j.$$

This shows (i). Next we have

$$\begin{aligned} W_j^* W_j^*(1 \otimes L_u) &= W_j^* \sum_{\sigma(v)=e_j} s_v \otimes R_v^* L_u \\ &= W_j^* \left(\sum_{\sigma(v)=e_j} s_v \otimes L_u R_v^* + \sum_{\sigma(v)=e_j} s_v \otimes (R_v^* L_u - L_u R_v^*) \right) \\ &= W_j^* \left(\sum_{\sigma(v)=e_j} s_v \otimes L_u R_v^* \right) + W_j^* \sum_{\sigma(v)=e_j} s_v \otimes (R_v^* L_u - L_u R_v^*) \\ &= W_j^* \left(\sum_{\sigma(v)=e_j} s_v \otimes L_u R_v^* \right) + \\ &\qquad + \sum_{\sigma(v)=\sigma(w)=e_j} s_w s_v \otimes R_w^*(R_v^* L_u - L_u R_v^*). \end{aligned}$$

The first summand in the last expression is $W_j^*(1 \otimes L_u) W_j^*$. The second summand is 0 since $R_w^*(R_v^* L_u - L_u R_v^*) = 0$ for any $u, v, w \in W$ with $\sigma(u) = \sigma(v) = \sigma(w) = e_i$. Indeed, then $R_v^* L_u \delta_x \neq L_u R_v^* \delta_x$ if and only if $\sigma(x)_j = 0$. But in this case $\sigma((R_v^* L_u - L_u R_v^*)\delta_x)_j = 0$ so that $R_w^*(R_v^* L_u - L_u R_v^*)\delta_x = 0$. □

Now let $W^{\sigma(u)} = W_1^{\sigma(u)_1} W_2^{\sigma(u)_2} \cdots W_r^{\sigma(u)_r}$ and $V_u = \left[W^{\sigma(u)}\right]^*(1 \otimes L_u)$.

Proposition 6.4. *$C^*(V_u \mid u \in W) \cong O_\Lambda$. Denoting generators of O_Λ by r_u the isomorphism is given by $V_u \mapsto r_u$.*

Proof. Since one checks that $V_{uw} = V_u V_w$ for all u, w (by 6.2.(iii) and 6.3) we only need to show that if $\sigma(u) = e_i$ then (a) $V_u^* V_u$ is a projection depending only on $a = t(u)$ which (b) equals $\sum_{\sigma(v)=e_j; o(v)=t(u)} V_v V_v^*$ for all j.

To show (a) using 6.2 and $W_i(1 \otimes L_u) = (1 \otimes L_u) W_i$ we obtain

$$\begin{aligned} V_u^* V_u &= (1 \otimes L_u^*) W_i W_i^* (1 \otimes L_u) = W_i^*(1 \otimes L_u^*)(1 \otimes L_u) W_i = W_i^*(1 \otimes P_a) W_i \\ &= \sum_{\sigma(u)=\sigma(v)=e_i} s_u s_v^* \otimes R_u^* P_a R_v = \sum_{\sigma(u)=e_i} s_u \otimes R_u^* P_a R_u \\ &= \sum_{b \in A} s_b \otimes (Q_b P_a). \end{aligned}$$

Thus indeed $V_u^* V_u = W_i^*(1 \otimes P_a) W_i$ does not depend on i but only on $a = t(u)$.

To show (b) observe

$$\begin{aligned} \sum_{\substack{o(v)=a \\ \sigma(v)=e_j}} V_v V_v^* &= \sum_{\substack{o(v)=a \\ \sigma(v)=e_j}} W_j^*(1 \otimes L_v)(1 \otimes L_v^*) W_j \\ &= W_j^*(1 \otimes (P_a - P_a^j)) W_j. \end{aligned}$$

But $W_j^*(1 \otimes P_a^j) = \sum_{\sigma(u)=e_j} (s_u \otimes R_u^* P_a^j) = 0$ and so

$$\sum_{\substack{o(v)=a \\ \sigma(v)=e_j}} V_v V_v^* = W_j^*(1 \otimes P_a) W_j = V_u^* V_u$$

as required. □

Now let $\mathcal{F}_k$ be the ideal in $C^*(W_j, V_u \mid u \in W, j \in \{1, \dots, r\})$ generated by the set $\{(W_k^* W_k - W_k W_k^*) V_u \mid u \in W\}$. Put

$$\begin{aligned} \mathcal{A}_{k-1} &= \mathcal{F}_k \cap \dots \cap \mathcal{F}_r \qquad \text{for } k \in \{1, \dots, r\}, \\ \mathcal{A}_r &= C^*(W_l, V_u \mid l \in \{1, \dots, r\}, u \in W) \quad \text{and} \\ \mathcal{A}_{r+1} &= C(\mathbb{T}^r) \otimes O_\Lambda. \end{aligned}$$

This sequence is analogous to the sequence $J_1 \cap \dots \cap J_r$, $J_2 \cap \dots \cap J_r, \dots, J_r$, $\mathcal{E}$, $\mathcal{E} / J_1 + \dots + J_r$ considered in section 3. Applying the same process as there we can define an $r+2$-term exact sequence

$$0 \to \widehat{\mathcal{A}_0} \to \widehat{\mathcal{A}_1} \to \dots \to \widehat{\mathcal{A}_r} \to \widehat{\mathcal{A}_{r+1}} \to 0$$

where

$$\begin{aligned} \widehat{\mathcal{A}_0} &= \mathcal{F}_1 \cap \dots \cap \mathcal{F}_r \\ \widehat{\mathcal{A}_1} &= \mathcal{F}_2 \cap \dots \cap \mathcal{F}_r \\ \widehat{\mathcal{A}_k} &= \mathcal{F}_{k+1} \cap \dots \cap \mathcal{F}_r \Big/ (\mathcal{F}_1 + \dots + \mathcal{F}_{k-1}) \cap \mathcal{F}_{k+1} \cap \dots \cap \mathcal{F}_r \\ \widehat{\mathcal{A}_r} &= \mathcal{A}_r \Big/ \mathcal{F}_1 + \dots + \mathcal{F}_{r-1} \\ \widehat{\mathcal{A}_{r+1}} &= \mathcal{A}_r \Big/ \mathcal{F}_1 + \dots + \mathcal{F}_r. \end{aligned}$$

Proposition 6.5. *$\mathcal{A}_0 \subseteq \mathcal{E}_0 \otimes O_\Lambda$, $\mathcal{A}_1 \subseteq \mathcal{E}_1 \otimes O_\Lambda$, $\mathcal{A}_k \subseteq (J_{k+1} \cap \dots \cap J_r) \otimes O_\Lambda$ for $k \in \{2, \dots, r-1\}$ so that $\widehat{\mathcal{A}_k} \subseteq \mathcal{E}_k \otimes O_\Lambda$. Moreover the following diagram is* *commutative with the vertical arrows isomorphisms*

$$\begin{array}{ccccccccc} 0\to & \widehat{\mathcal{A}_0} & \to & \widehat{\mathcal{A}_1} & \to\cdots\to & \widehat{\mathcal{A}_r} & \to & C(\mathbb{T}^r)\otimes O_\Lambda & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \| & \\ 0\to & \mathcal{T}_0\otimes O_\Lambda & \to & \mathcal{T}_1\otimes O_\Lambda & \to\cdots\to & \mathcal{T}_r\otimes O_\Lambda & \to & C(\mathbb{T}^r)\otimes O_\Lambda & \to 0. \end{array}$$

Proof. There are obvious inclusions $\mathcal{F}_k \subseteq J_k \otimes O_\Lambda$ since $[W_k^*, W_k] \in J_k \otimes O_\Lambda$ by Prop.6.2.(v) so that $\mathcal{A}_k \subseteq (J_{k+1} \cap \ldots \cap J_r) \otimes O_\Lambda$ and thus $\widehat{\mathcal{A}_k} \subseteq \mathcal{E}_k \otimes O_\Lambda$. The maps $i_k \otimes id_{O_\Lambda}$ can be restricted to $\widehat{\mathcal{A}_k}$ and the exactness of the top line in the above diagram follows as in Prop.3.1. W_l and V_u commute with each other for any $u \in W$ and $l \in \{1, \ldots, r\}$ so that the vertical arrows are defined by sending W_k to $S_k \otimes 1$ and V_u to $1 \otimes r_u$. Note that $\sum_{a\in A} s_a \otimes Q_a$ is a unit in each algebra $\mathcal{A}_k$. These arrows are isomorphisms because of universal properties of the Toeplitz algebra, $C(\mathbb{T})$ and O_Λ. □

Next we shall show that this newly defined sequence is a subsequence of the sequence formed by the $\mathcal{E}'_k$ and both give rise to the same class at the level of E-groups.

Proposition 6.6. *There is a commutative diagram*

$$\begin{array}{ccccccccc} 0\to & O_\Lambda\otimes\mathcal{E}_0 & \to & \mathcal{E}'_1 & \to\cdots\to & \mathcal{E}'_r & \to & C(\mathbb{T}^r)\otimes O_\Lambda & \to 0 \\ & \uparrow & & \uparrow & & \uparrow & & \| & \\ 0\to & \widehat{\mathcal{A}_0} & \to & \widehat{\mathcal{A}_1} & \to\cdots\to & \widehat{\mathcal{A}_r} & \to & C(\mathbb{T}^r)\otimes O_\Lambda & \to 0 \end{array}$$

where the vertical arrows represent inclusions. Moreover the inclusion $\mathcal{A}_0 \hookrightarrow O_\Lambda \otimes \mathcal{E}_0$ is an equivalence in E-theory.

Proof. Only the last statement is nontrivial. To prove it we shall find a subalgebra in $\mathcal{A}_0$ which is a full corner in both $\mathcal{A}_0$ and $\mathcal{E}_0 \otimes O_\Lambda$. Using (v) of Proposition 6.2 we have

$$(W_0^*W_0 - W_0W_0^*)\ldots(W_r^*W_r - W_rW_r^*) = \sum_{a\in A} s_a \otimes P_\Omega = 1 \otimes P_\Omega.$$

Moreover

$$V_u(1 \otimes P_\Omega) = s_u \otimes P_\Omega.$$

Thus the algebra $O_\Lambda \otimes P_\Omega$ is a full corner in both $\mathcal{A}_0$ and $\mathcal{E}_0 \otimes O_\Lambda$. By Lemma 6.1 it follows that both exact sequences in the above diagram indeed represent the same element in E^r. □

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