

Twisted Homology of $SL_q(2)$

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Differential calculi

• Let \mathcal{A} be a unital associative algebra over \mathbb{C} . A differential calculus (DC) over \mathcal{A} consists of

1. A \mathbb{N}_0 -graded algebra $\Omega = \bigoplus_{n \geq 0} \Omega_n$ with $\Omega_0 = \mathcal{A}$.

2. A linear map $d : \Omega \rightarrow \Omega$ such that

$$d(\Omega_n) \subset \Omega_{n+1},$$

$$\Omega_n = \text{span}\{a_0 da_1 \cdots da_n, da_1 \cdots da_n\},$$

$$d(\omega \cdot \eta) = d\omega \cdot \eta + (-1)^n \omega \cdot d\eta, \quad \omega \in \Omega_n, \eta \in \Omega_m,$$

$$d \circ d = 0.$$

• The elements of Ω_n are called the n -forms of the DC and d is called its differential.

Closed graded σ -traces

- By an automorphism of a DC (Ω, d) we mean an algebra automorphism σ of Ω such that

$$\sigma(\Omega_n) = \Omega_n, \quad \sigma \circ d = d \circ \sigma.$$

- If σ is an automorphism of a DC (Ω, d) , then a linear functional

$$\int : \Omega \rightarrow \mathbb{C}$$

is called a closed graded σ -trace, if

$$\int d\omega = 0 \quad \forall \omega \in \Omega,$$

$$\int \omega\eta = (-1)^{nm} \int \sigma(\eta)\omega \quad \omega \in \Omega_n, \eta \in \Omega_m.$$

- \int is said to be N -dimensional, if

$$\int \omega = 0 \text{ for } \omega \in \Omega_n, n \neq N.$$

Twisted cyclic cocycles from DC

- Kustermans, Murphy, Tuset: There is a “twisted” version $HC_{\sigma}^*(\mathcal{A})$ of cyclic cohomology depending on an automorphism σ of \mathcal{A} , such that the following result holds:

If \int is an N -dimensional closed graded σ -trace on (Ω, d) , then

$$\varphi(a_0, \dots, a_N) = \int a_0 da_1 \cdots da_N$$

is a σ -twisted cyclic N -cocycle, and any σ -twisted cyclic N -cocycle arises in this way.

- The corresponding result of Connes for $\sigma = \text{id}$ played a central role in the development of cyclic cohomology and leads to its interpretation as an extension of de Rham homology. A main motivation for the above generalization is the theory of covariant differential calculi over quantum groups. These are naturally equipped with closed graded σ -traces derived from the Haar functional of the quantum group.

Simplicial objects

- A simplicial object C_* in a category \mathcal{C} is a sequence of objects $C_n \in \mathcal{C}$ together with morphisms

$$d_i : C_n \rightarrow C_{n-1}, \quad s_j : C_n \rightarrow C_{n+1}, \quad 0 \leq i, j \leq n$$

satisfying

$$d_i \circ d_j = d_{j-1} \circ d_i, \quad s_j \circ s_i = s_i \circ s_{j-1}, \quad i < j,$$
$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & 0 < j - i < k \\ \text{id} & -1 \leq j - i \leq 0 \\ s_j \circ d_{i-1} & j - i < -1. \end{cases}$$

- With the obvious notion of morphism, the simplicial objects in a category form itself a category which we will denote by $\Delta(\mathcal{C})$.

Simplicial homology theories

- If C_* is a simplicial object in \mathcal{C} , define

$$b = \sum_{i=0}^n (-1)^i d_i : C_n \rightarrow C_{n-1}.$$

Then (C_*, b) is a complex, i.e., one has

$$b \circ b = 0.$$

- Hence any functor $\mathcal{C} \rightarrow \Delta(\mathcal{C})$ defines a homology theory, and many examples of homology theories are of this type.

Cyclic and paracyclic objects

- A cyclic object in a category \mathcal{C} is a simplicial object C_* together with an additional morphism

$$t : C_n \rightarrow C_n$$

satisfying

$$t^{n+1} = \text{id},$$

$$d_i \circ t = t \circ d_{i-1}, \quad i > 0,$$

$$d_0 \circ t = d_n,$$

$$s_j \circ t = t \circ s_{j-1}, \quad j > 0,$$

$$s_0 \circ t = t^2 \circ s_n.$$

- The same without $t^{n+1} = \text{id}$ is called a paracyclic object,

The (b, B) -bicomplex

- A cyclic object defines a bicomplex

$$\begin{array}{ccccc}
 & & b \downarrow & & b \downarrow & & b \downarrow & & \\
 & & & & & & & & \\
 & & C_2 & \xleftarrow{B} & C_1 & \xleftarrow{B} & C_0 & & \\
 & & b \downarrow & & b \downarrow & & & & \\
 & & C_1 & \xleftarrow{B} & C_0 & & & & \\
 & & b \downarrow & & & & & & \\
 & & C_0 & & & & & &
 \end{array}$$

with

$$B = (1 + (-1)^n t) \circ t \circ s_n \circ \sum_{j=0}^n (-1)^{nj} t^j.$$

- That this is a bicomplex means

$$b \circ b = B \circ B = b \circ B + B \circ b = 0.$$

Hence the diagonals $(T_* := \bigoplus_{k \geq 0} C_{*-2k}, b + B)$ form a complex. Its homologies are called the cyclic homologies of C_* .

The cyclic object of $HC_*^\sigma(\mathcal{A})$

- Let \mathcal{A} be an algebra and σ be an automorphism. Set $C_n = \mathcal{A}^{\otimes(n+1)}$ and abbreviate $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ by (a_0, a_1, \dots, a_n) . Define linear maps:

$$d_i(a_0, a_1, \dots, a_n) = (a_0, \dots, a_i a_{i+1}, \dots, a_n),$$

$$d_n(a_0, a_1, \dots, a_n) = (\sigma(a_n) a_0, a_1, \dots, a_{n-1}),$$

$$s_j(a_0, a_1, \dots, a_n) = (a_0, \dots, a_j, 1, a_{j+1}, \dots, a_n),$$

$$t(a_0, a_1, \dots, a_n) = (\sigma(a_n), a_0, \dots, a_{n-1}).$$

- For $\sigma = \text{id}$ this gives the standard cyclic object associated to \mathcal{A} leading to $HC_*(\mathcal{A})$. For general σ one obtains only a paracyclic object.
- To get a cyclic object, pass to the cokernels

$$C_n^\sigma = C_n / C_n^1, \quad C_n^1 = \text{im}(\text{id} - T), \quad T = t^{n+1}.$$

Following Kustermans et al. we call its cyclic homology the σ -twisted cyclic homology $HC_*^\sigma(\mathcal{A})$ of \mathcal{A} .

Twisted Hochschild homology (20)

- Let $HH_*^\sigma(\mathcal{A})$ denote the homologies of the columns of the (b, B) -bicomplex,

$$HH_n^\sigma(\mathcal{A}) = H_n(C_*^\sigma, b).$$

For $\sigma = \text{id}$ these are the Hochschild homologies of \mathcal{A} . Hence we call $HH_*^\sigma(\mathcal{A})$ as Kustermans et al. the σ -twisted Hochschild homology of \mathcal{A} .

- If X is a smooth affine variety, $\mathcal{A} = \mathbb{C}[X]$ and $\sigma = \text{id}$, then one has (Hochschild-Kostant-Rosenberg)

$$HH_n(\mathcal{A}) = \Omega_{\text{deRham}}^n(X).$$

In particular, the Hochschild dimension

$$\sup\{n \mid HH_n(\mathcal{A}) \neq 0\}$$

of \mathcal{A} equals $\dim(X)$.

Computation of $HH_*^\sigma(\mathcal{A})$

- The (b, B) -bicomplex with C_n^σ replaced by the original C_n is not a bicomplex, since

$$b \circ B + B \circ b = \text{id} - T.$$

The columns form a complex. It computes the Hochschild homology $H_*(\mathcal{A}, {}_\sigma\mathcal{A})$ with coefficients in the bimodule ${}_\sigma\mathcal{A} = \mathcal{A}$ with bimodule structure

$$a \triangleright b \triangleleft c = \sigma(a)bc.$$

- Define $C_n^0 = \ker(\text{id} - T)$. Then we have:

If $C_n = C_n^0 \oplus C_n^1$, then $H_*(\mathcal{A}, {}_\sigma\mathcal{A}) = HH_*^\sigma(\mathcal{A})$.

Proof: Since $[b, \text{id} - T] = 0$, $C_* = C_*^0 \oplus C_*^1$ as complexes, and we have $HH_*^\sigma(\mathcal{A}) = H_*(C_*^0, b)$ and $H_*(\mathcal{A}, {}_\sigma\mathcal{A}) = HH_*^\sigma(\mathcal{A}) \oplus H_*(C_*^1, b)$. But $(\text{id} - T)|_{C_*^1}$ is a bijection, and on C_*^1 we have

$$b \circ (\text{id} - T)^{-1} \circ B + (\text{id} - T)^{-1} \circ B \circ b = \text{id}.$$

Hence $(\text{id} - T)^{-1} \circ B$ is a contracting homotopy.

- This applies for example when σ is diagonalizable.

Hochschild homology for Hopf algebras

- Let \mathcal{A} be an algebra and \mathcal{M} be an \mathcal{A} -bimodule. Then

$$H_n(\mathcal{A}, \mathcal{M}) = \mathrm{Tor}_n^{\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}}(\mathcal{A}^{\mathrm{op}}, \mathcal{M}).$$

Explicit resolutions usually get lengthy!

- Feng, Tsygan: If \mathcal{A} is a Hopf algebra with coproduct, counit and antipode Δ, ε, S , then define a right \mathcal{A} -module $\mathcal{M}' = \mathcal{M}$ as a vector space with right action

$$m \blacktriangleleft a = \sum S(a_{(2)}) \triangleright m \triangleleft a_{(1)}, \quad a \in \mathcal{A}, m \in \mathcal{M}.$$

Consider $\mathbb{C} = \mathcal{A}/\ker \varepsilon$ as the trivial \mathcal{A} -module. Then

$$H_n(\mathcal{A}, \mathcal{M}) = \mathrm{Tor}_n^{\mathcal{A}}(\mathcal{M}', \mathbb{C}).$$

Proof of Feng-Tsygan

- $\text{Tor}_*^{\mathcal{A}}(\mathcal{M}', \mathbb{C})$ is computed from (C_*, d) with zeroth tensor component of C_* now being \mathcal{M}' and

$$d = \tilde{d}_0 + \sum_{i=1}^{n-1} (-1)^i d_i + (-1)^n \tilde{d}_n,$$

$$\tilde{d}_0(a_0, a_1, \dots, a_n) = (a_0 \blacktriangleleft a_1, a_2, \dots, a_n),$$

$$\tilde{d}_n(a_0, a_1, \dots, a_n) = (\varepsilon(a_n) a_0, a_1, \dots, a_{n-1}).$$

Define two linear maps $\xi, \xi' : C_n \rightarrow C_n$ by

$$\begin{aligned} & \xi(a_0, a_1, \dots, a_n) \\ = & (S((a_1 \dots a_n)_{(2)}) \triangleright a_0, (a_1)_{(1)}, \dots, (a_n)_{(1)}), \\ & \xi'(a_0, \dots, a_n) \\ = & ((a_1 \dots a_n)_{(2)}) \triangleright a_0, (a_1)_{(1)}, \dots, (a_n)_{(1)}. \end{aligned}$$

Then $\xi \circ \xi' = \xi' \circ \xi = \text{id}_{C_n}$. It is easily checked that ξ commutes with d_i for $1 \leq i \leq n-1$ and that $\xi \circ \tilde{d}_i = d_i \circ \xi$, $i = 0, n$. Hence $\xi \circ d = b \circ \xi$ and ξ is an isomorphism of complexes of vector spaces.

The HKR map

- Let $\pi : \mathcal{M}' \rightarrow H_0(\mathcal{A}, \mathcal{M}) = \mathcal{M}/(a \triangleright m - m \triangleleft a)$ be the canonical projection. Then

$$\pi(m \blacktriangleleft a) = \varepsilon(a)\pi(m),$$

and if we consider $H_0(\mathcal{A}, \mathcal{M})$ as trivial right \mathcal{A} -module, then $\pi \otimes \text{id}_{\mathcal{A}^{\otimes n}}$ induces a morphism

$$H_n(\mathcal{A}, \mathcal{M}) \rightarrow H_0(\mathcal{A}, \mathcal{M}) \otimes_{\mathbb{C}} \text{Tor}_n^{\mathcal{A}}(\mathbb{C}, \mathbb{C}).$$

- If \mathcal{A} is commutative and $\mathcal{M} = \mathcal{A}$ with the standard bimodule structure, then $H_0(\mathcal{A}, \mathcal{M}) = \mathcal{A}$, π is the identity, and the above map is the isomorphism of the HKR theorem.

Koszul complexes

- Let \mathcal{A} be an algebra and $\{x_{i,j}\} \in M_d(\mathcal{A})$ with

$$x_{i,j}x_{i-1,k} = x_{i,k}x_{i-1,j}.$$

- For $0 \leq n \leq d$ set $K_n = \mathcal{A}^{\binom{d}{n}}$. Embed K_n into $\mathcal{A}^d \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{A}^d$ (n factors) as

$$\text{span}_{\mathcal{A}}\{e_{i_1} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} e_{i_n} \mid 1 \leq i_1 < \dots < i_n \leq d\}.$$

Here e_i is a basis of \mathcal{A}^d . For $n > d$ set $K_n = 0$.

- Define for $m = 1, \dots, n$ \mathcal{A} -module maps

$$\begin{aligned} k_m & : K_n \rightarrow K_{n-1}, \\ e_{i_1} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} e_{i_n} & \mapsto \\ e_{i_1} \otimes_{\mathcal{A}} \cdots \hat{e}_{i_m} \cdots \otimes_{\mathcal{A}} e_{i_n} & \triangleleft x_{n,i_m}. \end{aligned}$$

Then for $r < s$ one has $k_r k_s - k_{s-1} k_r = 0$ and this implies that $k = \sum_{r=1}^n (-1)^r k_r$ satisfies $k \circ k = 0$. We call (K_*, k) the Koszul complex associated to the elements $x_{i,j}$.

Map to torsion complex

- In our application \mathcal{A} will be a Hopf algebra and $x_{1,j}$ ($1 \leq j \leq d$) generate $\ker \varepsilon$ as a left \mathcal{A} -module. The associated Koszul complex turns out to be acyclic, so it provides a resolution of $\mathcal{A}/\ker \varepsilon$, and $\mathrm{Tor}_*^{\mathcal{A}}(\mathcal{M}', \mathbb{C})$ are the homologies of $\mathcal{M}' \otimes_{\mathcal{A}} K_*$.
- The morphism to the standard complex calculating the torsion is

$$\begin{aligned}
 & e_{i_1} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} e_{i_n} \mapsto \\
 & x_{1,i_1} \wedge \cdots \wedge x_{1,i_n} \\
 = & \sum_{s \in S_n} (-1)^{|s|} x_{n,i_{s(n)}} \otimes \cdots \otimes x_{1,i_{s(1)}}.
 \end{aligned}$$

$SL_q(2)$

- From now on \mathcal{A} is the Hopf algebra $\mathbb{C}_q[SL(2)]$ with q not a root of unity. Thus \mathcal{A} has generators a, b, c, d satisfying

$$\begin{aligned} ab &= qba, & ac &= qca, & bd &= qdb, \\ cd &= qdc, & bc &= cb, \\ ad - qbc &= 1, & da - q^{-1}bc &= 1. \end{aligned}$$

- The following is a vector space basis of \mathcal{A} :

$$e_{\pm i, j, k} = a^{\pm i} b^j c^k, \quad i, j, k \in \mathbb{N}_0, \quad a^{-i} = d^i.$$

\mathcal{A} admits a \mathbb{Z} -grading and three separating decreasing \mathbb{N}_0 -filtrations

$$\mathcal{A} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_i^a, \quad \mathcal{A} = \mathcal{A}_0^x \supset \mathcal{A}_1^x \supset \dots, \quad x = b, c, bc,$$

where $\mathcal{A}_i^a = \text{span}\{e_{i, j, k}\}_{j, k}$ and \mathcal{A}_n^x is the span of $e_{i, j, k}$ with $j, k, j+k \geq n$ for $x = b, c, bc$, respectively.

The automorphism group of \mathcal{A}

- For $\lambda, \mu \in \mathbb{C} \setminus \{0\}$ there are unique automorphisms $\sigma_{\lambda, \mu}, \tau_{\lambda, \mu}$ of \mathcal{A} with

$$\sigma_{\lambda, \mu} : a, b, c, d \mapsto \lambda a, \mu b, \mu^{-1} c, \lambda^{-1} d,$$

$$\tau_{\lambda, \mu} : a, b, c, d \mapsto \lambda a, \mu^{-1} c, \mu b, \lambda^{-1} d.$$

This list of automorphisms is complete.

- All these automorphisms are diagonalizable. Hence one has $HH_n^\sigma(\mathcal{A}) = H_n(\mathcal{A}, \sigma\mathcal{A})$ for all $\sigma \in \text{Aut}(\mathcal{A})$.

The Haar functional

- The Hopf algebra \mathcal{A} is cosemisimple, that is, there is $h : \mathcal{A} \rightarrow \mathbb{C}$ satisfying $h(1) = 1$ and

$$(h \otimes \text{id})\Delta(x) = h(x)1 = (\text{id} \otimes h)\Delta(x) \quad \forall x \in \mathcal{A}.$$

We call h the Haar functional of \mathcal{A} .

- For any $x, y \in \mathcal{A}$, we have

$$h(xy) = h(y\sigma_{q^{-2}1}(x)).$$

Thus h is a $\sigma_{q^{-2}1}^{-1}$ -twisted cyclic 0-cocycle.

A Koszul resolution of $\mathcal{A}/\ker \varepsilon$

- $\ker \varepsilon$ is generated by $a - 1, b, c$. For these elements there exists a Koszul complex (K_*, k) as discussed above with $x_{i,j}$ given by

$$\begin{pmatrix} a - 1 & b & c \\ q^{-1}a - 1 & b & c \\ q^{-2}a - 1 & b & c \end{pmatrix}.$$

The Koszul complex is acyclic. Hence the left \mathcal{A} -module \mathbb{C} possesses a resolution (K_*, k) of the form

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{A}^3 \rightarrow \mathcal{A}^3 \rightarrow \mathcal{A} \rightarrow \mathbb{C} \rightarrow 0.$$

- In particular, $HH_n^\sigma(\mathcal{A}) = 0$ for all $n > 3$ and all $\sigma \in \text{Aut}(\mathcal{A})$.

$$HH_n^\sigma(\mathcal{A})$$

- We calculated $HH_n^\sigma(\mathcal{A})$ for $\sigma = \sigma_{\lambda, \mu}$. To give an overview of the results, we collect the dimensions of $HH_n^\sigma(\mathcal{A})$ for $n \leq 3$ as a vector space:

$\lambda, \mu (a, b \geq 0)$	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$q^{-(a+1)}, q^{\pm(b+1)}$	2	4	2	0
$q^{-(a+2)}, 1$	∞	∞	$a + 1$	$a + 1$
$\notin q^{-\mathbb{N}}, q^{\pm(a+1)}$ or $\neq 1, \notin q^{\mathbb{Z}}$	0	0	0	0
otherwise	∞	∞	0	0

$$HC_n^\sigma(\mathcal{A})$$

- We (in fact, Tom) calculated $HC_n^\sigma(\mathcal{A})$ using the spectral sequence associated to the (b, B) -bicomplex. To give an overview of the results, we collect the dimensions of $HC_n^\sigma(\mathcal{A})$ as a vector space:

$\lambda, \mu (a, b \geq 0)$	$n = 0$	1	$2, 2k + 3$	$2k + 4$
$q^{-(a+1)}, q^{\pm(b+1)}$	2	2	0	0
$q^{-(2a+2)}, 1$	∞	$2a + 1$	$2a + 1$	0
$q^{-(2a+3)}, 1$	∞	$2a + 3$	$2a + 3$	1
$\notin q^{-\mathbb{N}}, q^{\pm(a+1)}$ or $\neq 1, \notin q^{\mathbb{Z}}$	0	0	0	0
otherwise	∞	1	1	1

Spectral sequences - I

- Consider a first quadrant bicomplex

$$\begin{array}{ccccccc}
 & & & & & & \\
 & & & & & & \\
 b \downarrow & & & & b \downarrow & & & b \downarrow \\
 C_{10} & \xleftarrow{B} & C_{11} & \xleftarrow{B} & C_{12} & \xleftarrow{B} & \\
 b \downarrow & & & & b \downarrow & & & b \downarrow \\
 C_{00} & \xleftarrow{B} & C_{01} & \xleftarrow{B} & C_{02} & \xleftarrow{B} &
 \end{array}$$

Often one can compute the homologies of the total complex $(T_* = \bigoplus_{i+j=*} C_{ij}, b + B)$ by an “approximation” procedure:

- Set

$$E_{ij}^1 = H_i(C_{*j}, b), \quad E_{ij}^2 = H_j(E_{i*}^1, B).$$

One constructs subsequently differentials

$$d : E_{ij}^k \rightarrow E_{i+k-1, j-k}^k.$$

defining E_{ij}^{k+1} . For fixed ij the sequence of the E_{ij}^k becomes stable. Let E_{ij}^∞ denote this stable limit of the E_{ij}^k .

Spectral sequences - II

- Filter the total complex T_* by columns,

$$F^k T_n := \bigoplus_{i+j=n, j \leq k} C_{ij}.$$

This gives a filtration of its homologies H_* , and the spectral sequence $\{E_{ij}^k\}$ converges to H_* in the sense that

$$E_{ij}^\infty = F^j H_{i+j} / F^{j-1} H_{i+j}.$$