The Range of K-Invariants for C^*-Algebras of Infinite Graphs

WOJCIECH SZYMAŃSKI

ABSTRACT. It is shown that for any pair (K_0, K_1) of countable abelian groups, with K_1 free abelian, and any element $\Xi \in K_0$ there exists a purely infinite and simple, stable C^*-algebra $C^*(E)$ corresponding to a row-finite, transitive, directed graph E, and there exists a vertex $v \in E^0$ such that $K_i(C^*(E)) \cong K_i$ for $i = 0$, 1 and $[p_v] = \Xi$ in $K_0(C^*(E))$. A presentation of the corner $p_v C^*(E) p_v$ in terms of generators and relations is given.

0. INTRODUCTION

There are many reasons why Cuntz-Krieger algebras [3] are so important and interesting. Let us just mention the following three; their basic structure is rich and yet fairly well understood, their invariants are readily computable, and they admit a convenient presentation in terms of generators and relations. Not surprisingly, a number of generalizations of the now classical work of Cuntz and Krieger has emerged. In this article we deal with a class of generalized Cuntz-Krieger algebras associated with directed graphs [4, 7, 6, 1, 8]. Graphs are employed in order to conveniently encode relations among the generating families of partial isometries. These graph algebras enjoy the above mentioned properties of Cuntz-Krieger algebras. However, their class is much larger. It is the main purpose of the present article to determine exactly how large this class is, at least as far as purely infinite and simple C^*-algebras are concerned. It turns out that the well-known restriction on the K_1-group, it must be free abelian, is the only one. Namely, we prove in Theorem 1.2 below that any pair (K_0, K_1) of countable abelian groups, with K_1 free abelian, can occur as the K-theory of a C^*-algebra of a transitive graph. Such algebras are purely infinite and simple by [1, Propositions 5.1 and 5.3]. An obvious extension of [5, Lemma 2.13] to the case of row-finite graphs implies that the algebra is stable. A graph is row-finite if every vertex emits only finitely many edges and our result says, in particular, that such graphs suffice.

Indiana University Mathematics Journal ©, Vol. 51, No. 1 (2002)

In order to realize unital versions of these purely infinite and simple C^*-algebras we must cut $C^*(E)$ by a vertex projection p_v. By Theorem 1.2, it is possible to produce in this way the algebra with an arbitrary mark $[I]$ in K_0. It is therefore necessary to better understand the structure of such vertex corners. They were studied before in [7], through groupoids of pointed graphs, in connection with the Doplicher-Roberts algebras. We show that if E is finite, then $p_v C^*(E) p_v$ turns out to be another graph algebra (cf. Section 4 and Lemmas 2.1 and 2.2 below). This, however, is not the case, in general, for an infinite graph E. Nevertheless, a presentation of $p_v C^*(E) p_v$ in terms of generators and relations is always possible (cf. Section 7 and Theorem 2.3 below). Of course, the very existence of purely infinite and simple C^*-algebras with the desired K-groups has been known for some time already [9, Proposition 6.6]. However, Rørdam's argument involves a complicated inductive limit process, while our construction yields a clear-cut presentation. Such presentations are very useful in a variety of situations, for example in those involving questions about semiprojectivity [2].

1. THE MAIN RESULTS

We recall the definition of a graph algebra [7, 6]. Let $E = (E^0, E^1, r, s)$ be a countable directed graph with vertices E^0, edges E^1, and range and source functions $r,\ s : E^1 \to E^0$, respectively. In this article we only deal with row-finite graphs, in which every vertex emits but finitely many edges. $C^*(E)$ is by definition the universal C^*-algebra generated by families of mutually orthogonal projections $\{p_w \mid w \in E^0\}$ and partial isometries $\{s_e \mid e \in E^1\}$, subject to the following two relations:

(G1) $s_e^* s_e = p_{r(e)}$,

(G2) $p_w = \sum_{e \in E^1 | s(e) = w} s_e s_e^*$, provided $s^{-1}(w)$ is not empty.

Let V be the collection of all those vertices which emit at least one edge, and let $\mathbb{Z}V$ and $\mathbb{Z}E^0$ be free abelian groups on free generators V and E^0, respectively. According to [8, Theorem 3.2], the K-groups of $C^*(E)$ may be calculated as

$$K_0(C^*(E)) \cong \text{coker}(\Delta_E),$$
$$K_1(C^*(E)) \cong \ker(\Delta_E),$$

where $\Delta_E : \mathbb{Z}V \to \mathbb{Z}E^0$ is the map defined on generators as

$$\Delta_E(w) = \Big(\sum_{e \in E^1 | s(e) = w} r(e) \Big) - w.$$

Lemma 1.1. *For any countable abelian group K there exists a row-finite directed graph G with countably infinitely many vertices such that $K_0(C^*(G)) \cong K$ and $K_1(C^*(G)) = 0$. Furthermore, for any $\Xi \in K$ such G and $v \in G^0$ can be found that $[p_v] = \Xi$ in $K_0(C^*(G))$.*

Proof. Let B be a free abelian group with a countably infinite basis $\{b_i \mid i = 1, 2, \dots\}$. We denote by $\pi_i : B \to \mathbb{Z}$ the i'th coordinate map corresponding to this basis.

Since K is a countable abelian group it is not difficult to show that there exists an injective endomorphism $\varphi : B \to B$ such that $K \cong \operatorname{coker}(\varphi)$. For such a φ there are unique endomorphisms φ_+ and φ_- of B such that $\varphi(b) = \varphi_+(b) - \varphi_-(b)$ and $\pi_i(\varphi_+(b)) \geq 0$, $\pi_i(\varphi_-(b)) \geq 0$ for all $b \in B$ and all $i = 1, 2, \dots$. We denote by by $\Phi_\pm$ the matrices of $\varphi_\pm$ with respect to the basis $\{b_i \mid i = 1, 2, \dots\}$, and define Λ as the two-by-two block matrix

$$\Lambda = \begin{bmatrix} \Phi_+ & I \\ \Phi_- & I \end{bmatrix},$$

where I stands for the identity matrix. Now we define A as the matrix

$$A = \left(\begin{bmatrix} \Phi_+ & I \\ \Phi_- & I \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \right)^t,$$

where the superscript t denotes the transpose.

Since A is a row-finite matrix with non-negative integer entries there exists a countable, row-finite directed graph G such that A is the vertex matrix of G (cf. [7, Section 4]). We have $K_0(C^*(G)) \cong \operatorname{coker}(\Lambda)$ and $K_1(C^*(G)) \cong \ker(\Lambda)$. Thus, in order to complete the proof, it suffices to show that there exist elementary matrices S and T such that

$$S\Lambda T = \begin{bmatrix} \Phi & 0 \\ 0 & I \end{bmatrix},$$

where $\Phi = \Phi_+ - \Phi_-$ is the matrix of φ. Indeed, it suffices to set

$$S = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} I & 0 \\ -\Phi_- & I \end{bmatrix}.$$

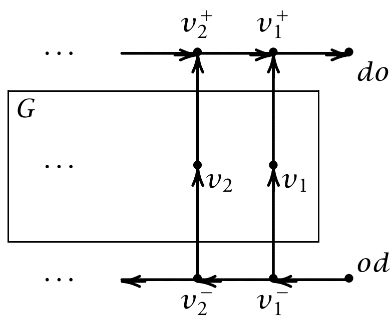
Suppose now that an element $\Xi \in K$ is given. Let $f : \operatorname{coker}(\varphi) \to K$ be an isomorphism, and let $f(\sum_i m_i b_i + \operatorname{im}(\varphi)) = \Xi$ for some $m_i \in \mathbb{Z}$. We replace B with $B \oplus \mathbb{Z}$ and φ with a map $\psi : B \oplus \mathbb{Z} \to B \oplus \mathbb{Z}$ such that

$$\psi(x, k) = \Big(\varphi(x) - k \sum_i m_i b_i, k \Big),$$

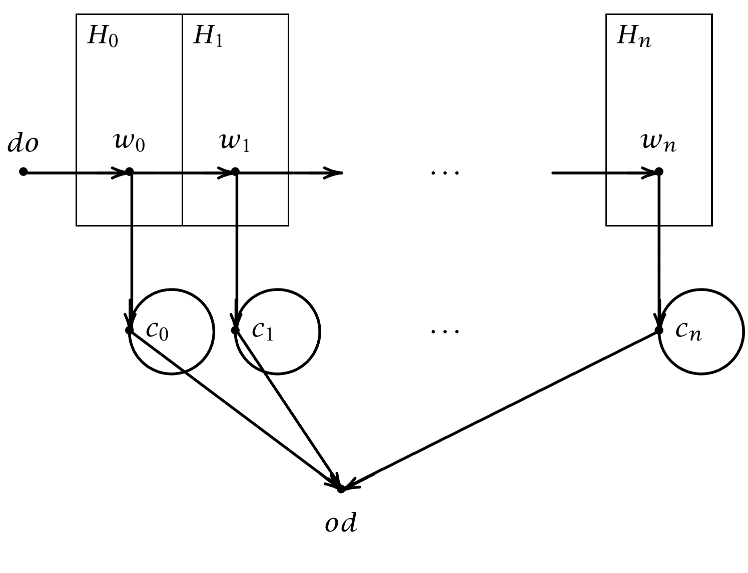
and repeat the above construction. The vertex v of the resulting graph corresponding to $(0, 1) \in B \oplus \mathbb{Z}$ has the required property. ◻

Theorem 1.2. *For any pair (K_0, K_1) of countable groups, with K_0 abelian and K_1 free abelian, there exists a row-finite,* transitive, *directed graph E with countably infinitely many vertices such that $K_i(C^*(E)) \cong K_i$ for $i = 0, 1$. Furthermore, for any $\Xi \in K_0$ such E and $v \in E^0$ can be found that $[p_v] = \Xi$ in $K_0(C^*(E))$.*

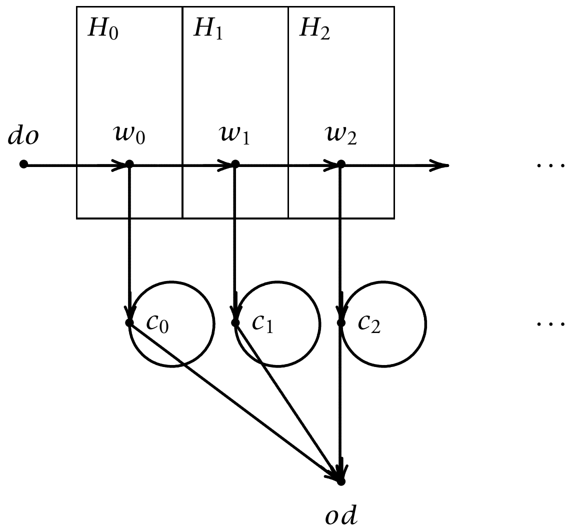
Proof. By Lemma 1.1 there exists a row-finite graph G with countably infinitely many vertices such that $K_0(C^*(G)) \cong K_0$ and $K_1(C^*(G)) = 0$. Let v_1, v_2, $\dots$ be the vertices of G. When representing G graphically, we will arrange them in a straight line. At first we define a larger graph Γ as



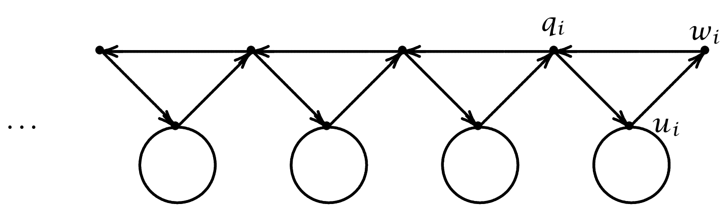
Let n be the rank of K_1. If n is finite, then we define a graph H as



If n is infinite, then we define H as



Where in both cases the subgraph H_i is



Finally, we define E as the union of the graphs Γ and H, with the vertices do on both Γ and H identified, and the vertices od on both graphs identified. By construction, E is countable, row-finite, and transitive. We must show that the K-groups are right.

Firstly, we consider $K_0(C^*(E))$, which is isomorphic to $\operatorname{coker}(\Delta_E)$. That $\operatorname{coker}(\Delta_E)$ is isomorphic to K_0 follows from

(i) $\mathbb{Z}E^0 = \mathbb{Z}G^0 + \operatorname{im}(\Delta_E)$, and
(ii) $\mathbb{Z}G^0 \cap \operatorname{im}(\Delta_E) = \operatorname{im}(\Delta_G)$.

To see (i) we first observe that $\Delta_E(\mathbb{Z}(H_i^0 \setminus \{w_i\}))$ contains H_i^0 (including w_i). Then, proceeding successively, we see that $\mathbb{Z}G^0 + (\sum_{i=0}^n \mathbb{Z}H_i^0) + \operatorname{im}(\Delta_E)$ contains; all v_i^+, do, H^0 (including od), all v_i^-, and hence $\mathbb{Z}E^0$, as claimed. To see (ii) let $x \in \mathbb{Z}E^0$ be such that $\Delta_E(x) \in \mathbb{Z}G^0$. We write $x = y + z + \sum m_i v_i^-$, where $m_i \in \mathbb{Z}$, $y \in \mathbb{Z}G^0$ and $z \in M$. Here M denotes the subgroup of $\mathbb{Z}E^0$ generated by all vertices not in $G^0 \cup \{v_i^- \mid i = 1, 2, \ldots\}$. We have

$$\Delta_E(x) = \Delta_G(y) + \sum_i m_i v_i + \sum_i m_i(v_{i+1}^- - v_i^-) + z' + kv_1^-,$$

where $k \in \mathbb{Z}$ and $z' \in M$. Since $\Delta_E(x) \in \mathbb{Z}G^0$, it follows that $m_i = 0$ for all $i = 1, 2, \ldots$ (otherwise consider the greatest i for which $m_i \neq 0$). It then follows that $k = 0$ and $z' = 0$. Thus $\Delta_E(x) = \Delta_G(y)$ and we have shown that $\mathbb{Z}G^0 \cap \operatorname{im}(\Delta_E) \subseteq \operatorname{im}(\Delta_G)$. To prove the reverse inclusion it suffices to notice that for any i

$$\Delta_G(v_i) = \Delta_E\Big(v_i + \sum_{j=1}^{i} v_j^+ + do - u_0\Big).$$

This gives (ii) and, consequently, $K_0(C^*(E)) \cong K_0$.

Secondly, we consider $K_1(C^*(E))$, which is isomorphic to $\ker(\Delta_E)$. Let $x \in \mathbb{Z}E^0$ be such that $\Delta_E(x) = 0$. We write

$$\begin{aligned} x = \sum_i r_i v_i + \sum_i m_i v_i^+ + \sum_i k_i v_i^- + a(do) + b(od) \\ + \sum_i \ell_i w_i + \sum_i t_i c_i + \sum_i h_i, \end{aligned}$$

where $r_i, m_i, k_i, a, b, \ell_i, t_i \in \mathbb{Z}$ and $h_i \in \mathbb{Z}(H_i^0 \setminus \{w_i\})$. We have

$$\begin{aligned} 0 = \Delta_E(x) = {} & \Big(\Delta_G\Big(\sum_i r_i v_i\Big) + \sum_i k_i v_i\Big) + \Big(bv_1^- + \sum_i k_i(v_{i+1}^- - v_i^-)\Big) \\ & + \Big(\sum_i r_i v_i^+ + \sum_{i\geq 2} m_i(v_{i-1}^+ - v_i^+) - m_1 v_1^+\Big) + (m_1 - a)do \\ & + \Big(\sum_i \ell_i(w_{i+1} - w_i) + aw_0\Big) + \Big(\sum_i t_i - b\Big)od \\ & + \sum_i \ell_i c_i + \sum_i \ell_i q_i + \sum_i \Delta_{H_i}(h_i). \end{aligned}$$

Since $\sum_i \ell_i c_i = 0$ we have $\ell_i = 0$ for $i = 0, \ldots, n$. Considering the greatest i for which $k_i \neq 0$ we see that $k_i = 0$ for all i, and $b = 0$ as well. Since $K_1(C^*(G)) \cong \ker(\Delta_G) = 0$, this implies that $\sum_i r_i v_i = 0$ and thus $r_i = 0$ for all i. Considering now the smallest i for which $m_i \neq 0$ we see that $m_i = 0$ for all i, and $a = 0$ as well. It is easy to verify that $\ker(\Delta_{H_i}) = 0$. Thus we have $h_i = 0$ for all $i = 0, \ldots, n$. Finally, we conclude that $x = \sum_i t_i c_i$ and $\sum_i t_i = 0$. Since any such x belongs to the kernel of Δ_E, we have proved that $K_1(C^*(E)) \cong K_1$, as required.

If an element $\Xi \in K_0$ is given, then choose, by Lemma 1.1, G and $v \in G^0$ with $[p_v] = \Xi$ in $K_0(C^*(G))$ and apply the above construction. □

2. CORNERS OF GRAPH ALGEBRAS

Let E be a *finite* directed graph with a distinguished vertex $v \in E^0$. If $\alpha, \mu \in E^*$, then we write $\mu \prec \alpha$ if μ is an initial subpath of α. We say that a path $\alpha = (e_1, \dots, e_k) \in E^*$ is simple if $s(e_i) \neq r(e_j)$ for $i \leq j$. For any $w \in E^0$ we select a path $\beta(w)$ from v to w (if such exists) with the smallest possible length. In particular, $\beta(v) = v$ is a path of length 0. We now define a new finite graph $E(v)$, as follows. $E(v)^0$ consists of all $\alpha \in E^*$ such that: $s(\alpha) = v$, μ is a simple path for any $\mu \prec \alpha$, $\mu \neq \alpha$, and either $r(\alpha)$ is a sink or α is not simple. For any $\alpha, \alpha' \in E(v)^0$ there is a (single) edge from α to α' if and only if $\alpha \neq \beta(r(\alpha)) \prec \alpha'$. We denote by P_α the projection in $C^*(E(v))$ corresponding to the vertex $\alpha \in E(v)^0$, and by $S_{\alpha,\alpha'}$ the partial isometry in $C^*(E(v))$ corresponding to the edge from α to α'.

Lemma 2.1. *If E is a finite directed graph with a distinguished vertex v, then*

$$p_v = \sum_{\alpha \in E(v)^0} s_\alpha s_\alpha^*.$$

Proof. To this end it suffices to show that for any $n = 0, 1, \dots$

$$p_v = \sum_{\alpha \in A_n} s_\alpha s_\alpha^* + \sum_{\gamma \in B_n} s_\gamma s_\gamma^*, \tag{2.1}$$

where

$$A_n = \{\alpha \in E(v)^0 : |\alpha| \leq n\}, \quad \text{and}$$
$$B_n = \{\gamma \in E^* : \gamma \text{ simple}, s(\gamma) = v,\ r(\gamma) \text{ not a sink}, |\gamma| = n\}.$$

We proceed by induction on n. Equation (2.1) is clearly satisfied when $n = 0$. Assume it holds for n. Applying condition (G2) (for graph E) to each vertex in $\{r(\gamma) \mid \gamma \in B_n\}$ and utilizing the inductive hypothesis we get

$$\begin{aligned} p_v &= \sum_{\alpha \in A_n} s_\alpha s_\alpha^* + \sum_{\gamma \in B_n} s_\gamma \Big(\sum_{e \in E^1 | s(e) = r(\gamma)} s_e s_e^* \Big) s_\gamma^* \\ &= \sum_{\alpha \in A_n} s_\alpha s_\alpha^* + \sum_{\alpha \in A_{n+1} \setminus A_n} s_\alpha s_\alpha^* + \sum_{\gamma \in B_{n+1}} s_\gamma s_\gamma^* \\ &= \sum_{\alpha \in A_{n+1}} s_\alpha s_\alpha^* + \sum_{\gamma \in B_{n+1}} s_\gamma s_\gamma^*. \end{aligned}$$

Thus equation (2.1) holds true for $n + 1$, as required. □

Lemma 2.2. *If E is a finite directed graph in which every loop has an exit, with a distinguished vertex v, and $E(v)$ is the corresponding graph, as constructed above, then there exists an isomorphism*

$$\varphi : C^*(E(v)) \to p_v C^*(E) p_v$$

such that

$$\varphi(P_\alpha) = s_\alpha s_\alpha^* \quad \text{and} \quad \varphi(S_{\alpha,\alpha'}) = s_\alpha s_{\beta(r(\alpha))}^* s_{\alpha'} s_{\alpha'}^*.$$

Proof. Projections $\{s_\alpha s_\alpha^*\}$ are pairwise orthogonal by Lemma 2.1. Likewise, partial isometries $\{s_\alpha s_{\beta(r(\alpha))}^* s_{\alpha'} s_{\alpha'}^*\}$ have mutually orthogonal ranges. Indeed, if $s_\alpha^* s_{\alpha'} \neq 0$, then either $\alpha \prec \alpha'$ or $\alpha' \prec \alpha$. This, however, is only possible if $\alpha = \alpha'$. Thus, universality of $C^*(E(v))$ implies existence of a homomorphism φ, as above, provided $\{s_\alpha s_\alpha^*, s_\alpha s_{\beta(r(\alpha))}^* s_{\alpha'} s_{\alpha'}^*\}$ satisfy conditions (G1), (G2) for the graph $E(v)$. Clearly, (G1) is fulfilled. If $\alpha = \beta(r(\alpha)) \in E(v)^0$, then α is a sink in $E(v)^0$. Otherwise Lemma 2.1 yields

$$s_\alpha s_{\beta(r(\alpha))}^* = \sum_{\alpha' \in E(v)^0} s_\alpha s_{\beta(r(\alpha))}^* s_{\alpha'} s_{\alpha'}^* \tag{2.2}$$
$$= \sum_{\alpha' \in E(v)^0 \mid \beta(r(\alpha)) \prec \alpha'} s_\alpha s_{\beta(r(\alpha))}^* s_{\alpha'} s_{\alpha'}^*.$$

Equality of the range projections of the partial isometries on the two sides of (2.2) yields

$$s_\alpha s_\alpha^* = \sum_{\alpha' \in E(v)^0 \mid \beta(r(\alpha)) \prec \alpha'} s_\alpha s_{\beta(r(\alpha))}^* s_{\alpha'} s_{\alpha'}^* s_{\beta(r(\alpha))} s_\alpha^*.$$

Thus (G2) is fulfilled.

In order to prove surjectivity of φ we must show that its range contains all $\{s_x s_y^* \mid x, y \in E^*, s(x) = s(y) = v, r(x) = r(y)\}$ (cf. [7, Section 3]). As $s_x s_y^* = (s_x s_{\beta(r(x))}^*)(s_y s_{\beta(r(x))}^*)^*$, we may assume $y = \beta(r(x))$. We proceed by induction on $|x|$. Case $|x| = 0$ follows from Lemma 2.1. So suppose that $|x| \geq 1$ and $\{s_y s_{\beta(r(y))}^* \mid y \in E^*, |y| < |x|, s(y) = v\} \subseteq \mathrm{im}(\varphi)$. We consider separately three cases, as follows. Firstly, if $x \in E(v)^0$, then $s_x s_{\beta(r(x))}^*$ is in $\mathrm{im}(\varphi)$ by (2.2). Secondly, if x is neither simple nor in $E(v)^0$, then $x = x_1 x_2$, with $x_1 \in E(v)^0$ and both x_1 and x_2 paths of positive length. Thus $r(x_1)$ is not a sink in E and we must have $|x_1| > |\beta(r(x_1))|$. Hence $s_x s_{\beta(r(x))}^* = (s_{x_1} s_{\beta(r(x_1))}^*)(s_{\beta(r(x_1))x_2} s_{\beta(r(x))}^*)$ is in $\mathrm{im}(\varphi)$ by the inductive hypothesis. Thirdly, suppose that x is simple and not in $E(v)^0$. An inductive argument very similar to the one from the proof of Lemma 2.1 yields

$$s_x s_{\beta(r(x))}^* = \sum_{\alpha \in E(v)^0 \mid x \prec \alpha} s_\alpha s_{\beta(r(x))\gamma(\alpha)}^*,$$

where $\alpha = x\gamma(\alpha)$. Thus

$$s_x s_{\beta(r(x))}^* = \sum_{\alpha \in E(v)^0 \mid x \prec \alpha} (s_\alpha s_{\beta(r(\alpha))}^*)(s_{\beta(r(x))\gamma(\alpha)} s_{\beta(r(\alpha))}^*)^*$$

is in $\mathrm{im}(\varphi)$ by (2.2).

In order to prove injectivity of φ we want to apply the Cuntz-Krieger uniqueness theorem [1, Theorem 3.1]. Since $\varphi(P_\alpha) \neq 0$ for all $\alpha \in E(v)^0$, it suffices to verify that every loop in $E(v)$ has an exit. So suppose that $\alpha_1, \ldots, \alpha_k \in E(v)^0$ and there is a loop L in $E(v)$ passing (in this order) through $\alpha_1, \alpha_2, \ldots, \alpha_k, \alpha_1$. We must have $\alpha_{i+1} = \beta(r(\alpha_i))\gamma_i$, for some path γ_i from $r(\alpha_i)$ to $r(\alpha_{i+1})$. Here we identify the indices $1, \ldots, k$ modulo k. Thus $(\gamma_1, \ldots, \gamma_k)$ is a loop in E. By hypothesis, this loop has an exit. Thus there are an $m \in \{1, \ldots, k\}$ and an $x \in E^*$ such that; $s(x) = r(\alpha_m)$, $|x| \leq |\gamma_m|$, and $x \not\prec \gamma_m$. Taking a suitably small initial subpath of x we may assume that $\beta(r(\alpha_m))x$ is either simple or belongs to $E(v)^0$. In the former case we may extend it to an element of $E(v)^0$. Thus there exists a $\mu \in E(v)^0$ such that $\beta(r(\alpha_m))x \prec \mu$. It follows that the edge in $E(v)$ from α_m to μ gives rise to an exit for L, as required. □

Let E be a row-finite and transitive directed graph with countably infinitely many vertices, with a distinguished vertex $v \in E^0$. For any $w \in E^0$ we select a path $\beta(w)$ from v to w with the smallest possible length. We denote by $\wp(E, v)$ the collection of all paths $\alpha \in E^*$ such that $s(\alpha) = v$ and μ is simple for any $\mu \prec \alpha$, $\mu \neq \alpha$. We define $C^*(E, v)$ to be the universal C^*-algebra generated by a family $\{T_\alpha \mid \alpha \in \wp(E, v)\}$ of Hilbert space operators, subject to the following relations;

(V1) $T_\alpha T_\alpha^* T_\gamma = T_\gamma$ if $\alpha \prec \gamma$,
(V2) $T_\alpha^* T_\gamma = 0$ if neither $\alpha \prec \gamma$ nor $\gamma \prec \alpha$,
(V3) $T_\alpha^* T_\alpha = T_{\beta(r(\alpha))}$,
(V4) $T_{\beta(w)} = \sum_{e \in s^{-1}(w)} T_{\beta(w)e} T_{\beta(w)e}^*$.

A proof of existence of such a universal algebra is routine. See for example [2, Section 1].

Theorem 2.3. *If E is a row-finite and transitive directed graph with countably infinitely many vertices and $v \in E^0$, then there exists an isomorphism*

$$\psi : C^*(E, v) \to p_v C^*(E) p_v$$

such that

$$\psi(T_\alpha) = s_\alpha s_{\beta(r(\alpha))}^* \quad \text{for } \alpha \in \wp(E, v).$$

Proof. It is clear that the family $\{s_\alpha s_{\beta(r(\alpha))}^* \in p_v C^*(E) p_v \mid \alpha \in \wp(E, v)\}$ satisfies conditions (V1)-(V4). Thus universality of $C^*(E, v)$ implies that there exists a C^*-algebra homomorphism $\psi : C^*(E, v) \to p_v C^*(E) p_v$ such that $\psi(T_\alpha) = s_\alpha s_{\beta(r(\alpha))}^*$ for any $\alpha \in \wp(E, v)$. We must construct its inverse.

Let W be a finite subset of E^0 containing v and let E_F be the subgraph of E consisting of all edges with sources in W (together with their source and range vertices). E_W is finite since E is row-finite, and every loop in E_W has exit since E is transitive. It follows from [8, Lemma 1.2] that the C^*-subalgebra of $C^*(E)$ generated by $\{s_e \mid e \in E_W^1\}$ is naturally isomorphic to $C^*(E_W)$. By virtue of Lemma

2.2 we may identify $p_v C^*(E_W)p_v$ with $C^*(E_W(v))$. Clearly, $E_W(v)^0$ is contained in $\wp(E,v)$. We claim that there exists a C^*-algebra homomorphism $f_W : C^*(E_W(v)) \to C^*(E,v)$ such that $f_W(P_\alpha) = T_\alpha T_\alpha^*$ and $f_W(S_{\alpha,\alpha'}) = T_\alpha T_{\alpha'} T_{\alpha'}^*$. To this end it suffices to verify that $\{T_\alpha T_\alpha^*\}$ and $\{T_\alpha T_{\alpha'} T_{\alpha'}^*\}$ satisfy conditions (G1), (G2) for graph $E_W(v)$. Indeed, (V1) and (V2) imply that $\{T_\alpha T_\alpha^*\}$ are mutually orthogonal projections and $\{T_\alpha T_{\alpha'} T_{\alpha'}^*\}$ are partial isometries with mutually orthogonal ranges. (G1) follows immediately from (V1) and (V3). To prove (G2) we first observe that

$$T_{\beta(r(\alpha))} = \sum_{\alpha' \in E_W(v)^0 | \beta(r(\alpha)) \prec \alpha'} T_{\alpha'} T_{\alpha'}^*,$$

for any $\alpha \in E_W(v)^0$. This equality follows from (V4) by an argument similar to the ones from the proofs of Lemmas 2.1 and 2.2 above. This, together with (V3), yields

$$T_\alpha T_\alpha^* = T_\alpha T_{\beta(r(\alpha))} T_\alpha^* = \sum_{\alpha' \in E_W(v)^0 | \beta(r(\alpha)) \prec \alpha'} T_\alpha T_{\alpha'} T_{\alpha'}^* T_\alpha^*,$$

and (G2) follows.

By construction, we have $\psi f_W = \mathrm{id}$. Thus the pointwise limit of the maps f_W, with larger and larger subsets W increasing to E^0, is the desired inverse of ψ. ◻

REFERENCES

[1] T. BATES, D. PASK, I. RAEBURN & W. SZYMAŃSKI, *The C^*-algebras of row-finite graphs*, New York J. Math. **6** (2000), 307-324.

[2] B. BLACKADAR, *Shape theory for C^*-algebras*, Math. Scand. **56** (1985), 249-275.

[3] J. CUNTZ & W. KRIEGER, *A class of C^*-algebras and topological Markov chains*, Invent. Math. **56** (1980), 251-268.

[4] M. ENOMOTO & Y. WATATANI, *A graph theory for C^*-algebras*, Math. Japonica **25** (1980), 435-442.

[5] J.V.B. HJELMBORG, *Purely infinite and stable C^*-algebras of graphs and dynamical systems*, Ergodic Theory Dynam. Systems **21** (2001), 1789-1808.

[6] A. KUMJIAN, D. PASK & I. RAEBURN, *Cuntz-Krieger algebras of directed graphs*, Pacific J. Math. **184** (1998), 161-174.

[7] A. KUMJIAN, D. PASK, I. RAEBURN & J. RENAULT, *Graphs, groupoids, and Cuntz-Krieger algebras*, J. Funct. Anal. **144** (1997), 505-541.

[8] I. RAEBURN & W. SZYMAŃSKI, *Cuntz-Krieger algebras of infinite graphs and matrices*, preprint, 1999.

[9] M. RØRDAM, *Classification of Cuntz-Krieger algebras*, *K*-Theory **9** (1995), 31-58.

Department of Mathematics
The University of Newcastle
Callaghan, NSW 2308
AUSTRALIA.
E-MAIL: `wojciech@frey.newcastle.edu.au`

Received: *December 16th, 1999; August 22nd, 2001.*