LHCphysics6

HIERARCHY PROBLEM

Let's discuss a scalar field theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4!}\varphi^4$$

We would like to calculate quantum corrections to the scalar mass

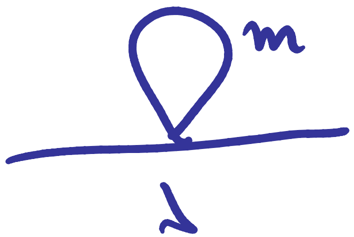
$$-i\Sigma =$$

· Remember how the propagator is modified

$$G = \frac{i}{k^2-m^2} + \frac{i}{k^2-m^2}\left[-i\,\Sigma(k)\right]\frac{i}{k^2-m^2} + \dots$$

$$= \frac{i}{k^2-m^2-\Sigma(k)}$$

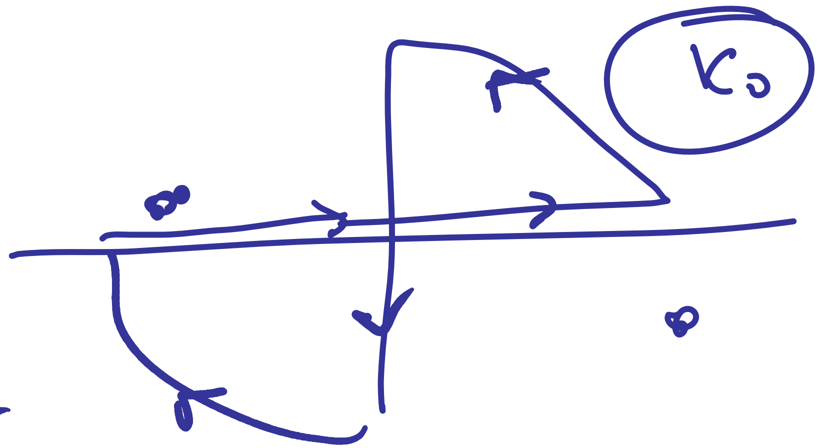
For the diagram

$-i\Sigma \equiv$ 

one gets

$$\frac{1}{2}(-i\lambda) \int \frac{d^4 k}{(2\pi)^n} \frac{i}{k^2 - m^2 + i\varepsilon}$$

$$\frac{1}{k^2 - m^2 + i\varepsilon} \Rightarrow$$



$$k_0 = \pm\left[\vec{k}^2 + m^2 - i\varepsilon\right]^{1/2}$$

Wick rotation: change the contour of integration to imaginary axis.

changing the variables

$$ik_0 \to k_0, \quad dk_0 \to -i\,dk_0, \quad k_0^2 - \vec{k}^2 \to -k_0^2 - \vec{k}^2$$

$$\int_{+i\infty}^{-i\infty} dk_0 \to +i\int_{-\infty}^{\infty} dk_0$$

$$k^2 - m^2 \to -(k^2 + m^2)$$

In Euclidean space

$$d^4k = \frac{1}{2} k^2 \, dk^2 \, d\Omega_4 , \quad \Omega_4 = 2\pi^2$$

$$\Sigma \sim \int_0^\infty dk^2 \frac{k^2}{k^2+m^2} =$$

$$= \int_0^{\mu^2} dk^2 + \int_{\mu^2}^\infty dk^2 \qquad (\text{for } \mu^2 \gg m^2)$$

divergent

Let's impose a renormalization condition

$$\int_{\mu^2}^{\infty} dk^2 = 0$$

so that

$$\Sigma^R(\mu \quad) = \int_0^{\mu^2} dk^2 \sim \mu^2$$

Then $m_F^2 = m_R^2(\mu) + \Sigma^R(\mu)$

Change the renormalization scheme

$$\mu_1^2 \to \mu_2^2$$

Then

$$m_R^2(\mu_2) \simeq m_R^2(\mu_1^2) + \int_{\mu_2^2}^{\mu_1^2} dk^2$$

$$= m_R^2(\mu_1^2) + (\mu_1^2 - \mu_2^2)$$

If $\mu_2^2 \ll \mu_1^2$, large fine tuning is necessary to keep $m_R^2(\mu_2) \ll \mu_1^2$

As long as there is only one physical scale (m) in the problem, the necessity of considering vastly different renormalization schemes is not so obvious.

However, let's consider a theory with two scalars, $m << M$.

As we shall see, renormalization at $\mu_1 = M$ and then its change to $\mu_2 = m$ is

convenient because it allows
for a decoupling of the heavy M
when calculating quantum corrections
to m

Let's discuss corrections to scalar masses in a theory with two scalars, with very different physical masses

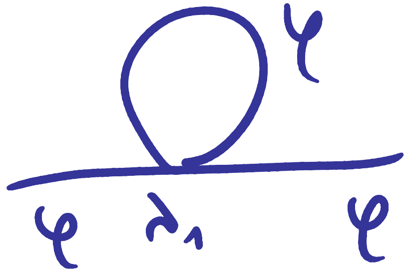
~~Let's see it using MS as the renormalisation scheme.~~

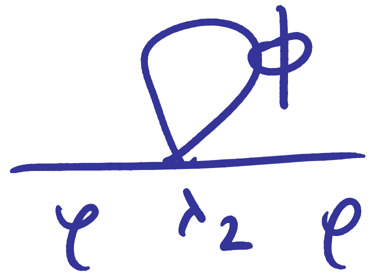
$$\mathcal{L} = \frac{1}{2}\left[(\partial\varphi)^2 + (\partial\phi)^2 - m^2\varphi^2 - M^2\phi^2\right]$$

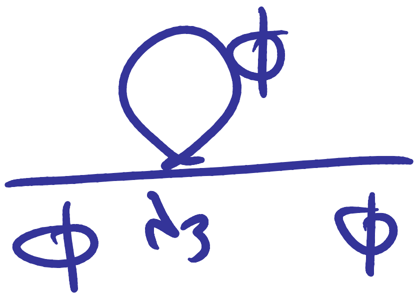
$$- \frac{\lambda_1}{4!}\varphi^4 - \frac{\lambda_2}{4!}\varphi^2\phi^2 - \frac{\lambda_3}{4!}\phi^4$$

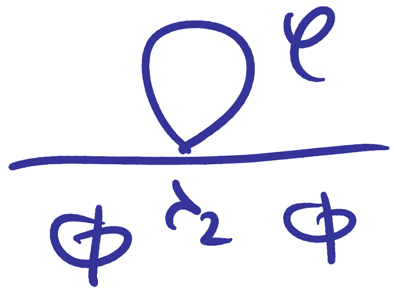
$$m \ll M$$

Among others, we get the diagrams

$$-i\,\Sigma_\varphi \equiv$$ 

$+$ 

$$-i\,\Sigma_\Phi \equiv$$ 

$+$ 

With two fields φ & Φ

$$\Sigma_\varphi \sim \lambda_1 \int_0^\infty dk^2 \frac{k^2}{k^2+m^2} + \lambda_2 \int_0^\infty dk^2 \frac{k^2}{k^2+M^2}$$

$$\simeq \lambda_1 \int_0^{M^2} dk^2 \frac{k^2}{k^2+m^2} + \lambda_2 \int_0^{M^2} dk^2 \frac{k^2}{k^2+M^2}$$

$$+ \int_{M^2}^\infty dk^2 k^2 \left(\lambda_1 \frac{1}{k^2+m^2} + \lambda_2 \frac{1}{k^2+M^2} \right)$$

Let's impose a renormalization condition

$$\int_{M}^{\infty} dk^2 k^2 (\ldots\ldots) = 0$$

so that

$$\Sigma_{\varphi}^{R}(\mu = M) = \int_{0}^{M^2} dk^2 k^2 \left(\frac{1}{k^2+m^2} + \frac{1}{k^2+M^2} \right)$$

Then $m_F^2 = m_R^2(\mu = M) + \Sigma_{\varphi}^{R}(\mu = M)$.

$\approx m_R^2(M) + M^2$! (fine tuning

It is useful to notice that I can organize this calculation also in another way:

Renormalizing at $\mu = m_F$ we put $\Sigma_\varphi(\mu = m_F) = \int_{m^2}^{\infty} dk^2 k^2 (\ldots) = 0$

so that $m_F^2 = m_R^2(\mu = m) =$

$$= m_R^2(\mu = M) + \int_0^{M^2} dk^2 \frac{1}{k^2+m^2} + \underbrace{\int_0^{M^2} dk^2 \frac{k^2}{k^2+M^2}}_{\approx M^2}$$

We can redefine

$$m_R^2(\mu=M) \to \tilde{m}_R^2(\mu=M) + \int_0^{M^2} dk^2 \frac{k^2}{k^2+M^2}$$

and we get

$$m_R^2(\mu=m) = \tilde{m}_R^2(\mu=M) + \int_0^{M^2} dk^2 \frac{k^2}{k^2+m^2}$$

We get a theory without the particle M and with the physical cut-off at M! (with some initial value $\tilde{m}_R^2(\mu=M)$)

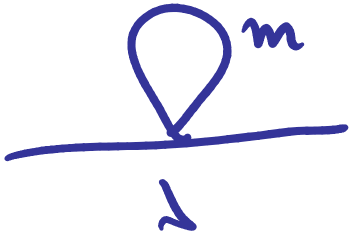
If we choose the cut-off M close to m, we have no fine-tuning problem in the effective theory of the particle m.

But we may get another fine-tuning problem in the theory of $(m+M)$ if there is a third scale $\hat{M} \gg M$.

And so on ...

This effect is often masked if we use minimal subtraction as our renormalization scheme (quadratic divergences are not manifest)

For a prototype diagram

$-i\Sigma \equiv$ 

one gets

$$\frac{1}{2}(-i\lambda)\mu^{\epsilon}\int\frac{d^n k}{(2\pi)^n}\,\frac{i}{k^2-m^2+i\varepsilon} =$$

$$= \frac{1}{2}(-i\lambda)\mu^{\epsilon}\frac{1}{(4\pi)^2}(4\pi)^{\epsilon/2}(m^2)^{1-\epsilon/2} \ *$$

$$* \ \Gamma\left(\tfrac{1}{2}\epsilon-1\right) =$$

$$= \frac{1}{2} i \frac{2\lambda m^2}{(4\pi)^2} \frac{1}{\varepsilon} - \frac{1}{2} i \frac{\lambda m^2}{(4\pi)^2} *$$

$$* \left(-1 + \gamma - \ln 4\pi + \ln \frac{m^2}{\mu^2}\right)$$

Useful formulae:

In n-dimensional Euclidean space

$$\int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2+b)^\alpha} = \frac{1}{(4\pi)^{n/2}} \frac{b^{n/2-\alpha}\,\Gamma(\alpha - \frac{1}{2}n)}{\Gamma(\alpha)}$$

$$\Gamma(x) \underset{x \to 0}{=} \frac{1}{x} - \gamma + O(x)$$

One more useful formula

$$a^{\epsilon} = e^{\varepsilon \ln a} = 1 + \varepsilon \ln a + O(\varepsilon^2)$$

and we get the final result quoted earlier.

Also, remember how the propagator is modified

$$G = \frac{i}{k^2-m^2} + \frac{i}{k^2-m^2}\left[-i\,\Sigma(k)\right]\frac{i}{k^2-m^2} + \dots$$

$$= \frac{i}{k^2-m^2-\Sigma(k)}$$

Renormalization ($\overline{MS}$)

$$\Sigma(k) \to \Sigma_R(k) = \frac{1}{2}\frac{\lambda m^2}{(4\pi)^2} \ln \frac{m^2}{\mu^2}$$

(After renormalization we put $\epsilon = 0$)

So

$$G = \frac{i}{k^2 - m_R^2(\mu) - \Sigma_R(k,\mu)}$$

The physical mass is defined by the equation $k^2 - m_R^2(\mu) - \Sigma_R(k,\mu) = 0$ for $k^2 = m_F^2$

We return to our model:

$$G_\varphi = \frac{i}{k^2 - m_R^2(\mu) - \frac{\lambda_1 m_R^2(\mu)}{(4\pi)^2} \ln \frac{m_R^2(\mu)}{\mu^2} - \frac{\lambda_2 M_R^2(\mu)}{(4\pi)^2} \ln \frac{M_R^2}{\mu^2}}$$

$$G_\Phi = \frac{i}{k^2 - M_R^2(\mu) - \frac{\lambda_3 M_R^2(\mu)}{(4\pi)^2} \ln \frac{M_R^2(\mu)}{\mu^2} - \frac{\lambda_2 m_R^2(\mu)}{(4\pi)^2} \ln \frac{m_R^2(\mu)}{\mu^2}}$$

Only logarithm terms depend on M^2, if we put $\mu^2 = M_R^2$

But even in this case we see the problem when we have three scales $m^2 \ll M^2 \ll \tilde{M}^2$ because we get then terms

$$M^2 \ln \frac{M^2}{\mu^2}, \quad \tilde{M}^2 \ln \frac{\tilde{M}^2}{\mu^2}$$

and we cannot make both corrections to m^2 small simultaneously

Another view at the hierarchy problem: take that model with two scalars, m & M, and suppose both of them develop vev's.

E.g. we want the first to break electroweak symmetry and the second to break the GUT group, so that we need $v \ll V$

Is this possible?

We look for solutions of two equations

$$\frac{\partial V}{\partial \varphi} = \left(-\frac{1}{2}m^2 + \frac{1}{3}\lambda_1|\varphi|^2 + \lambda_2|\phi|^2\right) = 0$$

$$\frac{\partial V}{\partial \phi} = \left(-\frac{1}{2}M^2 + \frac{1}{3}\lambda_3|\phi|^2 + \lambda_2|\varphi|^2\right) = 0$$

Replacing $\varphi \to v$, $\phi \to V$ and expanding in v/V we get (in the lowest order)

$$-3M^2 + 2\lambda_3 V^2 \simeq 0$$

$$-3m^2 + 2\lambda_1 v^2 + 6\lambda_2 V^2 = 0$$

$$\Rightarrow \quad 2\lambda_1 v^2 = 3m^2 + 6\lambda_2 \frac{3}{2\lambda_2} M^2 \ll M^2$$

huge tuning between the tree level parameters m^2, M^2 is necessary

Large cancellations between parameters m and M are necessary to keep the physical scalar mass small

Two solutions are known:

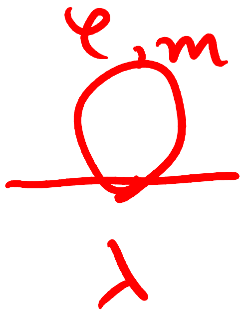
Solution $\underline{\mathrm{I}}$: supersymmetry

φ , ψ (fermion)

Couplings $\lambda \varphi^4$, $y \varphi \bar{\psi} \psi$

(Yukawa)

Correction to the mass of φ



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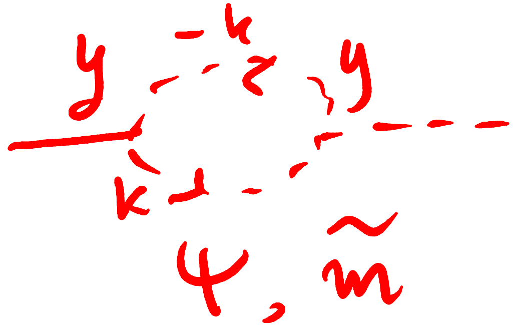


$$\lambda \int_0^\infty dk^2\, k^2 \frac{1}{k^2+m^2}$$

$$y^2 \int d^4k \frac{1}{\not{k}-\tilde{m}} \frac{1}{(-\not{k}-\tilde{m})} =$$

$$= -y^2 \int d^4k \frac{1}{k^2-\tilde{m}^2} \xrightarrow{} \text{euclid}$$

$$\to -y^2 \int dk^2 \frac{k^2}{k^2+\tilde{m}^2}$$

Cancellation, if $\lambda = y^2$, $m = \tilde{m}$.

This is the feature of supersymmetric theories (cancellation to all orders).

Imagine now that $\lambda = y^2$ but $m > \tilde{m}$ (softly broken supersymmetry)

$$\int dk^2\, k^2 \left(\frac{1}{k^2+m^2} - \frac{1}{k^2+\tilde{m}^2} \right) \simeq -(m^2 - \tilde{m}^2) \underbrace{\int_0^\infty \frac{dk^2}{k^2}}_{\text{logarithmically divergent}}$$

The emerging picture:

at low energy, we have non-supersymmetric theory of a fermion ψ and some scalar h (e.g. like in SM). There is a hierarchy problem for the scalar h.

It's solved by adding a scalar partner to ψ and a fermion partner to h with masses close to m_ψ, m_h.

The hierarchy problem of the non-supersymmetric model is then solved.
But we may still have a hierarchy problem of the full theory because of the terms

$$(m^2 - \tilde{m}^2) \int_{\tilde{m}^2}^{\infty} \frac{dk^2}{k^2} \; ; \text{ if there is}$$

one more physical scale, e.g. M_{GUT},
one gets $(m^2 - \tilde{m}^2) \ln \frac{M_{GUT}^2}{\tilde{m}^2}$; it can still be large

Solution II:

~~Exception~~ Goldstone boson

Toy model - $U(1)$ global symmetry

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - \frac{\lambda}{4}(\phi^* \phi)^2$$
$$+ i \psi_1 \bar{\sigma}_\mu \partial^\mu \psi_1 + i \psi_2 \bar{\sigma}_\mu \partial^\mu \psi_2$$
$$- g(\phi \psi_1 \psi_2 + \phi^* \bar{\psi}_1 \bar{\psi}_2)$$

ψ_1, ψ_2 - Weyl fermions; $U(1)$ charges

$\phi : +1$

$\psi_1 : -1 \quad , \psi_2 : 0$

Spontaneous symmetry breaking

$$\phi = \frac{1}{\sqrt{2}}\left(\varphi + v + i\chi\right)$$

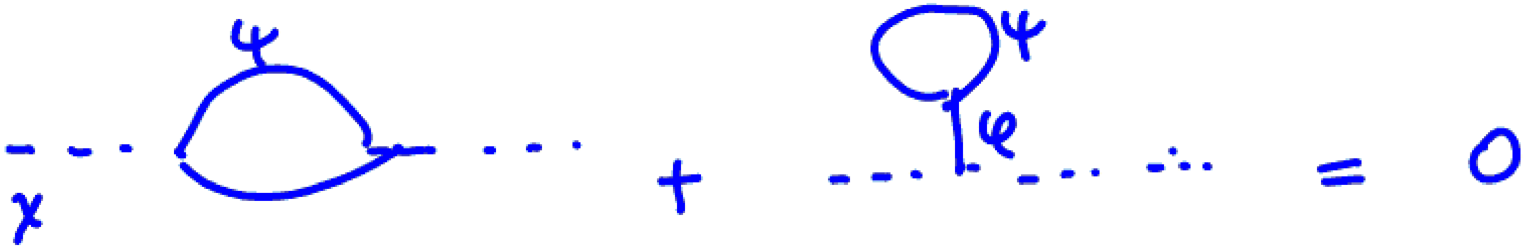
Dirac fermion $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}(\partial\chi)^2 + \bar{\psi}\,\partial\!\!\!/\,\psi - \frac{1}{4}\underbrace{\lambda v^2}_{m_\varphi^2}\varphi^2$$

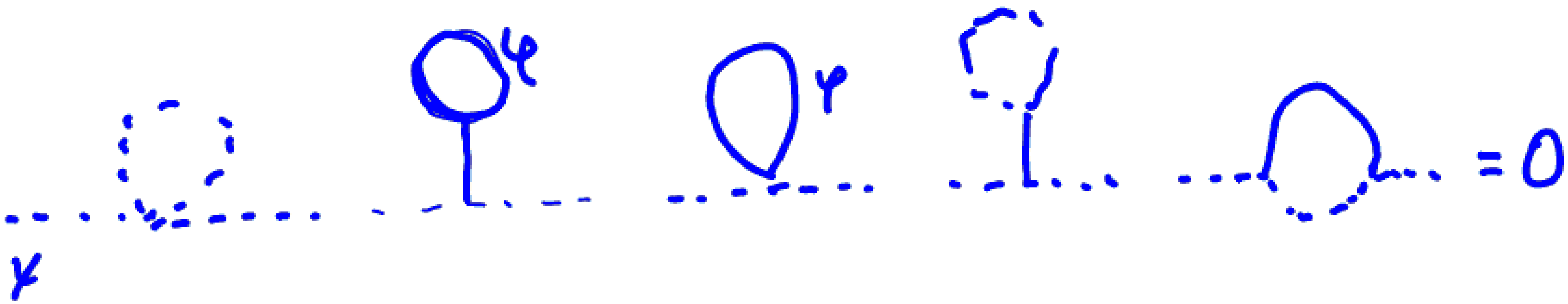
$$- \frac{\lambda v}{4}\varphi\left(\varphi^2 + \chi^2\right) - \frac{\lambda}{16}\left(\varphi^2 + \chi^2\right)^2$$

$$- \underbrace{\frac{gv}{\sqrt{2}}}_{m_\psi}\bar{\psi}\psi - \frac{1}{\sqrt{2}}g\varphi\bar{\psi}\psi - \frac{i}{\sqrt{2}}g\chi\bar{\psi}\gamma_5\psi$$

Corrections to the χ mass (Goldstone)



(no dependence on cut-off Λ !)



fermion - fermion
boson - boson

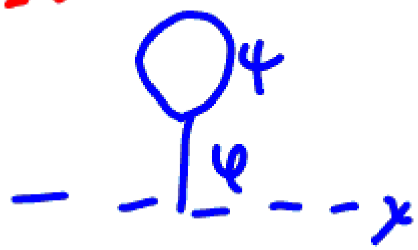
cancellations
(conspiracy of couplings)

With non-linear parametrization

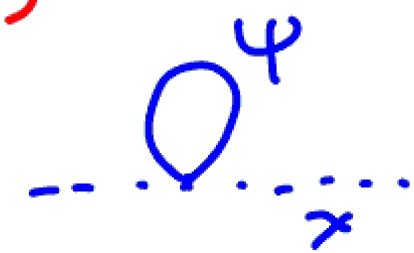
$$\phi = \frac{1}{\sqrt{2}}(\varphi + v)e^{i\chi/v}$$

$$-g(\phi\psi_1\psi_2 + \phi^*\bar\psi_1\bar\psi_2) = -\frac{gv}{\sqrt{2}}\left(e^{i\frac{\chi}{v}}\psi_1\psi_2 + e^{-i\frac{\chi}{v}}\bar\psi_1\bar\psi_2\right) = -\frac{ig}{\sqrt{2}}(\chi\psi_1\psi_2 - \chi\bar\psi_1\bar\psi_2)$$

$$+\frac{g}{\sqrt{2}v}\chi^2(\psi_1\psi_2 + \bar\psi_1\bar\psi_2) + \ldots$$



$\longrightarrow$



Corrections to a Goldstone boson mass vanish independently of the scale of new physics.

Fermion-boson cancellations of corrections to the Higgs boson mass : supersymmetry

Fermion-fermion & boson-boson cancellations : little Higgs models