Von Neumann algebras of discrete quantum groups, their invariants and the i.c.c. condition

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1. Von Neumann algebras of discrete groups
2. Compact (and discrete) quantum groups
3. Invariants related to $\tau^{\mathbb{G}}$ and $\sigma^{\boldsymbol{h}}$
4. Conjecture on the characterization of the Kac property
5. When does the conjecture hold?
6. I.c.c.-type conditions

- Let Γ be a discrete group and let $\Gamma \ni \gamma \mapsto \lambda_\gamma \in \mathrm{B}(\ell^2(\Gamma))$ be the regular representation of Γ.
- The von Neumann algebra generated by $\{\lambda_\gamma \,|\, \gamma \in \Gamma\}$ is the **group von Neumann algebra of** Γ.
- Notation: $L(\Gamma)$, $\mathrm{vN}(\Gamma)$, ...
- The group von Neumann algebra of Γ is a factor iff Γ is a **i.c.c. group**, i.e. all of its non-trivial conjugacy classes are infinite.
- The group von Neumann algebra always admits a faithful tracial state:

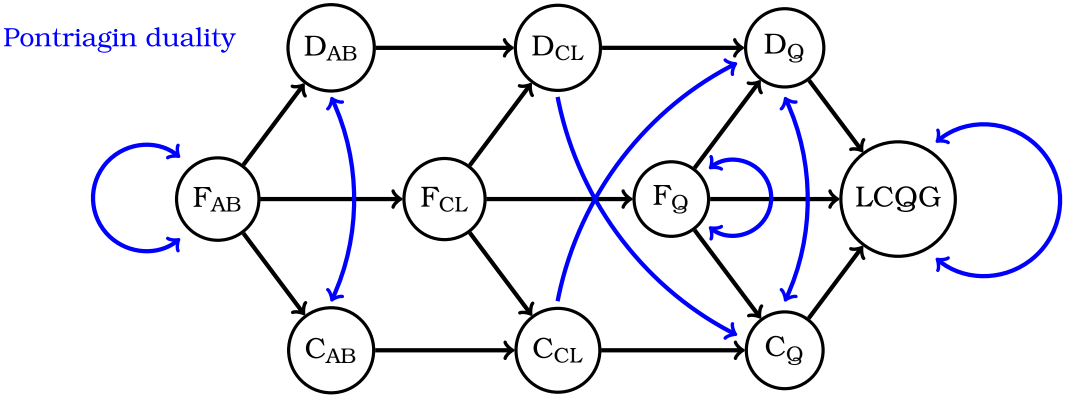
$$x \longmapsto \langle \delta_e | x\delta_e \rangle.$$

- In particular group von Neumann algebras of i.c.c. groups are always factors of type II_1.

- If Γ happens to be abelian, the group von Neumann algebra of Γ is naturally isomorphic to $L^\infty(\widehat{\Gamma})$, where $\widehat{\Gamma}$ is the Pontriagin dual of Γ:

$$L(\Gamma) = L^\infty(\widehat{\Gamma}).$$

- In other words the von Neumann algebras of discrete groups are all of the form $L^\infty(\mathbb{G})$ with $\mathbb{G}$ a compact quantum group which additionally is **cocommutative**.
- As we already mentioned, the possible factors we can obtain this way are all of type II_1.
- How about more general compact quantum groups?



D_{AB} = Discrete abelian, D_{CL} = Discrete classical, D_{Q} = Discrete quantum, F_{AB} = Finite abelian, F_{CL} = Finite classical, F_{Q} = Finite quantum, LCQG = Locally compact quantum, C_{AB} = Compact abelian, C_{CL} = Compact classical, C_{Q} = Compact quantum.

DEFINITION

A **compact quantum group** is an object $\mathbb{G}$ described by

- a unital C^*-algebra $\mathrm{C}(\mathbb{G})$ (usually non-abelian)
- a unital $*$-homomorphism $\Delta\colon \mathrm{C}(\mathbb{G}) \to \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G})$ s.t. $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ and $\Delta(\mathrm{C}(\mathbb{G}))(\mathbb{1} \otimes \mathrm{C}(\mathbb{G}))$ and $(\mathrm{C}(\mathbb{G}) \otimes \mathbb{1})\Delta(\mathrm{C}(\mathbb{G}))$ are dense in $\mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G})$.

EXAMPLES

1. Every compact group G is a compact quantum group $\mathbb{G} = G$ in the sense that $\mathrm{C}(\mathbb{G}) = \mathrm{C}(G)$ and $\Delta\colon \mathrm{C}(G) \to \mathrm{C}(G) \otimes \mathrm{C}(G) = \mathrm{C}(G \times G)$ is $\Delta(f)(x, y) = f(xy)$.
2. Every discrete group Γ gives rise to a compact quantum group $\widehat{\Gamma}$: $\mathrm{C}(\widehat{\Gamma}) = \mathrm{C}^*_{\mathrm{red}}(\Gamma)$ with $\Delta(\lambda_\gamma) = \lambda_\gamma \otimes \lambda_\gamma$.
3. Fix $q \in [-1, 1[\,\setminus\{0\}$ and let $\mathrm{C}(\mathrm{SU}_q(2))$ be the C^*-algebra generated by α, γ s.t.

$$\alpha\gamma = q\gamma\alpha, \quad \alpha^*\alpha + \gamma^*\gamma = \mathbb{1} = \alpha\alpha^* + q^2\gamma^*\gamma, \quad \gamma^*\gamma = \gamma\gamma^*.$$

Then define $\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma$ and $\Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma$.

THEOREM (S.L. WORONOWICZ)

Let $\mathbb{G}$ be a compact quantum group. Then there exists a unique state $\boldsymbol{h}$ on $\mathrm{C}(\mathbb{G})$ such that

$$(\boldsymbol{h} \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes \boldsymbol{h})\Delta(a) = \boldsymbol{h}(a)\mathbb{1}$$

for all $a \in \mathrm{C}(\mathbb{G})$.

- If $\mathbb{G} = G$ with G a compact group then $\boldsymbol{h}$ on is given by integration with respect to the normalized Haar measure.
- In general we call $\boldsymbol{h}$ the **Haar measure** or the **Haar state** of $\mathbb{G}$.
- If Γ is a discrete group then $\boldsymbol{h}$ on $\widehat{\Gamma}$ is the trace mentioned earlier.
- The von Neumann algebra $L^\infty(\mathbb{G})$ is defined to be the strong closure of the image of $\mathrm{C}(\mathbb{G})$ in the GNS representation defined by $\boldsymbol{h}$.

Theorem (J. Krajczok & P.M.S.)

1. For each $\lambda \in\,]0,1[$ there exist uncountably many pairwise non-isomorphic compact quantum groups $\mathbb{G}$ such that $L^\infty(\mathbb{G})$ is the injective factor of type III_λ.
2. There exists uncountably many compact quantum groups $\mathbb{G}$ such that $L^\infty(\mathbb{G})$ are pairwise non-isomorphic injective factors of type III_0.
3. There exists uncountably many pairwise non-isomorphic compact quantum groups $\mathbb{G}$ such that $L^\infty(\mathbb{G})$ is the injective factor of type III_1.
4. There does not exist a compact quantum group $\mathbb{G}$ with $L^\infty(\mathbb{G}) = N \oplus \mathrm{B}(\ell^2)$, where N is any von Neumann algebra or the zero vector space.

- In order to explain our constructions we need to introduce more structure.
- On $L^\infty(\mathbb{G})$ we have the modular group $\sigma^{\boldsymbol{h}}$ of the Haar measure $\boldsymbol{h}$.
- Furthermore there exists another one parameter group $\tau^{\mathbb{G}}$ of automorphisms of $L^\infty(\mathbb{G})$ called the **scaling group**.
- An exact description of this group requires a deeper dive into the theory, but for our present purposes the following information suffices:
  - the scaling group acts by quantum group automorphisms: $\Delta\circ\tau_t^{\mathbb{G}} = (\tau_t^{\mathbb{G}}\otimes\tau_t^{\mathbb{G}})\circ\Delta$,
  - the scaling and modular automorphisms commute,
  - the Haar measure is tracial iff $\tau_t^{\mathbb{G}} = \mathrm{id}$ for all t (in this case we say that $\mathbb{G}$ is of **Kac type**).

- We begin with the following theorem:

Theorem (J. Krajczok, M. Wasilewski)

Fix $\nu \in \mathbb{R}\setminus\{0\}$, $q \in \,]-1,1[\,\setminus\{0\}$ and let $r \in \mathbb{Q}$ act on $L^\infty(\mathrm{SU}_q(2))$ by $\tau_{\nu r}^{\mathrm{SU}_q(2)}$. Then the compact quantum group $\mathbb{H}_{\nu,q} = \mathbb{Q} \ltimes \mathrm{SU}_q(2)$ satisfies

- $L^\infty(\mathbb{H}_{\nu,q})$ is injective (because $\mathbb{H}_{\nu,q}$ is co-amenable),
- if $\nu \log|q| \notin \pi\mathbb{Q}$ then $L^\infty(\mathbb{H}_{\nu,q})$ is a factor,
- since there is a tracial weight on $L^\infty(\mathrm{SU}_q(2))$ invariant under the scaling group, the algebra $L^\infty(\mathbb{H}_{\nu,q})$ is not of type III.

Furthermore, assuming $\nu \log|q| \notin \pi\mathbb{Q}$, we have

- $\tau_t^{\mathbb{H}_{\nu,q}}$ is trivial iff $t \in \frac{\pi}{\log|q|}\mathbb{Z}$, so $\mathbb{H}_{\nu,q}$ is not of Kac type,
- consequently $L^\infty(\mathbb{H}_{\nu,q})$ is a factor of type II_∞.

- Our examples are constructed as infinite products
$$\mathbb{G}=\bigtimes_{n=1}^{\infty}\mathbb{H}_{q_n,\nu_n}$$
for certain sequences of parameters $(q_n)_{n\in\mathbb{N}}$ and $(\nu_n)_{n\in\mathbb{N}}$ s.t. $\nu_n\log|q_n|\notin\pi\mathbb{Q}$.
- Assume that one pair (q_n,ν_n) is repeated infinitely many times.
- Then $L^\infty(\mathbb{G})$ is an injective factor of type III and the invariant $T(L^\infty(\mathbb{G}))$ is given by
$$T\bigl(L^\infty(\mathbb{G})\bigr)=\Bigl\{t\in\mathbb{R}\Bigm|\sum_{n=1}^{\infty}\Bigl(1-\frac{1-q_n^2}{\bigl|1-|q_n|^{2+2it}\bigr|}\Bigr)<+\infty\Bigr\}.$$
- If $(\nu_n,q_n)=(\nu,q)$ for all n and some $(\nu,q)\in(\mathbb{R}\setminus\{0\})\times(]-1,1[\,\setminus\{0\})$ then $L^\infty(\mathbb{G})$ is the injective factor of type $\mathrm{III}_{|q|^2}$.
- Assume that there are two subsequences $(q_{n_p})_{p\in\mathbb{N}}$ and $(q_{m_p})_{p\in\mathbb{N}}$ such that $\{n_p\,|\,p\in\mathbb{N}\}\cap\{m_p\,|\,p\in\mathbb{N}\}=\varnothing$ and $q_{n_p}\xrightarrow[p\to\infty]{}r_1$, $q_{m_p}\xrightarrow[p\to\infty]{}r_2$ for some $r_1,r_2\in\,]-1,1[\,\setminus\{0\}$ such that $\frac{\pi}{\log|r_1|}\mathbb{Z}\cap\frac{\pi}{\log|r_2|}\mathbb{Z}=\{0\}$. Then $L^\infty(\mathbb{G})$ is the injective factor of type III_1.

- Now for $s \in \,]0,1[$ let
$$t_s = \sum_{p=1}^{\infty} \frac{\lfloor p^{1-s} \rfloor}{p!},$$
where $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}_{\geqslant 0}$.
- Define
$$l_k = \left\lfloor \exp(2\pi k!) k^{2s-1} \right\rfloor, \qquad k \in \mathbb{N}.$$
- Let $(q_n)_{n\in\mathbb{N}}$ be the sequence
$$\Bigl(\underbrace{\exp(-\pi 1!), \dots, \exp(-\pi 1!)}_{l_1 \text{ times}}, \underbrace{\exp(-\pi 2!), \dots, \exp(-\pi 2!)}_{l_2 \text{ times}}, \dots\Bigr)$$
and for each n choose $\nu_n \in \mathbb{R}\backslash\{0\}$ such that $\nu_n \log|q_n| \notin \mathbb{Q}$.
- Then, with $\mathbb{G}$ as before, $L^\infty(\mathbb{G})$ is an injective factor of type III_0 and its invariant T satisfies
  - $\mathbb{Q} \subset T(L^\infty(\mathbb{G}))$,
  - $t_{s'} \in T(L^\infty(\mathbb{G}))$ if and only if $s' > s$.

- We also analyzed examples of the form $\mathbb{K} = \Gamma \ltimes \mathbb{G}$, with $\mathbb{G}$ constructed as an infinite tensor product as before and Γ a countable subgroup of $\mathbb{R}$ (with discrete topology) acting via the scaling automorphisms.
- This way we can control which scaling automorphisms of the resulting quantum group are inner.
- We then used the set

$$T^{\tau}_{\mathrm{Inn}}(\mathbb{K}) = \left\{ t \in \mathbb{R} \,\middle|\, \tau^{\mathbb{K}}_t \in \mathrm{Inn}(L^{\infty}(\mathbb{K})) \right\}$$

to distinguish between the different examples.
- With $\mathbb{G} = \bigtimes\limits_{n=1}^{\infty} \mathbb{H}_{\nu,\sqrt{\lambda}}$ (and e.g. $\nu = \frac{2\pi^2}{\log\lambda}$) the algebra $L^{\infty}(\mathbb{K})$ is the injective factor of type III_{λ} and $T^{\tau}_{\mathrm{Inn}}(\mathbb{K}) = \Gamma + \frac{2\pi}{\log\lambda}\mathbb{Z}$.

- For general locally compact quantum groups we worked with the following invariants:

Definition

$$
\begin{aligned}
T^{\tau}(\mathbb{G}) &= \bigl\{t\in\mathbb{R}\,\big|\,\tau^{\mathbb{G}}_t=\mathrm{id}\bigr\},\\
T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) &= \bigl\{t\in\mathbb{R}\,\big|\,\tau^{\mathbb{G}}_t\in\mathrm{Inn}\bigl(L^\infty(\mathbb{G})\bigr)\bigr\},\\
T^{\tau}_{\overline{\mathrm{Inn}}}(\mathbb{G}) &= \bigl\{t\in\mathbb{R}\,\big|\,\tau^{\mathbb{G}}_t\in\overline{\mathrm{Inn}}\bigl(L^\infty(\mathbb{G})\bigr)\bigr\},\\
T^{\sigma}(\mathbb{G}) &= \bigl\{t\in\mathbb{R}\,\big|\,\sigma^{\varphi}_t=\mathrm{id}\bigr\},\\
T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) &= \bigl\{t\in\mathbb{R}\,\big|\,\sigma^{\varphi}_t\in\mathrm{Inn}\bigl(L^\infty(\mathbb{G})\bigr)\bigr\},\\
T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G}) &= \bigl\{t\in\mathbb{R}\,\big|\,\sigma^{\varphi}_t\in\overline{\mathrm{Inn}}\bigl(L^\infty(\mathbb{G})\bigr)\bigr\},\\
\mathrm{Mod}(\mathbb{G}) &= \bigl\{t\in\mathbb{R}\,\big|\,\delta^{\mathrm{i}t}=\mathbb{1}\bigr\},
\end{aligned}
$$

where δ is the modular element of $\mathbb{G}$.

Example: the quantum $E(2)$ group

Let $\mathbb{G} = \mathrm{E}_q(2)$ for some $q \in \,]0,1[$. Then we have

$$T^{\tau}(\mathbb{G}) = T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = T^{\tau}_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^{\sigma}(\mathbb{G}) = T^{\tau}(\widehat{\mathbb{G}}) = T^{\sigma}(\widehat{\mathbb{G}}) = \mathrm{Mod}(\widehat{\mathbb{G}}) = \tfrac{\pi}{\log q}\mathbb{Z},$$
$$T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) = T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^{\tau}_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^{\tau}_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = T^{\sigma}_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^{\sigma}_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = \mathrm{Mod}(\mathbb{G}) = \mathbb{R}.$$

Example: quantum "$az+b$" groups

Let $\mathbb{G}$ be the quantum "$az+b$" group for the deformation parameter q in one of the three cases:

1. $q = \mathrm{e}^{\frac{2\pi\mathrm{i}}{N}}$ with $N = 6, 8, \dots,$
2. $q \in \,]0,1[,$
3. $q = \mathrm{e}^{1/\rho}$ with $\operatorname{Re}\rho < 0$, $\operatorname{Im}\rho = \frac{N}{2\pi}$ with $N = \pm 2, \pm 4, \dots$.

Then

$$T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = T^{\tau}_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^{\tau}_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^{\tau}_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) = T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^{\sigma}_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^{\sigma}_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = \mathbb{R},$$

$$T^{\tau}(\mathbb{G}) = T^{\tau}(\widehat{\mathbb{G}}) = T^{\sigma}(\mathbb{G}) = T^{\sigma}(\widehat{\mathbb{G}}) = \operatorname{Mod}(\mathbb{G}) = \operatorname{Mod}(\widehat{\mathbb{G}}) = \begin{cases} \{0\} & \text{in cases } ① \text{ and } ③ \\ \frac{\pi}{\log q}\mathbb{Z} & \text{in case } ② \end{cases}.$$

- The sets $T^{\circ}_{\bullet}(\mathbb{G})$ are subgroups of $\mathbb{R}$ and are isomorphism invariants of the quantum group $\mathbb{G}$.
- $T^{\tau}(\mathbb{G}) = T^{\tau}(\widehat{\mathbb{G}})$.
- $T^{\bullet}(\mathbb{G})$, $T^{\bullet}_{\overline{\mathrm{Inn}}}(\mathbb{G})$, and $\mathrm{Mod}(\mathbb{G})$ are closed.
- $T^{\sigma}(\mathbb{G})$, $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G})$, and $T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G})$ are the same regardless of which Haar measure we choose.
- $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G})$ is equal to the Connes' invariant $T(L^{\infty}(\mathbb{G}))$. Consequently, $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G})$ depends only on the von Neumann algebra $L^{\infty}(\mathbb{G})$. It is also the case for $T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G})$.

Proposition

For any locally compact quantum group $\mathbb{G}$ we have

$$\begin{aligned}
T^{\sigma}(\mathbb{G}) &= T^{\tau}(\mathbb{G}) \cap \mathrm{Mod}(\widehat{\mathbb{G}}),\\
T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) \cap \mathrm{Mod}(\widehat{\mathbb{G}}) &= T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) \cap \mathrm{Mod}(\widehat{\mathbb{G}}),\\
T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G}) \cap \mathrm{Mod}(\widehat{\mathbb{G}}) &= T^{\tau}_{\overline{\mathrm{Inn}}}(\mathbb{G}) \cap \mathrm{Mod}(\widehat{\mathbb{G}}),\\
\mathrm{Mod}(\mathbb{G}) \cap \mathrm{Mod}(\widehat{\mathbb{G}}) &\subset \tfrac{1}{2}\,T^{\tau}(\mathbb{G}).
\end{aligned}$$

- The first equality above together with $T^{\tau}(\mathbb{G}) = T^{\tau}(\widehat{\mathbb{G}})$ reduces the list to 11 (invariants $T^{\sigma}(\mathbb{G})$, $T^{\sigma}(\widehat{\mathbb{G}})$ and $T^{\tau}(\widehat{\mathbb{G}})$ are determined by the remaining ones).
- If $\mathbb{G}$ is compact then $\mathrm{Mod}(\mathbb{G}) = T^{\tau}_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^{\sigma}_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^{\tau}_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = T^{\sigma}_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = \mathbb{R}$.
- If additionally $L^{\infty}(\mathbb{G})$ is semifinite then $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) = T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G}) = \mathbb{R}$.

EXAMPLE: U_F^+

Let $\mathbb{G}$ be the quantum group U_F^+. Then $L^\infty(\mathbb{G})$ is a full factor, so $\mathrm{Inn}(L^\infty(\mathbb{G})) = \overline{\mathrm{Inn}}(L^\infty(\mathbb{G}))$ (Vaes).

- $\mathbb{G}$ is compact, so $\mathrm{Mod}(\mathbb{G}) = T^\tau_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^\tau_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = T^\sigma_{\mathrm{Inn}}(\widehat{\mathbb{G}}) = T^\sigma_{\overline{\mathrm{Inn}}}(\widehat{\mathbb{G}}) = \mathbb{R}$.
- If $\mathbb{G}$ is not of Kac type ($\lambda F^*F \neq \mathbb{1}$) then

$$T^\tau_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^\tau_{\mathrm{Inn}}(\mathbb{G}) = T^\tau(\mathbb{G}) = \bigcap_{\Lambda\in\mathrm{Sp}(F^*F\otimes(F^*F)^{-1})\setminus\{1\}} \frac{2\pi}{\log(\Lambda)}\mathbb{Z},$$

while $\mathrm{Mod}(\widehat{\mathbb{G}}) = \bigcap_{\Lambda\in\mathrm{Sp}(F^*F)\setminus\{\lambda^{-1}\}} \frac{2\pi}{\log\lambda+\log(\Lambda)}\mathbb{Z}$, where $\lambda = \sqrt{\frac{\mathrm{Tr}((F^*F)^{-1})}{\mathrm{Tr}(F^*F)}}$.

- If $\mathbb{G}$ is not of Kac type then $L^\infty(\mathbb{G})$ is a type III_μ factor for some $\mu \in]0,1]$ and $T^\sigma_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^\sigma_{\mathrm{Inn}}(\mathbb{G}) = \frac{2\pi}{\log\mu}\mathbb{Z}$ (otherwise $T^\sigma_{\overline{\mathrm{Inn}}}(\mathbb{G}) = T^\sigma_{\mathrm{Inn}}(\mathbb{G}) = \mathbb{R}$).

- Given a semisimple compact connected Lie group G one can form a family of compact quantum groups $\{G_q\}_{q\in]0,1[}$ (this procedure generalizes the passage $\mathrm{SU}(2)\mapsto\mathrm{SU}_q(2)$).
- Since G_q is compact we again have

$$\mathrm{Mod}(G_q)=T^{\tau}_{\mathrm{Inn}}(\widehat{G_q})=T^{\tau}_{\overline{\mathrm{Inn}}}(\widehat{G_q})=T^{\sigma}_{\mathrm{Inn}}(\widehat{G_q})=T^{\sigma}_{\overline{\mathrm{Inn}}}(\widehat{G_q})=\mathbb{R}.$$

- Furthermore $T^{\sigma}_{\mathrm{Inn}}(G_q)=T^{\sigma}_{\overline{\mathrm{Inn}}}(G_q)=\mathbb{R}$ because $\mathrm{C}(G_q)$ is a C^*-algebra of type I.
- We have $T^{\tau}(G_q)=\frac{\pi}{\log q}\mathbb{Z}$ and

$$T^{\tau}_{\mathrm{Inn}}(G_q)=T^{\tau}_{\overline{\mathrm{Inn}}}(G_q)=\mathrm{Mod}(\widehat{G_q})=\frac{\pi}{\Upsilon_\Phi\log q}\mathbb{Z},$$

where Υ_Φ is a positive integer determined by the root system Φ of the complexified Lie algebra of G (see next two slides).

- Let $\Phi = \Phi_1 \cup \cdots \cup \Phi_l$ be the decomposition of Φ into irreducible parts. Then

$$\Upsilon_\Phi = \gcd(\Upsilon_{\Phi_1}, \ldots, \Upsilon_{\Phi_l}).$$

- We have

type	**group**	**range of n**	Υ_Φ	$T^{\tau}_{\mathrm{Inn}}(\mathbb{G}_q)$
A_n	$\mathrm{SU}(n+1)$	$n \geqslant 1$ odd	1	$\frac{\pi}{\log q}\mathbb{Z}$
		$n \geqslant 1$ even	2	$\frac{\pi}{2\log q}\mathbb{Z}$
B_n	$\mathrm{Spin}(2n+1)$	$n \geqslant 2$ odd	1	$\frac{\pi}{\log q}\mathbb{Z}$
		$n \geqslant 2$ even	2	$\frac{\pi}{2\log q}\mathbb{Z}$
C_n	$\mathrm{Sp}(2n)$	$n \geqslant 3$	2	$\frac{\pi}{2\log q}\mathbb{Z}$
D_n	$\mathrm{Spin}(2n)$	$n \geqslant 4$ odd	2	$\frac{\pi}{2\log q}\mathbb{Z}$
		$n \geqslant 4$ even	1	$\frac{\pi}{\log q}\mathbb{Z}$

- And for the exceptional cases we have
  - type E_6: $\Upsilon_\Phi = 2$ and $T^\tau_{\mathrm{Inn}}(G_q) = \frac{\pi}{2\log q}\mathbb{Z}$,
  - type E_7: $\Upsilon_\Phi = 1$ and $T^\tau_{\mathrm{Inn}}(G_q) = \frac{\pi}{\log q}\mathbb{Z}$,
  - type E_8: $\Upsilon_\Phi = 2$ and $T^\tau_{\mathrm{Inn}}(G_q) = \frac{\pi}{2\log q}\mathbb{Z}$,
  - type F_4: $\Upsilon_\Phi = 2$ and $T^\tau_{\mathrm{Inn}}(G_q) = \frac{\pi}{2\log q}\mathbb{Z}$,
  - type G_2: $\Upsilon_\Phi = 2$ and $T^\tau_{\mathrm{Inn}}(G_q) = \frac{\pi}{2\log q}\mathbb{Z}$.

- Consider the compact quantum group $\mathrm{SU}_q(3)$.
- Then $\Upsilon_\Phi = 2$, so
$$T^{\tau}_{\mathrm{Inn}}\bigl(\mathrm{SU}_q(3)\bigr) = \tfrac{\pi}{2\log q}\mathbb{Z},$$
while $T^{\tau}\bigl(\mathrm{SU}_q(3)\bigr) = \tfrac{\pi}{\log q}\mathbb{Z}$.
- This means that there are non-trivial inner scaling automorphisms.
- $\mathrm{SU}_q(3)$ does not have non-trivial one-dimensional representations, so these scaling automorphisms are not implemented by a group-like element.

Proposition

Let G be such that $\Upsilon_\Phi = 2$. Then a unitary implementing the scaling automorphism for $t = \tfrac{\pi}{2\log q}$ does not belong to $\mathrm{C}(G_q)$. In particular, the restriction of this automorphism to $\mathrm{C}(G_q)$ is not inner.

- We saw that we do not always have the equality

$$T^{\tau}(\mathbb{G}) = T^{\tau}_{\mathrm{Inn}}(\mathbb{G}).$$

- However, all compact quantum groups we considered so far (e.g. G_q or U_F^+) belong to the class for which the following statement is true:

Conjecture (∗)

If $\mathbb{G}$ is a second countable compact quantum group and $T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = \mathbb{R}$ then $\mathbb{G}$ is of Kac type.

- In other words, if $T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = \mathbb{R}$ then $T^{\tau}(\mathbb{G}) = \mathbb{R}$.
- We were able to prove this conjecture for several classes of compact quantum groups.

Theorem

Let $\mathbb{G}$ be a compact quantum group which has a two-dimensional representation whose quantum dimension is strictly larger then 2. Then conjecture $(*)$ holds for $\mathbb{G}$.

- This is done by proving that $T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) \neq \mathbb{R}$.
- This is achieved by first constructing a sequence of irreps $\{U^n\}_{n\in\mathbb{N}}$ of $\mathbb{G}$ such that
$$\Gamma(U^n) \xrightarrow[n\to\infty]{} +\infty \qquad \text{and} \qquad \inf_{n\in\mathbb{N}} \left(\frac{1}{\gamma(U^n)\,\mathrm{dim_q}U^n} \frac{\Gamma(U^n)}{\mathrm{dim_q}U^n} \right) > 0,$$
where $\Gamma(U^n)$ and $\gamma(U^n)$ are the largest and smallest eigenvalue of the modular matrix ρ_{U^n} of U^α and $\mathrm{dim_q}U^\alpha$ is its quantum dimension: $\mathrm{dim_q}U^n = \mathrm{Tr}(\rho_{U^n})$.
- Out of matrix elements of U^n we construct certain elements of $L^\infty(\mathbb{G}) \mathbin{\overline{\otimes}} L^\infty(\mathbb{G})$ which help prove that it is impossible that all scaling automorphisms of $\mathbb{G}$ are inner.

- Let $\mathbb{\Gamma}$ be a discrete quantum group.
- Thus $\mathbb{\Gamma}$ is the dual (in the sense of Pontriagin duality for locally compact quantum groups) of a compact quantum group $\widehat{\mathbb{\Gamma}}$.
- We say that $\mathbb{\Gamma}$ is **type** I if $\mathrm{C^u}(\widehat{\mathbb{\Gamma}})$ is a C*-algebra of type I.
- Using the direct integral decomposition of $L^\infty(\widehat{\mathbb{\Gamma}})$ and a corresponding expression for $\boldsymbol{h}$ we can prove the following:

Theorem

Let $\mathbb{\Gamma}$ be a type I discrete quantum group. Then conjecture $(*)$ holds for $\widehat{\mathbb{\Gamma}}$.

- In fact we were able to show that if $\widehat{\mathbb{\Gamma}}$ is not of Kac type then $T^\tau_{\mathrm{Inn}}(\widehat{\mathbb{\Gamma}})$ is at most countable.

Proposition

Let Γ be a discrete group. Then the following are equivalent:

1. Γ is i.c.c.,
2. $L(\Gamma)$ is a factor,
3. $\Delta_{\hat{\Gamma}}^{(n)}\bigl(L(\Gamma)\bigr)' \cap \underbrace{L(\Gamma)\mathbin{\overline{\otimes}}\cdots\mathbin{\overline{\otimes}}L(\Gamma)}_{n+1} = \mathbb{C}\mathbb{1}$ for some $n \in \mathbb{N}$,
4. $\Delta_{\hat{\Gamma}}^{(n)}\bigl(L(\Gamma)\bigr)' \cap \underbrace{L(\Gamma)\mathbin{\overline{\otimes}}\cdots\mathbin{\overline{\otimes}}L(\Gamma)}_{n+1} = \mathbb{C}\mathbb{1}$ for all $n \in \mathbb{N}$.

Proposition

Let $\mathbb{G}$ be a locally compact quantum group and assume that

$$\Delta_{\mathbb{G}}^{(n)}\bigl(L^\infty(\mathbb{G})\bigr)' \cap \underbrace{L^\infty(\mathbb{G})\,\overline{\otimes}\cdots\overline{\otimes}\,L^\infty(\mathbb{G})}_{n+1} = \mathbb{C}\mathbb{1}$$

for some $n \in \mathbb{N}$. Then $L^\infty(\mathbb{G})$ is a factor.

Definition

Let $\mathbb{\Gamma}$ be a discrete quantum group. We say that $\mathbb{\Gamma}$ is n-i.c.c. if

$$\Delta_{\widehat{\mathbb{\Gamma}}}^{(n)}\bigl(L^\infty(\widehat{\mathbb{\Gamma}})\bigr)' \cap \underbrace{L^\infty(\widehat{\mathbb{\Gamma}})\,\overline{\otimes}\cdots\overline{\otimes}\,L^\infty(\widehat{\mathbb{\Gamma}})}_{n+1} = \mathbb{C}\mathbb{1}.$$

Proposition

Let $\mathbb{\Gamma}$ be a discrete quantum group. If $\mathbb{\Gamma}$ is n-i.c.c. for some n then $\mathbb{\Gamma}$ is m-i.c.c. for all natural $m \leqslant n$.

Theorem

Let $\mathbb{G}$ be a second countable compact quantum group whose dual is 1-i.c.c. then conjecture $(*)$ holds for $\mathbb{G}$.

THEOREM

Let $\mathbb{G}$ be a second countable compact quantum group whose dual is 1-i.c.c. and such that $T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = \mathbb{R}$. Then $\mathbb{G}$ is of Kac type.

- We have $\tau^{\mathbb{G}}_t = \mathrm{Ad}(b^{\mathrm{i}t})$ for some positive self-adjoint operator b (Kallman).
- Furthermore, for any $x \in L^\infty(\mathbb{G})$ and any $t \in \mathbb{R}$

  $$(b^{-\mathrm{i}t} \otimes b^{-\mathrm{i}t})\Delta_{\mathbb{G}}(b^{\mathrm{i}t})\Delta_{\mathbb{G}}(x)\Delta_{\mathbb{G}}(b^{-\mathrm{i}t})(b^{\mathrm{i}t} \otimes b^{\mathrm{i}t}) = (\tau^{\mathbb{G}}_{-t} \otimes \tau^{\mathbb{G}}_{-t})\Delta_{\mathbb{G}}\bigl(\tau^{\mathbb{G}}_t(x)\bigr) = \Delta(x),$$

  so $(b^{-\mathrm{i}t} \otimes b^{-\mathrm{i}t})\Delta(b^{\mathrm{i}t}) \in \Delta_{\mathbb{G}}(L^\infty(\mathbb{G}))' \cap L^\infty(\mathbb{G}) \mathbin{\overline{\otimes}} L^\infty(\mathbb{G}) = \mathbb{C}\mathbb{1}$.
- Thus $(b^{-\mathrm{i}t} \otimes b^{-\mathrm{i}t})\Delta(b^{\mathrm{i}t}) = z_t\mathbb{1}$ for some scalars z_t. Moreover $t \mapsto z_t$ is a continuous homomorphism, so $z_t = \lambda^{\mathrm{i}t}$ for some $\lambda > 0$.
- Put $B = \lambda b$. Then still $\tau^{\mathbb{G}}_t = \mathrm{Ad}(B^{\mathrm{i}t})$ and, additionally, $\Delta_{\mathbb{G}}(B^{\mathrm{i}t}) = B^{\mathrm{i}t} \otimes B^{\mathrm{i}t}$ for all $t \in \mathbb{R}$.

Theorem

Let $\mathbb{G}$ be a second countable compact quantum group whose dual is 1-i.c.c. and such that $T^{\tau}_{\mathrm{Inn}}(\mathbb{G})=\mathbb{R}$. Then $\mathbb{G}$ is of Kac type.

- Next we calculate

$$(\boldsymbol{h}\otimes\mathrm{id})\Delta_{\mathbb{G}}\left(\int\limits_{t-\frac{1}{n}}^{t+\frac{1}{n}}B^{\mathrm{i}s}\,\mathrm{d}s\right)=(\boldsymbol{h}\otimes\mathrm{id})\int\limits_{t-\frac{1}{n}}^{t+\frac{1}{n}}(B^{\mathrm{i}s}\otimes B^{\mathrm{i}s})\,\mathrm{d}s=\int\limits_{t-\frac{1}{n}}^{t+\frac{1}{n}}\boldsymbol{h}(B^{\mathrm{i}s})B^{\mathrm{i}s}\,\mathrm{d}s$$

- Multiplying by $2n$ and taking $\lim\limits_{n\to\infty}$ we obtain $\boldsymbol{h}(B^{\mathrm{i}t})\mathbb{1}=\boldsymbol{h}(B^{\mathrm{i}t})B^{\mathrm{i}t}$, so $B=\mathbb{1}$.
- It follows that $\tau^{\mathbb{G}}_t=\mathrm{id}$ for all t.

- Recall that $\operatorname{Irr}\mathrm{U}_F^+ = \mathbb{Z}_+ \star \mathbb{Z}_+$ with the two copies of $\mathbb{Z}_+$ generated by the class α of the defining representation and $\beta = \overline{\alpha}$.
- For $x \in \mathbb{Z}_+ \star \mathbb{Z}_+$ put

$$D_{x,n} = \begin{cases} \|\rho_x^2 - \mathbb{1}\| \frac{\|\rho_x\|^{2(n+1)}-1}{\|\rho_x\|^2-1} & \rho_x \neq \mathbb{1} \\ 0 & \rho_x = \mathbb{1} \end{cases}.$$

- Let $D_n = \max\{D_{\alpha\beta,n}, D_{\beta\alpha,n}, D_{\alpha^2\beta,n}\}$.

THEOREM

If $D_n < 1 - \frac{1}{\sqrt{2}}$ and $\frac{2(7-4D_n)D_n}{2(1-D_n)^2-1} < \frac{1}{\sqrt{n+1}}$ then $\widehat{\mathrm{U}_F^+}$ is n-i.c.c.

THEOREM

Take $n \in \mathbb{N}$ and write $c = \max\Bigl\{\bigl\|\lambda F^* F - \mathbb{1}\bigr\|, \bigl\|(\lambda F^* F)^{-1} - \mathbb{1}\bigr\|\Bigr\}$, where $\lambda = \sqrt{\frac{\operatorname{Tr}((F^*F)^{-1})}{\operatorname{Tr}(F^*F)}}$. If

$$\sqrt{n}(n+1)c(2+c)(1+c)^{4+6n} < \tfrac{1}{72}$$

then $\widehat{\mathrm{U}_F^+}$ is n-i.c.c.

1. Are the n-i.c.c. conditions really different?
2. Does the conjecture hold for all second countable compact quantum groups?
3. Which type III_0 factors do we obtain?
4. In particular are they the ITPFI_2?

Thank you for your attention