Compact and discrete quantum groups
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- SU(2) is the group of unitary 2×2 matrices with determinant 1.
- We have
$$\mathrm{SU}(2)=\left\{\begin{bmatrix}a & -\overline{c}\\ c & \overline{a}\end{bmatrix}\,\middle|\,a,c\in\mathbb{C},\ |a|^2+|c|^2=1\right\}.$$
- Writing $a=x_1+\mathrm{i}x_2$ and $c=x_3+\mathrm{i}x_4$ we immediately find that $\mathrm{SU}(2)=\mathbb{S}^3$ as topological spaces (even as differential manifolds).
- Clearly the group operations
$$\left(\begin{bmatrix}a & -\overline{c}\\ c & \overline{a}\end{bmatrix},\begin{bmatrix}a' & -\overline{c'}\\ c' & \overline{a'}\end{bmatrix}\right)\longmapsto\begin{bmatrix}aa'-\overline{c}c' & -\overline{(ca'+\overline{a}c')}\\ ca'+\overline{a}c' & \overline{aa'-\overline{c}c'}\end{bmatrix}$$
and
$$\begin{bmatrix}a & -\overline{c}\\ c & \overline{a}\end{bmatrix}\longmapsto\begin{bmatrix}\overline{a} & \overline{c}\\ -c & a\end{bmatrix}$$
are continuous (in fact smooth).

- It is known that a compact space (like $\mathrm{SU}(2)$) can be completely described by its Banach $*$-algebra of continuous functions (with pointwise multiplication, supremum norm and involution $f \mapsto \overline{f}$).
- Thus the study of $\mathrm{SU}(2)$ is equivalent to the study of the Banach algebra $\mathrm{C}(\mathrm{SU}(2))$.
- Let α and γ be the elements of $\mathrm{C}(\mathrm{SU}(2))$ defined by

$$\alpha\colon \mathrm{SU}(2) \ni \begin{bmatrix} a & -\overline{c} \\ c & \overline{a} \end{bmatrix} \longmapsto a \in \mathbb{C}, \qquad \gamma\colon \mathrm{SU}(2) \ni \begin{bmatrix} a & -\overline{c} \\ c & \overline{a} \end{bmatrix} \longmapsto c \in \mathbb{C}.$$

- Furthermore let $\mathrm{Pol}(\mathrm{SU}(2))$ be the $*$-subalgebra of $\mathrm{C}(\mathrm{SU}(2))$ generated by α and γ.
- $\mathrm{Pol}(\mathrm{SU}(2))$ contains the unit because $\alpha^*\alpha + \gamma^*\gamma = \mathbb{1}$.
- By the Stone-Weierstrass theorem $\mathrm{Pol}(\mathrm{SU}(2))$ is dense in $\mathrm{C}(\mathrm{SU}(2))$.

- The multiplication map $\mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SU}(2)$ can be transferred to $\mathrm{C}(\mathrm{SU}(2))$ like this
$$\mathrm{C}(\mathrm{SU}(2)) \ni f \longmapsto \Delta(f) \in \mathrm{C}(\mathrm{SU}(2) \times \mathrm{SU}(2)),$$
where
$$\Delta(f)(A, B) = f(AB), \qquad A, B \in \mathrm{SU}(2).$$
- Since $\mathrm{C}(\mathrm{SU}(2) \times \mathrm{SU}(2)) = \mathrm{C}(\mathrm{SU}(2)) \otimes \mathrm{C}(\mathrm{SU}(2))$ (completed tensor product) we obtain $\Delta\colon \mathrm{C}(\mathrm{SU}(2)) \to \mathrm{C}(\mathrm{SU}(2)) \otimes \mathrm{C}(\mathrm{SU}(2))$.
- It is easy to see that Δ is a unital $*$-homomorphism of Banach algebras (in fact it is isometric).
- Associativity of multiplication in $\mathrm{SU}(2)$ implies the **coassociativity** of Δ:
$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta.$$
- Δ is called the **comultiplication**.

- Since for
$$A=\begin{bmatrix}a&-\overline{c}\\c&\overline{a}\end{bmatrix}\quad\text{and}\quad B=\begin{bmatrix}a'&-\overline{c'}\\c'&\overline{a'}\end{bmatrix}$$
we have
$$AB=\begin{bmatrix}aa'-\overline{c}c'&-\overline{(ca'+\overline{a}c')}\\ca'+\overline{a}c'&\overline{aa'-\overline{c}c'}\end{bmatrix},$$
the functions $\Delta(\alpha)$ and $\Delta(\gamma)$ (of two variables) are equal
$$\Delta(\alpha)=\alpha\otimes\alpha-\gamma^*\otimes\gamma\quad\text{and}\quad\Delta(\gamma)=\gamma\otimes\alpha+\alpha^*\otimes\gamma.$$

- If we set $u=\begin{bmatrix}\alpha&-\gamma^*\\\gamma&\alpha^*\end{bmatrix}$ then u is a unitary element of $\mathrm{Mat}_2(\mathrm{C}(\mathrm{SU}(2)))=\mathrm{Mat}_2(\mathbb{C})\otimes\mathrm{C}(\mathrm{SU}(2))$. Moreover, for any i,j we have
$$\Delta(u_{i,j})=\sum_{k=1}^{2}u_{i,k}\otimes u_{k,j}.\qquad(\star)$$

- The fact that $\mathrm{SU}(2)$ is a group (every element is invertible) can be encoded in a slightly counter-intuitive way: the multiplication obeys cancellation laws from both sides:
$$\bigl(AB = AC\bigr) \Longrightarrow \bigl(B = C\bigr) \quad \text{and} \quad \bigl(BA = CA\bigr) \Longrightarrow \bigl(B = C\bigr)$$
for any $A, B, C \in \mathrm{SU}(2)$.
- On the level of $\mathrm{C}(\mathrm{SU}(2))$ and Δ this translates to the fact that the sets
$$\bigl\{\Delta(f)(\mathbb{1} \otimes g) \bigm| f, g \in \mathrm{C}(\mathrm{SU}(2))\bigr\} \quad \text{and} \quad \bigl\{(f \otimes \mathbb{1})\Delta(g) \bigm| f, g \in \mathrm{C}(\mathrm{SU}(2))\bigr\}$$
are linearly dense in $\mathrm{C}(\mathrm{SU}(2)) \otimes \mathrm{C}(\mathrm{SU}(2))$.

- How to see the density of $\{\Delta(f)(\mathbb{1}\otimes g)\,|\,f,g\in \mathrm{C}(\mathrm{SU}(2))\}$?
- Let us again consider the unitary matrix

$$u=\begin{bmatrix}\alpha & -\gamma^*\\ \gamma & \alpha^*\end{bmatrix}\in \mathrm{Mat}_2(\mathrm{C}(\mathrm{SU}(2)))=\mathrm{Mat}_2(\mathbb{C})\otimes \mathrm{C}(\mathrm{SU}(2)).$$

- Equation ($\star$) says that $(\mathrm{id}\otimes\Delta)u=u_{12}u_{13}$, where

$$u_{12}=\begin{bmatrix}\alpha\otimes\mathbb{1} & -\gamma^*\otimes\mathbb{1}\\ \gamma\otimes\mathbb{1} & \alpha^*\otimes\mathbb{1}\end{bmatrix}\quad\text{and}\quad u_{13}=\begin{bmatrix}\mathbb{1}\otimes\alpha & -\mathbb{1}\otimes\gamma^*\\ \mathbb{1}\otimes\gamma & \mathbb{1}\otimes\alpha^*\end{bmatrix}.$$

- Thus elements of the matrix $u_{12}=\big((\mathrm{id}\otimes\Delta)u\big)u_{13}^*$ belong to $\{\Delta(f)(\mathbb{1}\otimes g)\,|\,f,g\in \mathrm{C}(\mathrm{SU}(2))\}$.
- It follows that for any $f\in\mathrm{C}(\mathrm{SU}(2))$ the elements

$$\alpha\otimes f,\gamma\otimes f,\alpha^*\otimes f,\gamma^*\otimes f$$

belong to $\{\Delta(f)(\mathbb{1}\otimes g)\,|\,f,g\in \mathrm{C}(\mathrm{SU}(2))\}$.

- We can try the same game with the matrix

$$u \textcircled{\scriptsize T} u = \begin{bmatrix} \alpha^2 & -\alpha\gamma^* & -\gamma^*\alpha & \gamma^{*2} \\ \alpha\gamma & \alpha\alpha^* & -\gamma^*\gamma & -\gamma^*\alpha^* \\ \gamma\alpha & -\gamma\gamma^* & \alpha^*\alpha & -\alpha^*\gamma^* \\ \gamma^2 & \gamma\alpha & \alpha^*\gamma & \alpha^{*2} \end{bmatrix}$$

  to find that

$$\alpha^2 \otimes f, -\alpha\gamma^* \otimes f, -\gamma^*\alpha \otimes f, \gamma^{*2} \otimes f, \alpha\gamma \otimes f, \alpha\alpha^* \otimes f, -\gamma^*\gamma \otimes f, -\gamma^*\alpha^* \otimes f,$$
$$\gamma\alpha \otimes f, -\gamma\gamma^* \otimes f, \alpha^*\alpha \otimes f, -\alpha^*\gamma^* \otimes f, \gamma^2 \otimes f, \gamma\alpha \otimes f, \alpha^*\gamma \otimes f, \alpha^{*2} \otimes f$$

  belong to $\left\{\Delta(f)(\mathbb{1} \otimes g) \,\middle|\, f, g \in \mathrm{C}(\mathrm{SU}(2))\right\}$ for any f.
- Continuing along these lines we can show that for any $P \in \mathrm{Pol}(\mathrm{SU}(2))$ and for any $f \in \mathrm{C}(\mathrm{SU}(2))$ we have $P \otimes f \in \left\{\Delta(f)(\mathbb{1} \otimes g) \,\middle|\, f, g \in \mathrm{C}(\mathrm{SU}(2))\right\}$.

Question

Can $\mathrm{SU}(2)$ be "deformed" in the sense that there exists other compact groups which are "close" to $\mathrm{SU}(2)$ in some sense (perhaps forming a continuous family – again – in some sense)?

- One way to ask this more precisely is to choose a basis in the Lie algebra of $\mathrm{SU}(2)$ and consider other compact Lie groups with a choice of basis in their Lie algebras whose structure constants are within ε of the ones for $\mathrm{SU}(2)$.
- Unfortunately any compact Lie group which is close to $\mathrm{SU}(2)$ in this sense is isomorphic to $\mathrm{SU}(2)$.
- It remains to look for other ways to deform $\mathrm{SU}(2)$.

- Recall that $\mathrm{C}(\mathrm{SU}(2))$ is the Banach $*$-algebra generated by two elements α and γ such that $\alpha^*\alpha + \gamma^*\gamma = \mathbb{1}$ and α, γ, α^* and γ^* commute.
- Moreover the norm and involution of $\mathrm{C}(\mathrm{SU}(2))$ are related by

$$\|f^*f\| = \|f\|^2, \qquad f \in \mathrm{C}(\mathrm{SU}(2))$$

which we abbreviate by saying that $\mathrm{C}(\mathrm{SU}(2))$ is a C***-algebra**.

DEFINITION

Fix $q \in [-1, 1[\setminus \{0\}$ and let $\mathrm{C}(\mathrm{SU}_q(2))$ be the C*-algebra generated by two elements α and γ satisfying

$$\begin{aligned} \alpha^*\alpha + \gamma^*\gamma &= \mathbb{1}, & \alpha\gamma &= q\gamma\alpha, \\ \alpha\alpha^* + q^2\gamma^*\gamma &= \mathbb{1}, & \gamma\gamma^* &= \gamma^*\gamma. \end{aligned}$$

- $\mathrm{C}(\mathrm{SU}_q(2))$ is not commutative! In particular there does not exist a space X such that $\mathrm{C}(\mathrm{SU}_q(2)) \cong \mathrm{C}(X)$ and "$\mathrm{C}(\mathrm{SU}_q(2))$" is only a notational convention.

THEOREM

1. There exists a unique unital $*$-homomorphism $\Delta\colon\ \mathrm{C}(\mathrm{SU}_q(2)) \to \mathrm{C}(\mathrm{SU}_q(2)) \otimes \mathrm{C}(\mathrm{SU}_q(2))$ such that
$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma \quad \text{and} \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma,$$
2. Δ is coassociative: $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$,
3. the sets $\{\Delta(a)(\mathbb{1} \otimes b) \,|\, a, b \in \mathrm{C}(\mathrm{SU}_q(2))\}$ and $\{(a \otimes \mathbb{1})\Delta(b) \,|\, a, b \in \mathrm{C}(\mathrm{SU}_q(2))\}$ are linearly dense in $\mathrm{C}(\mathrm{SU}_q(2)) \otimes \mathrm{C}(\mathrm{SU}_q(2))$.

- We say that $\mathrm{C}(\mathrm{SU}_q(2))$ is the algebra of continuous functions on the **quantum group** $\mathrm{SU}_q(2)$.
- As we already mentioned, this is an abuse of language – there does not exist a space X such that $\mathrm{C}(\mathrm{SU}_q(2))$ is isomorphic to the algebra of functions on X (because $\mathrm{C}(\mathrm{SU}_q(2))$ is not commutative).

REMARK

If G is a compact group then the integration with respect to the Haar measure $h\colon\ \mathrm{C}(G) \ni f \mapsto \int\limits_G f\,\mathrm{d}\mu \in \mathbb{C}$ is a positive functional of norm 1 (a **state**) with the property that

$$(h \otimes \mathrm{id})\Delta(f) = h(f)\mathbb{1} = (\mathrm{id} \otimes h)\Delta(f)$$

for any $f \in \mathrm{C}(G)$.

THEOREM

There exists a unique state $\boldsymbol{h}$ on $\mathrm{C}(\mathrm{SU}_q(2))$ such that

$$(\boldsymbol{h} \otimes \mathrm{id})\Delta(a) = \boldsymbol{h}(a)\mathbb{1} = (\mathrm{id} \otimes \boldsymbol{h})\Delta(a)$$

for any $a \in \mathrm{C}(\mathrm{SU}_q(2))$.

- The functional $\boldsymbol{h}$ is called the **Haar measure** of $\mathrm{SU}_q(2)$.

- Let α and γ be operators on $\ell^2(\mathbb{Z}_+ \times \mathbb{Z})$ defined by

$$\alpha e_{n,k} = \sqrt{1 - q^{2n}} e_{n-1,k}, \qquad \gamma e_{n,k} = q^n e_{n,k+1}$$

  where $\{e_{n,k}\}_{n \in \mathbb{Z}_+, k \in \mathbb{Z}}$ is the standard basis of $\ell^2(\mathbb{Z}_+ \times \mathbb{Z})$.
- There exists a unique $*$-homomorphism $\pi\colon \mathrm{C}(\mathrm{SU}_q(2)) \to \mathrm{B}(\ell^2(\mathbb{Z}_+ \times \mathbb{Z}))$ such that $\pi(\alpha) = \alpha$ and $\pi(\gamma) = \gamma$.
- We have

$$\boldsymbol{h}(a) = (1 - q^2) \sum_{n=0}^{\infty} q^{2n} \left\langle e_{n,0} \middle| \pi(a) e_{n,0} \right\rangle$$

  for all $a \in \mathrm{C}(\mathrm{SU}_q(2))$.
- Moreover π is injective.

DEFINITION

A matrix $U \in \mathrm{Mat}_n(\mathbb{C}) \otimes \mathrm{C}(\mathrm{SU}_q(2))$ is a **unitary representation** of $\mathrm{SU}_q(2)$ if

1. U is unitary,
2. we have $(\mathrm{id} \otimes \Delta)U = U_{12}U_{13}$, where $U_{12} = U \otimes \mathbb{1} \in \mathrm{Mat}_n(\mathbb{C}) \otimes \mathrm{C}(\mathrm{SU}_q(2)) \otimes \mathrm{C}(\mathrm{SU}_q(2))$ and $U_{13} = (\mathrm{id} \otimes \mathrm{flip})U_{12}$.

- Condition 2 is equivalent to

$$\Delta(U_{i,j}) = \sum_{k=1}^{n} U_{i,k} \otimes U_{k,j}$$

for all i, j.

- Let $U \in \mathrm{Mat}_n(\mathbb{C}) \otimes \mathrm{C}(\mathrm{SU}_q(2))$ and $V \in \mathrm{Mat}_m(\mathbb{C}) \otimes \mathrm{C}(\mathrm{SU}_q(2))$ be unitary representations of $\mathrm{SU}_q(2)$.
- We say that $T \in \mathrm{B}(\mathbb{C}^n, \mathbb{C}^m)$ **intertwines** U and V if $(T \otimes \mathbb{1})U = V(T \otimes \mathbb{1})$.
- U and V are **equivalent** if there is a unitary intertwining U and V.
- U is **irreducible** if any projection P intertwining U with itself is either $\mathbb{1}$ or 0.
- The representation

$$U \oplus V = \begin{bmatrix} U & \\ & V \end{bmatrix} \in \mathrm{Mat}_{n+m}(\mathbb{C}) \otimes \mathrm{C}(\mathrm{SU}_q(2))$$

  is called the **direct sum** of U and V.
- The representation

$$U \circledT V = U_{13} V_{23} \in \mathrm{Mat}_n(\mathbb{C}) \otimes \mathrm{Mat}_m(\mathbb{C}) \otimes \mathrm{C}(\mathrm{SU}_q(2)) = \mathrm{Mat}_{nm}(\mathbb{C}) \otimes \mathrm{C}(\mathrm{SU}_q(2))$$

  (with $V_{23} = \mathbb{1} \otimes V$ and $U_{13} = (\mathrm{flip} \otimes \mathrm{id})U_{23}$) is called the **tensor product** of U and V.

Example

The matrix $u = \begin{bmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{bmatrix}$ is a two-dimensional irreducible representation of $\mathrm{SU}_q(2)$. Its tensor square

$$u \textcircled{\scriptsize T} u = \begin{bmatrix} \alpha^2 & -q\alpha\gamma^* & -q\gamma^*\alpha & q^2\gamma^{*2} \\ \alpha\gamma & \alpha\alpha^* & -q\gamma^*\gamma & -q\gamma^*\alpha^* \\ \gamma\alpha & -q\gamma\gamma^* & \alpha^*\alpha & -q\alpha^*\gamma^* \\ \gamma^2 & \gamma\alpha & \alpha^*\gamma & \alpha^{*2} \end{bmatrix}$$

is equivalent to the direct sum of the one-dimensional **trivial representation** $\mathbb{1} \in \mathrm{C}(\mathrm{SU}_q(2)) = \mathrm{Mat}_1(\mathbb{C}) \otimes \mathrm{C}(\mathrm{SU}_q(2))$ and the irreducible three-dimensional representation

$$\begin{bmatrix} \alpha^2 & -\sqrt{1+q^2}\alpha\gamma^* & q^2\gamma^{*2} \\ \sqrt{1+q^2}\gamma\alpha & \mathbb{1} - (1+q^2)\gamma^*\gamma & -\sqrt{1+q^2}\gamma^*\alpha^* \\ \gamma^2 & \sqrt{1+q^2}\alpha^*\gamma & \alpha^{*2} \end{bmatrix}$$

Theorem

The equivalence classes of irreducible representations of $\mathrm{SU}_q(2)$ are labeled by $\frac{1}{2}\mathbb{Z}_+$. We can choose representatives $\{U^s\}_{s\in\frac{1}{2}\mathbb{Z}_+}$ so that $U^0=\mathbb{1}$, $U^{\frac{1}{2}}=\begin{bmatrix}\alpha & -q\gamma^*\\ \gamma & \alpha^*\end{bmatrix}$ and

$$U^s \mathbin{\textcircled{\tiny T}} U^{s'} \approx U^{|s-s'|}\oplus U^{|s-s'|+1}\oplus\dots\oplus U^{s+s'}.$$

for all $s,s'\in\frac{1}{2}\mathbb{Z}_+$.

- A **permutation matrix** is an element U of $\mathrm{Mat}_n(\mathbb{C})$ such that
  1. $U_{i,j} \in \{0, 1\}$ for all i,j,
  2. for each i we have $\sum_{j=1}^{n} U_{i,j} = 1$,
  3. for each j we have $\sum_{i=1}^{n} U_{i,j} = 1$.
- The symmetric group S_n can be interpreted as the set of all $n \times n$ permutation matrices.

Definition

We define $\mathrm{C}(\mathrm{S}_n^+)$ as the C*-algebra generated by $\{U_{i,j}\}_{i,j=1,\dots,n}$ satisfying the following relations:

1. for all i,j the element $U_{i,j}$ is a projection (self-adjoint idempotent),
2. for each i we have $\sum_{j=1}^{n} U_{i,j} = \mathbb{1}$,
3. for each j we have $\sum_{i=1}^{n} U_{i,j} = \mathbb{1}$.

- The abelianization of $\mathrm{C}(\mathrm{S}_n^+)$ coincides with the finite-dimensional algebra of all functions on the group S_n.

Theorem

1. There exists a unique $*$-homomorphism $\Delta\colon\ C(S_n^+) \to C(S_n^+) \otimes C(S_n^+)$ such that fo all i,j
$$\Delta(U_{i,j}) = \sum_{k=1}^{n} U_{i,k} \otimes U_{k,j},$$
2. Δ is coassociative,
3. the sets $\{\Delta(a)(\mathbb{1} \otimes b) \,|\, a, b \in C(S_n^+)\}$ and $\{(a \otimes \mathbb{1})\Delta(b) \,|\, a, b \in C(S_n^+)\}$ are linearly dense in $C(S_n^+) \otimes C(S_n^+)$.

Definition

We call S_n^+ the **quantum permutation group** of n points.

- For $n = 1, 2, 3$ we have $C(S_n^+) = C(S_n)$.
- For $n \geqslant 4$ the C*-algebra $C(S_n^+)$ is non-commutative and infinite-dimensional.

- Notice that
$$P=\begin{bmatrix}U_{1,1}&\cdots&U_{1,n}\\ \vdots&\ddots&\vdots\\ U_{n,1}&\cdots&U_{n,n}\end{bmatrix}\in\mathrm{Mat}_n(\mathbb{C})\otimes\mathrm{C}(\mathbb{S}_n^+)$$
is a unitary representation of $\mathbb{S}_n^+$.
- Clearly P is not irreducible.

Theorem

The irreducible representations of $\mathbb{S}_n^+$ are $\{V^n\}_{n\in\mathbb{Z}_+}$ with $V^0=\mathbb{1}$, $P=V^0+V^1$ and
$$V^1 \circledT V^n=V^{n+1}\oplus V^n\oplus V^{n-1},\qquad n\in\mathbb{N}.$$

- There is also a combinatorial formula for the Haar measure of any polynomial in the matrix entries of $\{V^n\}_{n\in\mathbb{Z}_+}$.

Definition

A **compact quantum group** $\mathbb{G}$ is described by a unital C*-algebra denoted by $\mathrm{C}(\mathbb{G})$ together with a coassociative unital $*$-homomorphism $\Delta\colon\ \mathrm{C}(\mathbb{G}) \to \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G})$ such that the sets

$$\bigl\{\Delta(a)(\mathbb{1}\otimes b)\,\big|\,a,b\in\mathrm{C}(\mathbb{G})\bigr\}\quad\text{and}\quad\bigl\{(a\otimes\mathbb{1})\Delta(b)\,\big|\,a,b\in\mathrm{C}(\mathbb{G})\bigr\}$$

are linearly dense in $\mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G})$.

Theorem (S.L. Woronowicz)

Let $\mathbb{G}$ be a compact quantum group. Then there exists a unique state $\boldsymbol{h}$ on $\mathrm{C}(\mathbb{G})$ such that

$$(\boldsymbol{h}\otimes\mathrm{id})\Delta(a) = \boldsymbol{h}(a)\mathbb{1} = (\mathrm{id}\otimes\boldsymbol{h})\Delta(a)$$

for all $a\in\mathrm{C}(\mathbb{G})$.

Example

Let Γ be a discrete group and let $\lambda\colon\Gamma\to\mathrm{B}(\ell^2(\Gamma))$ be the left regular representation of Γ:

$$\big(\lambda(s)\psi\big)(t)=\psi(s^{-1}t),\qquad s,t\in\Gamma,\ \psi\in\ell^2(\Gamma).$$

Let $\mathrm{C}(\hat{\Gamma})$ be the **reduced group** C^***-algebra** of Γ, i.e. the C^*-subalgebra of $\mathrm{B}(\ell^2(\Gamma))$ generated by the range of λ. It follows that

- there exists a unique $*$-homomorphism $\Delta\colon\ \mathrm{C}(\hat{\Gamma})\to\mathrm{C}(\hat{\Gamma})\otimes\mathrm{C}(\hat{\Gamma})$ such that $\Delta(\lambda(s))=\lambda(s)\otimes\lambda(s)$ for all $s\in\Gamma$,
- Δ is coassociative and $\hat{\Gamma}$ is a compact quantum group.
- the Haar measure of $\hat{\Gamma}$ is the **von Neumann trace**:

$$\mathrm{C}(\hat{\Gamma})\ni a\longmapsto\langle\delta_e|a\delta_e\rangle\in\mathbb{C}.$$

Note that the range of Δ is contained in the space of symmetric tensors (we say that $\hat{\Gamma}$ is **co-commutative**). Irreducible representations of $\hat{\Gamma}$ are $\{\lambda(s)\}_{s\in\Gamma}$.

- Compact quantum groups generalize compact groups.
- Any discrete group Γ gives rise to the compact quantum group $\widehat{\Gamma}$.
- A simple example of this interplay is $\Gamma = \mathbb{Z}$, $\widehat{\Gamma} = \mathbb{T}$.

Definition

Let $\mathbb{G}$ be a compact quantum group and let $\operatorname{Irr}\mathbb{G}$ denote the set of equivalence classes of irreducible unitary representations of $\mathbb{G}$. Let $\{u^\alpha\}_{\alpha\in\operatorname{Irr}\mathbb{G}}$ be a choice representatives and for each α let n_α be the dimension of u^α (i.e. n such that $u^\alpha \in \mathrm{Mat}_n(\mathbb{C}) \otimes \mathrm{C}(\mathbb{G})$). We define $\mathrm{c}_0(\widehat{\mathbb{G}})$ to be the the c_0-direct sum

$$\mathrm{c}_0(\widehat{\mathbb{G}}) = \bigoplus_{\alpha\in\operatorname{Irr}\mathbb{G}} \mathrm{Mat}_{n_\alpha}(\mathbb{C}).$$

and

$$W = \bigoplus_{\alpha\in\operatorname{Irr}\mathbb{G}} u^\alpha.$$

- Since $\|u^\alpha\| = 1$ for all α, in general $W \notin c_0(\widehat{\mathbb{G}}) \otimes C(\mathbb{G})$. However W belongs to the **multiplier algebra** $M(c_0(\widehat{\mathbb{G}}) \otimes C(\mathbb{G}))$ of $c_0(\widehat{\mathbb{G}}) \otimes C(\mathbb{G})$.
- In particular, it makes sense to state the following theorem:

Theorem

There exists a unique $\widehat{\Delta}\colon c_0(\widehat{\mathbb{G}}) \to M(c_0(\widehat{\mathbb{G}}) \otimes c_0(\widehat{\mathbb{G}}))$ such that

$$(\widehat{\Delta} \otimes \mathrm{id})W = W_{23}W_{13},$$

where $W_{23} = \mathbb{1} \otimes W \in M(c_0(\widehat{\mathbb{G}}) \otimes c_0(\widehat{\mathbb{G}}) \otimes C(\mathbb{G}))$ and $W_{13} = (\mathrm{flip} \otimes \mathrm{id})W_{23}$.

- The object $\widehat{\mathbb{G}}$ is an example of what is known as a **locally compact quantum group**.
- $\widehat{\mathbb{G}}$ is in fact **discrete** which simply means that the corresponding C*-algebra is a direct sum of matrix algebras.
- There is a perfect duality between compact and discrete quantum groups – in a sense these are two pictures of the same class of objects.

- Let $\mathbb{G}$ be a compact quantum group.
- The Haar measure $\boldsymbol{h}$ of $\mathbb{G}$ gives rise to a pre-Hilbert space structure on $\mathrm{C}(\mathbb{G})/\mathcal{N}$, where $\mathcal{N} = \{a \in \mathrm{C}(\mathbb{G}) \,|\, \boldsymbol{h}(a^*a) = 0\}$, via $\langle a + \mathcal{N} | b + \mathcal{N} \rangle = \boldsymbol{h}(a^*b)$.
- Let $L^2(\mathbb{G})$ be the completion of this pre-Hilbert space.
- We note that often $\mathcal{N} = \{0\}$, but not always.
- $\mathrm{C}(\mathbb{G})$ is represented on $L^2(\mathbb{G})$ via λ as follows:

$$\lambda(a)(b + \mathcal{N}) = ab + \mathcal{N}, \qquad a, b \in \mathrm{C}(\mathbb{G}).$$

- In case $\mathbb{G} = \widehat{\Gamma}$, the range of this map is the reduced group C^*-algebra of of Γ.
- We define $L^\infty(\mathbb{G})$ as the strong closure of the range of λ. In case $\mathbb{G} = \widehat{\Gamma}$ this is the **group von Neumann algebra** of Γ (often denoted by $L(\Gamma)$).
- The comultiplication Δ passes to the image of λ and can then be extended to $L^\infty(\mathbb{G})$.

- The most important von Neumann algebras the **factors**, i.e. those whose center is trivial (equal to the set of scalar multiples of $\mathbb{1}$).

Proposition

The group von Neumann algebra of a discrete group Γ is a factor if and only if Γ is i.c.c. (all the non-trivial conjugacy classes in Γ are infinite).

Proof: The map $\eta\colon L(\Gamma)\ni a\mapsto a\delta_e\in\ell^2(\Gamma)$ is injective and maps multiples of $\mathbb{1}$ to functions with support $\{e\}$. Assume that Γ is i.c.c., a is a non-zero central element of $L(\Gamma)$ and take $s,t\in\Gamma$. Denoting by ρ the right regular representation of Γ we have

$$\eta(a)(s^{-1}ts)=\langle\delta_{s^{-1}ts}|a\delta_e\rangle=\left\langle\lambda(s^{-1}t)\delta_s\middle|a\delta_e\right\rangle=\left\langle\delta_s\middle|\lambda(t^{-1}s)a\delta_e\right\rangle=\left\langle\delta_s\middle|a\lambda(t^{-1}s)\delta_e\right\rangle$$
$$=\left\langle\delta_s\middle|a\rho(s^{-1}t)\delta_e\right\rangle=\left\langle\delta_s\middle|\rho(s^{-1}t)a\delta_e\right\rangle=\left\langle\rho(t^{-1}s)\delta_s\middle|a\delta_e\right\rangle=\langle\delta_t|a\delta_e\rangle=\eta(a)(t).$$

A non-zero square integrable function cannot be constant on an infinite set. Conversely, if $C\subset\Gamma$ is a finite conjugacy class then $\mathbb{C}\mathbb{1}\not\ni\sum\limits_{s\in C}\lambda(s)$ is central. $\square$

- In the next proposition $\Delta^{(n)}$ is $(\Delta\otimes\underbrace{\mathrm{id}\otimes\cdots\otimes\mathrm{id}}_{n-1})\circ\cdots\circ(\Delta\otimes\mathrm{id})\circ\Delta$.
- For a set of operators A on a Hilbert space the symbol A' denotes the **commutant** of A, i.e. the set of all operators which commute with all operators from the set A.

Proposition

Let Γ be a discrete group and let Δ be the comultiplication on $L^\infty(\widehat{\Gamma})$. Then the following are equivalent:

1. Γ is i.c.c.,
2. $L(\Gamma)$ is a factor,
3. $\Delta^{(n)}\bigl(L(\Gamma)\bigr)'\cap\underbrace{L(\Gamma)\bar{\otimes}\cdots\bar{\otimes}L(\Gamma)}_{n+1}=\mathbb{C}\mathbb{1}$ for some $n\in\mathbb{N}$,
4. $\Delta^{(n)}\bigl(L(\Gamma)\bigr)'\cap\underbrace{L(\Gamma)\bar{\otimes}\cdots\bar{\otimes}L(\Gamma)}_{n+1}=\mathbb{C}\mathbb{1}$ for all $n\in\mathbb{N}$.

Definition

Let $\mathbb{G}$ be a compact quantum group. We say that the discrete quantum group $\widehat{\mathbb{G}}$ is ***n*-i.c.c.** if

$$\Delta^{(n)}\big(L(\Gamma)\big)' \cap \underbrace{L(\Gamma)\bar{\otimes}\cdots\bar{\otimes}L(\Gamma)}_{n+1} = \mathbb{C}\mathbb{1}.$$

- It is known that if $\widehat{\mathbb{G}}$ is n-i.c.c. then $L^\infty(\mathbb{G})$ is a factor.
- It is also known that if $\mathbb{G}$ is n-i.c.c. for some n then it is m-i.c.c. for all $m < n$.

Questions

1. Which discrete quantum groups are n-i.c.c. and for what n?
2. Are all i.c.c. conditions equivalent?
3. What are other consequences of i.c.c. conditions?

Thank you for your attention