Quantum groups from analytic viewpoint

Piotr M. Sołtan

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❶ **From groups to quantum groups – motivation**

❷ **Examples**

❸ **Typical problems**

❹ **Multiplicative unitaries**

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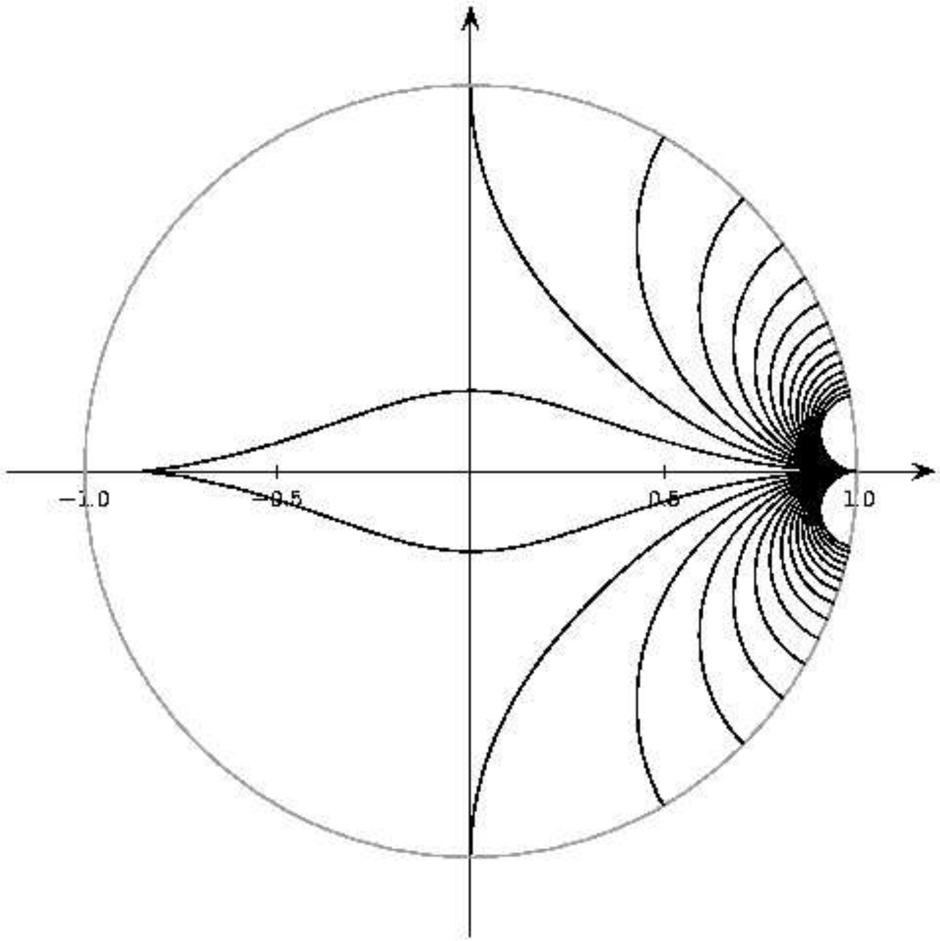

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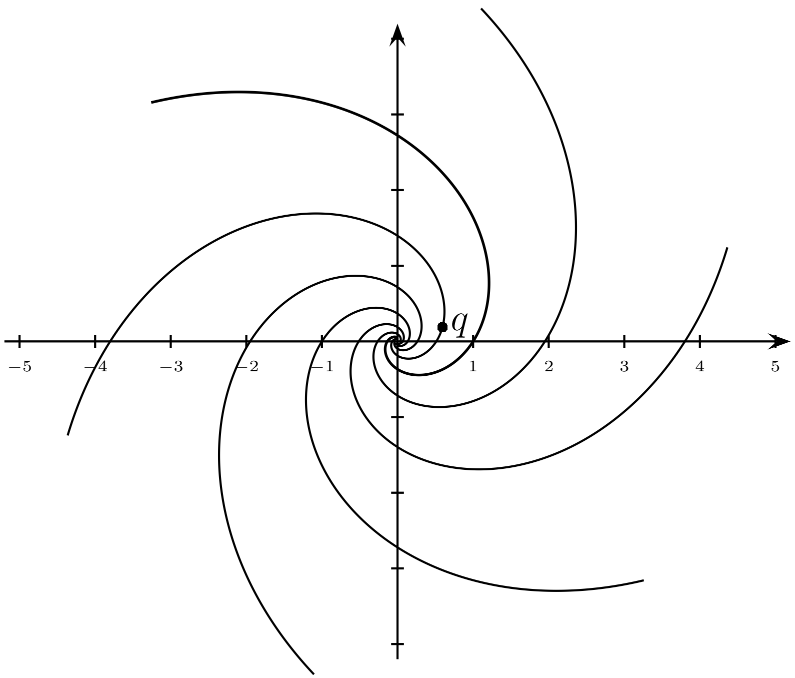

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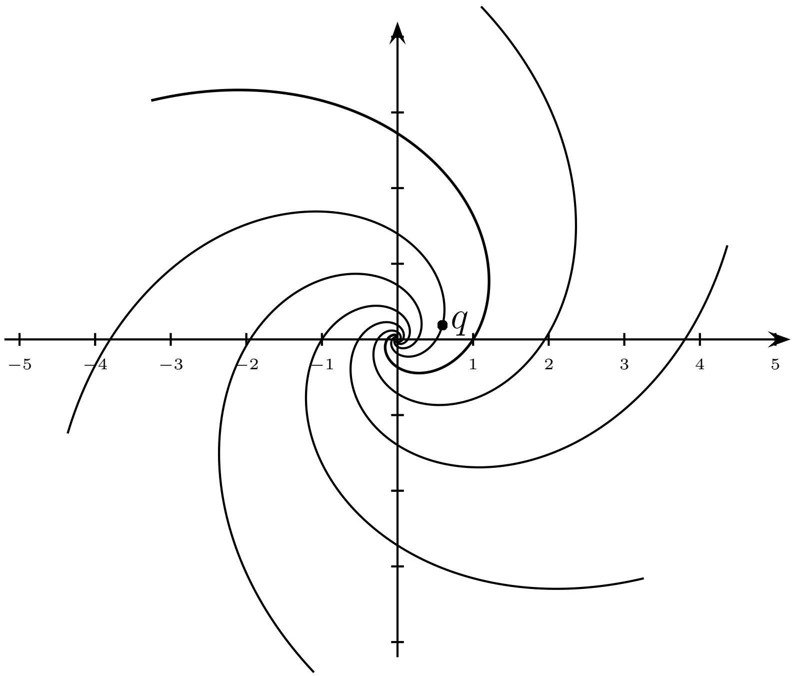

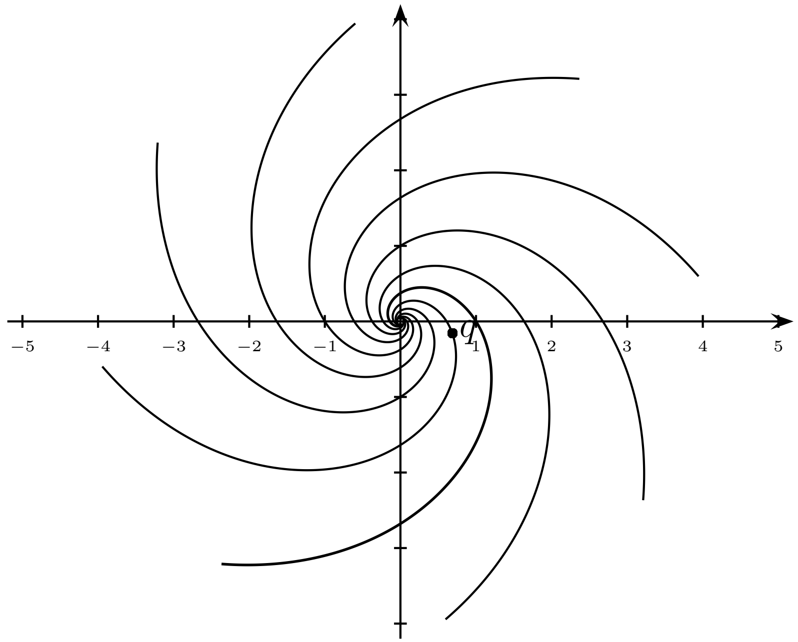

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➪ Need for advanced technical tools of functional analysis

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- ➪ the operator $a\otimes b + b\otimes I$ is not closed and thus not normal
- ➪ it is closable and its closure $a\otimes b \dotplus b\otimes I$ is normal
- ➪ it would not have any normal extensions if spectra of a and b were different

❸

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➪ Coinverse: $\begin{cases} \kappa(\alpha) = \alpha^*, \\ \kappa(\gamma) = -q\gamma, \\ \kappa(\alpha^*) = \alpha, \\ \kappa(\gamma^*) = -q^{-1}\gamma^* \end{cases}$

Fact: Quantum $\mathrm{SU}(1,1)$ does not exist!

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The problem has since been (partially) solved by Korogodsky, Woronowicz, Kustermans, Koelink.

❹

Multiplicative unitaries

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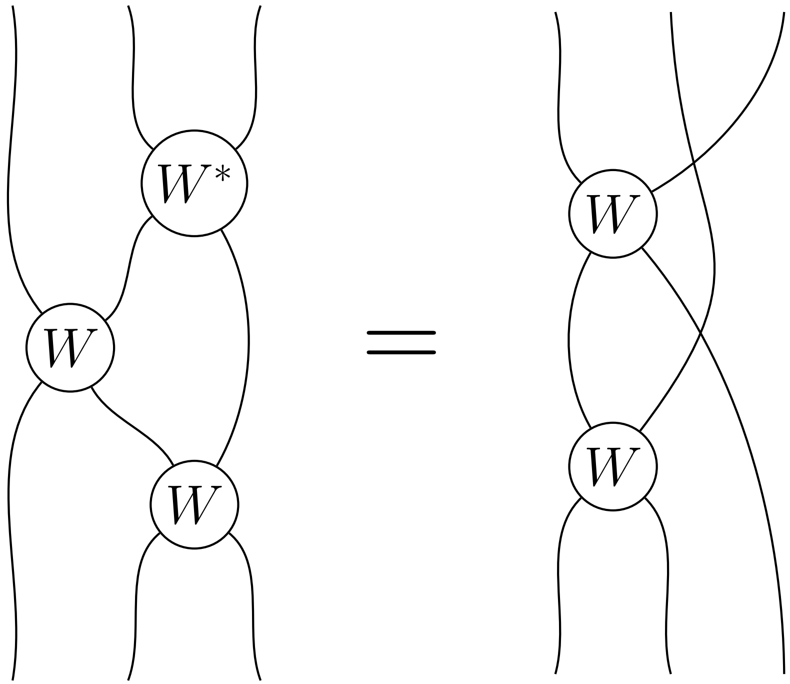

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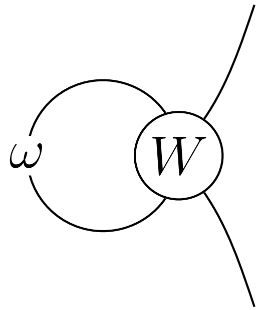
$$A = \Big\{$$


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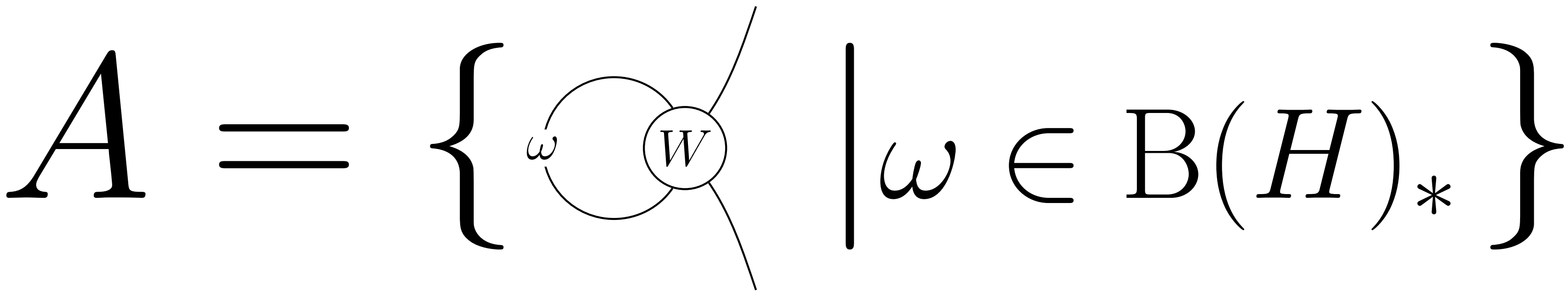

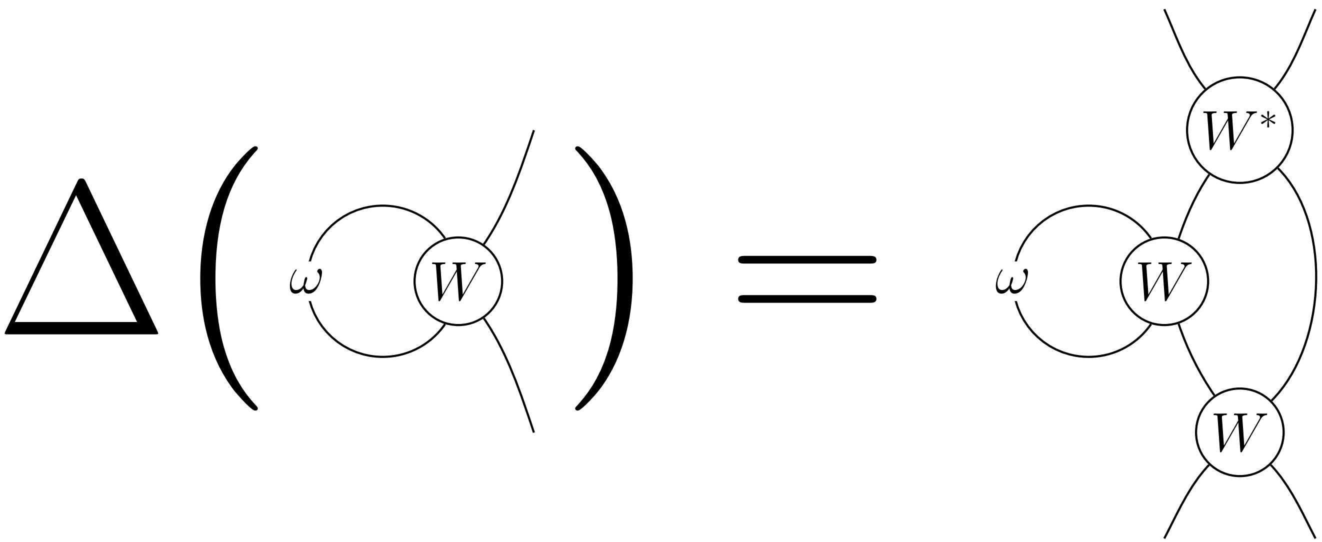

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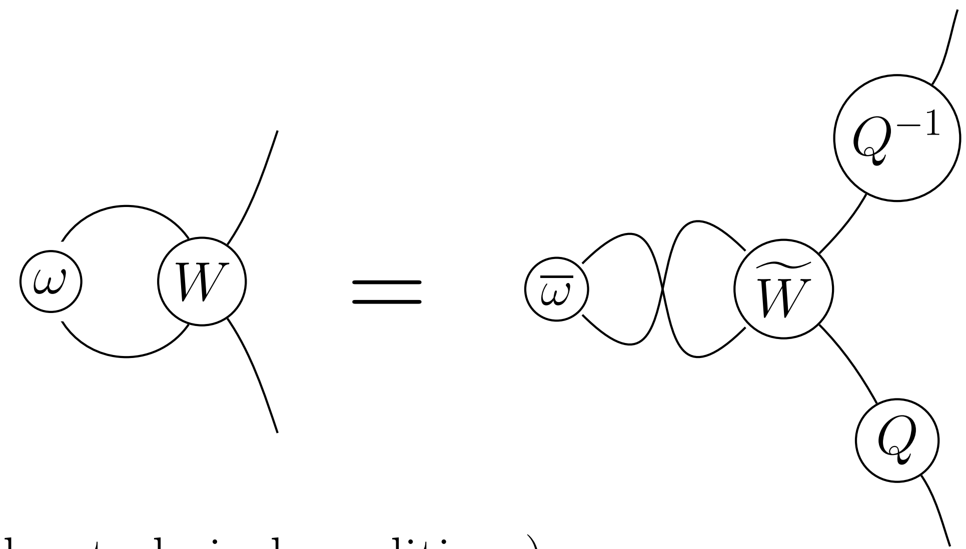
$\exists$ unitary $\widetilde{W}$ and positive $Q = Q^*$ such that



(plus some other technical conditions)

❹

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- The map

$$\Delta : A \ni a \longmapsto W(a \otimes I)W^* \in \mathrm{M}(A \otimes A)$$

is the standard comultiplication on $\mathrm{C}_0(\mathbb{G})$.

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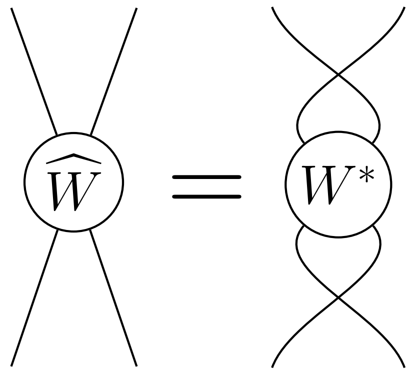

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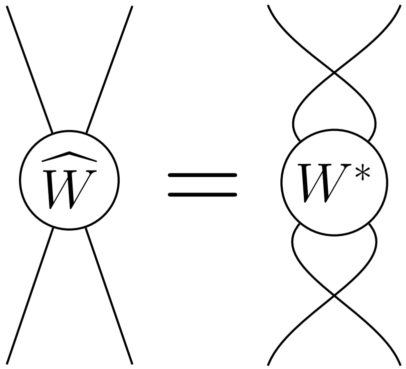

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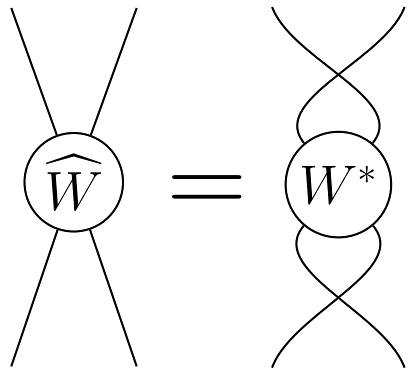

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- the dual of quantum "$az+b$" is its opposite quantum group

❹

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- ➪ Standard methods of constructing new examples from old ones can be applied on the level of multiplicative unitaries
- ➪ Representation theory can be studied in this language (Woronowicz, P.M.S.)
- ➪ Modularity gives a new framework to study existence of Haar measures (Haar weights)

Thank you