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The thermodynamic efficiency of heat engines with friction

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The presence of the work done against friction is incorporated into the analysis of the efficiency of heat engines based on the first and second laws of thermodynamics. We obtain the efficiencies of Stirling and Brayton engines with friction and recover results known from finite-time thermodynamics. We show that $\eta_{\rm fric}/\eta \approx (1 - W_{\rm fric}/W)$, where $\eta_{\rm fric}/\eta$ is the ratio of the efficiencies with and without friction and $W_{\rm fric}/W$ is the fraction of the work W performed by the working fluid which is spent against friction forces.

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I. INTRODUCTION

Friction is unavoidably present in any heat engine or power plant and has been treated in the framework of finite-time thermodynamics.[1–6] However, the picture that emerges from the existing literature on friction and thermodynamics is neither general nor unified.[3–6] Friction is usually introduced in a model-dependent manner, suitable for a particular problem.[3–6] Friction has long been of interest in fundamental thermodynamics,[7,8] but only recently has an attempt been made to include friction from first principles in the formulation of macroscopic thermodynamics.[9–11] The aim is to bring classical thermodynamics closer to actual devices, rather than the ideal, frictionless formulation found in textbooks.[12–15]

Thermodynamics has always been concerned with the efficiency of cyclic heat engines.[12–15] In this article, we propose general expressions for the efficiency of heat engines for which friction is present. Our analysis also allows friction to be included in the analysis of refrigerators, heat pumps, and power or refrigeration plants. The point of view we have adopted is a practical, engineering-minded one, in which the efficiency of heat engines is introduced as a "return-over-investment" ratio, drawing from notions of daily life. One reason for this point of view is to make the analysis accessible to teachers of thermodynamics in undergraduate courses, and to show that thermodynamics is not restricted to the study of idealized Carnot engines.[12–15] The situation in which the working fluid interacts with a continuum of reservoirs is briefly addressed in the Appendix, and results from finite-time thermodynamics are recovered.[3,5]

To avoid ambiguities regarding the meaning of heat, we refrain from using the latter as a noun.[16,17] If there is any doubt, the meaning of heat is simply the difference between the change in a system's internal energy and the macroscopic work done on it.[18] Also, when reference is made to a frictionless engine, we are not necessarily referring to a Carnot engine nor one with Carnot efficiency.[12–15] We adopt the usual sign convention for heat transfers and work done, which is to take them as intrinsically positive quantities. Their algebraic signs are related to the direction in which their transfers take place, as indicated by the arrows in the diagrams.

II. HEAT ENGINES WITH FRICTION: THE PHYSICAL MODEL

The model we adopt to analyze a heat engine with friction functioning between two reservoirs is schematized in Fig. 1 and is a straightforward extension of the usual textbook analysis,[13–15] to which heat and work transfers related to frictional losses have been added. $Q_{\rm exch,h}$ is the amount of energy exchanged between the hot reservoir with absolute temperature $T_{\rm h}$ and the working fluid in the device M,[20] part of which is used to produce work W; $Q_{\rm exch,c}$ is the part directly transferred from the fluid to the cold reservoir at $T_{\rm c}$. The major effect due to friction is to subtract from W the amount of frictional work $W_{\rm fric}$, thus reducing the engine's output to $W - W_{\rm fric}$.[9,10]

It has been known since Joule's famous paddle wheel experiment that $W_{\rm fric}$ is dissipated and heats either the system or the surroundings, or both.[9,10,13] No assumption is made on where dissipation takes place, so the coefficients $\alpha_{\rm h}$ and $\alpha_{\rm c}$ are introduced to divide $W_{\rm fric,h}$ and $W_{\rm fric,c}$ between the fluid and the hot and cold reservoirs, respectively.[5,9,10] The values of $\alpha_{\rm h}$ and $\alpha_{\rm c}$ give the fractions of $W_{\rm fric,h}$ and $W_{\rm fric,c}$ that are dissipated in the system, and are assumed to be known, either from the experiment or a theoretical model.

The subscripts h and c denote quantities of that part of the engine's cycle where one or the other of the two reservoirs is present and do not imply any isothermal nature, be it at $T_{\rm h}$ or $T_{\rm c}$, of the processes occurring in the fluid. For instance, we cannot assume that the heat transfers $Q_{\rm exch,h}$, $\alpha_{\rm h}$ $W_{\rm fric,h}$, and $(1 - \alpha_{\rm h})W_{\rm fric,h}$ occur while the fluid is at the constant temperature $T_{\rm h}$, and similarly for the heat transfers with subscript c.[21] The working fluid is not required to be characterized by a single temperature nor by a single pressure at every point of the cycle, except at the endpoints because, for the process to be cyclic, the fluid's initial and final states must be the same, which would be nearly impossible if they were not equilibrium states. That is, it is not assumed that the processes in the cycle are quasistatic, so its points, with the exception of the initial and final ones, need not correspond to equilibrium states.

Unlike what Fig. 1 might wrongly suggest, it is not required that frictional dissipation into the working fluid occurs only when the latter is in contact with one of the reservoirs, which would seem to rule out application of the present formulation to adiabatic processes, hence to a Carnot cycle. The definition here of an adiabatic transformation with friction requires only that there be no direct exchange of energy between the fluid and the reservoir, meaning the appropriate $Q_{\rm exch}$ must vanish,[9,10] which is not the same as saying that the reservoir cannot receive a fraction $(1 - \alpha)W_{\rm fric}$ of the energy dissipated by friction. Even if a stricter definition of an adiabatic process were used, where not only the system and surroundings would be thermally insulated from each other but also the latter would not be there, so that

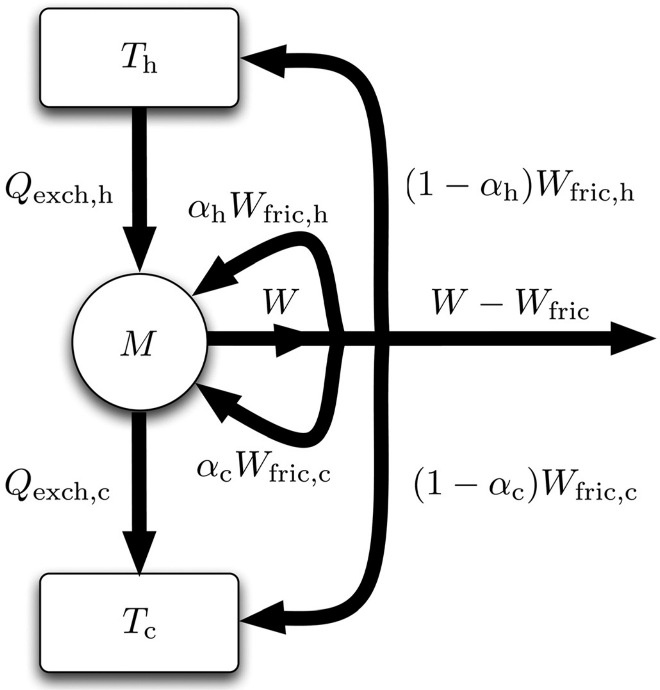

Fig. 1. Schematic diagram of a cyclic heat engine with friction. In one cycle, the hot reservoir at absolute temperature T_h delivers energy $Q_\mathrm{exch,h}$ to the working fluid in the mechanical device M, which performs work W and transfers energy $Q_\mathrm{exch,c}$ to the cold reservoir at absolute temperature T_c. The engine's work output per cycle is the difference between W and the total work W_fric produced against friction forces, the latter being the sum of $W_\mathrm{fric,h}$ and $W_\mathrm{fric,c}$, which are the work done against friction for those parts of the cycle where the hot and cold reservoirs are present, respectively, and whose fractions α_h and α_c are dissipated in the fluid. The remaining fractions $1-\alpha_\mathrm{h}$ and $1-\alpha_\mathrm{c}$ are recirculated to the respective reservoirs.[19]

frictional dissipation would take place entirely in the fluid, Fig. 1 and the associated analysis would continue to apply if the α were set equal to one during the corresponding part of the cycle. In addition, when comparing engines with and without friction based on a same thermodynamic cycle performed by the working fluid, if one of the branches of the frictionless cycle is an adiabatic process, we must ensure the latter occurs in the engine with friction such that $|Q_\mathrm{exch}|$ equals αW_fric, so that there is no net heat transfer into or out of the fluid. We can check in Fig. 1 that this absence of heat transfer into or out of the fluid would be the same as setting the appropriate α and Q_exch both equal to zero, implying that the reservoir would have to absorb the energy W_fric produced by friction.

III. FIRST AND SECOND LAW ANALYSIS: THE THERMODYNAMIC EFFICIENCY

The thermodynamic efficiency η_fric of heat engines can be expressed as the ratio between "what we get" and "what we have to pay to get it." "What we get" from a heat engine is the mechanical work W_0, which is effectively extracted from it to be delivered for useful purposes to its surroundings, denoted by the subscript 0. If there is frictional dissipation, W_0 is less than the work W produced by the fluid in the device, which equals the work output of a frictionless engine operating between the same two reservoirs. The difference stems from the work W_fric performed against friction according to[9,10]

$$W_0 = W - W_\mathrm{fric}. \tag{1}$$

"What we have to pay to get it" includes the energy $Q_\mathrm{exch,h}$ received due to direct exchange with the reservoir at T_h. As discussed, frictional work ends up as dissipated energy, a fraction α_h of which is transferred to the working fluid during its interaction with the hot reservoir; the remaining fraction $1-\alpha_\mathrm{h}$ goes to the latter.[5,9,10] The engine thus extracts $Q_\mathrm{exch,h}$ from the reservoir and returns to it $(1-\alpha_\mathrm{h})W_\mathrm{fric,h}$, so "we only have to pay" for the difference

$$Q_\mathrm{0,h} = Q_\mathrm{exch,h} - (1-\alpha_\mathrm{h})W_\mathrm{fric,h}, \tag{2}$$

which is the net energy loss of the hot reservoir.[9,10] Therefore,

$$\eta_\mathrm{fric} \equiv \frac{W_0}{Q_\mathrm{0,h}} = \frac{W - W_\mathrm{fric}}{Q_\mathrm{exch,h} - (1-\alpha_\mathrm{h})W_\mathrm{fric,h}}, \tag{3}$$

which agrees with the conclusion of a more formal analysis based on a refinement of the Clausius inequality to include friction.[10]

The total frictional work is given by

$$W_\mathrm{fric} = W_\mathrm{fric,h} + W_\mathrm{fric,c}. \tag{4}$$

We apply the first law, or energy conservation, by following the arrows in Fig. 1, and obtain

$$W - W_\mathrm{fric} = Q_\mathrm{exch,h} - (1-\alpha_\mathrm{h})W_\mathrm{fric,h} - Q_\mathrm{exch,c} - (1-\alpha_\mathrm{c})W_\mathrm{fric,c}. \tag{5}$$

Hence, from Eqs. (3) and (5) we have

$$\eta_\mathrm{fric} = 1 - \frac{Q_\mathrm{exch,c} + (1-\alpha_\mathrm{c})W_\mathrm{fric,c}}{Q_\mathrm{exch,h} - (1-\alpha_\mathrm{h})W_\mathrm{fric,h}} = 1 - \frac{Q_\mathrm{0,c}}{Q_\mathrm{0,h}}, \tag{6}$$

where $Q_\mathrm{0,h}$ and $Q_\mathrm{0,c}$ are the net energy transfers from and into the hot and cold reservoirs, respectively.

We next consider the second law and keep in mind that the engine works cyclically, so that the working fluid returns to its initial state after one cycle and its entropy change for this cycle vanishes. The total change of the entropy in the universe, which is the fluid plus the two reservoirs, is bounded according to

$$\Delta S = \frac{Q_\mathrm{0,c}}{T_\mathrm{c}} - \frac{Q_\mathrm{0,h}}{T_\mathrm{h}} = (\eta_\mathrm{carnot} - \eta_\mathrm{fric})\frac{Q_\mathrm{0,h}}{T_\mathrm{c}} \geq 0, \tag{7}$$

where Eq. (6) has been used, and

$$\eta_\mathrm{carnot} \equiv 1 - \frac{T_\mathrm{c}}{T_\mathrm{h}}. \tag{8}$$

Equation (7) implies that

$$\eta_\mathrm{fric} \leq \eta_\mathrm{carnot}, \tag{9}$$

and we thus recover Carnot's theorem on the maximum efficiency. This recovery is not surprising because Eqs. (6)–(9) show that, using the net energy transfers to and from the reservoirs and surroundings, defined by Eqs. (1), (2), and (6),

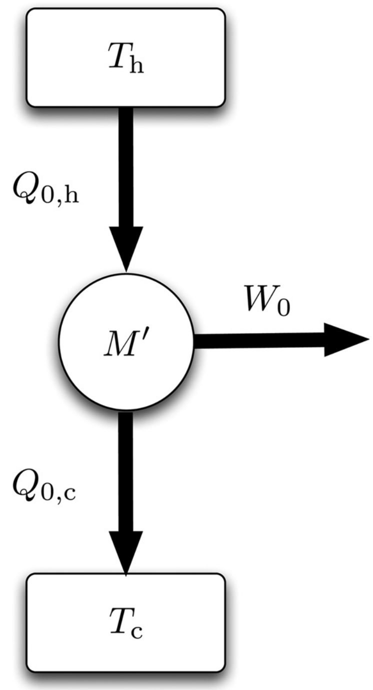

Fig. 2. Schematic diagram of a heat engine with friction in terms of the net energy transfers to and from the reservoirs and surroundings. In one cycle, $Q_{0,h}$ and $Q_{0,c}$ are the total energies lost and gained by the hot and cold reservoirs, respectively, which, as given in Eqs. (2) and (6), are the energies directly exchanged with the working fluid plus the frictional losses dissipated in the reservoirs. W_0 is the effective work delivered to the surroundings which, as indicated in Eq. (1), is the difference between the total work produced by the fluid and the work done against friction. The device M' incorporates not only the original mechanical device M, where the fluid undergoes the thermodynamic processes, but all frictional losses as well.

and introducing a device M' to account not only for M but also for all friction losses, Fig. 1 can be replaced by Fig. 2, which looks like the usual diagram for a frictionless engine.[13–15]

IV. FRICTION AND FRICTIONLESS ENGINES: A COMPARISON

If there are no frictional losses, we can define the efficiency as

$$\eta \equiv \frac{W}{Q_{\mathrm{exch,h}} + \alpha_{\mathrm{h}} W_{\mathrm{fric,h}}} = 1 - \frac{Q_{\mathrm{exch,c}} - \alpha_{\mathrm{c}} W_{\mathrm{fric,c}}}{Q_{\mathrm{exch,h}} + \alpha_{\mathrm{h}} W_{\mathrm{fric,h}}}. \tag{10}$$

It might seem strange to see the term $W_{\mathrm{fric,h}}$ in the definition of the frictionless quantity η. We have assumed that the processes undergone by the working fluid are the same in both cases, which implies the net heat and work transfers into and out of it must also be the same.[22] More precisely, and as seen in Fig. 1, for M to produce the output work W, it needs a total heat transfer

$$Q_{\mathrm{h}} = Q_{\mathrm{exch,h}} + \alpha_{\mathrm{h}} W_{\mathrm{fric,h}}, \tag{11}$$

which would have to come entirely from the hot reservoir if there were no friction, in which case it would also have to transfer to the cold reservoir the energy[23]

$$Q_{\mathrm{c}} = Q_{\mathrm{exch,c}} - \alpha_{\mathrm{c}} W_{\mathrm{fric,c}}. \tag{12}$$

A more formal rationale for the definition in Eq. (10) can be provided by using Q_{h} and Q_{c} of Eqs. (11) and (12) to rewrite η_{fric} in Eqs. (3) or (6) and η in Eq. (10) as

$$\eta_{\mathrm{fric}} = \frac{W - W_{\mathrm{fric}}}{Q_{\mathrm{h}} - W_{\mathrm{fric,h}}} = 1 - \frac{Q_{\mathrm{c}} + W_{\mathrm{fric,c}}}{Q_{\mathrm{h}} - W_{\mathrm{fric,h}}} \tag{13}$$

and

$$\eta = \frac{W}{Q_{\mathrm{h}}} = 1 - \frac{Q_{\mathrm{c}}}{Q_{\mathrm{h}}}, \tag{14}$$

which shows that Eq. (14), which is equivalent to Eq. (10), can be obtained from Eq. (13) by eliminating the quantities explicitly identified with friction.

It is expected that frictional work degrades engine efficiency, but such a conclusion might not seem obvious from Eq. (3), because frictional losses decrease both the numerator and denominator. The engine's work output must be less because of dissipation, but this effect is partly compensated for because "we do not have to pay as much for it." This compensation can only be partial as is apparent from Eq. (13), if we keep in mind that $W_{\mathrm{fric,h}}$ cannot be greater than W_{fric}. If we compare Eqs. (13) and (14), an alternative explanation for the reduction in engine efficiency due to friction comes from realizing that dissipation makes the energy decrease of the hot reservoir in one engine cycle less than it would be otherwise, while making the energy increase of the cold reservoir greater than it would be otherwise. We can combine Eqs. (13) and (14) to obtain

$$\eta_{\mathrm{fric}} = 1 - (1-\eta)\frac{1 + W_{\mathrm{fric,c}}/Q_{\mathrm{c}}}{1 - W_{\mathrm{fric,h}}/Q_{\mathrm{h}}} \le \eta, \tag{15}$$

and hence

$$0 \le \eta_{\mathrm{fric}} \le \eta, \tag{16}$$

as expected.[10] We can also combine Eqs. (13) and (14) in the form[24]

$$\eta_{\mathrm{fric}} = \eta \frac{1 - W_{\mathrm{fric}}/W}{1 - \eta W_{\mathrm{fric,h}}/W}. \tag{17}$$

Because η_{fric} in Eq. (17) is a monotonically decreasing function of W_{fric},[25] we can see that the limiting values for η_{fric} in Eq. (16) follow from the values between which W_{fric} may vary, namely, zero and W. Writing η_{fric} as in Eqs. (13) or (17), with no explicit dependence on α_{h} and α_{c}, might lead to the misleading conclusion that these parameters are of little consequence. This conclusion would be a mistake because we would be forgetting that there is an implicit dependence on α_{h} and α_{c} via Q_{h} or η, as well as via W, as can be checked in Eqs. (5), (10), and (11).[26]

If we assume that $W_{\mathrm{fric,h}}$ and $W_{\mathrm{fric,c}}$ are the same, we have from Eq. (4)

$$W_{\mathrm{fric,h}} = \frac{W_{\mathrm{fric}}}{2}. \tag{18}$$

We use this assumption to generate a contour plot of $\eta_{\mathrm{fric}}/\eta$ as a function of η and W_{fric}/W, as is shown in Fig. 3. With the exception of the region where η is close to unity,[27] the plots for $\eta_{\mathrm{fric}}/\eta$ deviate little from equally spaced horizontal lines, indicating that η_{fric} behaves almost linearly on η and W_{fric}/W, following the leading term in the series expansion in η of Eq. (17):

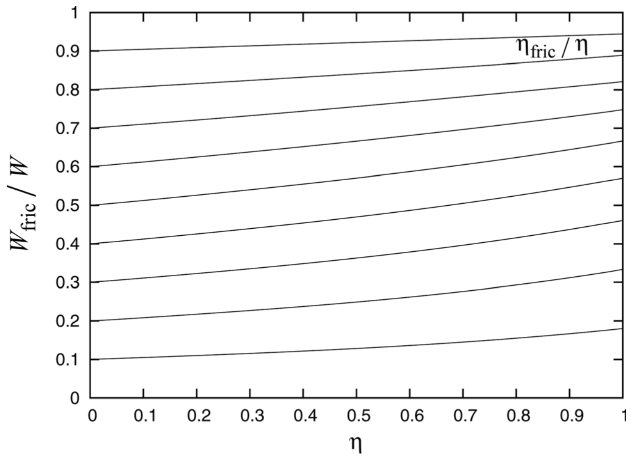

Fig. 3. Contour plot of the ratio η_{fric}/η of the thermodynamic efficiencies with and without friction as a function of η and of the fraction of frictional losses W_{fric}/W, as given by Eq. (17) for $W_{fric,h} = W_{fric}/2$. Contour levels are shown for $\eta_{fric}/\eta = 0.1n$, with n an integer between 0 and 10. The values $\eta_{fric}/\eta = 0$ and $\eta_{fric}/\eta = 1$ coincide with the lines $W_{fric}/W = 1$ and $W_{fric}/W = 0$, respectively.

$$\eta_{fric} = \left(1 - \frac{W_{fric}}{W}\right)\eta + \left(1 - \frac{W_{fric}}{W}\right)\frac{W_{fric,h}}{W}\eta^2 + O(\eta^3). \tag{19}$$

The derivation of Eq. (19) is similar to the derivation of the expansion for the efficiency at maximum power of finite-time Carnot engines.[28] Figure 3 is a quantitative confirmation that the linear relation

$$\eta_{fric} \approx \left(1 - \frac{W_{fric}}{W}\right)\eta \tag{20}$$

is a very good approximation to the effect of the percentage of frictional work on the efficiency of a heat engine. Note that, from Eqs. (19) and (20), this linear behavior does not depend on the assumption of Eq. (18). Equation (20) conveys all the features we could expect for a heat engine with friction: η_{fric} is less than η, it approaches the latter when W_{fric} goes to zero, and vanishes as W_{fric} approaches W.

Other definitions of the efficiency of a frictionless engine might have been chosen as an alternative to Eq. (10). For instance, an intuitive choice of η is the ratio $W/Q_{exch,h}$. However, by using Eq. (5) we can write this ratio in terms of the energy transfers into and out of the reservoirs. We can show that it can be expressed as $1 - (Q_{exch,c} - \alpha_h W_{fric,h} - \alpha_c W_{fric,c})/Q_{exch,h}$, in which frictional quantities appear explicitly, as in Eq. (10). If instead of Eq. (10), η was defined as $W/Q_{exch,h}$, the right-hand side of Eq. (17) would be $\eta(1 - W_{fric}/W)/[1 - (1 - \alpha_h)\eta W_{fric,h}/W]$, which would not have changed the discussion of Fig. 3 and Eqs. (19) and (20).

Another plausible definition would be to write η as $1 - Q_{exch,c}/Q_{exch,h}$, as suggested by Eq.(6). But using Eq. (5), this ratio can be expressed as $(W + \alpha_h W_{fric,h} + \alpha_c W_{fric,c})/Q_{exch,h}$, with frictional quantities again present. Both alternative choices would have increased the efficiency without friction even more than Eq. (10), but would have made the comparison more difficult to understand because the working fluid would not perform the same cycle in the friction and frictionless cases. The difficulty of defining a unique η to which η_{fric} can be compared comes from the fact that only two out of the three energy flows in a frictionless engine can be fixed independently to two of the quantities $Q_{exch,h}$, $Q_{exch,c}$, and W; the third quantity is determined by the first law according to Eq. (5). We can circumvent this difficulty if we note that Q_h, Q_c, and W can remain unchanged for the purpose of comparing the friction and frictionless engines. Hence, the frictionless efficiency defined in Eqs. (10) and (14) has the advantage that the fluid goes through the same thermodynamic processes whether friction is present or not.[29]

V. ISOTHERMALS AND ISOCHORICS: THE STIRLING ENGINE

It is instructive to calculate the efficiency including friction of some well-known engines. Consider the Stirling cycle for an external-combustion engine, which comprises isothermal compression, isochoric heating, isothermal expansion, and isochoric cooling; the isochoric branches are realized using a perfect regenerative heat exchanger to keep the processes as close as possible to the idealized case,[14,15] but for friction. We assume that the working fluid consists of n moles of an ideal gas, and hence the isothermal work it performs must equal the total heating energy it receives. We also assume that the isothermal processes are quasistatic, so the work done by the gas is given by[12–15]

$$Q_{exch,h} + \alpha_h W_{fric,h} = nRT_h \ln r. \tag{21}$$

In Eq. (21), R is the usual gas constant and r the engine's compression ratio. A similar expression holds for the low temperature isothermal. We note that the second of our two assumptions puts Eq. (21) outside the realm of finite-time thermodynamics and makes it representative of sliding friction between the piston and the cylinder walls.[7,9,10] Equations (6) and (21) yield

$$\eta_{fric} = 1 - \frac{T_c}{T_h}\frac{1 + W_{fric,c}/nRT_c \ln r}{1 - W_{fric,h}/nRT_h \ln r}, \tag{22}$$

which follows from Eq. (15) as well if we recall that η for an ideal frictionless Stirling engine is equal to η_{carnot} in Eq. (9).[14,15] If we further suppose that the friction force per unit cross-sectional area of the piston is a constant P_{fric},[7,9,10] Eq. (22) can be expressed as

$$\eta_{fric} = 1 - \frac{T_c}{T_h}\frac{1 + P_{fric}\Delta v/RT_c \ln r}{1 - P_{fric}\Delta v/RT_h \ln r}, \tag{23}$$

where Δv is the change in molar volume per cycle.

We can use this example to see how α_h can affect η_{fric}. We assume that the heat $Q_{exch,h}$ exchanged between the hot reservoir and the working fluid is kept constant, together with, say, the minimum molar volume v_{min} in the cycle. For the same friction model that led to Eq. (23), Eqs. (21) and (23) become

$$Q_{exch,h} + \alpha_h n P_{fric}(r - 1)v_{min} = nRT_h \ln r \tag{24}$$

and

$$\eta_{fric} = 1 - \frac{T_c}{T_h}\frac{1 + P_{fric}(r-1)v_{min}/RT_c \ln r}{1 - P_{fric}(r-1)v_{min}/RT_h \ln r}, \tag{25}$$

which gives, via r, η_{fric} as an implicit function of α_h. This dependence of η_{fric} on α_h can be understood by noting from Eqs. (21) or (24) that, if α_h becomes larger, more energy goes into the fluid, thus forcing it to perform more work during the high-temperature isothermal process, making it expand more and simultaneously increase the compression ratio and the amount of frictional dissipation.

The Stirling cycle also allows us to discuss in more detail the implications of the work W_{fric} associated with friction. Recall that α_{h} and α_{c} ensure that all possible ways of allocating this work to the fluid or the heat reservoirs, or to both, are included in the present formulation.[5,9,10] For the low-temperature isothermal, where there might be an apparent conflict between the energy that must be lost to the cold reservoir during compression and the energy $\alpha_{\mathrm{c}} W_{\mathrm{fric,c}}$ which enters the fluid due to friction, we might think that $\alpha_{\mathrm{c}} W_{\mathrm{fric,c}}$ would immediately be transferred to the cold reservoir to keep the fluid at the proper temperature, in which case there would be no need for an α_{c}. Or we might imagine that $\alpha_{\mathrm{c}} W_{\mathrm{fric,c}}$ remains in the fluid, which would apparently cause its temperature to rise, thus questioning the assumption that there is good thermal contact with the reservoir. Or we might hypothesize that the excess energy $\alpha_{\mathrm{c}} W_{\mathrm{fric,c}}$ would leave the fluid in the form of work, which would eventually lead to the replacement of the isochorics by volume-changing processes. These hypotheses ignore the fact that the net decrease in the energy of the fluid must be $Q_{\mathrm{exch,c}} - \alpha_{\mathrm{c}} W_{\mathrm{fric,c}}$ to ensure a constant temperature, as follows from Eq. (21) adapted to the cold isothermal. For a given amount of work delivered to the fluid during an isothermal process, the greater the energy dissipated in the latter due to frictional work, the greater the amount of energy it must transfer to the reservoir. Inside the working fluid, we cannot distinguish the energy that is transferred to the reservoir from that entering due to friction. What we can determine is how much energy enters and leaves the fluid, which is all we need to apply the first and second laws. The final outcome of keeping track of the energy transfers is $Q_{\mathrm{exch,c}} - \alpha_{\mathrm{c}} W_{\mathrm{fric,c}}$. To accept that these two terms adjust themselves during an isothermal process of the Stirling cycle is no more demanding than the assumptions we have to make when imagining, for instance, the fluid going through an ideal Carnot cycle.

VI. ADIABATICS AND ISOBARICS: THE BRAYTON ENGINE

We next consider the Brayton cycle, also named after Joule, which describes jet engines as well as gas turbines and consists of adiabatic compression, isobaric heating, adiabatic expansion, and isobaric cooling processes.[14,15] We assume that work is performed quasistatically in all four processes. For the purpose of comparison, we assume that the fluid undergoes the same thermodynamic processes in the friction and frictionless cases, which implies that it does not transfer any energy due to heating or cooling in the adiabatic processes. Hence, during the latter we set α and Q_{exch} equal to zero, so that energy from frictional heating ends up entirely in the reservoirs, thus making them necessary participants even in adiabatic processes. For an ideal gas undergoing a quasistatic adiabatic process we have[12–15]

$$TP^{(1-\gamma)/\gamma} = (T + \Delta T)(P + \Delta P)^{(1-\gamma)/\gamma} \tag{26}$$

for an isentropic of an ideal gas,[30] where

$$\gamma \equiv \frac{c_P}{c_P - R}, \tag{27}$$

and c_P is the molar specific heat at constant pressure P. For the present analysis, T_{h} and T_{c} are taken as the highest and lowest values attained by the gas temperature T during the cycle, which corresponds to the endpoints of the isobaric heating and cooling processes, respectively.

The total energy transferred to the fluid from the hot reservoir during the hot isobaric process is given by

$$Q_{\mathrm{exch,h}} + \alpha_{\mathrm{h}} W_{\mathrm{fric,h}} = n c_P \Delta T_{\mathrm{h}}, \tag{28}$$

with ΔT_{h} the corresponding temperature change.[31] An analogous result holds for the cold isobaric. If we combine Eqs. (6) and (28), and use Eq. (26) rewritten as

$$\frac{T_{\mathrm{h}}}{T_{\mathrm{c}} + \Delta T_{\mathrm{c}}} = \frac{T_{\mathrm{h}} - \Delta T_{\mathrm{h}}}{T_{\mathrm{c}}} = p^{(1-\gamma)/\gamma}, \tag{29}$$

with p the engine's pressure ratio, we obtain

$$\eta_{\mathrm{fric}} = 1 - \frac{T_{\mathrm{c}}}{T_{\mathrm{h}} - \Delta T_{\mathrm{h}}} \frac{1 + W_{\mathrm{fric,c}}/n c_P \Delta T_{\mathrm{c}}}{1 - W_{\mathrm{fric,h}}/n c_P \Delta T_{\mathrm{h}}}. \tag{30}$$

We can use Eq. (29) to write Eq. (30) as

$$\eta_{\mathrm{fric}} = 1 - p^{(1-\gamma)/\gamma} \frac{1 + W_{\mathrm{fric,c}}/n c_P (T_{\mathrm{h}} p^{(1-\gamma)/\gamma} - T_{\mathrm{c}})}{1 - W_{\mathrm{fric,h}}/n c_P (T_{\mathrm{h}} - T_{\mathrm{c}} p^{(1-\gamma)/\gamma})}, \tag{31}$$

which can also be obtained from Eq. (15) by noting that the frictionless efficiency η for a standard Brayton or Joule cycle is $1 - p^{(1-\gamma)/\gamma}$.[14,15] Let us assume that there is a constant friction force per unit area whose ratio to the pressure of the hot isobaric is p_{fric}. If we also assume that the hot reservoir is involved only during this same process, all other processes involving interaction with the colder environment, we can use Eq. (29) to obtain

$$W_{\mathrm{fric,c}} = nRp_{\mathrm{fric}}[T_{\mathrm{h}}(2p^{1/\gamma} - 1) - T_{\mathrm{c}} p^{(\gamma-1)/\gamma}] \tag{32}$$

and

$$W_{\mathrm{fric,h}} = nRp_{\mathrm{fric}}[T_{\mathrm{h}} - T_{\mathrm{c}} p^{(\gamma-1)/\gamma}], \tag{33}$$

so that Eq. (31) becomes

$$\eta_{\mathrm{fric}} = 1 - p^{(1-\gamma)/\gamma} \frac{1 + Rp_{\mathrm{fric}}[T_{\mathrm{h}}(2p^{1/\gamma} - 1) - T_{\mathrm{c}} p^{(1-\gamma)/\gamma}]/c_P (T_{\mathrm{h}} p^{(1-\gamma)/\gamma} - T_{\mathrm{c}})}{1 - Rp_{\mathrm{fric}}/c_P}. \tag{34}$$

This analysis of the Brayton cycle illustrates how the model presented here to address the efficiency of heat engines with friction has no limitations regarding its applicability and allows for any type of heat exchange between the fluid and the reservoirs including, in particular, all situations where such exchange is not isothermal.

VII. THE SECOND LAW AND FRICTION: ADDITIONAL THOUGHTS

It is useful to discuss more explicitly the implications that the work associated with friction has on entropy and the second law. If we expand Eq. (7) using Eq. (2), or the implicit definitions for $Q_{0,\mathrm{c}}$ and $Q_{0,\mathrm{h}}$ in Eq. (6), and define the entropy change due to the direct exchange of energy between the working fluid and the reservoirs,

$$\Delta S_{\mathrm{exch}} \equiv \frac{Q_{\mathrm{exch,c}}}{T_{\mathrm{c}}} - \frac{Q_{\mathrm{exch,h}}}{T_{\mathrm{h}}}, \tag{35}$$

and the non-negative entropy production resulting from dissipation of the frictional work,

$$\Delta S_{\mathrm{fric}} \equiv \frac{(1-\alpha_{\mathrm{c}})W_{\mathrm{fric,c}}}{T_{\mathrm{c}}} + \frac{(1-\alpha_{\mathrm{h}})W_{\mathrm{fric,h}}}{T_{\mathrm{h}}} \geq 0, \tag{36}$$

we obtain[32]

$$\Delta S = \Delta S_{\mathrm{exch}} + \Delta S_{\mathrm{fric}} \geq 0. \tag{37}$$

Equation (37) is a form of the Clausius inequality with friction considered explicitly and indicates that entropy production arising from friction lowers the minimum value imposed on ΔS_{exch} by the second law, a value which is allowed to become negative.[10,11,33] We next rewrite Eqs. (35)–(37) to obtain

$$\Delta S = \frac{Q_{\mathrm{exch,c}} - \alpha_{\mathrm{c}} W_{\mathrm{fric,c}}}{T_{\mathrm{c}}} - \frac{Q_{\mathrm{exch,h}} + \alpha_{\mathrm{h}} W_{\mathrm{fric,h}}}{T_{\mathrm{h}}} + \frac{W_{\mathrm{fric,c}}}{T_{\mathrm{c}}} + \frac{W_{\mathrm{fric,h}}}{T_{\mathrm{h}}}. \tag{38}$$

The difference between the first two terms on the right-hand side of Eq. (38) is the non-negative entropy change for the equivalent frictionless engine. We conclude that, if there is friction, ΔS is strictly positive and so the equality sign in Eqs. (7) and (9) will not hold.

It is still possible, by using Eqs. (8) and (10), to recast Eq. (38) in the more appealing form

$$\Delta S = (\eta_{\mathrm{carnot}} - \eta)\frac{Q_{\mathrm{exch,h}} + \alpha_{\mathrm{h}} W_{\mathrm{fric,h}}}{T_{\mathrm{c}}} + \frac{W_{\mathrm{fric,c}}}{T_{\mathrm{c}}} + \frac{W_{\mathrm{fric,h}}}{T_{\mathrm{h}}}. \tag{39}$$

Equation (39) shows that the first term on the right-hand side of Eq. (39) cannot be negative, because η_{carnot} cannot be less than η, and so ΔS must be positive if the other two terms do not vanish. Equation (39) also illustrates the basic entropy-production sources in the model: the first term on its right-hand side has to do with the deviations from a Carnot cycle in the original, frictionless engine, and would be there even if there were no friction; the other two terms are equivalent to the entropy change due to dissipative work at temperatures T_{c} and T_{h}.

If α_{h} and α_{c} are both equal to unity, so that the work produced against the dissipative forces goes entirely to the fluid, Eq. (36) might seem to misleadingly imply that there is no increase in the entropy of the universe due to friction in this case, which cannot be. According to the model schematized in Fig. 1, recall that the energy from dissipation goes to one of four places: the hot fluid, the cold fluid, the cold reservoir, or the hot reservoir. The last two are directly accounted for in ΔS_{fric} of Eq. (36), but the first two are apparently absent in ΔS_{exch} in Eq. (35). The reason is that, in a cyclic engine, the entropy change of the fluid in one cycle vanishes and so does not contribute directly to ΔS of the universe. Nonetheless, with both α_{h} and α_{c} equal to one, the first two terms on the right-hand side of Eq. (38), which give the entropy production of the equivalent frictionless engine, yield $(Q_{\mathrm{exch,c}} - W_{\mathrm{fric,c}})/T_{\mathrm{c}} - (Q_{\mathrm{exch,h}} + W_{\mathrm{fric,h}})/T_{\mathrm{h}}$, a smaller amount than the right-hand side of Eq. (35). Therefore, dissipation in the hot and cold fluids manifests itself indirectly in ΔS_{exch}, thus contributing to increase the entropy of the universe.

VIII. CONCLUDING REMARKS

The derivation of general expressions for the thermodynamic efficiency of heat engines with friction has been addressed from a practical point of view. The result agrees with that of a more formal analysis.[10] The efficiency is expressed as the ratio of the effective work delivered to the environment, which is the actual work performed by the engine fluid minus the work done against friction, to the net energy lost by the hot reservoir, which is that part directly transferred to the fluid subtracted by the part corresponding to the frictional losses dissipated in the reservoir. The numerator of this ratio, which is the effective output or power, has been properly treated in finite-time thermodynamics,[3–6] but the denominator has been treated somewhat casually, and a detailed discussion of how friction affects it was still lacking. With the exception of internally dissipative friction,[5] it has been either stated that the dissipated energy is not returned to the fluid[4] or implicitly assumed that dissipation takes place in the cold reservoir.[3,5,6] All of these known cases are successfully recovered in the Appendix.

The general framework we have proposed was used to obtain the efficiencies for both Stirling and Brayton engines with friction. This analysis also illustrated the universal nature of the underlying model by applying it to the four basic thermodynamic processes, namely, adiabatic, isothermal, isochoric, and isobaric. It was found that, to a good approximation, the efficiency of an engine with friction is proportional to the frictionless efficiency and to one minus the amount of frictional losses, quantified as the fraction of the net work performed by the working fluid which is lost to friction. The procedure we have followed can be extended to define general expressions for the coefficient of performance of refrigerators and heat pumps with friction, and is as general as the treatments found in textbooks on thermodynamics.[12–15]

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APPENDIX: CONTINUUM FORMULATION

If instead of interacting with only two reservoirs, the working fluid exchanges energy with a continuum of reservoirs, Eq. (6) must be rewritten as

$$\eta_{\rm fric} = 1 - \frac{\oint [\dot{Q}_{\rm exch,c} + (1-\alpha_{\rm c})\dot{W}_{\rm fric,c}]dt}{\oint [\dot{Q}_{\rm exch,h} - (1-\alpha_{\rm h})\dot{W}_{\rm fric,h}]dt}, \qquad \text{(A1)}$$

where a dot means a time derivative. We define[34]

$$\dot{Q}_{\rm exch} = \dot{Q}_{\rm exch,h} - \dot{Q}_{\rm exch,c} \qquad \text{(A2)}$$

as the instantaneous thermal power directly exchanged with the reservoirs. Because these quantities vary continuously during one cycle, there is no need for the subscripts h and c, and we set $\alpha_{\rm h}$ and $\alpha_{\rm c}$ equal to α. If the period of the engine is τ, out of which a time $\tau_{\rm h}$ is spent in thermal contact with the hot reservoirs,[35] Eq. (A1) can be written as

$$\eta_{\rm fric} = \frac{\int_0^{\tau}[\dot{Q}_{\rm exch} - (1-\alpha)\dot{W}_{\rm fric}]dt}{\int_0^{\tau_{\rm h}}[\dot{Q}_{\rm exch} - (1-\alpha)\dot{W}_{\rm fric}]dt}. \qquad \text{(A3)}$$

A continuum formulation might also be helpful, even for an engine working between a hot and a cold reservoir with constant temperatures, if the friction characteristics change during the cycle undergone by the working fluid. In such a case $\alpha_{\rm h}$ should be replaced in Fig. 1 and in the various equations by an average $\bar{\alpha}_{\rm h}$ defined as

$$\bar{\alpha}_{\rm h} = \frac{\int_0^{\tau_{\rm h}} \alpha_{\rm h}\dot{W}_{\rm fric,h}dt}{\int_0^{\tau_{\rm h}} \dot{W}_{\rm fric,h}dt}. \qquad \text{(A4)}$$

A similar expression applies to $\bar{\alpha}_{\rm c}$.

The form of the efficiency in Eq. (A3) has been used to describe internally dissipative friction,[5] after substituting detailed forms for $\dot{Q}_{\rm exch}$ and $\dot{W}_{\rm fric}$.[36] Care should be exercised when concluding that, in this type of engine, it is better to dissipate to the surroundings than directly in the fluid,[5] in apparent agreement with Eq. (3), which shows, all other quantities in Eq. (3) being constant, that decreasing $\alpha_{\rm h}$ leads to increasing $\eta_{\rm fric}$. A seemingly opposite conclusion may be extracted from Eqs. (6) or (A1), which indicate that $\eta_{\rm fric}$ increases with both $\alpha_{\rm h}$ and $\alpha_{\rm c}$. Using either Eq. (3) or Eqs. (6) or (A1) to maximize $\eta_{\rm fric}$ in terms of $\alpha_{\rm h}$ and $\alpha_{\rm c}$ for the same amount of friction, we are implicitly assuming that either W or $Q_{\rm exch,c}$ and $Q_{\rm exch,h}$ are kept constant in the process, which is not the same because these three quantities are coupled to $\alpha_{\rm h}$ and $\alpha_{\rm c}$, and to each other via Eq. (5). Such an apparent contradiction indicates that the conclusion depends as much on the model of friction adopted, as on the optimization procedure.[37]

To see how some additional results can be obtained from the general formalism we have introduced, we derive general expressions for the efficiency of what are known as endoreversible or externally dissipative engines, in which frictional dissipation does not occur within the system but, for instance, might occur by an external mechanical linkage.[3,5,6] In this case $\alpha_{\rm h}$ and $\alpha_{\rm c}$ are both equal to zero and, if we assume that there are no losses during interaction with the heating source, the dissipated energy is totally transferred to the colder reservoir, which is here the surroundings, and $W_{\rm fric,h}$ vanishes.[3,5,6] Hence, Eq. (6) becomes[3,6,38]

$$\eta_{\rm fric} = \frac{Q_{\rm exch,h} - Q_{\rm exch,c} - W_{\rm fric,c}}{Q_{\rm exch,h}}, \qquad \text{(A5)}$$

or, in a continuum form resembling Eq. (A3),[5,39]

$$\eta_{\rm fric} = \frac{\int_0^{\tau}(\dot{Q}_{\rm exch} - \dot{W}_{\rm fric})dt}{\int_0^{\tau_{\rm h}} \dot{Q}_{\rm exch}\, dt}. \qquad \text{(A6)}$$

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[19]The symmetry of Fig. 1 might lead to the incorrect conclusion that the two reservoirs are connected by an arrow representing energy flow between them. The directions of the arrows indicate that there is no possibility of a direct transfer of energy from one to the other, and the two are effectively isolated from each other.
[20]A device is a part of an engine, which consists of two reservoirs and the mechanical device M where the processes undergone by the working fluid take place, as in Ref. 13.
[21]As an example, we may think of the valve-exhaust process after the power stroke in an internal combustion engine described by a Diesel or a Otto

cycle, during which an isochoric, non-isothermal cooling takes place while energy is removed from the fluid to the cold reservoir, as discussed in Refs. 13–15.
[22]Although the processes undergone by the working fluid are assumed to be the same in this comparison, the engines with and without friction are different because the net amount of energy coming out from the hot reservoir is not identical in the two cases, as well as the net amount of energy transferred to the cold reservoir.
[23]To avoid confusion, remember that $Q_{\rm exch,h}$ and $Q_{\rm exch,c}$ are the energies exchanged between the fluid and the two reservoirs; $Q_{0,\rm h}$ and $Q_{0,\rm c}$ are the net amounts of energy leaving the hot reservoir and going to the cold reservoir, respectively; $Q_{\rm h}$ and $Q_{\rm c}$ are the net energies transferred into and out of the fluid. In the absence of friction these three sets of quantities reduce to a single one, as shown by Eqs. (2), (6), (11), and (12).
[24]Equation (17) is the same as Eq. (24) in Ref. 10.
[25]If we assume that $W_{\rm fric,h}$ and $W_{\rm fric}$ are proportional to one another, we can show that $(W/\eta)(\partial\eta_{\rm fric}/\partial W_{\rm fric})$ is equal to $-(1-\eta\, W_{\rm fric,h}/W_{\rm fric})/(1-\eta\, W_{\rm fric,h}/W)^2$ and is thus negative, because η and $W_{\rm fric,h}/W_{\rm fric}$ are both less than unity.
[26]From a more fundamental point of view, the need for $\alpha_{\rm h}$ and $\alpha_{\rm c}$ stems from the second law, as explained in Ref. 9 in connection with Eqs. (20) and (21).
[27]Deviations from Eq. (20) in the region where η approaches one are not a problem because engine operation with efficiencies close to unity is hindered by the second law.
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[30]Note that if the appropriate α is not zero and the friction force per unit area $P_{\rm fric}$ is constant, Eq. (26) would be written with $TP^{-1}[P \pm (1-\gamma^{-1})\alpha P_{\rm fric}]^{1/\gamma}$ instead of $TP^{(1-\gamma)/\gamma}$, as follows from Eqs. (31) and (36) in Ref. 9.
[31]To avoid confusion, recall that ΔT in Eq. (26) represents the temperature change in the quasistatic adiabatic processes of the Brayton cycle, in contrast to $\Delta T_{\rm h}$ and $\Delta T_{\rm c}$, which are the temperature changes for its hot and cold isobarics, respectively.
[32]Recall that the working fluid and the components in these heat engines operate cyclically so, when ΔS is calculated, the focus is on the entropy change of the reservoirs.
[33]Compare, for instance, Eqs. (35)–(37) with Eq. (16) in Ref. 10.
[34]$\dot{Q}_{\rm exch}$ is an exception to the sign convention adopted earlier in the paper, because it may be either positive or negative.
[35]As in Ref. 5, $t_{\rm h}$ is sometimes interpreted as a switching time.
[36]In Ref. 5, $\dot{Q}_{\rm exch}$ is assumed to be governed by a general heating source in addition to Newtonian conduction, whereas $\dot{W}_{\rm fric}$ is taken to be proportional to $\dot{Q}^2_{\rm exch}$, consistent with a friction force being linear in the engine speed, as in a well-lubricated system according to Ref. 4.
[37]For instance, the quantity optimized in Ref. 5 is output power, more precisely, average power or output work per cycle, which amounts to maximizing the numerator in Eq. (A3), an increasing function of α.
[38]In Refs. 3 and 6, additional energy loss terms were considered which are not discussed in the present work, which focuses only on friction. With this distinction in mind, Eq. (A5) is equivalent to Eq. (19) in Ref. 6.
[39]In Ref. 3, where the tricycle description has been developed, the cycle-averaged power loss $W_{\rm fric}/\tau$ is taken to be proportional to $[(Q_{\rm exch,h}-Q_{\rm exch,c})/\tau]^2$, while in Ref. 5, where instantaneous powers are used, $\int_0^\tau \dot{W}_{\rm fric}dt$ is assumed to be proportional to $\left(\int_0^\tau \dot{Q}_{\rm exch}dt\right)^2$.