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Integrability Conditions and Carathéodory's Theorem

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The theorem of Carathéodory is proved by showing that the necessary and sufficient condition for the existence of points neighboring to an arbitrary point which are inaccessible from the latter along solution curves of a linear total differential equation is that the conditions of integrability of the equation be satisfied.

1. INTRODUCTION

Let[1]

$$P_i(x_1,\cdots,x_n)dx_i=0 \qquad (1)$$

be a linear total differential equation in $n(\geqslant 3)$ variables, the P_i being supposed to possess continuous first derivatives. Most treatments of this type of equation[2] center around the *conditions of integrability*, that is, it is shown that if, and only if, the expressions W_{ijk} given by[3]

$$W_{ijk}=P_i(P_{j,k}-P_{k,j})+P_j(P_{k,i}-P_{i,k})+P_k(P_{i,j}-P_{j,i}) \qquad (2)$$

vanish identically then there exist functions $q(x_1,\cdots,x_n)$ and $F(x_1,\cdots,x_n)$ such that

$$P_i dx_i \equiv q dF, \qquad (3)$$

and the integral equivalent of Eq. (1) is therefore the *single* algebraic equation

$$F=\text{const.} \qquad (4)$$

In terms of a representative space[4] R_n one may look upon Eq. (1) as asserting the orthogonality at x_i of a line element dx_i and the given vector P_i. In the first place, therefore, the geometrical object representing the solution of Eq. (1) is a set of *curves*. Accordingly, if the equation is *integrable* then all solution curves (i.e., curves satisfying Eq. (1)) which pass through a given point x_i^0 of R_n must lie on a certain surface, *viz.*, just the surface

$$F(x_1,\cdots,x_n)=F(x_1^0,\cdots,x_n^0). \qquad (5)$$

Now the theorem of Carathéodory (which plays an important part in the development of the consequences of the second law of thermodynamics) is also concerned with the integrability of linear total differential equations. It may be stated here as follows: In the neighborhood of any arbitrary point of R_n there are points inaccessible along solution curves of Eq. (1) if, and only if, the equation is integrable.

Neither of the existing proofs[5,6] of this theorem, however, makes any explicit reference to the conditions of integrability (2), and in this sense the theory of the type of equation under consideration suffers from a certain incoherence. In this paper, therefore, I give an alternative proof which remedies this defect. It is based upon a method used by Hamilton in the course of his work on the calculus of principal relations.[7] The case of any number of variables is treated from the outset, as the restriction to only three variables brings with it few formal advantages from the viewpoint of simplicity, provided an appropriate notation be used.

2. PROOF OF THE THEOREM

Let C be an arbitrary solution curve of Eq. (1) through the (arbitrary) initial point A_0 and let A' be some other point on C. If the equation of C be written as[8]

$$x_i=x_i(t), \qquad (6)$$

where t is a variable parameter, then in virtue of the given equation the n functions $x_i(t)$ must

[1] A subscript i takes all values from 1 to n, and when it is repeated summation over this range is implied.

[2] E.g., (i) E. R. Forsyth, *Differential Equations* (Macmillan, London, 1943), 6th ed., Chap. VIII, pp. 309 ff; (ii) E. L. Ince, *Ordinary Differential Equations* (Longmans Green, London, 1927), Chap. II, pp. 52 ff.

[3] Subscripts $i, j, \cdots$ following a comma denote differentiation with respect to $x_i, x_j, \cdots$.

[4] Cf. H. A. Buchdahl, Am. J. Phys. **17**, 212 (1949), Sec. 2a.

[5] C. Carathéodory, Math. Ann., **67**, 355 (1909), particularly pp. 369–70.

[6] H. A. Buchdahl, Am. J. Phys., **17**, 44 and 212 (1949).

[7] *The Mathematical Papers of Sir W. R. Hamilton* (Cambridge University Press, 1940), Vol. II, pp. 297 ff.

[8] For the purposes of this proof it suffices to consider only solution curves which are such that the functions x_i and ξ_i which occur in their parametric representations (6), (8) possess continuous first derivatives.

identically satisfy the equation

$$P_i\dot{x}_i=0, \tag{7}$$

a dot denoting differentiation with respect to t. It is convenient to lay down that Eq. (6) shall give the coordinates of A', A_0 when t has the values t', t_0, respectively, $(t'<t_0)$. The arbitrariness of C means that $n-1$ of the n functions $x_i(t)$ may be chosen arbitrarily, the nth then being given by the integral of the ordinary first-order equation (7).

Consider an arbitrary solution curve C^* of Eq. (1) which passes through A' and which lies in the neighborhood of C, i.e., a curve whose equation can be written in the form

$$x_i=x_i(t)+\epsilon\xi_i(t), \tag{8}$$

where ϵ is a small constant and the ξ_i are such that

$$\xi_i(t')=0. \tag{9}$$

Of the n functions $\xi_i(t)$ $n-1$ may be chosen arbitrarily; the nth is then determined by the requirement that Eq. (8) must satisfy Eq. (7). Selecting ξ_k as the function to be thus determined, where k is now an arbitrary but *fixed* index, it follows on substituting Eq. (8) in Eq. (7) and neglecting terms involving powers of ϵ higher than the first that ξ_k must satisfy the equation[9]

$$P_k\dot{\xi}_k+P_{i,k}\dot{x}_i\xi_k=-(P_j\dot{\xi}_j+P_{i,j}\dot{x}_i\xi_j). \tag{10}$$

It is supposed, without loss of generality, that P_k is not identically zero.

Let λ be an integrating factor of the ordinary linear first-order equation (10); i.e., let λ be chosen such that

$$d(\lambda P_k)/dt=\lambda P_{i,k}\dot{x}_i,$$

whence

$$P_k\dot{\lambda}=\lambda(P_{i,k}-P_{k,i})\dot{x}_i. \tag{11}$$

Keeping Eq. (9) in mind, Eq. (10) then gives

$$\lambda P_k\xi_k=-\int_{t'}^{t}\lambda(P_j\dot{\xi}_j+P_{i,j}\dot{x}_i\xi_j)dt. \tag{12}$$

[9] A subscript j only takes the values $1, \cdots, k-1, k+1, \cdots, n$, and when it is repeated summation over this range is implied.

The first term on the right may be transformed as follows:

$$-\int_{t'}^{t}\lambda P_j\dot{\xi}_j dt=-\lambda P_j\xi_j+\int_{t'}^{t}\xi_j(\lambda\dot{P}_j+P_j\dot{\lambda})dt$$

$$=-\lambda P_j\xi_j+\int_{t'}^{t}\lambda\xi_j\dot{x}_i \times[P_{j,i}+(P_{i,k}-P_{k,i})P_j/P_k]dt,$$

in view of Eq. (11). Equation (12) therefore gives

$$\lambda P_k\xi_k=-\lambda P_j\xi_j+\int_{t'}^{t}(\lambda/P_k)\xi_j\dot{x}_i \times[(P_{j,i}-P_{i,j})P_k+(P_{i,k}-P_{k,i})P_j]dt. \tag{13}$$

In view of Eq. (7) the group of terms in the integrand enclosed in square brackets may be replaced by $-W_{ijk}$. Hence finally

$$\xi_k=-P_j\xi_j/P_k -(\lambda P_k)^{-1}\int_{t'}^{t}(\lambda/P_k)W_{ijk}\dot{x}_i\xi_j dt. \tag{14}$$

That the integrability of Eq. (1) is a *sufficient* condition for the existence of inaccessible points in the neighborhood of $A_0(x_1^0,\cdots,x_n^0)$ is obvious, for all points not lying on the surface Eq. (5) are inaccessible; that it is also a necessary condition may be inferred as follows.

Taking $t=t_0$ in Eq. (14), the $\epsilon\xi_i(t_0)$ are the components of the small displacement of points A_1 in the neighborhood of A_0 relative to the latter. Unless the integrand in Eq. (14) vanishes for every choice of the independent functions $\xi_j(t)$, there will be no necessary connection between $\xi_k(t_0)$ and the $\xi_j(t_0)$, and consequently all neighboring points A_1 will be accessible from A_0, *viz.*, along the route $A_0\rightarrow A'\rightarrow A_1$. The existence of inaccessible points therefore requires that the condition

$$W_{ijk}\dot{x}_i=0 \tag{15}$$

be satisfied. Since the $\dot{x}_j$ may be chosen arbitrarily, and $\dot{x}_k$ does not in fact appear in Eq. (15), W_{ijk} being antisymmetrical in i and k, it therefore follows that W_{ijk} must vanish identically, i.e., the conditions of integrability must be satisfied.