AMERICAN JOURNAL of PHYSICS

Elementary introduction to the Green's function

R. C. Whitten and P. T. McCormick

Citation: Am. J. Phys. **43**, 541 (1975); doi: 10.1119/1.9799
View online: http://dx.doi.org/10.1119/1.9799
View Table of Contents: http://ajp.aapt.org/resource/1/AJPIAS/v43/i6
Published by the American Association of Physics Teachers

Elementary introduction to the Green's function

R. C. Whitten
Ames Research Center
National Aeronautics and Space Administration
Moffett Field, California 94035

P. T. McCormick
University of Santa Clara
Santa Clara, California 95053
(Received 13 September 1974)

A technique, using the method of variation of parameters for solving differential equations, is developed for introducing Green's functions early in an undergraduate curriculum. Various examples are presented.

The Green's function approach to boundary value problems is a very powerful technique. Unfortunately, its introduction in the curriculum is usually delayed until the graduate level, where its presentation frequently evokes a sense of mysticism as to its origin and its physical meaning. We suggest that it can be introduced early in a physics student's education, in a physically meaningful way, by using a very elementary technique—the solution of ordinary differential equations by the method of variation of parameters,[1] which students encounter in elementary differential equations courses. The only additional information required is a definition of the delta function, which is not usually covered in such a course. We illustrate the approach by means of the equation for the vibrating string and summarize some of the other cases (such as the circular membrane) at the end of this article.

We shall first demonstrate the connection between the Green's function and a solution to the inhomogeneous equation

$$\frac{d^2v(x)}{dx^2}+k^2v(x)=-f(x), \qquad (1)$$

where $f(x)$ is a forcing term that is distributed over the string. We also introduce the corresponding equation for which a unit impulse (represented by a delta function) is applied at point x_0:

$$\frac{d^2g(x,x_0)}{dx^2}+k^2g(x,x_0)=-\delta(x-x_0). \qquad (2)$$

To develop the connection between v and g, first multiply Eq. (1) by g, Eq. (2) by v, and subtract

$$\frac{d^2v(x)}{dx^2}g(x,x_0)-v(x)\frac{d^2g(x,x_0)}{dx^2}$$

$$=-f(x)g(x,x_0)+v(x)\delta(x-x_0). \qquad (3)$$

Integrating over the length of the string (0 to L) yields

$$\left[g(x,x_0)\frac{dv(x)}{dx}-v(x)\frac{dg(x,x_0)}{dx}\right]\Bigg|_0^L$$

$$=v(x_0)-\int_0^L f(x)g(x,x_0)\,dx. \qquad (4)$$

If the string is fixed at both ends, g and v vanish at the boundaries, and we have

$$v(x)=\int_0^L f(x')g(x',x)\,dx' \qquad (5)$$

as the integral equation which replaces Eq. (1). However, it does not give any additional insight into the problem unless one can find an explicit form for $g(x,x_0)$. This we now do via the method of variation of parameters.

The solution of Eq. (2) is written in terms of solutions to the corresponding homogeneous equation as

$$g(x,x_0)=\alpha(x,x_0)\cos(kx)+\beta(x,x_0)\sin(kx). \qquad (6)$$

Then we have (dropping arguments on α and β)

$$\frac{dg(x,x_0)}{dx}=-k[\alpha\sin(kx)-\beta\cos(kx)]$$
$$+\frac{d\alpha}{dx}\cos(kx)+\frac{d\beta}{dx}\sin(kx), \qquad (7)$$

to which we apply the usual constraint

$$\frac{d\alpha}{dx}\cos(kx)+\frac{d\beta}{dx}\sin(kx)=0. \qquad (8)$$

Adding the second derivative of $g(x,x_0)$,

$$\frac{d^2g(x,x_0)}{dx^2}=-k^2[\alpha\cos(kx)+\beta\sin(kx)]$$
$$-k\left(\frac{d\alpha}{dx}\sin(kx)-\frac{d\beta}{dx}\cos(kx)\right), \qquad (9)$$

to $k^2g(x,x_0)$ yields

$$\frac{d\alpha}{dx}\sin(kx)-\frac{d\beta}{dx}\cos(kx)=\frac{1}{k}\delta(x-x_0). \qquad (10)$$

Solving the simultaneous set of equations, (8) and (10), for

$d\alpha/dx$ and $d\beta/dx$ results in

$$\frac{d\alpha}{dx}=\frac{1}{k}\;\frac{\begin{vmatrix}\delta(x-x_0) & -\cos(kx)\\ 0 & \sin(kx)\end{vmatrix}}{\begin{vmatrix}\sin(kx) & -\cos(kx)\\ \cos(kx) & \sin(kx)\end{vmatrix}}$$
$$=(1/k)\sin(kx)\delta(x-x_0), \tag{11a}$$

$$\frac{d\beta}{dx}=-\frac{1}{k}\cos(kx)\delta(x-x_0), \tag{11b}$$

which, when integrated, yield

$$\alpha(x,x_0)=c_1+\begin{cases}k^{-1}\sin(kx_0), & x>x_0\\ 0, & x<x_0,\end{cases} \tag{12a}$$

$$\beta(x,x_0)=c_2-\begin{cases}k^{-1}\cos(kx_0), & x>x_0\\ 0, & x<x_0,\end{cases} \tag{12b}$$

and

$$g(x,x_0)=c_1\cos(kx)+c_2\sin(kx)$$
$$-\frac{1}{k}\begin{cases}\sin[k(x-x_0)], & x>x_0\\ 0, & x<x_0.\end{cases} \tag{13}$$

The determinant in the denominator of the first expression of Eq. (11a) is the *Wronskian,* which takes a simple value (in this case unity) for all equations of interest.

We now apply the boundary conditions which specify that the string ends are to be fixed.

$$g(0,x_0)=0,\qquad g(L,x_0)=0. \tag{14}$$

These conditions require that $c_1=0$, $c_2=k^{-1}[\cos(kx_0)-\cot(kL)\sin(kx_0)]$, so that

$$g(x,x_0)=\frac{\sin(kx)}{k\sin(kL)}\sin[k(L-x_0)],\quad x<x_0$$
$$=\frac{\sin(kx_0)}{k\sin(kL)}\sin[k(L-x)],\quad x>x_0. \tag{15}$$

We have thus obtained an explicit form for $g(x,x_0)$.

Let us now try to obtain the expression (5) for $v(x)$ in terms of the forcing term $f(x)$ and the Green's function $g(x,x')$ by establishing a connection between $g(x,x')$ and the solution to Eq. (1) that was obtained by the method of variation of parameters. The solution to Eq. (1) is again expressed in terms of the parameters $\alpha(x)$ and $\beta(x)$, which now take the forms

$$\alpha(x)=c_1+k^{-1}\int_0^x f(x')\sin(kx')\,dx', \tag{16a}$$

Equation	Boundary Conditions	Wronskian	Green's Function
$\frac{d^2g}{dx^2}+k^2g=-\delta(x-x_0)$ Harmonic oscillator equation	$g(0)=0$ $g(L)=0$	1	$\frac{\sin kx \sin k(L-x_0)}{k\sin kL}$ $x<x_0$ $\frac{\sin kx_0 \sin k(L-x)}{k\sin kL}$ $x>x_0$
$\frac{d^2g}{dx^2}-k^2g=-\delta(x-x_0)$	$g(0)=0$ $g(L)=0$	1	$\frac{\sinh kx \sinh k(L-x_0)}{k\sinh kL}$ $x<x_0$ $\frac{\sinh kx_0 \sinh k(L-x)}{k\sinh kL}$ $x>x_0$
$\frac{d^2g}{dx^2}+2\gamma\frac{dg}{dx}+(k^2-\gamma^2)g$ $=-\delta(x-x_0)$ Damped harmonic oscillator equation	$g(0)=0$ $g(L)=0$	1	$\frac{\exp^{[-\gamma(x-x_0)]}\sin kx \sin k(L-x_0)}{k\sin kL}$ $x<x_0$ $\frac{\exp^{[-\gamma(x-x_0)]}\sin kx \sin k(L-x)}{k\sin kL}$ $x>x_0$
$\frac{d^2g}{dx^2}+\frac{1}{x}\frac{dg}{dx}+\left(1-\frac{m^2}{x^2}\right)g$ $=-\frac{1}{x}\delta(x-x_0)$ Bessel's equation (circular membrane)	$g(0)$ finite $g(R)=0$	$\frac{2}{\pi x}$	$\frac{2}{\pi J_m(R)}\begin{cases}[N_m(R)J_m(x_0)-J_m(R)N_m(x_0)]J_m(x) & x<x_0\\ [N_m(R)J_m(x)-J_m(R)N_m(x)]J_m(x_0) & x>x_0\end{cases}$ J_m and N_m are Bessel functions of the first and second kinds.[3]
$x\frac{d^2g}{dx^2}+(b-x)\frac{dg}{dx}-ag=-\delta(x-x_0)$ Confluent hypergeometric equation	$g(0)=0$ $g(L)=0$	$-\Gamma(b)\,z^{-b}\,e^z/\Gamma(a)$	$\frac{\Gamma(a)}{\Gamma(b)}\frac{1}{M(a,b;L)}\begin{cases}[M(a,b;L)U(a,b;z_0)-U(a,b;L)M(a,b;z_0)]M(a,b;z) & z<z_0\\ [M(a,b;L)U(a,b;z)-U(a,b;L)M(a,b;z)]M(a,b;z_0) & z>z_0\end{cases}$ $M(a,b;z)$ and $U(a,b;z)$ are confluent hypergeometric functions of the first and second kinds.[4]
$\frac{d}{dx}\left[p(x)\frac{dg}{dx}\right]+\lambda[r(x)-q(x)]g$ $=-\delta(x-x_0)$ General equation	$g(0)$, $g(L)$ specified	$\frac{1}{p(x)}$	$g(x,x_0)=\frac{1}{\Delta}\{g(0)[f_1(x)f_2(L)-f_1(L)f_2(x)]$ $+g(L)[f_1(0)f_2(x)-f_1(x)f_2(0)]$ $+[f_1(0)f_2(x_0)-f_1(x_0)f_2(0)]$ $\times[f_1(x)f_2(L)-f_1(L)f_2(x)]\}$, $x>x_0$ $=\frac{1}{\Delta}\{g(0)[f_1(x)f_2(L)-f_1(L)f_2(x)]$ $+g(L)[f_1(0)f_2(x)-f_1(x)f_2(0)]$ $+[f_1(0)f_2(x)-f_1(x)f_2(0)]$ $\times[f_1(x_0)f_2(L)-f_1(L)f_2(x_0)]\}$ $x<x_0$ $\Delta=f_1(0)f_2(L)-f_1(L)f_2(0)$ where f_1 and f_2 are solutions to the inhomogeneous equation.

Fig. 1. Examples of one-dimensional Green's functions.

$$\beta(x) = c_2 - k^{-1} \int_0^x f(x') \cos(kx')\, dx'. \qquad (16b)$$

Application of the boundary conditions yields

$$v(x) = [k \sin(kL)]^{-1} \times \{ \int_x^L \sin(kx) \sin[k(L - x')] f(x')\, dx' - \int_0^x \sin(kx') \sin[k(L - x)] f(x')\, dx' \}. \qquad (17)$$

In terms of the Green's function, Eq. (17) takes the form

$$v(x) = \int_0^L f(x') g(x', x)\, dx'. \qquad (18)$$

which was obtained earlier [Eq. (5)]. Thus, we have established that Eq. (15) is indeed the Green's function for the vibrating string.

The foregoing derivation applies only to one-dimensional problems, of course. However, it is perfectly adequate for introducing the Green's function, its origin, and its meaning on a very elementary level. Having done so, it should be quite feasible to extend the approach to the multidimensional case further on in an undergraduate's education and to defer more rigorous treatments[2] until the graduate level. A table of Green's functions for other equations is shown in Fig. 1.

[1]W. E. Boyce and R. C. DiPrima, *Introduction to Differential Equations* (Wiley, New York, 1970).
[2]P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Vol. 1, Chap. 7.
[3]M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (U.S. GPO, Washington, DC, 1964), Chap. 9.
[4]Reference 3, Chap. 13.

WHO DECIDES WHAT RESEARCH TO DO?

Research cannot be conducted according to the rules of efficiency. Research must be lavish of ideas, enthusiasm, and time. The best advice is don't quit easily, don't assume the worst, especially don't take any advice from any commercial person or financial expert and, finally, if you really don't know what to do, flip for it. The best persons to decide what research should be done are the ones doing the research. The next best is the head of the section. After that you leave the field of the best persons and meet increasingly worse groups. The first of these is the research director, who is probably wrong more than half the time. Then comes a research committee, which is wrong most of the time. Finally there is the head committee which is wrong all the time.

—Anonymous