

# Lab VII

## Playing the Quantum Prisoner Dillemma

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## Definitions: Quantum Prisoner's Dilemma

In the Quantum Prisoner's Dilemma, both players start with their qubits in the cooperate state  $|C\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ . The defect state is defined as  $|D\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ .

The operators representing the defect and quantum strategies are:

$$\hat{D} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \hat{Q} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

We implement (up to an unimportant phase) with one qubit gates:

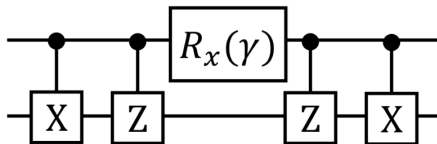
$$\hat{D} \equiv Y \quad \hat{Q} \equiv Z.$$

# Constructing the entangling gate $J$

The entangling operator  $\hat{J}$  generates entanglement between the qubits:

$$\hat{J}(\gamma) = \exp\left(-i\frac{\gamma}{2}\hat{D} \otimes \hat{D}\right)$$

where  $\gamma \in [0, \pi/2]$  controls the degree of entanglement.



$$\hat{J} = \exp(-i \gamma \hat{D} \otimes \hat{D} / 2)$$

## Task 1 (5 pts.): Quantum Prisoner's Dilemma pay-offs for Alice

The full game circuit applies  $\hat{J}$ , then the players' chosen strategies  $\hat{U}_A \otimes \hat{U}_B$ , then the disentangling gate  $\hat{J}^\dagger$ .

Finally, the state is measured in the  $\{|CC\rangle, |CD\rangle, |DC\rangle, |DD\rangle\}$  basis.

The expected payoff for Player A is defined as:

$$\$_A = rP_{CC} + tP_{DC} + sP_{CD} + pP_{DD}$$

with parameters  $r = 3$ ,  $t = 5$ ,  $s = 0$ , and  $p = 1$ .

( $P_{XY}$  is the probability of outcome  $XY$ , calculated from measurements count).

Note the convention: "0" is  $C$ , "1" is  $D$ .

# Task 1 continued

- Plot the expected payoff for a range  $\gamma \in [0, \pi/2]$  (use array of 50 points).
- Produce a figure with two side-by-side panels showing the payoff curves for different strategy pairs ( $\hat{U}_A \times \hat{U}_B$ ):
  - **Left panel ( $\hat{D} \times \hat{D}$  and  $\hat{Q} \times \hat{D}$ ):** Quantum tricky Alice against a classical defecting Bob.
  - **Right panel ( $\hat{D} \times \hat{Q}$  and  $\hat{Q} \times \hat{Q}$ ):** Alice playing against a quantum aware Bob.
- Locate and examine the transition thresholds on the plots: mark vertical lines at  $\gamma_{th1} = \arcsin(\sqrt{1/5})$  and  $\gamma_{th2} = \arcsin(\sqrt{2/5})$ .

# Extra credit (1 pt) & presentation topic

## Hardware execution: Theory vs. Reality

Test the Quantum Prisoner's Dilemma on an actual quantum computer and investigate the effects of hardware noise.

### Tasks:

- Choose a specific strategy pair (e.g.,  $\hat{Q} \times \hat{Q}$  or  $\hat{Q} \times \hat{D}$ ) and select 4 to 5 distinct values of  $\gamma$ .
- Execute these circuits on a real IBM Quantum backend (via Qiskit Runtime).
- Plot the hardware-sampled payoffs alongside your ideal simulator results.
- **Discuss:** How do real-world gate errors and readout noise affect the expected payoffs? Does hardware noise destroy the quantum advantage or the Nash equilibrium?

# Extra topic (alternative): the "Miracle Move" (1 pt)

## The always-winning quantum strategy

Can Alice guarantee a win if Bob restricts himself to classical strategies? Yes! She can use the "Miracle Move", a purely quantum strategy that has no classical counterpart. **The Operators:**

- **Alice's Miracle Move ( $\hat{M}$ ):**

$$\hat{M} = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix}$$

- **Bob's classical strategy ( $\hat{U}_B(\theta)$ ):** A probabilistic mixture of Cooperation and Defection, parameterized by  $\theta \in [0, \pi]$ :

$$\hat{U}_B(\theta) = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$

**Task:** Set the game to maximal entanglement ( $\gamma = \pi/2$ ). Plot Alice's expected payoff  $\$A$  as a function of Bob's strategy parameter  $\theta$ . **Discuss:** What is Alice's minimum guaranteed payoff regardless of what Bob does? Does Bob have any viable classical counter-strategy?