Lasers

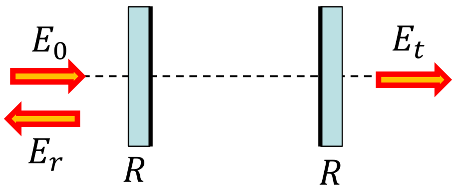
lecture 5

Czesław Radzewicz

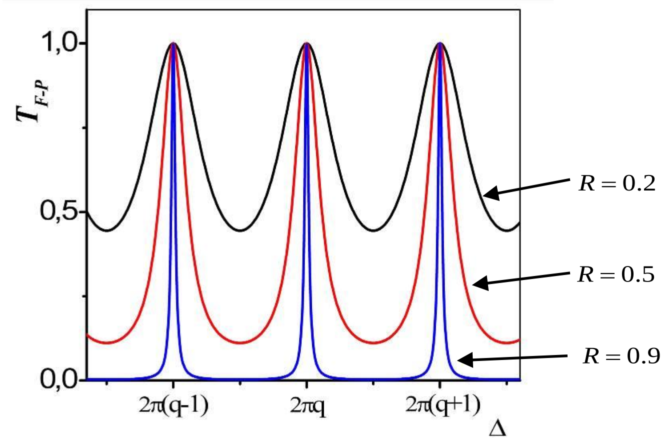
optical resonators, optical cavities

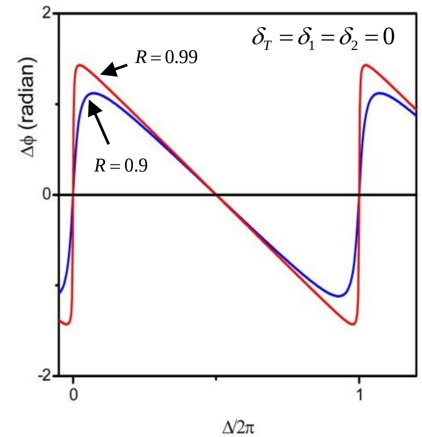
open resonators:

- Fabry-Perot

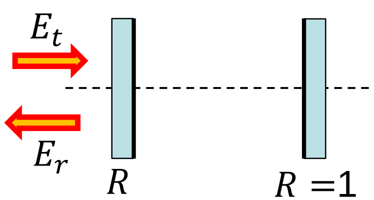


$$E_t = \sqrt{T} E_0 e^{i\Delta\phi}$$

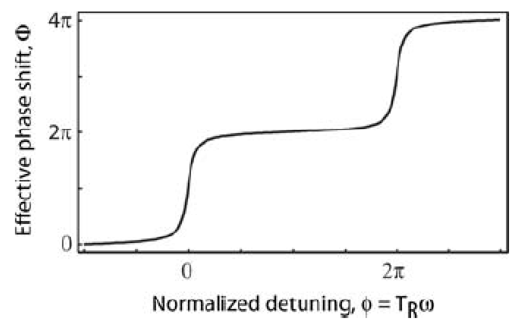




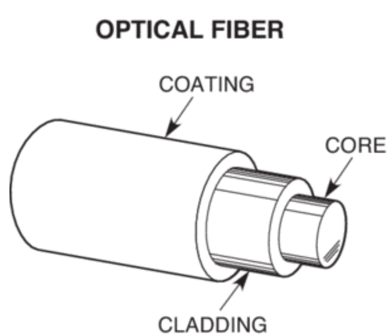
- Gires-Tournois



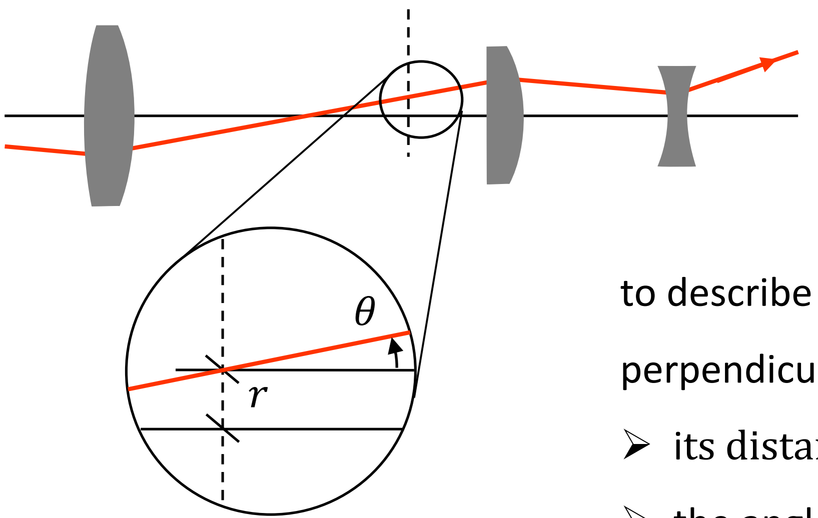
$$E_r = E_0 e^{i\phi}$$



fiber resonators:



a reminder – ABCD matrices in geometrical optics

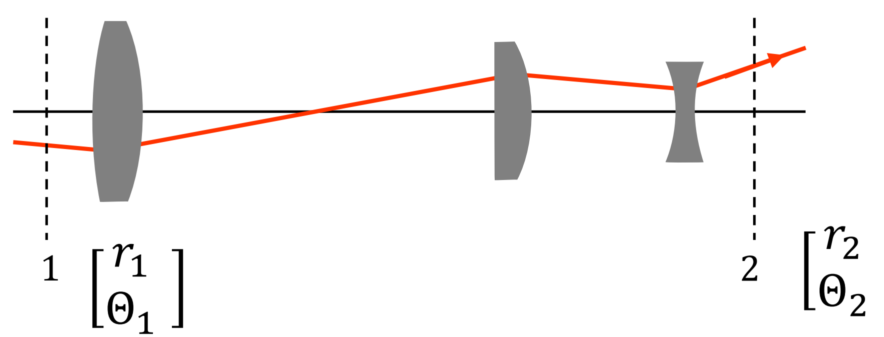


to describe an optical ray in our system in any given plane perpendicular to the axis of the system we need two parameters:

- its distance from the optic axis r (real value)
- the angle between the ray and axis Θ (real value)

for paraxial systems:

$$\begin{bmatrix} r_2 \\ \Theta_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_1 \\ \Theta_1 \end{bmatrix}$$

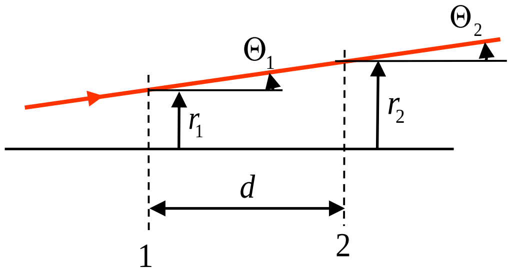


$$\det T \equiv AD - BC = {}^{n_1}/_{n_2}$$

n_1 - refractive index in plane 1

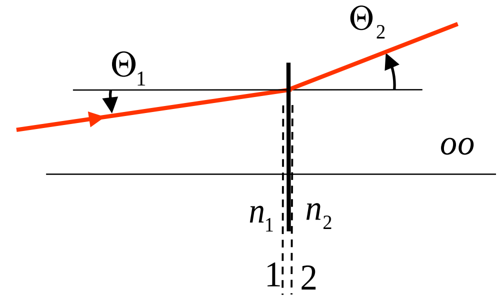
n_2 - Refractive index in plane 2

some ABCD matrices

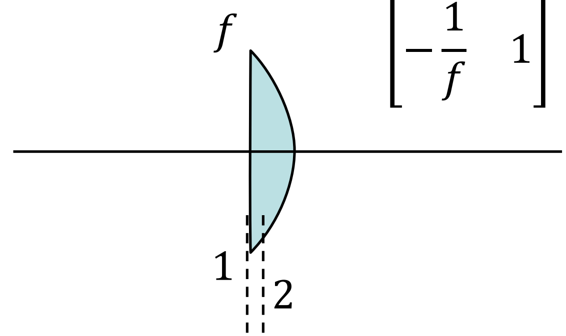


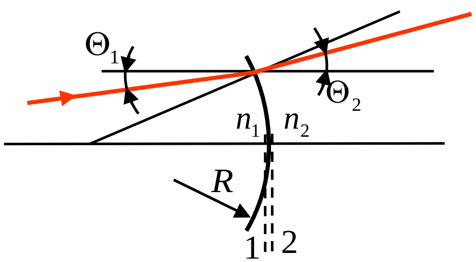
$$r_2 = r_1 + d\Theta_1 \qquad \Theta_2 = \Theta_1 \quad \Rightarrow \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & {n_1}/{n_2} \end{bmatrix}$$



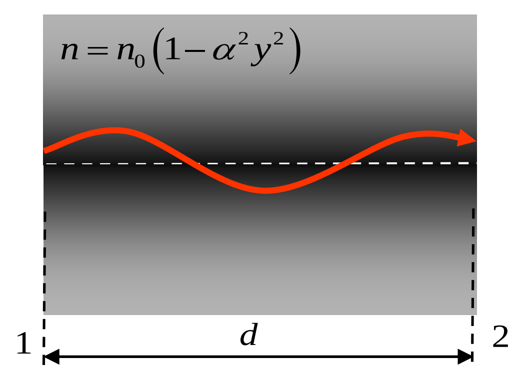
$$\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$



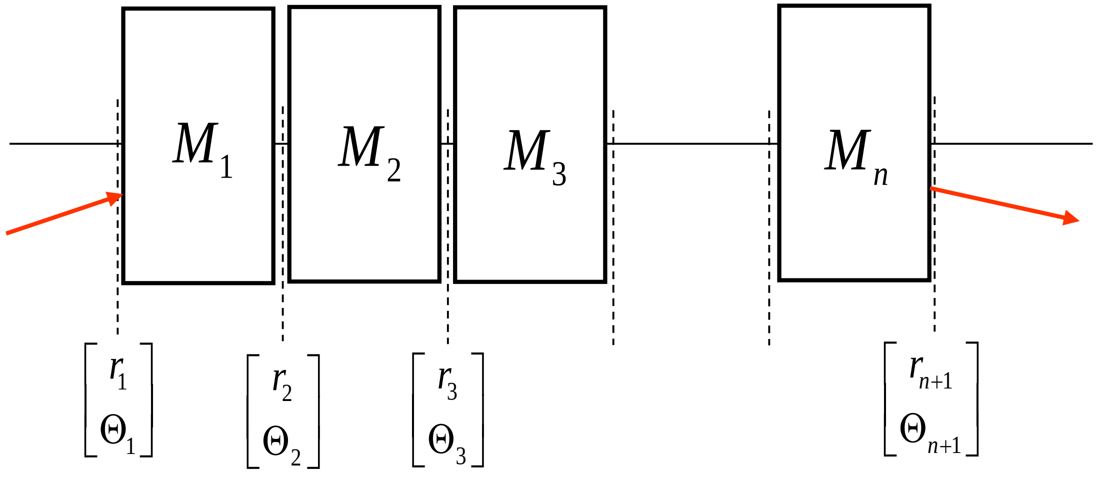


$$\begin{bmatrix} 1 & 0 \\ -\frac{n_1 - n_1}{n_2 R} & {n_1}/{n_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & {n_1}/{n_2} \end{bmatrix}$$

$$\begin{bmatrix} \cos(\alpha d) & \frac{1}{\alpha}\sin(\alpha d) \\ -\alpha\sin(\alpha d) & \cos(\alpha d) \end{bmatrix}$$



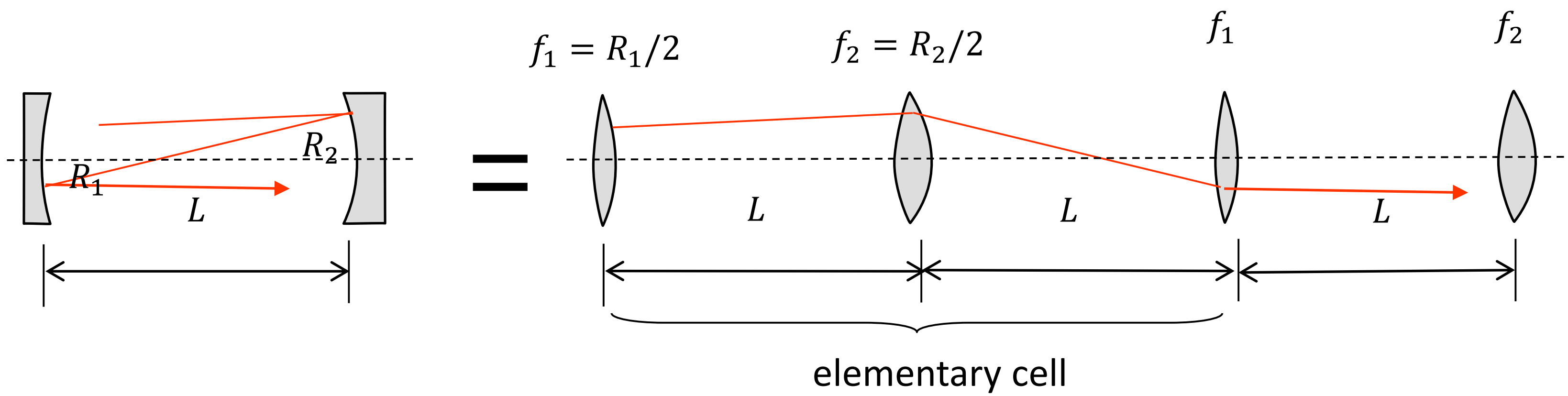
multiplication of ABCD matrices



$$\begin{bmatrix} r_{n+1} \\ \Theta_{n+1} \end{bmatrix} = M_n \cdot M_{n-1} \cdot \quad \cdot M_2 \cdot M_1 \begin{bmatrix} r_1 \\ \Theta_1 \end{bmatrix}$$

optical resonators in the geometrical optics approximation

an example: a two-mirror resonator



geometrical optics: the resonator is stable when the ray is trapped inside it.

for our example the ABCD matrix of the elementary cell is

$$M = \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f_1} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f_2} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - \dfrac{L}{f_2} & L + L\left(1 - \dfrac{L}{f_2}\right) \\ -\dfrac{1}{f_1} - \dfrac{1}{f_2}\left(1 - \dfrac{L}{f_1}\right) & \left(1 - \dfrac{L}{f_1}\right)\left(1 - \dfrac{L}{f_2}\right) - \dfrac{L}{f_1} \end{bmatrix}$$

after *n* round-trips: $\begin{bmatrix} r_{n+1} \\ \Theta_{n+1} \end{bmatrix} = M \begin{bmatrix} r_n \\ \Theta_n \end{bmatrix}$

$$\begin{aligned} r_{n+1} &= A r_n + B\Theta_n \\ \Theta_{n+1} &= C r_n + D\Theta_n \end{aligned} \quad \Rightarrow \quad r_{n+2} - (A + D) r_{n+1} + r_n = 0$$

optical resonators in the geometrical ..., 2

$$r_{n+2} - (A+D)r_{n+1} + r_n = 0$$

we search for oscillating solutions: $r_n = r_0 e^{in\Theta}$ $\Rightarrow$ $r_0 e^{in\Theta} \underbrace{\left[\left(e^{in\Theta}\right)^2 - 2\frac{A+D}{2}e^{in\Theta} + 1\right]}_{\text{quadratic equation} \quad x = e^{in\Theta}}$

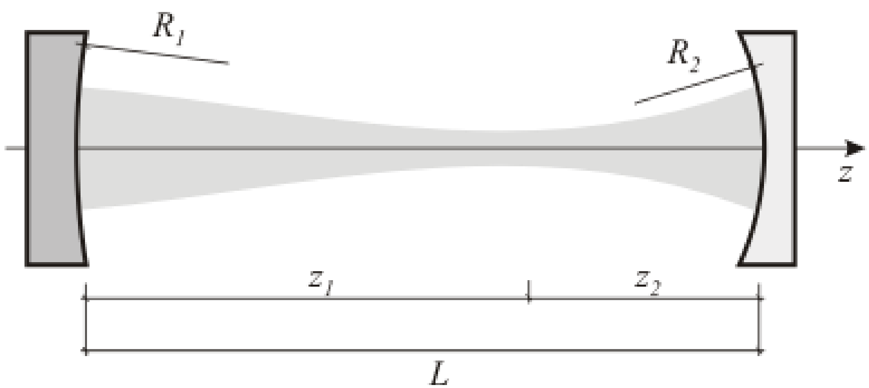
$$\Delta = 4\left[\left(\frac{A+D}{2}\right)^2 - 1\right]$$

x is complex only for $\Delta < 0 \Leftrightarrow \underbrace{-2 < A+D < 2}_{\text{stability condition}}$

solution:

$$x = e^{i\Theta} = \frac{A+D}{2} \pm i\sqrt{1 - \left(\frac{A+D}{2}\right)^2}$$

optical resonators in the geometrical ..., 3



back to the two-mirror F-P...

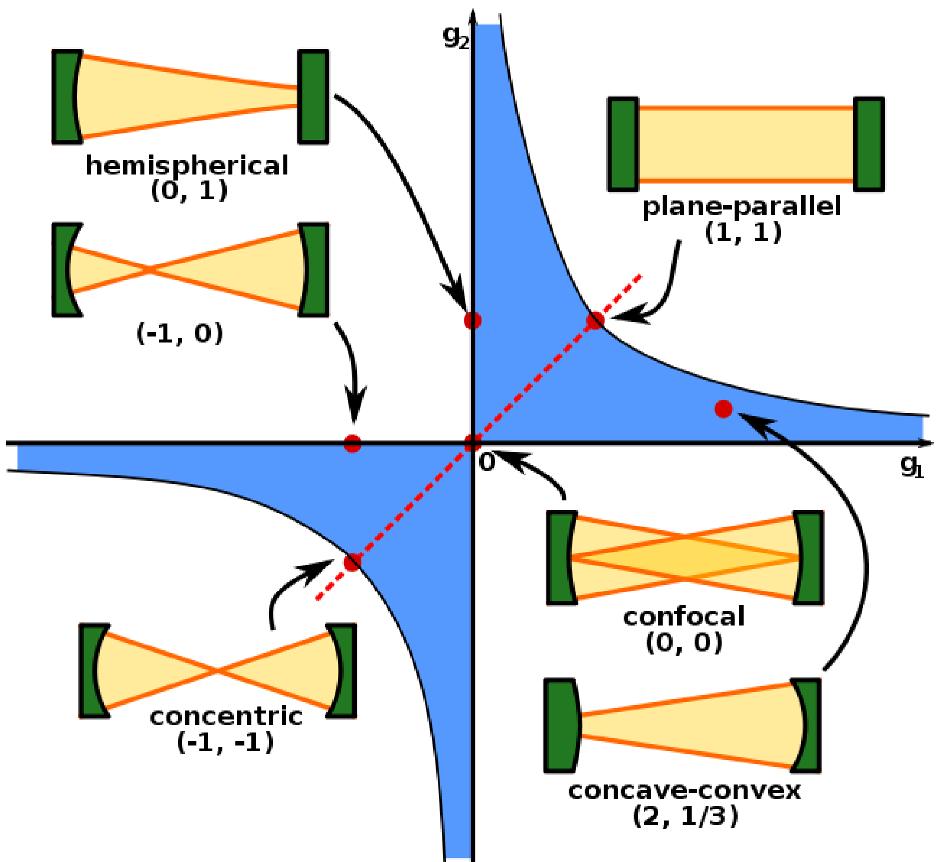
$$g_i \equiv 1 - {}^{L}/_{R_i}, \qquad i = 1,2$$

$$M = \begin{bmatrix} 1 - \frac{L}{f_2} & L\left(2 - \frac{L}{f_2}\right) \\ -\frac{1}{f_1} - \frac{1}{f_2} + \frac{L}{f_1 f_2} & 1 - \frac{2L}{f_1} - \frac{L}{f_2} + \frac{L^2}{f_1 f_2} \end{bmatrix}$$

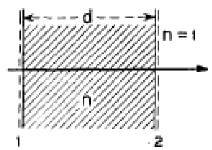
$$-2 \leq A + D \leq 2$$

...

$$\boxed{0 \leq \left(1 - \frac{L}{R_1}\right)\left(1 - \frac{L}{R_2}\right) \leq 1}$$



optical resonators Bible

6		$\begin{vmatrix} 1 & d/n \\ 0 & 1 \end{vmatrix}$

Laser Beams and Resonators

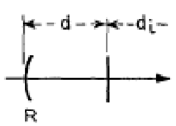
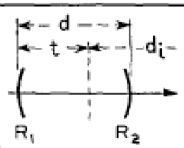
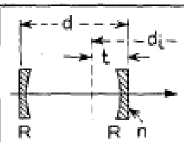
H. KOGELNIK AND T. LI

Abstract—This paper is a review of the theory of laser beams and resonators. It is meant to be tutorial in nature and useful in scope. No attempt is made to be exhaustive in the treatment. Rather, emphasis is placed on formulations and derivations which lead to basic understanding and on results which bear practical significance.

TABLE II

FORMULAS FOR THE CONFOCAL PARAMETER AND THE LOCATION OF BEAM WAIST FOR VARIOUS OPTICAL STRUCTURES

1550 APPLIED OPTICS / Vol. 5, No. 10 / October 1966

NO	OPTICAL SYSTEM	$\frac{1}{2}b = \pi w_0^2/\lambda$	t
1		$\sqrt{d(R-d)}$	–
2		$\frac{1}{2}\sqrt{d(2R-d)}$	$\frac{1}{2}d$
3		$\frac{\sqrt{d(R_1-d)(R_2-d)(R_1+R_2-d)}}{R_1+R_2-2d}$	$\frac{d(R_2-d)}{R_1+R_2-2d}$
4		$\frac{R\sqrt{d(2R-d)}}{2R+d(n^2-1)}$	$\frac{ndR}{2R+d(n^2-1)}$

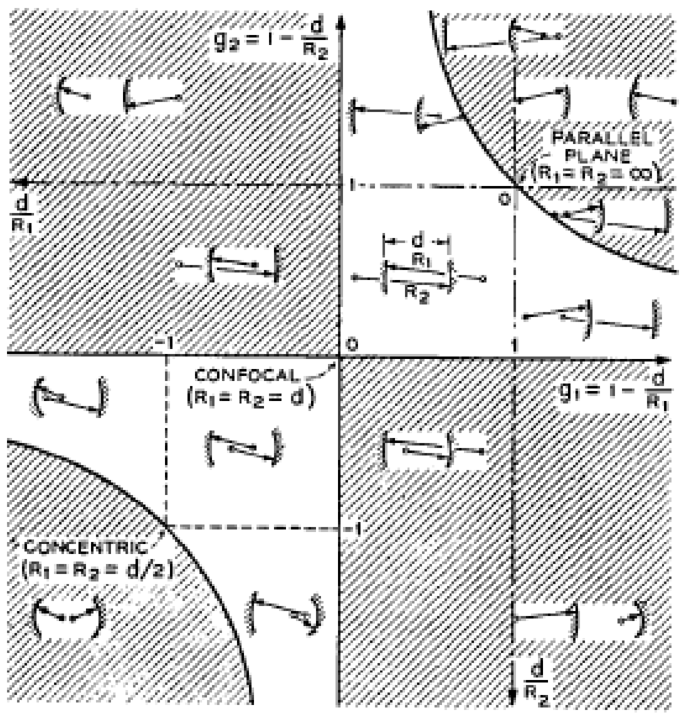
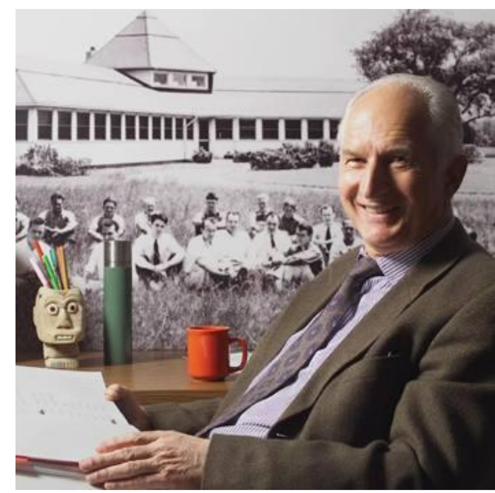


Fig. 4. Stability diagram. Unstable resonator systems lie in shaded regions.



Herwig Kogelnik

Gaussian beam

paraxial approximation: $E(x,y,z) = \psi(x,y,z)e^{-ikz}$

Helmholtz equation: $\Delta_r\psi - 2ik\psi = 0$ $\qquad \Delta_r = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$

Trial solution: $\psi = \psi_0 e^{-i\left[P(z)+\frac{kr^2}{2q(z)}\right]}$

algebra …

$$q(z) = iz_0 + z = z_0(i+\zeta) \qquad \zeta \equiv {}^{z}/_{z_0}$$

$$e^{-iP(z)} = \frac{1}{\sqrt{1+\zeta^2}}e^{i\tan^{-1}\zeta}$$

the final result:

$$E(x,y,z) = \underbrace{\frac{\psi_0}{\sqrt{1+\zeta^2}}}_{\text{amplitude}} e^{i[-kz+\tan^{-1}\zeta]} e^{-i\underbrace{\frac{k(x^2+y^2)}{2q(z)}}}$$

plane wave phase

Guoy phase

phase front
+
Intensity distribution

the beam is defined by two real parameters: z_0 and λ

Gaussian beam, 2

$$E(x,y,z) = \frac{\psi_0}{\sqrt{1+\zeta^2}} e^{i[-kz+\tan^{-1}\zeta]} e^{-i\frac{kr^2}{2q(z)}}$$

the physical interpretation of the q parameter:

$$-i\frac{kr^2}{2q(z)} = -i\frac{kr^2}{2(iz_0+z)} = -\frac{kz_0r^2}{2(z^2+{z_0}^2)} - i\frac{kzr^2}{2(z^2+{z_0}^2)} = -\frac{r^2}{w^2(z)} - i\frac{kr^2}{2R(z)}$$

$$w^2(z) = \frac{\lambda_0 z_0}{n\pi}\left[1+\left({}^{z}/_{z_0}\right)^2\right]$$

$$R(z) = z\left[1+\left({}^{z_0}/_{z}\right)^2\right]$$

$$e^{-i\frac{kr^2}{2q(z)}} = e^{-z\frac{r^2}{w^2(z)}} + e^{-i\frac{kr^2}{2R(z)}}$$

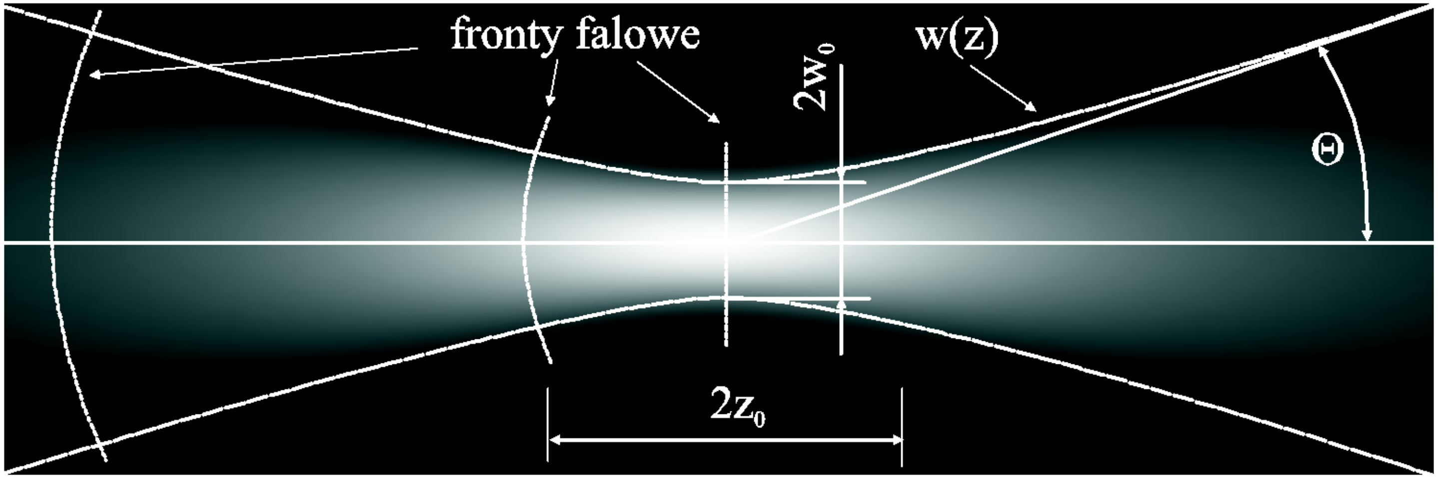
$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda_0}{\pi w^2(z)}$$

phase front radius of curvature

beam radius

Gaussian beam, 3

$$E(x,y,z) = \underbrace{\psi_0 \frac{w_0}{w(z)} \cdot e^{-\frac{r^2}{w^2(z)}}}_{\text{Gaussian intensity distribution}} \cdot \underbrace{e^{-i\frac{r^2}{2R(z)}}}_{\text{spherical phase fronts}} \cdot \underbrace{e^{i[-kz+\tan^{-1}\zeta]}}_{\text{phase on the axis}}$$



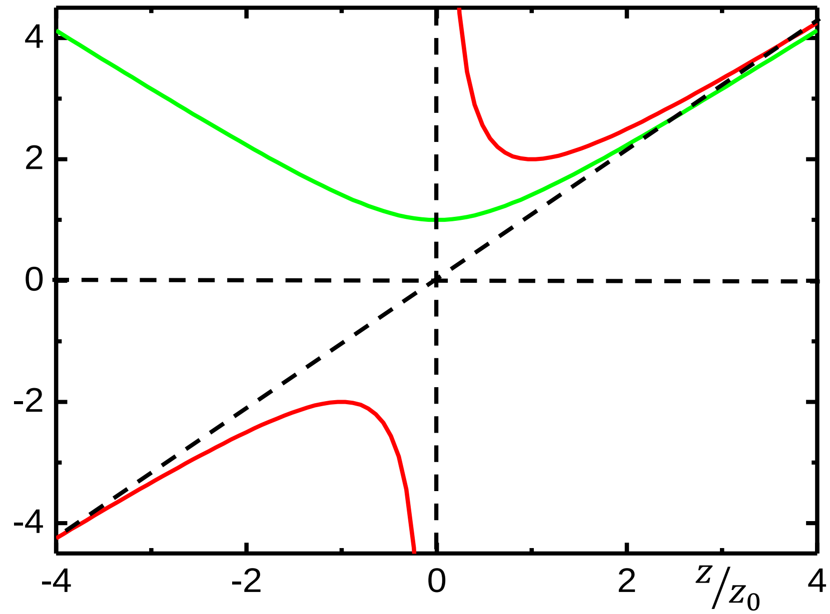
$$I(x,y,z) = \psi_0{}^2 \left[\frac{w_0}{w(z)}\right]^2 e^{-\frac{2r^2}{w^2(z)}}$$

$$w^2(z) = \frac{\lambda_0 z_0}{n\pi}[1+(z/z_0)^2] = w_0{}^2[1+(z/z_0)^2], \qquad w_0{}^2 = w(0) = \frac{\lambda_0 z_0}{n\pi} = \frac{\lambda z_0}{\pi}$$

$$\lim_{z\to\infty} w(z) = w_0 \cdot z/z_0 = \Theta z \qquad \Theta = w_0/z_0 = \frac{\lambda_0}{n\pi w_0}$$

Rayleigh range: $2z_0$ $\qquad w(z_0) = \sqrt{2} z_0$

Gaussian beam, 4



$$\frac{w(z)}{w_0} = \sqrt{1 + \left(^{z}/_{z_0}\right)^2}$$

$$\frac{R(z)}{z_0} = \frac{z}{z_0}\left[1 + \left(^{z_0}/_{z}\right)^2\right]$$

Gauss-Hermite beams

In Cartesian coordinate system:

$$E^{GH}{}_{m,n}(x,y,z) = H_m\left[\frac{\sqrt{2}x}{w(z)}\right] \cdot H_n\left[\frac{\sqrt{2}y}{w(z)}\right] \cdot \psi_0 \, \frac{w_0}{w(z)} \cdot e^{-\frac{r^2}{w^2(z)}} \cdot e^{-i\frac{r^2}{2R(z)}} \cdot e^{i[-kz+(1+m+n)\tan^{-1}\zeta]}$$

which are called TEM_{mn} beams.

Hermite polynomial:

$$H_n(x) \equiv (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

and some low order Hermite polynomials:

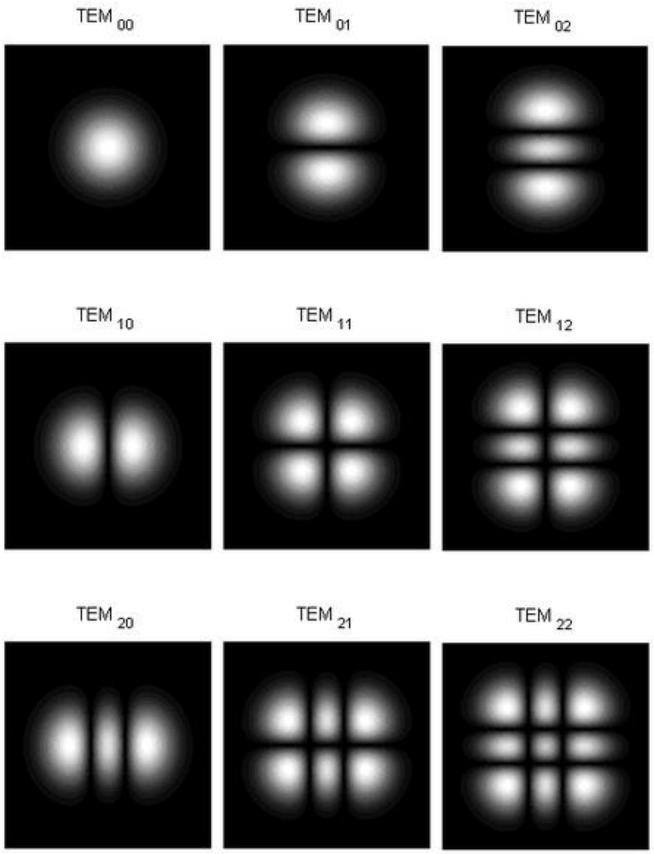
$H_0(x) = 1$

$H_1(x) = x$

$H_2(x) = x^2 - 1$

$H_3(x) = x^3 - 3x$

.....



TEM_{mn} beams are ortho-normal and form a complete basis for paraxial beam

Gauss-Laguerre beams

Cylindrical symmetry; r, ϕ, z coordiantes

$$E(r,\phi,z) = \psi_0 \frac{e^{-in\phi}}{w(z)}\left(\frac{r}{w(z)}\right)^n L_m{}^n\left(\frac{2r^2}{w^2(z)}\right)\cdot e^{-\frac{r^2}{w^2(z)}}\cdot e^{-i\frac{r^2}{2R(z)}}\cdot e^{i\left[-kz+(1+\mathrm{m}+\mathrm{n})\tan^{-1}\left(z/z_0\right)\right]}$$

$L_m{}^n$- Laguerre polynomial

Laguerre polynomial:

$$L_m{}^n(x) = \frac{x^{-n}e^x}{m!}\frac{d^m}{dx^m}\left(e^{-x}x^{n+m}\right)$$

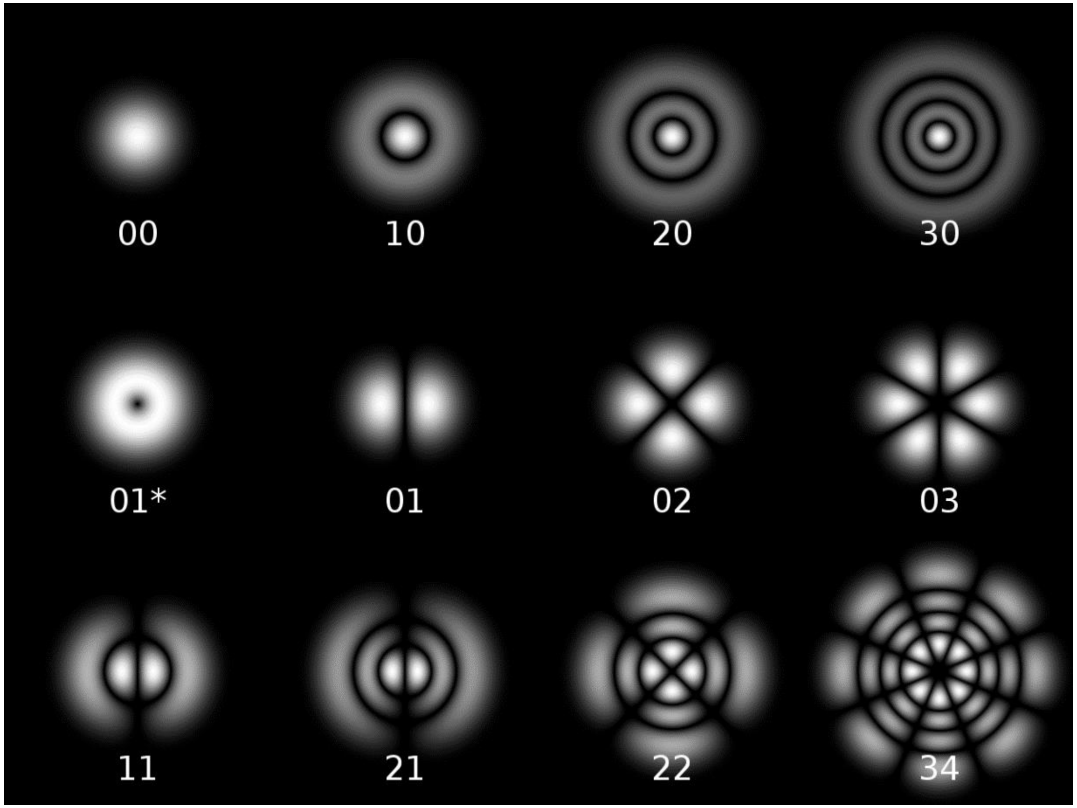
and a few of them:

$$L_0{}^l(x) = 1$$

$$L_1{}^l(x) = l + 1 - x$$

$$L_2{}^l = \frac{1}{2}(l+1)(l+2) - (l+2)x + \frac{1}{2}x^2$$

....



Gauss-Laguerre beams are ortho-normal and form a complete basis for paraxial beam

non-zero orbital momentum

Gauss-Ince beams

144 OPTICS LETTERS / Vol. 29, No. 2 / January 15, 2004

Ince–Gaussian beams

Miguel A. Bandres and Julio C. Gutiérrez-Vega

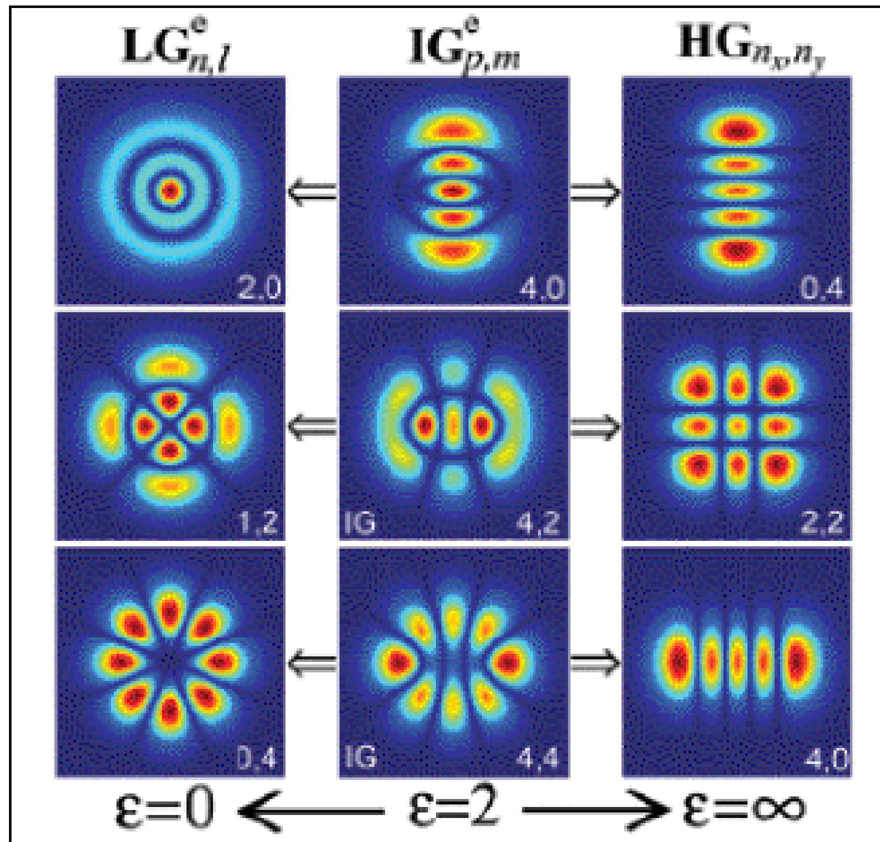
Photonics and Mathematical Optics Group, Tecnológico de Monterrey, Monterrey, N.L., 64849, Mexico

Received July 7, 2003

We demonstrate the existence of the Ince–Gaussian beams that constitute the third complete family of exact and orthogonal solutions of the paraxial wave equation. Their transverse structure is described by the Ince polynomials and has an inherent elliptical symmetry. Ince–Gaussian beams constitute the exact and continuous transition modes between Laguerre and Hermite–Gaussian beams. The propagating characteristics are discussed as well.

OCIS codes: 260.1960, 350.5500, 140.3300, 050.1960, 140.3410.

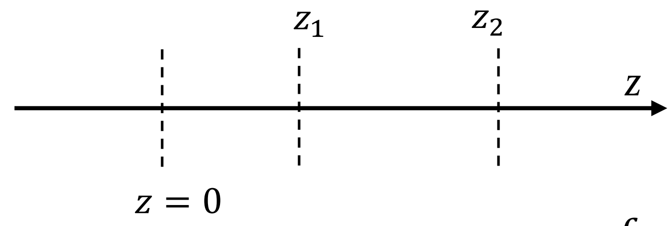
$$\mathrm{IG}^{e}_{p,m}(\mathbf{r}) = \frac{Dw_0}{w(z)} C_p^m(i\xi,\epsilon) C_p^m(\eta,\epsilon) \exp\left[\frac{-r^2}{w^2(z)}\right] \times \exp\left[ikz + i\,\frac{kr^2}{2R(z)} - i(p+1)\arctan\left(\frac{z}{z_R}\right)\right], \qquad (5)$$

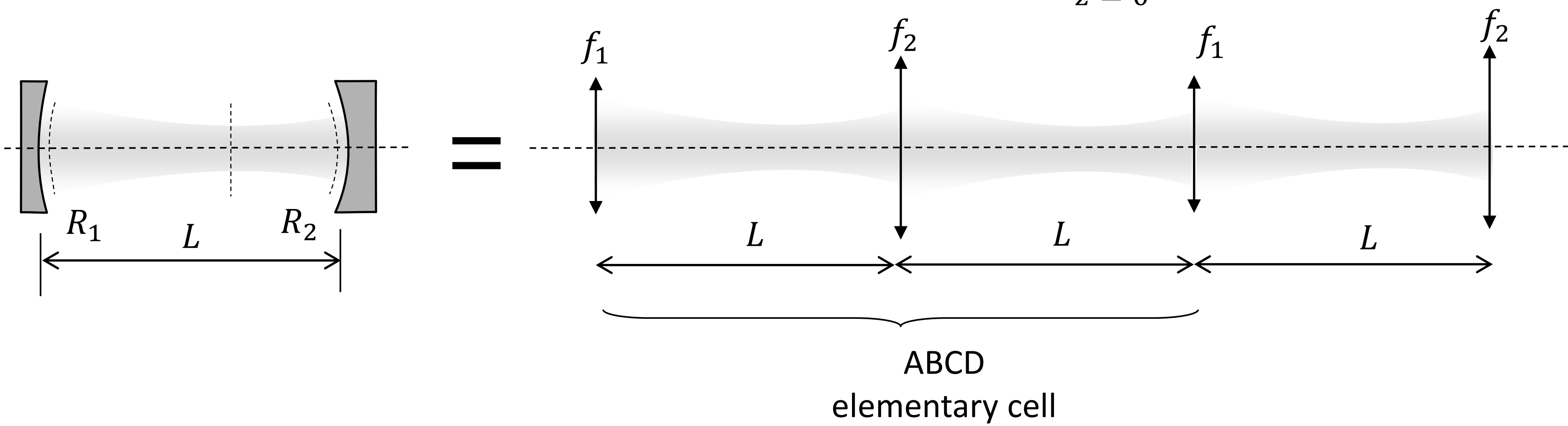


Gauss-* beams in optical resonators

Gaussian beam propagation

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}$$





The beam can be a mode of the resonator if:

$$q = \frac{Aq + B}{Cq + D} \Rightarrow B\left(\frac{1}{q}\right)^2 + (A - D)\frac{1}{q} - C = 0$$

....

$$\frac{1}{q} = -\frac{A - D}{2B} - i\frac{\sqrt{1 - \left(\frac{A + D}{2}\right)^2}}{B} = \frac{1}{R} - i\frac{\lambda}{\pi w^2}$$

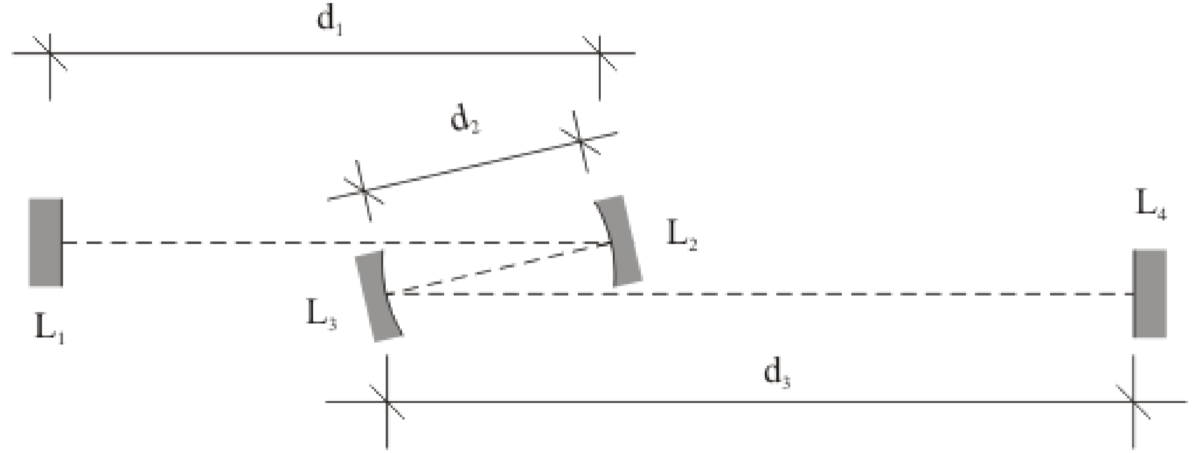
at the beginning of the elementary cell

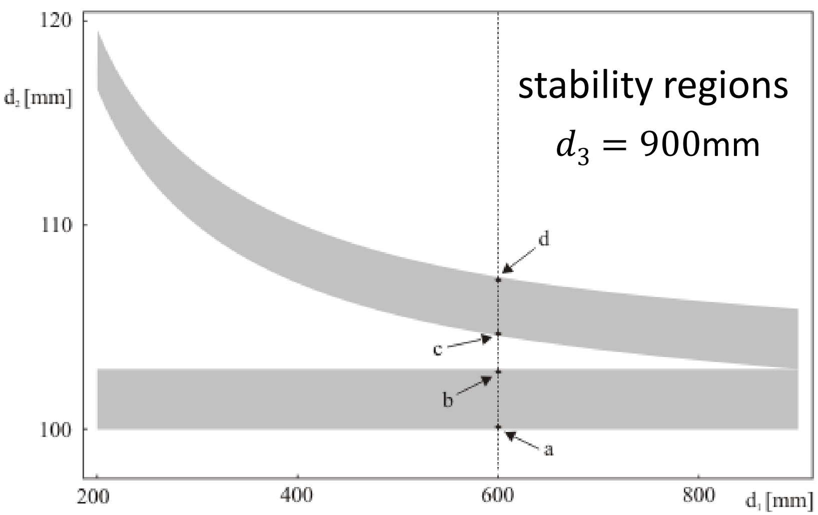
the procedure for transverse mode calculation:

- select the elementary cell
- calculate ABCD matrix of the elementary cell
- check for stability: $-2 < A + D < 2$
- calculate ${}^1/_q$ at the beginning of elem. cell
- propagate ${}^1/_q$ to obtain beam parameters at any plane inside the resonator

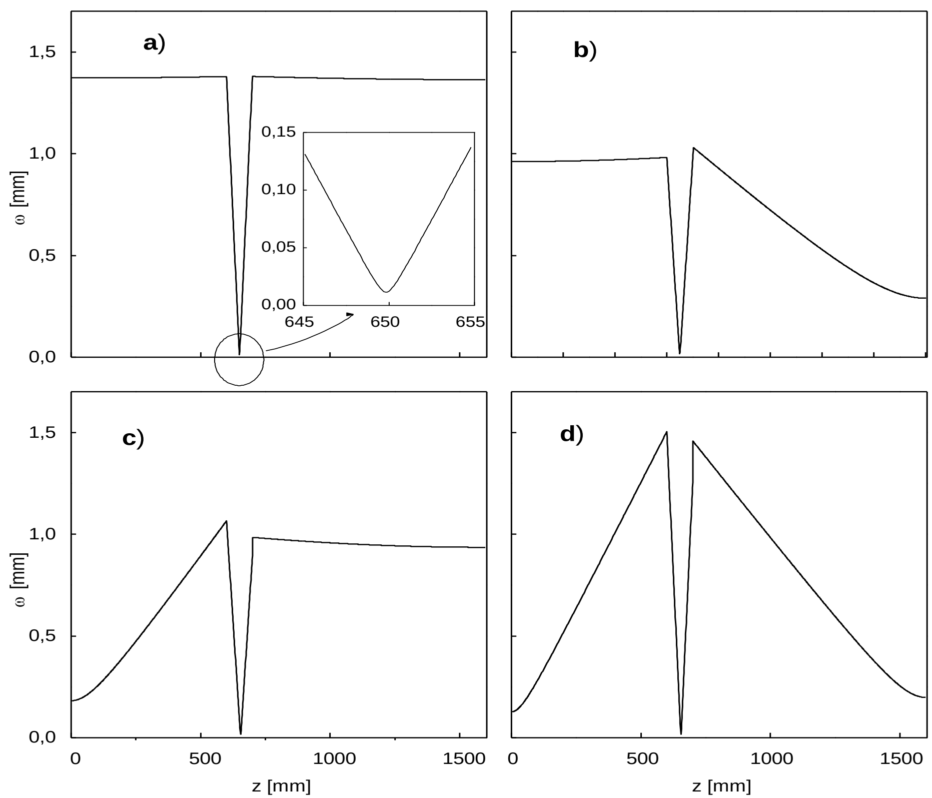
Transverse mode

Z-resonator

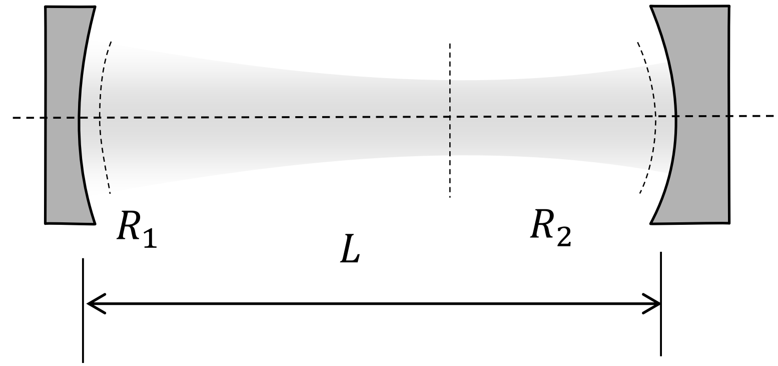




beam radius inside the resonator



mode frequencies for an open resonator



an example: TEM_{mn} modes

$$E^{GH}{}_{m,n}(x,y,z) = H_m\left[\frac{\sqrt{2}x}{w(z)}\right]\cdot H_n\left[\frac{\sqrt{2}y}{w(z)}\right]\cdot \psi_0\,\frac{w_0}{w(z)}\cdot e^{-\frac{r^2}{w^2(z)}}\cdot e^{-i\frac{r^2}{2R(z)}}\cdot e^{i[-kz+(1+\mathrm{m}+\mathrm{n})\tan^{-1}\zeta]}$$

for the beam to be a mode of the resonator we need the round-trip phase to be a multiple of 2π:

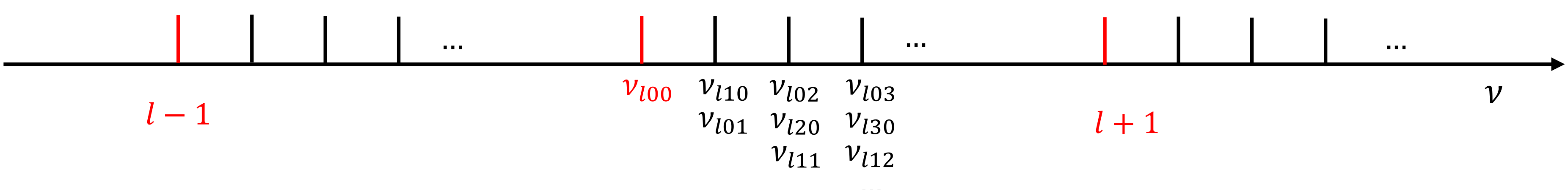
$kL-(1+\mathrm{m}+\mathrm{n})\tan^{-1}({}^{L}/{}_{z_0}) = l\pi,$ $\quad$ l – natural number indexing longitudinal modes

$$\nu_{lmn} = \frac{c}{2L}\left[l+\frac{1}{\pi}(1+\mathrm{m}+\mathrm{n})\tan^{-1}\left({}^{L}/{}_{z_0}\right)\right]$$

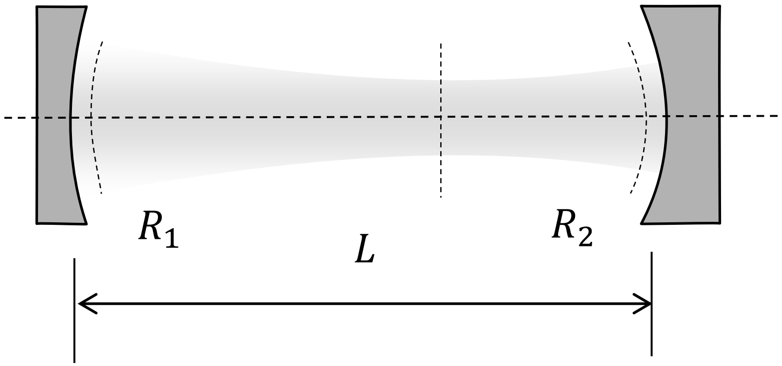
the procedure for any stable resonator:

- select the elementary cell
- calculate ABCD matrix of the elementary cell
- check for stability c$-2 < A+D < 2$
- calculate z_0
- from the equation on the left calculate frequencies

typical structure: tranverse and longitudinal modes



mode frequencies for an open resonator, 2



an example: two-mirror F-P resonator

$$\nu_{lmn} = \frac{c}{2L}\left[l + \frac{1}{\pi}(1 + \mathrm{m} + \mathrm{n})\tan^{-1}\left({}^{L}\!/_{z_0}\right)\right] \quad \Rightarrow$$

$$\nu_{lmn} = \frac{c}{2L}\left[l + \frac{1}{\pi}(1 + \mathrm{m} + \mathrm{n})\cos^{-1}\sqrt{\left(1 - {}^{L}\!/_{R_1}\right)\left(1 - {}^{L}\!/_{R_2}\right)}\right]$$

Specific cases

- plane-parrallel Fabry-Perot $\quad R_1 = R_2 = \infty, \quad \nu_{lmn} = l\frac{c}{2L}$
- confocal symmetric $\quad R_1 = R_2 = L \quad \nu_{lmn} = l\frac{c}{4L}$
- spherical symmetric $\quad R_1 = R_2 = L/2 \quad \nu_{lmn} = l\frac{c}{2L}$

open resonators with diffraction losses

Laser Beams and Resonators

H. KOGELNIK AND T. LI

1550 APPLIED OPTICS / Vol. 5, No. 10 / October 1966

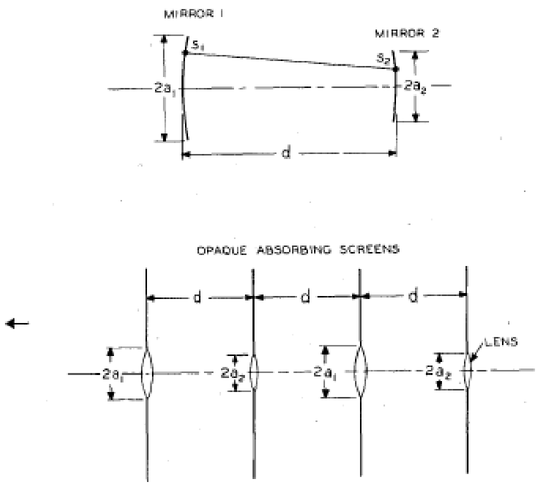


Fig. 16. Geometry of a spherical-mirror resonator with finite mirror apertures and the equivalent sequence of lenses set in opaque absorbing screens.

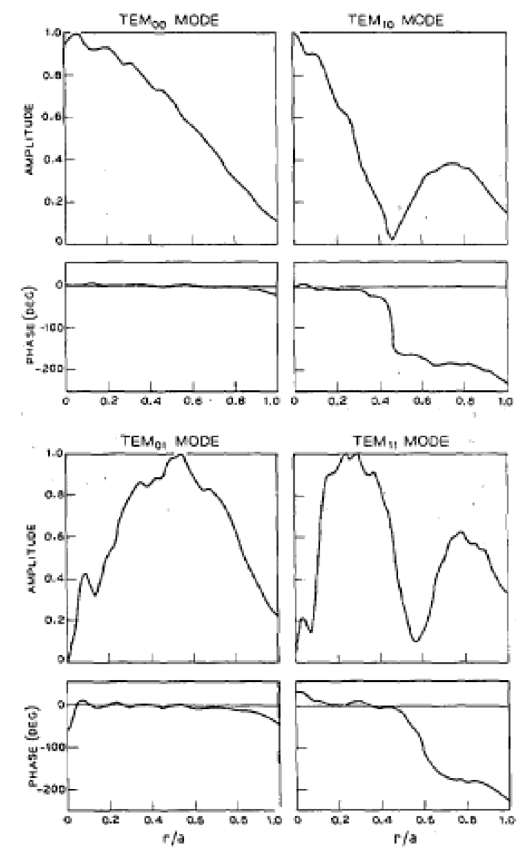


Fig. 21. Relative field distributions of four of the low order modes of a Fabry-Perot resonator with (parallel-plane) circular mirrors ($N = 10$).

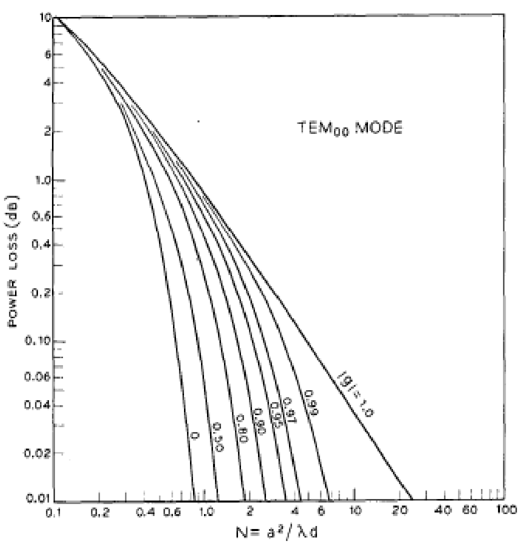


Fig. 22. Diffraction loss per transit (in decibels) for the TEM$_{00}$ mode of a stable resonator with circular mirrors.

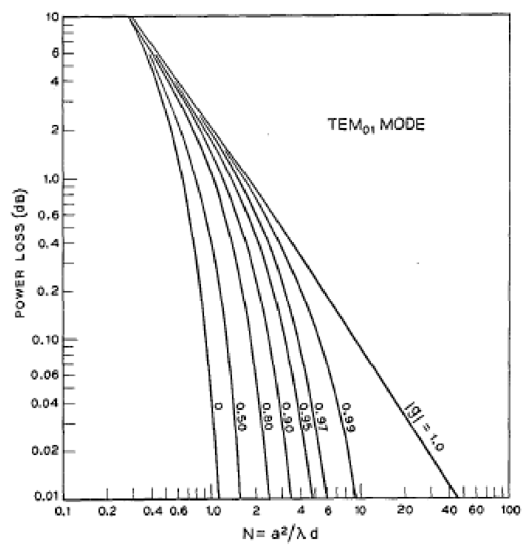
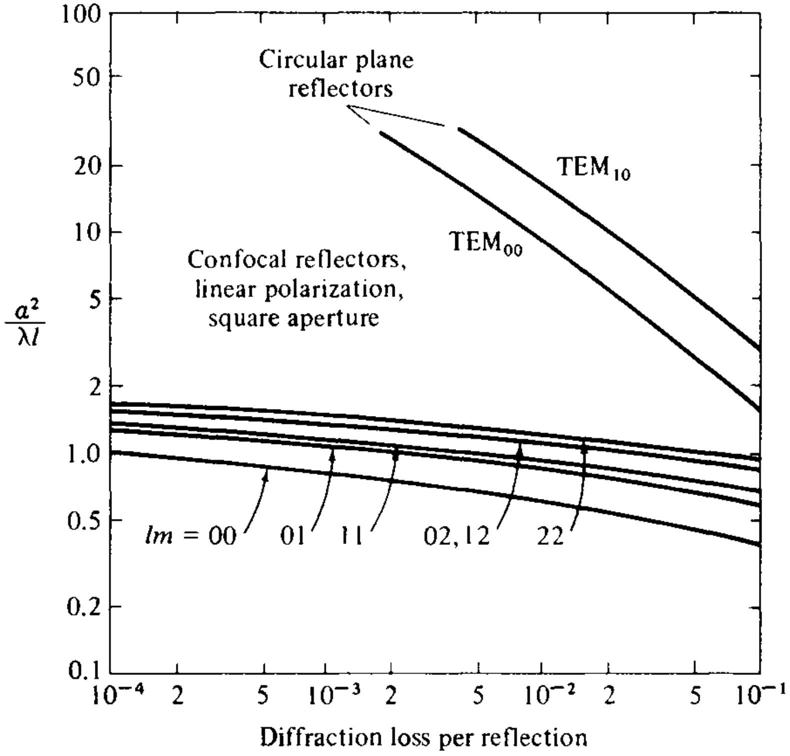


Fig. 23. Diffraction loss per transit (in decibels) for the TEM$_{01}$ mode of a stable resonator with circular mirrors.

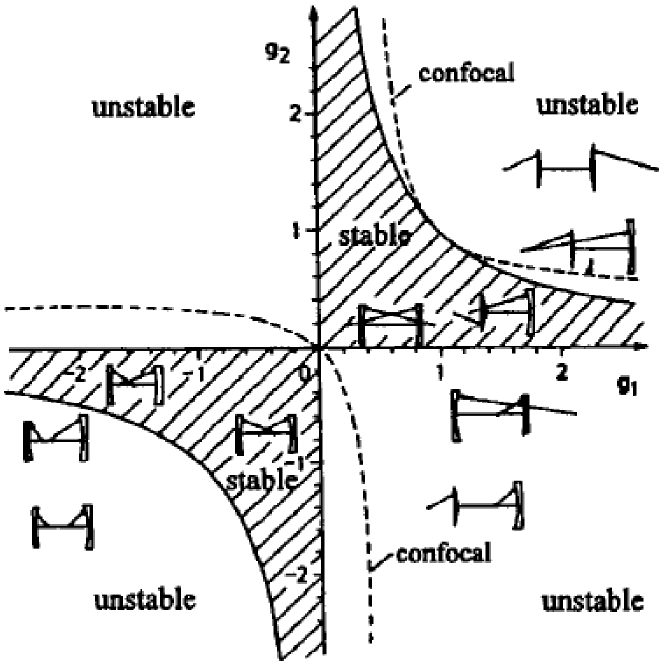
TEM$_{00}$ mode selection

open resonators with diffraction losses, 2



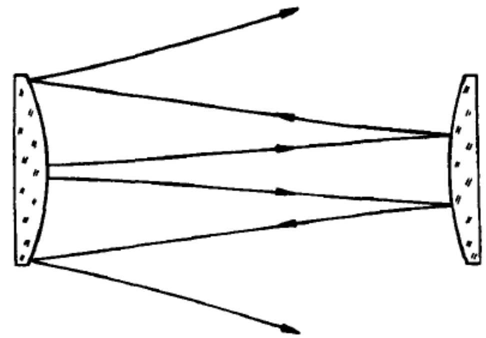
selekcja modu TEM$_{00}$

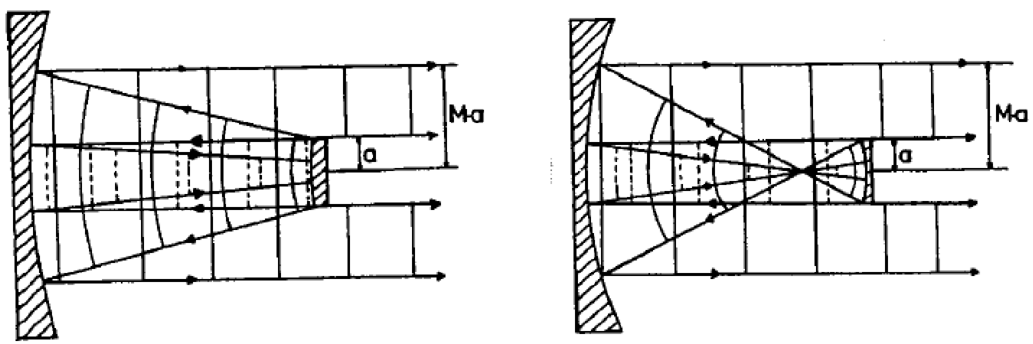
astable resonators



$$G = 2g_1g_1 - 1$$

stable resonators	:	*0 < |G| < 1*
resonators on the stability limits	:	*|G| = 1*
unstable resonators	:	*|G| > 1*





a) b)

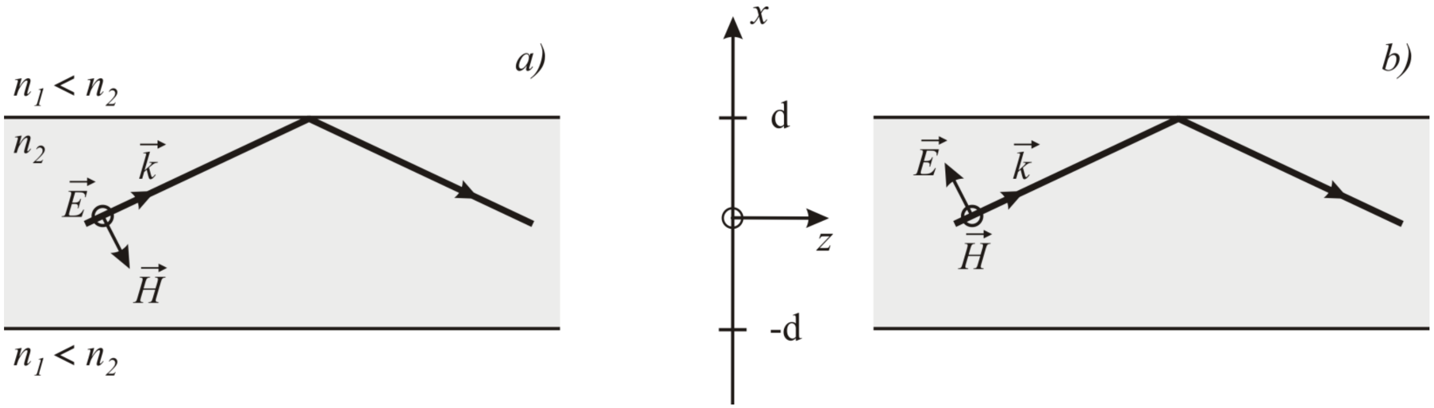
Fig. 7.7 Beam propagation in confocal unstable resonators with magnification |M|=2. a) positive branch, b) negative branch.

optical wave-guides

different geometries:

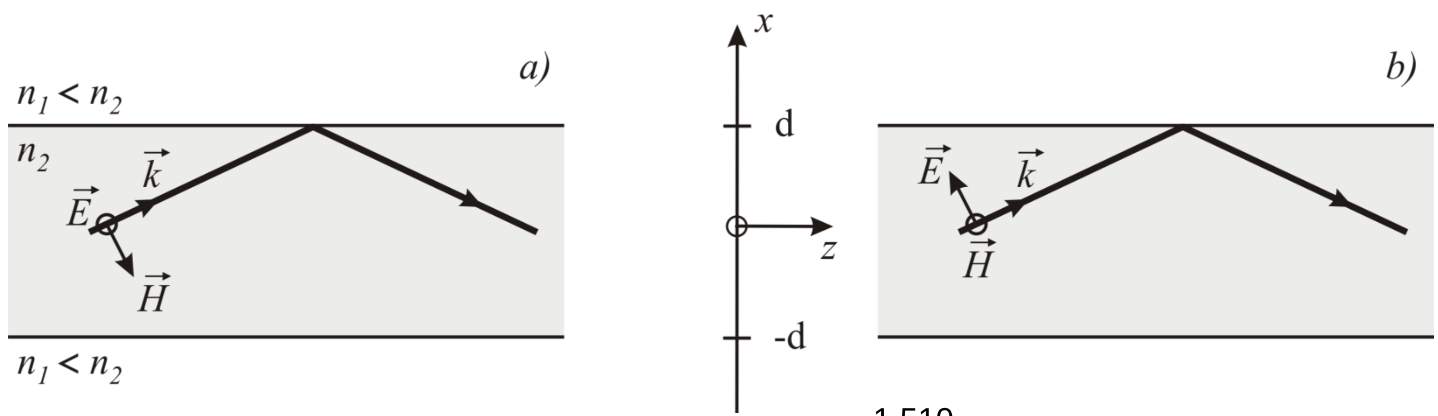
- ✓ flat, one-dimensional – a sheet
- ✓ flat 2D – a rectangle
- ✓ round standard (telecommunications)
- ✓ photonic

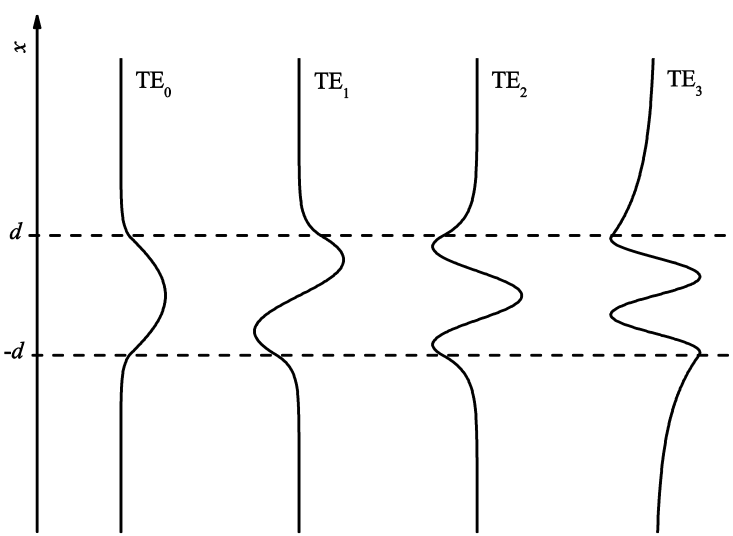
The light propagates along z. The field is given by $E(x,y,z) = A_n(x,y)e^{i\beta_n z}$ with n refractive index (a set of indices). Always, discrete solutions polarization-dependent
Rough classification: single-mode vs multimode

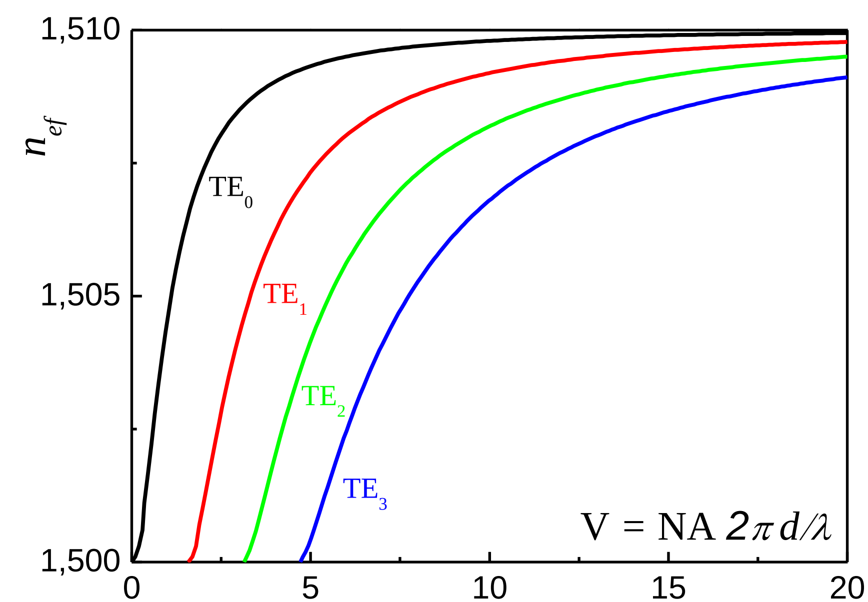


optical wave-guides, 2

example 1: flat symmetrical waveguide (1D), two families of solutions: TE and TM. An important parameter numerical aperture of the waveguide; $NA = \sqrt{{n_2}^1 - {n_1}^2}$





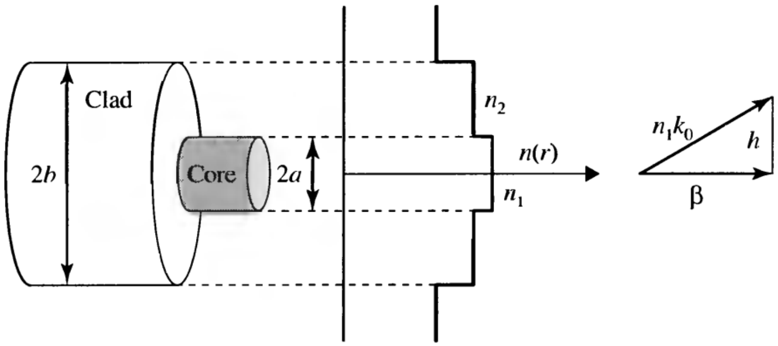


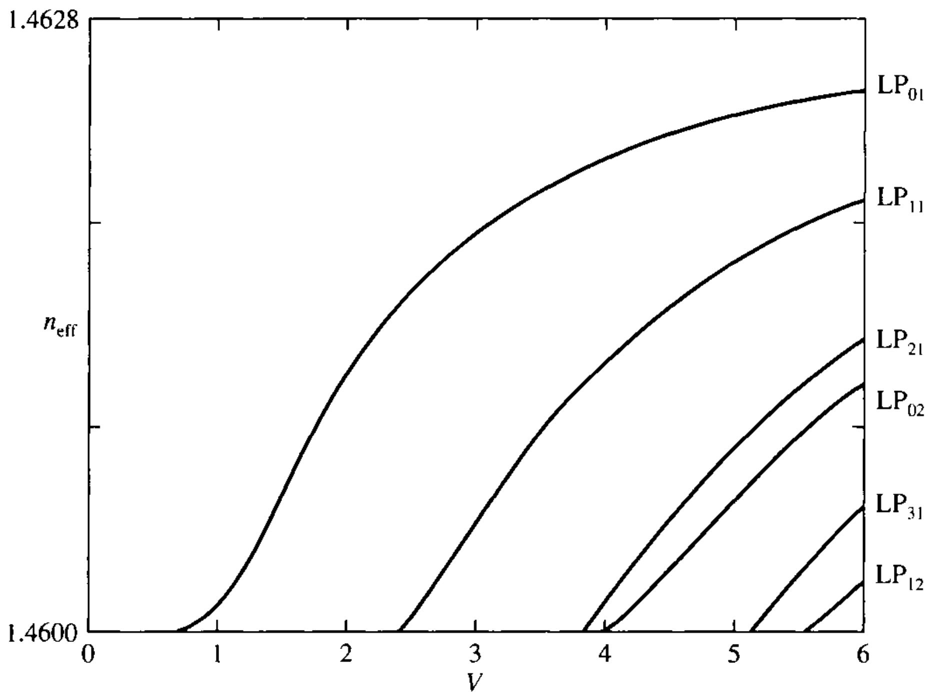
$$\beta = \frac{2\pi n_{eff}}{\lambda}$$

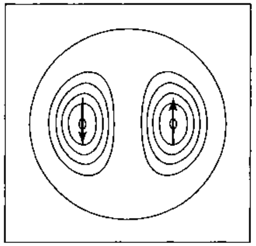
optical wave-guides, 2

example 2: cylindrical optical fiber
with a step index

$$V = \frac{2\pi a}{\lambda} NA$$

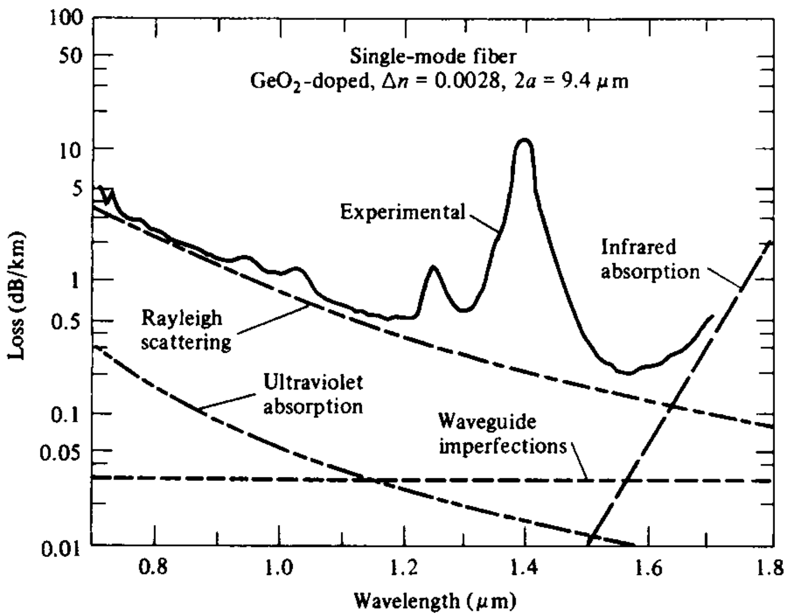






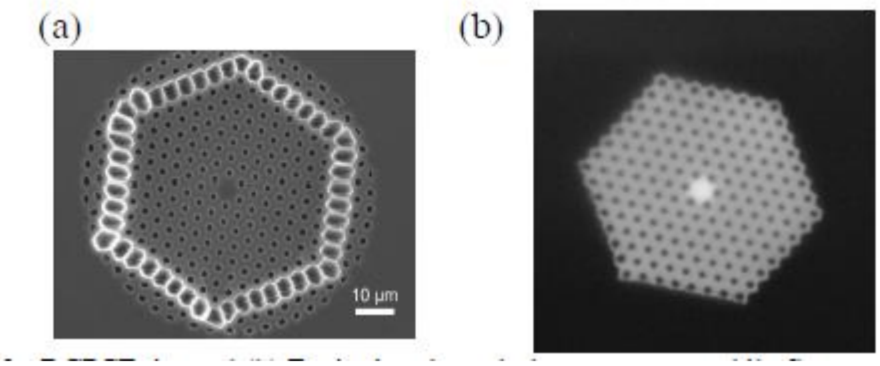
LP_{11}

$$\beta = \frac{2\pi n_{eff}}{\lambda}$$

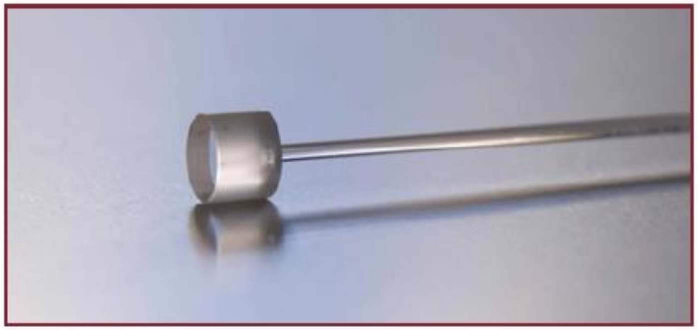


photonic fibers

Example 3: a fiber with double clad and doped core



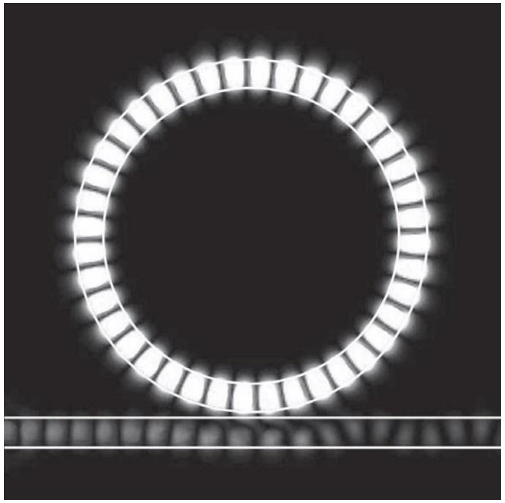
Schematic of the fiber geometry showing the bow-tie configured stress elements and the step index core.

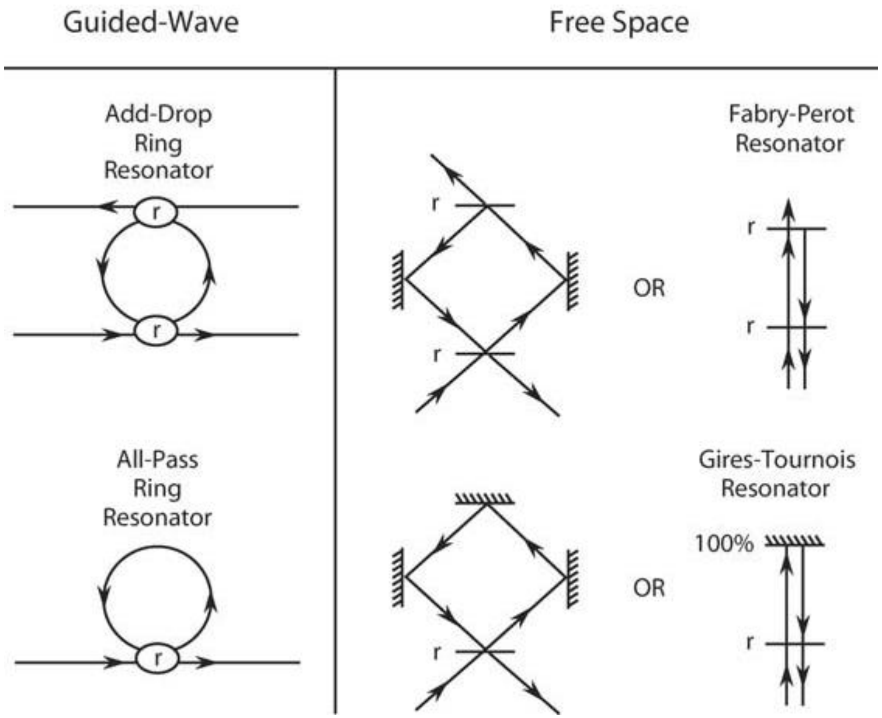


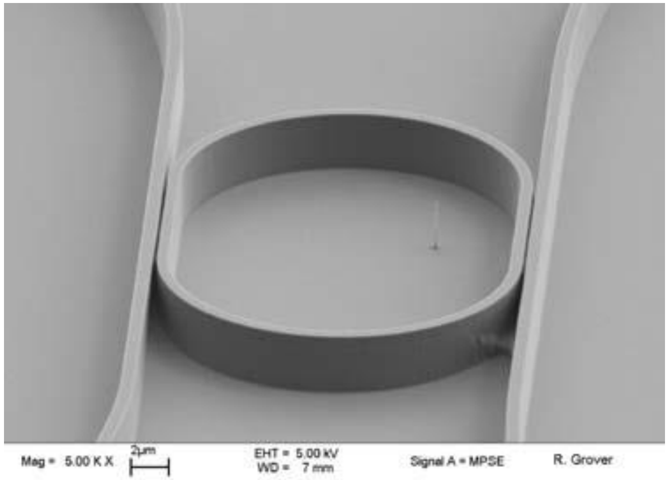
Optical properties	
Signal core	
Mode field diameter	76 ± 5 µm
Mode field area	4500± 200 µm²
NA @ 1060 nm	~ 0.02
Multimode pump core	
Numerical aperture @ 950 nm	0.6 ± 0.05
Pump absorption @ 920 nm	~ 10 dB/m
Pump absorption @ 976 nm	~ 30 dB/m
Slope efficiency	~60%

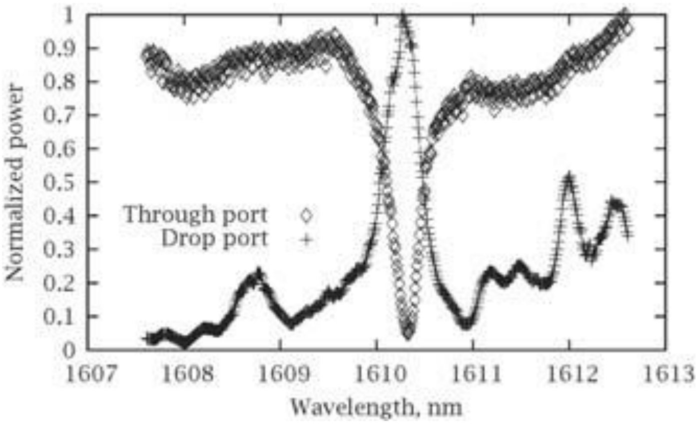
Physical properties	
Core material	Yb-doped silica
Outer cladding diameter	1.7 ± 0.1 mm
Coating	None
Signal core diameter	100 ± 5 µm
Pump-cladding diameter	285 ± 10 µm
Pump-cladding shape	Circular

wave-guide micro-resonators









Fiber/waveguide optical resonators

- transverse mode = mode of the fiber
- standing wave condition for a mode with index n: $\beta_n L = l\pi$ (l is a natural number) is used to find the frequencies of (longitudinal modes).
- numerical calculations.