MAŁE DRGANIA

UKŁAD Z WIĘZAMI HOLONOMICZNYMI 2-STRONNYMI, BEZ TARCIA (NA POCZĄTEK). n STOPNI SWOBODY

OGÓLNIE

$$L = \sum_{i,j=1}^{n} A_{ij}(q_1,\dots,q_n)\dot{q}_i\dot{q}_j - V(q_1,\dots,q_n)$$

← bo startujemy z $\frac{1}{2}m\bar{v}^2$

R. RUCHU:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

$$\frac{d}{dt}\sum_j A_{ij}\dot{q}_j - \frac{\partial A_{kl}}{\partial q_i}\dot{q}_k\dot{q}_l + \frac{\partial V}{\partial q_i} = 0$$

POŁOŻENIE RÓWNOWAGI → $q_i = const$ SĄ ROZW. RÓWNAŃ RUCHU. $\dot{q}_i$ ZNIKAJĄ ⇒ R. RUCH DAJĄ

$$\frac{\partial V}{\partial q_i} = 0 \quad \text{— EKSTREMA POTENCJAŁU.}$$

ROZPATRZMY PRZYPADEK <u>MINIMUM</u> POTENCJAŁU I ~~"MAŁE~~ RUCH ("MAŁE DRGANIA") WOKÓŁ MINIMUM q_i^0

$$V(q_1,\dots,q_n) = V(q_1^0,\dots,q_n^0) + (q_i - q_i^0)\cancel{\frac{\partial V}{\partial q_i}\Big|_{q_i^0}} + \frac{1}{2}(q_i-q_i^0)(q_j-q_j^0)\frac{\partial^2 V}{\partial q_i \partial q_j}$$

$$+ \frac{1}{3!}(q_i-q_i^0)(q_j-q_j^0)(q_k-q_k^0)\frac{\partial^3 V}{\partial q_i\partial q_j\partial q_k} + \dots$$

"MAŁOŚĆ" DRGAŃ – KOLEJNE WYRAZY SZYBKO MALEJĄ, – PRZYKŁAD 1-WYM:

$$\frac{1}{2}\Delta q^2 \frac{\partial^2 V}{\partial q^2} \gg \frac{1}{3!}\Delta q^3 \frac{\partial^3 V}{\partial q^3}$$

$$|\Delta q| \ll 3 \left| \frac{\frac{\partial^2 V}{\partial q^2}}{\frac{\partial^3 V}{\partial q^3}} \right|$$

DLA MAŁYCH DRGAŃ ZANIEDBUJEMY WYŻSZE WYRAZY, WPROWADZAMY $\xi_i = q_i - q_i^0$, $\dot{\xi}_i = \dot{q}_i$

$$V \triangleq V_0 + \frac{1}{2}\xi_i \xi_j \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{q_i^0, q_j^0} \longrightarrow \frac{1}{2} B_{ij} \xi_i \xi_j$$

$$B_{ij} = \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{\substack{q_i = q_i^0 \\ q_j = q_j^0}}$$

$$T = A_{ij}(q_1, q_j, \ldots)\, \dot{q}_i \dot{q}_j \stackrel{\sim}{=} A_{ij}(q_1^0, \ldots, q_n^0)\, \dot{\xi}_i \dot{\xi}_j$$

$$L \simeq \frac{1}{2}\left(A_{ij}\dot{\xi}_i\dot{\xi}_j - B_{ij}\xi_i\xi_j\right)$$

A, B – MACIERZE RZECZYWISTE SYMETRYCZNE, A, B DODATNIO OKREŚLONA

R. RUCHU

$$\frac{d}{dt} A_{ij} \dot{\xi}_j + B_{ij} \xi_j = 0$$

$$A_{ij} \ddot{\xi}_j + B_{ij} \xi_j = 0$$

MACIERZOWO

$$\hat{A} \ddot{\bar{\xi}} + \hat{B} \bar{\xi} = 0$$

A DODATNIO OKREŚLONA, B TEŻ.

ISTNIEJE A^{-1}, $A^{1/2}$, $A^{-1/2}$

$A^{1/2}$ – ~~...~~ JEŻELI $U^+ A U = D$, TO $A = U D U^T =$

$$= U D^{1/2} U^T U D^{1/2} U^T \equiv A^{1/2} \cdot A^{1/2}$$

$$A^{1/2} \ddot{\bar{\xi}} + A^{-1/2} B A^{-1/2} A^{1/2} \bar{\xi} = 0$$

PODSTAWMY $\bar{\eta} = A^{1/2} \xi$, $C = A^{-1/2} B A^{-1/2}$

$$\ddot{\bar{\eta}} + C \bar{\eta} = 0$$

POSZUKAJMY ROZW. POSTACI $\bar{\eta} = \bar{\eta}_0 e^{i\omega t}$, WSZYSTKIE ~~WSP.~~ STOPNIE SWOBODY DRGAJĄ Z TĄ SAMĄ CZĘSTOŚCIĄ – TZW. DRGANIA WŁASNE

$$(-\omega^2 \mathbb{1} + C) \bar{\eta}_0 = 0$$

$$\det(C - \omega^2 \mathbb{1}) = 0 = \det(A^{-1/2} B A^{-1/2} - \omega^2)$$

$$\det\left(A^{-1/2} B A^{-1/2} - \omega^2\right) = \det A^{-1/2} \det\left(B - \omega^2 A\right) \det A^{-1/2}$$

$$\det\left(B - \omega^2 A\right) = 0$$

$$\bar{\eta} = \bar{\eta}_0 e^{i\omega t} \Longrightarrow \bar{\xi} = A^{-1/2} \bar{\eta}_0 e^{i\omega t} \equiv \bar{\xi}_0 e^{i\omega t}$$

$$\left(B - \omega^2 A\right)\bar{\xi}_0 = 0$$

NIECH MACIERZ U DIAGONALIZUJE

MACIERZ C

$$U C U^T = \begin{pmatrix} \omega_1^2 & & \\ & \ddots & \\ & & \omega_n^2 \end{pmatrix} \equiv \Omega^2 \qquad U U^T = 1$$

$$L = \frac{1}{2} \dot{\xi}^T A \dot{\xi} - \frac{1}{2} \xi B \xi = \frac{1}{2} \left(A^{1/2} \dot{\xi}\right)^T \left(A^{1/2} \dot{\xi}\right) - \frac{1}{2} \left(A^{1/2} \xi\right)^T A^{-1/2} B A^{-1/2} A^{1/2} \xi$$

$$= \frac{1}{2} \dot{\eta}^T \dot{\eta} - \frac{1}{2} \eta^T C \eta$$

$$= \frac{1}{2} \dot{\eta}^T U^T U \dot{\eta} - \frac{1}{2} \eta^T U^T U C U^T U \eta$$

$$= \frac{1}{2} \left(U\dot{\eta}\right)^T \left(U\dot{\eta}\right) - \frac{1}{2} \left(U\eta\right) \Omega^2 U\eta$$

OZNACZMY $\rho = U\eta$ – "WSP. NORMALNE"

$$\rho = U\eta = U A^{-1/2} \xi$$

$$L = \cancel{\frac{1}{2} \dot{\rho}^T \rho} \; \frac{1}{2} \sum_{i=1}^{n} \left(\dot{\rho}_i^2 - \omega_i^2 \rho_i^2\right) = \frac{1}{2} \dot{\rho}^T \dot{\rho} - \frac{1}{2} \rho^T \Omega^2 \rho$$

DRGANIA WE WSP. NORMALNYCH SIĘ SEPARUJĄ!

OGÓLNA POSTAĆ RUCHU

$$\cancel{\xi_i(t)} =$$

$$\bar{\xi}(t) = \sum_{i=1}^{n} D_i \bar{\xi}_0^{(i)} \cos(\omega_i t + \varphi_i)$$

$$= \sum_{i=1}^{n} \bar{\xi}_0^{(i)} (a_i \cos\omega_i t + b_i \sin\omega_i t)$$

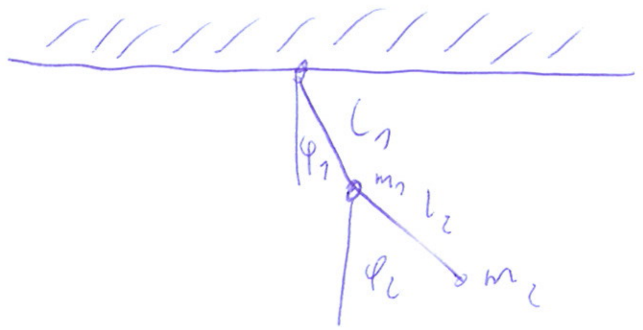
D_i, φ_i ALBO a_i, b_i Z WAR POCZĄTKOWYCH

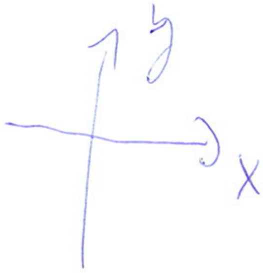
$$\bar{\xi}_0^{(i)} = \begin{pmatrix} \xi_1^{(i)} \\ \xi_2^{(i)} \\ \vdots \\ \xi_n^{(i)} \end{pmatrix}$$

$$\xi_i(t) = \sum_{j=1}^{n} \xi_i^{(j)} (a_j \cos\omega_j t + b_j \sin\omega_j t)$$

PRZYKŁADY, RUCH DRGAJĄCY

A) WAHADŁO PODWÓJNE





$$x_1 = l_1 \sin\varphi_1 \qquad x_2 = l_1 \sin\varphi_1 + l_2 \sin\varphi_2$$

$$y_1 = -l_1 \cos\varphi_1 \qquad y_2 = -l_1 \cos\varphi_1 - l_2 \cos\varphi_2$$

$$E_p = m_1 g y_1 + m_2 g y_2$$

$$= -g\left[(m_1+m_2) l_1 \cos\varphi_1 + m_2 l_2 \cos\varphi_2\right]$$

$$\dot{x}_1 = l_1 \cos\varphi_1 \dot{\varphi}_1 \qquad \dot{x}_2 = l_1 \cos\varphi_1 \dot{\varphi}_1 + l_2 \cos\varphi_2 \dot{\varphi}_2$$

$$\dot{y}_1 = l_1 \sin\varphi_1 \dot{\varphi}_1 \qquad \dot{y}_2 = l_1 \sin\varphi_1 \dot{\varphi}_1 + l_2 \sin\varphi_2 \dot{\varphi}_2$$

$$E_k = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) = E_{k1} + E_{k2}$$

$$E_{k1} = \frac{1}{2} m_1 l_1^2 \dot{\varphi}_1^2$$

$$E_{k2} = \frac{1}{2} m_2 \left[l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2 + 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)\right]$$

$$L = \frac{1}{2}(m_1+m_2) l_1^2 \dot\varphi_1^2 + \frac{1}{2} m_2 l_2^2 \dot\varphi_2^2$$
$$+ m_2 l_1 l_2 \cos(\varphi_1-\varphi_2)\dot\varphi_1\dot\varphi_2 + (m_1+m_2) g l_1 \cos\varphi_1$$
$$+ m_2 g l_2 \cos\varphi_2$$

POT. RÓWNOWAGI

$$\frac{\partial E_p}{\partial \varphi_1} = \frac{\partial E_p}{\partial \varphi_2} = 0 \qquad \frac{\partial^2 E_p}{\partial^2 \varphi_i} > 0$$

$$\Longrightarrow \varphi_1 = \varphi_2 = 0$$

MAŁE $\varphi_1, \varphi_2 \Rightarrow \cos\varphi \cong 1 - \frac{\varphi^2}{2}, \ \sin\varphi \approx \varphi$

$$L \cong \frac{1}{2}(m_1+m_2) l_1^2 \dot\varphi_1^2 + \frac{1}{2} m_2 l_2^2 \dot\varphi_2^2 + m_2 l_1 l_2 \dot\varphi_1 \dot\varphi_2$$
$$- \frac{1}{2}(m_1+m_2) g l_1 \frac{\varphi_1^2}{\cancel{2}} - \frac{1}{2} m_2 g l_2 \frac{\varphi_2^2}{\cancel{2}}$$

$$= (\dot\varphi_1, \dot\varphi_2) \begin{bmatrix} \frac{1}{2}(m_1+m_2) l_1^2 & \frac{1}{2} m_2 l_1 l_2 \\ \frac{1}{2} m_2 l_1 l_2 & \frac{1}{2} m_2 l_2^2 \end{bmatrix} \begin{pmatrix} \dot\varphi_1 \\ \dot\varphi_2 \end{pmatrix}$$

(φ_1, φ_2)

$$- \frac{1}{2} \begin{bmatrix} (m_1+m_2) g l_1 & 0 \\ 0 & m_2 g l_2 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \frac{1}{2}\left(\dot\varphi^T A \dot\varphi - \varphi B \varphi\right)$$

$$A = \begin{bmatrix} (m_1+m_2) l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{bmatrix} \qquad B = \begin{bmatrix} (m_1+m_2) g l_1 & 0 \\ 0 & m_2 g l_2 \end{bmatrix}$$

ROZW. DLA $l_1 = l_2 = l$, $m_1 = m_2 = m$

$$A = ml^2\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix} \qquad B = mgl\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$$

$$\det(\omega^2 A - B) = 0$$

$$\cancel{\det\begin{bmatrix}2ml^2 - 2mgl\,\omega^2 & ml^2\\ ml^2 & -mgl\,\omega^2 + mgl\end{bmatrix} = 0}$$

$$\det\begin{bmatrix}2ml^2\omega^2 - 2mgl & ml^2\omega^2\\ ml^2\omega^2 & ml^2\omega^2 - mgl\end{bmatrix} = 0$$

$$\Rightarrow \quad \det\begin{bmatrix}2\omega^2 - 2\frac{g}{l} & \omega^2\\ \omega^2 & \omega^2 - \frac{g}{l}\end{bmatrix} = 0$$

$$\omega_0^2 = \frac{g}{l}$$

$$2(\omega^2 - \omega_0^2)^2 - \omega^4 = 0$$

$$\omega^4 - 4\omega_0^2\omega^2 + 2\omega_0^4 = 0$$

$$\Delta = 16 - 8 = 8\omega_0^4$$

$$\omega^2 = \frac{4\omega_0^2 \pm 2\omega_0^2\sqrt{2}}{2} = \omega_0^2(2 \pm \sqrt{2}) = \frac{g}{l}(2 \pm \sqrt{2})$$

$$\omega_{\pm}^2 = \frac{g}{l}(2 \pm \sqrt{2})$$

$$\left[ml^2\frac{g}{l}\begin{bmatrix}2(2\pm\sqrt{2}) & (2\pm\sqrt{2})\\ (2\pm\sqrt{2}) & (2\pm\sqrt{2})\end{bmatrix} - mgl\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\right]\begin{bmatrix}\varphi_1^{\pm}\\ \varphi_2^{\pm}\end{bmatrix} = 0$$

$$\begin{bmatrix} 2\pm 2\sqrt{2} & 2\pm\sqrt{2} \\ 2\pm\sqrt{2} & 1\pm\sqrt{2} \end{bmatrix} \begin{bmatrix} \varphi_1^{\pm} \\ \varphi_2^{\pm} \end{bmatrix} = 0$$

$$2(1\pm\sqrt{2})\varphi_1^{\pm} + (2\pm\sqrt{2})\varphi_2^{\pm} = 0$$

$$\varphi_2^{\pm} = \frac{-2(1\pm\sqrt{2})}{2\pm\sqrt{2}}\varphi_1^{\pm} = \frac{-2(1\pm\sqrt{2})(2\mp\sqrt{2})}{4-2}\varphi_1^{\pm}$$

$$= -(\cancel{2} \mp\sqrt{2} \pm 2\sqrt{2} - \cancel{2}) = -(\pm\sqrt{2}) = \mp\sqrt{2}$$

$$\varphi^{+} = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \qquad \varphi^{-} = \begin{pmatrix} 1 \\ +\sqrt{2} \end{pmatrix}$$

ZNORMALIZOWANE $\varphi^{+} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix} \qquad \varphi^{-} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix}$

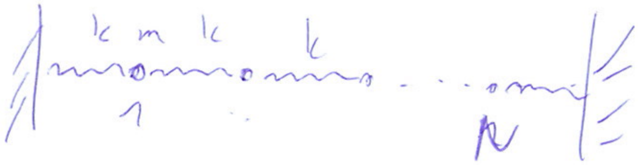
OGÓLNE ROZW.

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}(t) = A_{+}\begin{pmatrix} \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix}\cos(\omega_{+}t + \varphi_{+}) + A_{-}\begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix}\cos(\omega_{-}t + \varphi_{-})$$

$A_{\pm}, \varphi_{\pm}$ - war. początkowe

$$\omega_{\pm} = \sqrt{\frac{g}{l}}\sqrt{2\pm\sqrt{2}}$$

6) UKŁADY O N STOPNIACH SWOBODY



WSP: $x_1 \dots x_N$

$E_k = \frac{1}{2} m(\dot{x}_1^2 + \dots + \dot{x}_n^2)$

~~$E_p = \frac{1}{2} k x_1^2 + \frac{1}{2}$~~ DŁUGOŚĆ SPOCZYNKOWA

~~POŁ. RÓWNOWAG.~~ SPRĘŻYNEK l_0

ODLEGŁOŚĆ MIĘDZY ŚCIANKAMI: L

POŁ. RÓWNOWAGI MASY i: $\frac{Li}{N+1}$

ODCHYLENIE OD POŁ. RÓWNOWAGI $y_i = x_i - \frac{Li}{N+1}$

$\dot{y}_i = \dot{x}_i$

E_{pot}:

1 SPRĘŻYNKA $\frac{1}{2} k(x_1 - l_0)^2 = \frac{1}{2} k\left(y_1 - l_0 + \frac{L}{N+1}\right)^2$

N+1 —||— $\frac{1}{2} k(L - x_N - l_0)^2 = \frac{1}{2} k\left(L - l_0 - y_N - \frac{LN}{N+1}\right)^2$

$= \frac{1}{2} k\left(y_N + l_0 - \frac{L}{N+1}\right)^2$

i SPRĘŻYNKA $\frac{1}{2} k(x_i - x_{i-1} - l_0)^2 = \frac{1}{2} k\left(y_i + \frac{Li}{N+1} - y_{i-1} - \frac{L(i-1)}{N+1} - l_0\right)^2$

$= \frac{1}{2} k\left(y_i - y_{i-1} + \frac{L}{N+1} - l_0\right)^2$

OZNACZMY $l = l_0 - \frac{L}{N+1}$

$$E_{p1} = \frac{1}{2}k(y_1 - l)^2$$

$$E_{pi} = \frac{1}{2}k(y_i - y_{i-1} - l)^2$$

$$E_{pN+1} = \frac{1}{2}k(y_N + l)^2$$

$$E_p = \frac{1}{2}ky_1^2 - \cancel{ky_1l} + \frac{1}{2}k(y_2^2 + y_1^2 - 2y_2y_1) - ky_2l + \cancel{ky_1l} + \dots$$

$$= k(y_1^2 + \dots + y_N^2) - k(y_1y_2 + \dots + y_{N-1}y_N)$$

$$A_{n\times n} = m\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

$$B_{N\times N} = k\begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$$

$$\det\begin{bmatrix} 2k - m\omega^2 & -k & & \\ -k & \ddots & \ddots & \\ & \ddots & \ddots & -k \\ & & -k & 2k - m\omega^2 \end{bmatrix} = 0 \qquad \omega_0^2 = \frac{k}{m}$$

$$\det\begin{bmatrix} 2 - \frac{\omega^2}{\omega_0^2} & -1 & & \\ -1 & \ddots & \ddots & -1 \\ & & -1 & 2 - \frac{\omega^2}{\omega_0^2} \end{bmatrix} = 0 \qquad x = \frac{\omega^2}{\omega_0^2}$$

$$\det\begin{bmatrix} 2-x & -1 & & \\ -1 & 2-x & -1 & \\ & \ddots & \ddots & -1 \\ & & -1 & 2-x \end{bmatrix} = 0 \qquad = D_N$$

$$D_N = (2-x)\,D_{N-1} + \begin{bmatrix} \cancel{1-x} & \cancel{0} & \cancel{0} & \\ -1 & -1 & 0 & \\ 0 & 2-x & -1 & \\ \vdots & -1 & 2-x & \ddots \\ \vdots & \vdots & -1 & \ddots \\ 0 & 0 & 0 & & -1 \end{bmatrix}$$

$$= (2-x)\,D_{N-1} - D_{N-2}$$

$$D_N = r^N$$

$$r^2 = (2-x)\,r - 1$$

$$r^2 - (2-x)\,r + 1 = 0$$

$$\Delta = (2-x)^2 - 4$$

$$r_\pm = \frac{(2-x) \pm \sqrt{4 + (2-x)^2}}{2} = \cancel{\frac{1-x}{2} \pm \sqrt{1 + \left(\frac{1-x}{2}\right)^2}}$$

$$\cancel{1 + \cos\varphi = 2\cos^2\frac{\varphi}{2}}$$

$$\cancel{D_N = A\,r_+^N + B\,r_-^N}$$

$$\cancel{D_1 = 1-x = A\,r_+ + B\,r_-}$$

$$D_2 = \cancel{(1-x)^2 - 1 = x^2 - 2x} = A\,r_+^2 + B\,r_-^2$$

ÓGÓLNE ROZW.

$$D_N = A r_+^N + B r_-^N$$

~~WA~~ STAŁE A, B Z

$$D_1 = (2-x)^2$$

$$D_2 = (2-x)^2 - 1$$

ALE PODSTAWMY

$$x = 4\sin^2\frac{\varphi}{2}$$

$$2-x = 2\left(1-2\sin^2\frac{\varphi}{2}\right) = 2\cos\varphi$$

$$r_\pm = \frac{2\cos\varphi \pm \sqrt{4\cos^2\varphi - 4}}{2} = \cos\varphi \pm i\sin\varphi = e^{\pm i\varphi}$$

$$\begin{cases} A e^{i\varphi} + B e^{-i\varphi} = \cancel{1} e^{i\varphi} + e^{-i\varphi} \\ A e^{2i\varphi} + B e^{-2i\varphi} = 4\cos^2\varphi - 1 = e^{2i\varphi} + e^{-2i\varphi} + 1 \end{cases}$$

$$A e^{2i\varphi} + B = e^{2i\varphi} + 1$$

$$B\left(1 - e^{-2i\varphi}\right) = \cancel{e^{2i\varphi}} + \cancel{1} - \cancel{e^{2i\varphi}} - e^{-2i\varphi} - \cancel{1}$$

$$B = \frac{-e^{-2i\varphi}}{1-e^{-2i\varphi}} = \frac{-e^{-i\varphi}}{e^{i\varphi} - e^{-i\varphi}}$$

$$A = 1 + e^{-2i\varphi} - B e^{-2i\varphi} = 1 + e^{-2i\varphi} + \frac{e^{-3i\varphi}}{e^{i\varphi} - e^{-i\varphi}} = \frac{e^{i\varphi} - \cancel{e^{-i\varphi}} + \cancel{e^{-i\varphi}} - \cancel{e^{-3i\varphi}} + \cancel{e^{-3i\varphi}}}{e^{i\varphi} - e^{-i\varphi}}$$

$$\therefore A = \frac{e^{i\varphi}}{e^{i\varphi}-e^{-i\varphi}}$$

$$B = \frac{-e^{+i\varphi}}{e^{i\varphi}-e^{-i\varphi}}$$

$$D_N = \frac{e^{i\varphi}e^{iN\varphi} - e^{-i\varphi}e^{-iN\varphi}}{e^{i\varphi}-e^{-i\varphi}} = \frac{\sin(N+1)\varphi}{\sin\varphi}$$

$$D_N = 0 \Leftrightarrow (N+1)\varphi = k\pi \quad , \quad k = 1, \dots, N$$

$$\varphi_k = \frac{k\pi}{N+1}$$

$$\omega_k = 2\omega_0 \sin\frac{k\pi}{2(N+1)} \qquad k = 1, \dots, N$$

$$\vec{x}^{(i)} = \sqrt{\frac{2}{N+1}} \begin{bmatrix} \sin\varphi_i \\ \vdots \\ \sin N\varphi_i \end{bmatrix}$$

$$\vec{x}(t) = \sum_i A_i \vec{x}^{(i)} \cos(\omega_i t + \varphi_i)$$

GRANICA CIĄGŁA

$$x_i \to x(i) \to x(l_i)$$

$$l_i = \frac{iL}{N}$$

$$x_{i+1} - x_i = x(l_{i+1}) - x(l_i) \simeq \frac{\partial x}{\partial l}\Delta l$$

$$x_{i+1} - 2x_i + x_{i-1} = \frac{\partial x(l_{i+1})}{\partial l}\Delta l - \frac{\partial x(l_i)}{\partial l}\Delta l$$

$$\simeq \frac{\partial^2 x}{\partial l^2}\Delta l^2$$

$$\ddot{x}_i \to \frac{\partial^2 x(l,t)}{\partial t^2}$$

$$m\ddot{x}_i + k(2x_i - x_{i+1} - x_{i-1}) = 0$$

$$\frac{\partial^2 x}{\partial t^2} - \frac{k}{m}\Delta l^2 \frac{\partial^2 x}{\partial l^2} = 0$$

$$m = \frac{M}{N}$$

$$\Delta l = \frac{L}{N}$$

$$k = \frac{k_0}{\cancel{M}} N$$

$$\frac{k}{m}\Delta l^2 = \underbrace{\frac{k_0 L^2}{M}}_{E} \simeq \text{MODUŁ YOUNGA}$$

$$\frac{\partial^2 x}{\partial t^2} - E\frac{\partial^2 x}{\partial l^2} = 0$$

FALE STOJĄCE $x = e^{i(kl - \omega t)}$

$\omega^2 = k^2 E$, $\omega = k\sqrt{E}$ – liniowe, jak małe q_k

REZONANS

70A

$$L = \frac{1}{2}\dot{\xi}^T A \dot{\xi} - \frac{1}{2}\xi^T B \xi$$

$f_i = h_i \sin \Omega t$ – PERIODYCZNA SIŁA WYMUSZAJĄCA

WSP. NORMALNE

$$\rho = U\eta = U A^{1/2}\xi = V\xi \qquad \xi = V^{-1}\rho = A^{-1/2}U^T$$

$$U A^{-1/2} B A^{-1/2} U^T \equiv \omega^2$$

$$A\ddot{\vec{\xi}} + B\vec{\xi} = \vec{h}\sin\Omega t$$

$$A V^{-1}\ddot{\vec{\rho}} + B V^{-1}\vec{\rho} = \vec{h}\sin\Omega t$$

$$A V^{-1} = A^{1/2}U^T$$

$$B V^{-1} = B A^{-1/2} U^T = A^{1/2} U^T U A^{-1/2} B A^{-1/2} U^T$$

$$= A^{1/2} U^T \omega^2$$

$$\ddot{\vec{\rho}} + \omega^2\vec{\rho} = (A^{1/2}U^T)^{-1}\vec{h}\sin\Omega t$$

$$= U A^{-1/2}\vec{h}\sin\Omega t = \vec{g}\sin\Omega t$$

WNIOSEK → REZONANS W KAŻDYM MODZIE NORMALNYM ODDZIELNIE

$$\ddot{\rho} + \omega^2 \rho = g_0 \sin \omega t$$

$$\rho = (At + B) \sin(\omega t + \varphi)$$

$$\dot{\rho} = A \sin(\omega t + \varphi) + \omega (At + B) \cos(\omega t + \varphi)$$

$$\ddot{\rho} = A\omega \cos(\omega t + \varphi) + \omega A \cos(\omega t + \varphi) - \omega^2 (At + B) \sin(\omega t + \varphi)$$

$$\ddot{\rho} + \omega^2 \rho = 2\omega A \cos(\omega t + \varphi) = g_0 \sin \omega t$$

$$A = \frac{g_0}{2\omega} \qquad \varphi = -\frac{\pi}{2}$$

$$\rho(t) = -\frac{g_0}{2\omega} t \cos \omega t + \rho_0 \sin(\omega t + \varphi_0)$$

WIELE CZĘSTOŚCI WYMUSZAJĄCYCH

→ UKŁAD LINIOWY, SUMA REZONANSÓW.

TŁUMIENIE

7a

SIŁY $f_i(q,\dot q)$ → ROZWIJAMY DO WYRAZÓW LINIOWYCH W q I $\dot q$

$$\boxed{A\ddot{\vec{\xi}} + B\vec{\xi} + C\dot{\vec{\xi}} = 0}$$

$$\vec{\xi} = \vec{\xi}_0 e^{\lambda t}$$

$$(A\lambda^2 + B + C\lambda)\vec{\xi}_0 = 0$$

$\det(A\lambda^2 + B + C\lambda) = 0 \Rightarrow$ WYLICZAMY λ

SIŁY GIROSKOPOWE → $\perp$ DO PRĘDKOŚCI

DYSSYPATYWNE → ~~...~~ $\parallel$ DO PRĘDKOŚCI

1 SIŁY GIROSKOPOWE, NP. SIŁA LORENTZA

$$\vec{F} = \vec{v} \times \vec{c}$$

POT. UOGÓLNIONY $U = \frac{1}{2}\vec{c}(\vec{v}\times\vec{r}) = \frac{1}{2}\epsilon_{ijk}c_i\dot{x}_j x_k$

$U \equiv c_{jk}\dot{x}_j x_k$, $c_{jk} = \frac{1}{2}\epsilon_{ijk}c_i = -c_{kj}$

SIŁY GIROSKOPOWE WE WSP. UOGÓLNIONYCH (70D)

$$U = \frac{1}{2} C_{ij} \zeta_i \dot{\zeta}_j = \frac{1}{2} \vec{\zeta}^T C \dot{\vec{\zeta}}$$

$$C_{ij} = -C_{ji}$$

INNE SIŁY - POTENCJALNE ($\to$ MACIERZ B)

$$L = E_k - E_p - U$$

~~A ζ~~

$$\frac{d}{dt} \frac{\partial}{\partial \dot{\vec{\zeta}}} (-U) = -\frac{1}{2} \dot{\vec{\zeta}}^T C$$

$$\frac{\partial}{\partial \vec{\zeta}} (-U) = -\frac{1}{2} C \dot{\vec{\zeta}}$$

$$A \ddot{\vec{\zeta}} - \frac{1}{2} \dot{\vec{\zeta}}^T C = -B \vec{\zeta} - \frac{1}{2} C \dot{\vec{\zeta}}$$

$$C^T = -C \Rightarrow A \ddot{\vec{\zeta}} + B \vec{\zeta} + C \dot{\vec{\zeta}} = 0$$

DWA ST. SWOBODY, A i B DIAGONALNE

$$\begin{vmatrix} \lambda^2 + \omega_1^2 & c_{12} \\ -c_{12} & \lambda^2 + \omega_2^2 \end{vmatrix} = 0$$

$$\lambda^4 + \lambda^2 (\omega_1^2 + \omega_2^2 + \gamma^2) + \omega_1^2 \omega_2^2 = 0 \qquad \gamma = |c_{12}|$$

$$\lambda_{12} = -\frac{1}{2} \left[\omega_1^2 + \omega_2^2 + \gamma^2 \pm \sqrt{(\omega_1^2 - \omega_2^2)^2 + 2\gamma^2 (\omega_1^2 + \omega_2^2) + \gamma^4} \right]$$

$$\lambda_{12} < 0 \Rightarrow \sqrt{\lambda_{12}} = i\sqrt{-\lambda_{12}}$$

(7CF)

→ NIE MA ZANIKANIA DRGAŃ!

→ CZĘSTOŚCI ω_1, ω_2 SIĘ "ODPYCHAJĄ"

TO SAMO DLA WIĘKSZEJ ILOŚCI ST. SWOBODY.

(2) SIŁY DYSSYPATYWNE

→ FUNKCJA RAYLEIGHA

$$R = \frac{1}{2} C_{ij} \dot{\xi}_i \dot{\xi}_j$$ → DODATNIO OKREŚLONA

→ SIŁY "ŚCIŚLE DYSSYPATYWNE"

SIŁY ROZPRASZAJĄCE $f_i = -\frac{\partial R}{\partial \dot{\xi}_i}$

$$\frac{dE}{dt} = f_i \dot{q}_i = -2R$$

$$(A\lambda^2 + B + C\lambda)\vec{\xi}_0 = 0$$

$$\xi_0^+ A \xi_0 \lambda^2 + \xi_0^+ B \xi_0 + \xi_0^+ C \xi_0 \lambda = 0$$

R. KWADRATOWE O DODATNICH WSPÓŁCZYNNIKACH

λ - PIERWIASTEK → λ^* TEŻ PIERWIASTEK

$$\lambda + \lambda^* = -\frac{b}{a} = -\frac{\xi_0^+ C \xi_0}{\xi^+ A \xi_0} \leq 0$$

WE WSP. NORMALNYCH

$$E_k = \frac{1}{2}\sum_i \dot{\rho}_i^2 \qquad E_p = \frac{1}{2}\sum_i \omega_i^2 \rho_i^2$$

$$R = \frac{1}{2}\gamma_{ij}\dot{\rho}_i\dot{\rho}_j$$

$$\ddot{\vec{\rho}} + \hat{\Omega}^2\vec{\rho} + \Gamma\dot{\vec{\rho}} = 0$$

$$\vec{\rho} = \vec{\rho}_0 e^{\lambda t}$$

$$\det(\lambda^2 + \Omega^2 + \Gamma\lambda) = 0 = \begin{vmatrix} \lambda^2 + \omega_1^2 + \gamma_{11}, & \gamma_{12}\lambda, & \dots & \gamma_{1f}\lambda \\ \vdots & & & \vdots \\ \vdots & \dots & & \end{vmatrix}$$

POMIJAMY WYRAZY $\sim \gamma^2$

$$(\lambda^2 + \omega_1^2 + \gamma_{11}\lambda) \dots (\lambda^2 + \omega_f^2 + \gamma_{ff}\lambda) = 0$$

$$\lambda_k \cong \pm i\omega_k - \frac{1}{2}\gamma_{kk}$$

WEŹMY $\lambda = \lambda_1$

$$\rho_0^k = \frac{\mp i\gamma_{k1}\omega_1\eta_1}{\pm i\omega_1(\gamma_{kk} - \gamma_{11}) + \omega_k^2 - \omega_1^2} \cong \pm\frac{i\gamma_{k1}\omega_1}{\omega_k^2 - \omega_1^2}\eta_1$$

ETC ...

MAŁE DRGANIA WOKÓŁ RUCHU STACJONARNEGO

7a

WAHADŁO KULISTE

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

$$= \frac{1}{2} m R^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\varphi}^2) - mgR\cos\theta$$

R. LAGRANGE'A

$$R(\ddot{\theta} - \dot{\varphi}^2 \sin\theta \cos\theta) - g\sin\theta = 0$$

$$\frac{d}{dt}(\sin^2\theta\, \dot{\varphi}) = 0$$

$$\begin{cases} \dot{\varphi} = \dfrac{A}{\sin^2\theta} \\ \frac{1}{2} R(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) + g\cos\theta = B \end{cases}$$

$$\dot{\theta}^2 + \frac{A^2}{\sin^2\theta} + \frac{2g}{R}\cos\theta = \frac{2B}{R}$$

OZNACZAMY $A^2 = \frac{2g\alpha}{R}$ $B = g\beta$ $\alpha \geq 0$ $\beta \geq -1$

$$\dot{\theta}^2 + \frac{2g}{R}\left(\frac{\alpha}{\sin^2\theta} + \cos\theta - \beta\right) = 0$$

TAM

"WAHADŁO STOŻKOWE" $\longrightarrow \dot{\Theta} = 0$

$$\Theta = \mathrm{const} = \Theta_0 \qquad \varphi = \frac{2g\lambda}{R\sin^2\Theta_0}\, t$$

$$\cancel{\frac{\pi}{2} < \alpha < \pi \qquad \dot{\varphi} = \dot{\varphi}_0 = t}$$

$$\beta = \frac{\lambda}{\sin^2\Theta_0} + \cos\Theta_0$$

GELFAND FOMIN

RACH. WARIACYJNY