ELEMENTY MECHANIKI RELATYWISTYCZNEJ

DOŚWIADCZENIE MICHELSONA – PRĘDKOŚĆ ŚWIATŁA W PRÓŻNI TAKA SAMA W KAŻDYM INERCJALNYM UKŁ. ODNIESIENIA

BEZWZGLĘDNA? NIE!

ROZWIĄZANIE: STW; ZAKŁADAMY ŻE W KAŻDYM UKŁ. INERCJALNYM

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = CONST$$

$\sqrt{ds^2}$ – przedział czasoprzestrzenny.

$x_0 = ct$ $\qquad ds^2 = (dx_0)^2 - (dx_1)^2 - \ldots = (dx_0')^2 - \ldots$

PRZEJŚCIE $x_\alpha \to x_\alpha'$ (TRANSF. LORENTZA)

- LINIOWE
- DLA $V \ll c$ REDUKUJE SIĘ DO TRANSF. GALILEUSZA
- ZACHOWUJE ds^2
- WYZNACZNIK 1

WYBIERZMY OSIE T. ŻE $V \parallel x \parallel x'$

$$x' = Ax + Bx_0$$

$$x_0' = Bx + Cx_0 \qquad y' = y;\ z' = z$$

$$A^2 - D^2 = 1$$

$$C^2 - B^2 = 1$$

$$AB - CD = 0$$

STĄD $A = C = \sqrt{1+D^2}$ ZNAK + → NIE MA ODBIĆ

$B = D$

$A, C \geq 1$ — PARAMETRYZACJA

$$A = C = (1-\beta^2)^{-1/2} = \gamma$$

$$B = D = \pm\beta(1-\beta^2)$$

$$\begin{cases} x' = \gamma(x \pm \beta x_0) \\ x_0' = \gamma(x_0 \pm \beta x) \end{cases}$$

$v \ll c \Rightarrow x' = x - Vt \quad (x = x' + Vt)$

STĄD $\beta = \frac{V}{c}$

$$\begin{cases} x' = \gamma(x - \beta x_0) \\ x_0' = \gamma(x_0 - \beta x) \end{cases} \qquad \begin{matrix} y' = y \\ z' = z \end{matrix}$$

OGÓLNIE $x' = Lx$, L – złożenie 3 obrotów i 3 boostów

$$L^{\alpha}{}_{\mu}\, g^{\alpha\beta} L_{\beta\nu} = g_{\mu\nu} = L^{\beta}{}_{\mu} L_{\beta\nu}$$

"GRUPA LORENTZA" $\quad g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = g^{\mu\nu}$

SKŁADANIE PRĘDKOŚCI

$$dx' = \gamma(dx - V\,dt)$$

$$dt' = \gamma\left(dt - V\frac{dx}{c^2}\right)$$

$$v'_{\parallel} = \frac{dx'}{dt'} = \frac{v_{\parallel} - V}{1 - \frac{V v_{\parallel}}{c^2}}$$

$$v'_{\perp} = \frac{d}{dt'}\sqrt{z'^2 + y'^2} = \sqrt{\left(\frac{dz}{dt'}\right)^2 + \left(\frac{dy}{dt'}\right)^2} =$$

$$= \frac{v_{\perp}}{\gamma\left(1 - \frac{V v_{\parallel}}{c^2}\right)}$$

$$v'^2 = v'^2_{\perp} + v'^2_{\parallel} = c^2\left[1 - \frac{\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{V^2}{c^2}\right)}{\left(1 - \frac{V v_{\parallel}}{c^2}\right)^2}\right]$$

$$v = c \Rightarrow v' = c\,!$$

POZA TYM – DYLATACJA CZASU, SKRÓCENIE DŁUGOŚCI, PARADOKS BLIŹNIĄT, ...

OGÓLNA TRANSFORMACJA, OSIE DOWOLNIE

$$\vec{r}' = \vec{r} + \vec{V}\left[\frac{\vec{r}\vec{V}}{V^2}(\gamma - 1) - \gamma t\right]$$

$$t' = \gamma\left[t - \frac{\vec{r}\vec{V}}{c^2}\right]$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}$$

DEFINIUJEMY

$x^\mu = (ct, \vec{r})$ — "kontra"

$x_\mu = (ct, -\vec{r})$ —

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \Rightarrow \begin{cases} x_\mu = g_{\mu\nu} x^\nu \\ x^\mu = g^{\mu\nu} x_\nu \end{cases}$$

WTEDY $ds^2 = dx^\mu dx_\mu = dx_\mu dx^\mu = dx_\mu g^{\mu\nu} dx_\nu =$

$= dx^\nu g_{\mu\nu} dx^\mu$

ds^2 NIEZMIENNICZY:

$x_\mu' = L_\mu{}^\nu x_\nu \qquad L_\mu{}^\nu = L_{\mu\alpha} g^{\alpha\nu}$

$ds^2 = dx_\mu' g^{\mu\nu} dx_\nu' = L_\mu{}^\alpha dx_\alpha g^{\mu\nu} L_\nu{}^\beta dx_\beta$

$$= \underbrace{L_\mu{}^\alpha L_\nu{}^\beta g^{\mu\nu}}_{g^{\alpha\beta}} dx_\alpha dx_\beta$$

CZYLI $L_\mu{}^\alpha L_\nu{}^\beta g^{\mu\nu} = g^{\alpha\beta} =$

$A_\mu' A'^\mu = L_\mu{}^\nu A_\nu L^\mu{}_\gamma A^\gamma = A_\mu' A_\nu' g^{\mu\nu}$

$= L_\mu{}^\alpha L_\nu{}^\beta A_\alpha A_\beta g^{\mu\nu} = g^{\alpha\beta} A_\alpha A_\beta = A_\mu A^\mu$

90A

$$L = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ BOOST $Ox \to Ox'$

BOOSTY NIE TWORZĄ GRUPY!

~~W JEDN~~ WZDŁUŻ JEDNEJ OSI – TWORZĄ

$$L_{V_2} L_{V_1} = L_V = L_{V_1} L_{V_2}$$

$$V = (V_1 + V_2)\left(1 + \frac{V_1 V_2}{c^2}\right)^{-1}$$

ZRÓBMY BOOST $Ox \to L(e_1, V)$ ORAZ

$Oy \to L(e_2, V)$

POTEM $\varphi =$ $\cos\varphi = \sqrt{(1-\beta^2)(2-\beta^2)}$ ($\sin\varphi = -\gamma\cos\varphi$)

$R(\varphi)$ – OBRÓT WOKÓŁ Oz. WTEDY

$$L(e_1, V)\, L(e_2, V)\, R(\varphi)\, L\left(-e_1, V\sqrt{2-\beta^2}\right) =$$

$$= R\left(\tfrac{1}{2}\pi - \varphi\right)$$ – OBRÓT PRZESTRZ.

4-PRĘDKOŚĆ $u^\mu = \frac{dx^\mu}{d\tau}$ $\quad d\tau = dt\sqrt{1 - \frac{v^2}{c^2}}$

"CZAS WŁASNY"

$$u^0 = \frac{c}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$u^k = \frac{v^k}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \boxed{u^2 = c^2}$$

4-WEKTOR PĘDU

$$p^\mu = m_0 u^\mu$$

$$p^0 = \frac{m_0 c}{\sqrt{1-\frac{\bar{v}^2}{c^2}}} \qquad \bar{p} = \frac{m\bar{v}}{\sqrt{1-\frac{\bar{v}^2}{c^2}}}$$

$$\bar{p} = m(v)\,\bar{v} \qquad p^2 = m_0^2 c^2$$

ZDEFINIUJMY ENERGIĘ RELATYWISTYCZNĄ E_r

$$E_r = c\,p^0 = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$E_r = m_0 c^2 + \frac{1}{2} m\bar{v}^2 + \frac{3}{8} m_0 c^2 \left(\frac{v^2}{c^2}\right)^2 + \dots$$

$$E_r = E_0 + \frac{1}{2} m\bar{v}^2 + \dots \triangleq \underbrace{m_0 c^2}_{\text{ENERGIA SPOCZYNKOWA}} + E_{\text{KIN. GAL.}}$$

$$E_r = m(v)\,c^2$$

BARDZO DUŻE $v \sim c$

$$E_r = c|p| + \frac{m_0 c^2}{|p|} + \dots$$

CZĄSTKA BEZMASOWA, (ALE MA ENERGIĘ!) 92

~~WAW~~ $E_r = c|\bar{p}|$

OGÓLNIE $\vec{v} = \frac{\bar{p}}{E_r} c^2$

CZĄSTKA BEZMASOWA $v = c$!

ŚR. MASY

$$\vec{V} = c^2 \frac{\sum \bar{p_i}}{\sum E_{r_i}}$$ — NIE JEST SKŁADOWĄ 4-WEKTORA! MNIEJ UŻYTECZNY..

4-SIŁA:

$$K^\mu = \frac{dp^\mu}{d\tau}$$

$$\frac{dp^k}{dt} = \frac{dp^k}{d\tau}\frac{d\tau}{dt} = K^k\sqrt{1-\frac{v^2}{c^2}}$$

$$K^k = F^k\left(\sqrt{1-\frac{v^2}{c^2}}\right)^{-1} = \gamma F^k$$

$$K^\mu u_\mu = \frac{dp^\mu}{d\tau}u_\mu = m_0 u_\mu \frac{du^\mu}{d\tau} = m_0 \frac{d\, u_\mu u^\mu}{d\tau} = 0$$

$$K^0 = \frac{\bar{K}\bar{u}}{u^0} = \frac{\bar{F}\bar{v}}{c\sqrt{1-v^2/c^2}} = \gamma \bar{\beta} \bar{F}$$

$$K = (\gamma \bar{\beta} \bar{F}, \gamma \bar{F})$$

$$\frac{dp^0}{dt} = \frac{dp^0}{d\tau} \frac{d\tau}{dt} = \sqrt{1-\frac{v^2}{c^2}} \frac{1}{\sqrt{}} \bar{\beta} \bar{F} = \frac{\bar{v}\bar{F}}{c} = \frac{1}{c} \frac{dE_r}{dt}$$

$$\frac{dE_r}{dt} = \bar{F}\bar{v}$$

MOMENT PĘDU, MOMENT SIŁY

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$$

$$D^{\mu\nu} = x^\mu K^\nu - x^\nu K^\mu$$

$$\frac{d}{d\tau} J^{\mu\nu} = D^{\mu\nu}$$

SIŁA LORENTZA:

$$\vec{B} = \mathrm{rot}\,\vec{A}$$

$$E = -\mathrm{grad}\,\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} =$$

ZDEFINIUJMY

$$A^{\mu} = (\varphi, \vec{A})$$

~~WTEDY~~ ORAZ $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$

WTEDY:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

NIEPOTRZEBNE

R. RUCHU Z SIŁĄ LORENTZA: $\boxed{m\frac{d^2x^{\mu}}{d\tau^2} = \frac{q}{c} F^{\mu}{}_{\nu} \frac{dx^{\nu}}{d\tau}}$

$$m\frac{d^2\vec{r}}{d\tau^2} = \frac{e}{c}\left(\vec{E}u_0 + c\,\vec{u}\times\vec{B}\right)$$

$$m\frac{d^2x_0}{d\tau^2} = \frac{e}{c}\vec{E}\vec{u} \quad \text{(PRAWO ZACH. ENERGII)}$$

FORMALIZM LAGRANGE'A

NIERELATYWISTYCZNIE $L = E_k - V$

RELATYWISTYCZNIE $E_r - V$? ŹLE!

SPRÓBUJMY

$$L = -m_0 c^2 \sqrt{1 - \frac{\vec{v}^2}{c^2}} - V$$

$$\frac{d}{dt}\frac{\partial L}{\partial \vec{v}} - \frac{\partial L}{\partial \vec{r}} = 0$$

$$-m_0 c^2 \frac{d}{dt}\frac{\partial}{\partial \vec{v}}\sqrt{1 - \frac{\vec{v}^2}{c^2}} = -\frac{\partial V}{\partial \vec{r}} + \frac{d}{dt}\frac{\partial V}{\partial \vec{v}} = \vec{F}(\vec{r}, \vec{v})$$

$$\frac{\partial}{\partial \vec{v}}\sqrt{1 - \frac{\vec{v}^2}{c^2}} = \frac{1}{2\sqrt{1 - \frac{\vec{v}^2}{c^2}}} \cdot \frac{-2\vec{v}}{c^2}$$

$$\frac{d}{dt}\frac{m_0 \vec{v}}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} = \vec{F}$$ OK! $p_{\text{ŚR}} \rightarrow \frac{m_0 \vec{v}}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}}$

PRAWA ZACHOWANIA

1) TRANSLACJA W PRZESTRZENI

$$\frac{\partial L}{\partial \vec{v}} = \text{CONST} = \vec{p}_i$$

2) TRANSLACJA W CZASIE

$$\vec{v}\frac{\partial L}{\partial \vec{v}} - L = \text{CONST} = \vec{p}\vec{v} - L = \frac{m_0 c^2}{\sqrt{}} - m_0 c^2 \sqrt{} - L = E_r$$

PRZYKŁAD

RUCH KEPLERA

$$L = -m_0 c^2 \sqrt{1-\frac{v^2}{c^2}} + \frac{\alpha}{r}$$

$$\frac{m c^2}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{\alpha}{r} = E$$

$$v^2 = \dot{r}^2 + r^2 \dot{\varphi}^2$$

$$\frac{\partial L}{\partial \varphi} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\varphi}} = p_\varphi = \text{const} = -m_0 c^2 \frac{\partial}{\partial \dot{\varphi}} \sqrt{1-\frac{\dot{r}^2}{c^2} - \frac{r^2\dot{\varphi}^2}{c^2}}$$

$$= -m_0 \cancel{c^2} \frac{1}{2\sqrt{1-\frac{v^2}{c^2}}} \left(\frac{-2r^2\dot{\varphi}}{\cancel{c^2}} \right) = \frac{m r^2 \dot{\varphi}}{\sqrt{1-\frac{v^2}{c^2}}} = J$$

$$\frac{dr}{dt} = \frac{dr}{d\varphi}\dot{\varphi} \qquad \dot{\varphi}^2 = \frac{J^2\left(1-\frac{v^2}{c^2}\right)}{m^2 r^4}$$

$$v^2 = (r^2 + r'^2)\dot{\varphi}^2 = \frac{J^2(r^2+r'^2)}{m^2 r^4}\left(1-\frac{v^2}{c^2}\right)$$

$$v^2 = \frac{\frac{J^2(r^2+r'^2)}{m^2 r^4}}{1 + \frac{J^2(r^2+r'^2)}{m^2 c^2 r^4}} = \frac{1}{\frac{1}{c^2} + \frac{m^2 r^4}{J^2(r^2+r'^2)}}$$

$$\cancel{1 - \frac{v^2}{c^2}} \qquad \frac{v^2}{c^2} = \frac{1}{1 + \frac{m^2 r^4 c^2}{J^2(r^2+r'^2)}}$$

$$1-\frac{v^2}{c^2} = 1-\frac{1}{1+x} = \frac{x}{1+x} = \frac{1}{1+1/x} \quad \to \quad 96$$

$$1-\frac{v^2}{c^2} = \frac{1}{1+\frac{J^2(r^2+r'^2)}{m^2r^4c^2}}$$

$$r=\frac{1}{u} \qquad \frac{r^2+r'^2}{r^4} = u^4\left(\frac{1}{u^2}+\frac{u'^2}{u^4}\right) = u^2+u'^2$$

$$mc^2\sqrt{1+\frac{J^2}{m^2c^2}\left(u^2+u'^2\right)} - \alpha u = E$$

$$-Jcu' = \sqrt{(E+\alpha u)^2 - m^2c^4 - J^2c^2u^2}$$

$$\varphi-\varphi_0 = \frac{Jc}{\sqrt{J^2c^2-\alpha^2}}\arcsin\frac{(\alpha^2-J^2c^2)u+\alpha E}{c\sqrt{J^2(E^2-m^2c^4)+\alpha^2m^2c^2}}$$

$$r = \frac{p}{1+\varepsilon\cos\varkappa(\varphi-\varphi_0)}$$

$$\varkappa = \sqrt{1-\frac{\alpha^2}{J^2c^2}} < 1$$

$$\Delta\varphi_{per} = -2\pi+\frac{2\pi}{\varkappa} = -2\pi\left(1-\frac{1}{\varkappa}\right) \simeq -2\pi\left(1-\left(1+\frac{\alpha^2}{2J^2c^2}\right)\right)$$

$$\simeq \frac{\pi\alpha^2}{J^2c^2} = 7''/100\,\text{lat} \; - \; \frac{1}{6}\ \text{OTW!}$$

RUCH KEPLERA W OTW

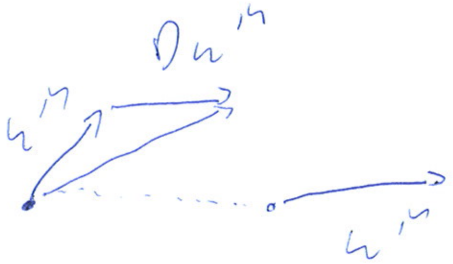
97A

DLA CZĄSTEK SWOBODNYCH

$$\frac{du^\mu}{d\tau} = 0 \Leftrightarrow du^\mu = 0$$

WSP. KRZYWOLINIOWE

$$Du^\mu = 0 \rightarrow$$ KRZYWA PRZESTRZEŃ ZAMIAST POLA GRAW.



$$Du^\mu = du^\mu - \delta u^\mu = du^\mu - \sum_{\alpha,\lambda} \Gamma^\mu_{\alpha\lambda} u^\alpha \delta x^\lambda$$

$$\boxed{\frac{du^\mu}{d\tau} + \sum_{\alpha,\lambda} \Gamma^\mu_{\alpha\lambda} u^\alpha u^\lambda = 0}$$

POLE SCHWARZSCHILDA

$$ds^2 = c^2\left(1 - \frac{2R}{r}\right)dt^2 - \left(\delta_{kl} + \frac{2R}{r-2R}\right)\frac{x_k x_l}{r^2}\, dx_k dx_l$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} \qquad \Gamma^2_{33} = -\sin\theta\cos\varphi$$

$$\frac{d^2\theta}{d\tau^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\theta}{d\tau} - \sin\theta\cos\theta\left(\frac{d\varphi}{d\tau}\right)^2 = 0$$

97B

$\theta_0 = \frac{\pi}{2}$, $\left.\frac{d\theta}{d\tau}\right|_{\tau=0} = 0$ – NIE MA ZALEŻNOŚCI OD α, $\theta(\tau) = \frac{\pi}{2}$

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{d}{dr}\ln\left(1 - \frac{2GM}{rc^2}\right) \qquad \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$$

$$\begin{cases} \frac{d^2x^0}{d\tau^2} + \frac{d}{dr}\ln\left(1 - \frac{2GM}{rc^2}\right)\frac{dr}{d\tau}\frac{dx^0}{d\tau} = 0 \\ \frac{d^2\varphi}{d\tau^2} + \frac{2}{r}\frac{d\varphi}{d\tau}\frac{dr}{d\tau} = 0 = \frac{d}{d\tau}\left(r^2\frac{d\varphi}{d\tau}\right) \end{cases}$$

CZYLI $r^2\frac{d\varphi}{d\tau} = C_1$

TAKŻE

$$\left(1 - \frac{2GM}{rc^2}\right)\frac{dx^0}{d\tau} = C_2$$

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)(dx^0)^2 - \frac{dr^2}{1 - \frac{2GM}{rc^2}} - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$= c^2 d\tau^2$$

97

RÓWNANIE TRAJEKTORII:

$$C_1^{-2} c^2 r^4 d\varphi^2 = \frac{c_2^2 r^4 d\varphi^2}{c_1^2\left(1-\frac{2GM}{rc^2}\right)} - \frac{dr^2}{1-\frac{2GM}{rc^2}} - r^2 d\varphi^2$$

$$u = \frac{1}{r}$$

$$\left(\frac{du}{d\varphi}\right)^2 = -u^2\left(1-\frac{2GMu}{c^2}\right) - \frac{c^2}{C_1^2}\left(1-\frac{2GMu}{c^2}\right) + \frac{C_2^2}{C_1^2}$$

$$\frac{d^2u}{d\varphi^2} = -u + \frac{3GM}{c^2}u^2 + \frac{GM}{C_1^2}$$

$$u = p^{-1}\left(1+\varepsilon\cos(\varkappa\varphi)\right)$$

POMIJAMY ε^2, ZOSTAJE

$$-\varkappa^2 + 1 - \frac{6GM}{pc^2} = 0$$

$$\frac{1}{p} = \frac{3GM}{c^2p^2} + \frac{GM}{C_1^2}$$

STĄD $\frac{1}{p} \approx \frac{GM}{C_1^2}\left(1+\frac{3G^2M^2}{c^2C_1^2}\right)$

$$\varkappa = 1 - \frac{3G^2M^2}{c^2C_1^2}$$

THEORY = 42.9''
EXP = 42.56''

!