An example of a braided locally compact group

S.L. Woronowicz*
Department of Mathematical Methods in Physics,
Faculty of Physics, University of Warsaw,
Hoża 74, 00-682 Warszawa, Poland

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Abstract

The notions of the crossed product and the braided category are discussed within the theory of C*-algebras. Like in the purely algebraic approach of S. Majid these notions are used to generalize the quantum group theory. We give an example of a braided group. It coincides with the closed subset $\overline{\mathbf{C}}^q = \{z : |z| = \dots, q^{-2}, q^{-1}, 1, q, q^2, \dots, 0\}$ of the complex plane equipped with the q^2-braiding: $z_2 z_1 = q^2 z_1 z_2$, $z_2 \overline{z_1} = q^2 \overline{z_1} z_2$. The deformation parameter $0 < q < 1$. $\overline{\mathbf{C}}^q$ is closed under braided addition. The braided group $\overline{\mathbf{C}}^q$ is selfdual and the universal bicharacter describing the selfduality is found.

0 Introduction

In the usual notion of the tensor product $A \otimes B$ of two algebras A and B one assumes that the copies A and B in $A \otimes B$ do commute: for any $a \in A$ and $b \in B$,

$$j_2(b) j_1(a) = j_1(a) j_2(b),$$

where $j_1(a) = a \otimes I_B$ and $j_2(b) = I_A \otimes b$. Replacing this simple low by a more complicated one we arrive to the notion of crossed product of algebras. For instance, considering $\mathbf{Z}_2$-graded algebras and introducing in the right hand side of the above equation $\pm$ sign depending on the grades of homogeneous elements a and b we arrive to the notion of supersymmetrical tensor product. S. Majid proposed to use the rather general commutation rule:

$$j_2(b) j_1(a) = \sum j_1(a_k) j_2(b_k),$$

where $\sum a_k \otimes b_k = R(b \otimes a)$ and the mapping $R : B \otimes A \longrightarrow A \otimes B$ is given in advance. Considering a collection of algebras equipped with the mappings R he arrived to the notion of braided category (to have the associativity of the crossed product one has to assume that the mappings R satisfy the braid equation). Next S. Majid used this

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notion in the theory of quantum groups: using the crossed product as the target of the comultiplication map he introduced the notion of the braided quantum group (see [1] and references [1] – [10] in this paper).

In the present paper we develop the similar scheme within the theory of C*-algebra theory. In Section I we present the general notion of the crossed product of C*-algebras. The reader should notice that it contains the crossed products considered so far (e.g: crossed product of an algebra by a locally compact automorphism group). Section II contains the outline of the theory of braided categories of C*-algebras. Instead of using the braiding mappings, we take the crossed product as a primary concept. The approach starting with the braiding mappings R is not general enough. According to our axioms the sets

$$j_1(A)j_2(B) = \left\{ \sum j_1(a_k)j_2(b_k) : \begin{array}{c} a_1, a_2, \ldots, a_n \in A \\ b_1, b_2, \ldots, b_n \in B \end{array} \right\}$$

and

$$j_2(B)j_1(A) = \left\{ \sum j_2(b_k)j_1(a_k) : \begin{array}{c} a_1, a_2, \ldots, a_n \in A \\ b_1, b_2, \ldots, b_n \in B \end{array} \right\}$$

are dense in the crossed product of A and B. However there is no guarantee that they have a non-trivial intersection. In other words, the braiding R may have too small domain to contain the information necessary to reconstruct the crossed product. The examples presented in Sections I and II serve as environment for the braided group considered in the last section.

On the algebraic level, the *-algebra $\mathcal{A}$ of polynomial functions on this group is generated by a single normal (i.e: commuting with its hermitian adjoint) element λ and the comultiplication is introduced by the formula: $\Delta(\lambda) = j_1(\lambda) + j_2(\lambda)$, whereas braiding is fixed by $j_2(\lambda)j_1(\lambda) = q^2 j_1(\lambda)j_1(\lambda)$ and $j_2(\lambda)j_1(\lambda^*) = j_1(\lambda^*)j_2(\lambda)$.

Let Λ be a closed subset of $\mathbf{C}$ that is not contained in any (real) algebraic curve. Then $\mathcal{A}$ may be identified with the *-algebra of polynomials on Λ. It is interesting to know whether the braiding and the comultiplication related to $\mathcal{A}$ have natural extensions to the C*-algebra $C_\infty(\Lambda)$.

The answer is given in this paper. It depends on the shape of Λ. To include $C_\infty(\Lambda)$ into the suitable braided category, Λ must be invariant under rotations and homotheties with factor being integer powers of q. To prove the existence of Δ on the level of $C_\infty(\Lambda)$, Λ must be of the form $\tau\overline{\mathbf{C}}^q$, where

$$\overline{\mathbf{C}}^q = \left\{ z \in \mathbf{C} : z = 0 \text{ or } |z| \in q^{\mathbf{Z}} \right\}.$$

and τ is a strictly positive number (cf Remark 3.6).

For non-unital C*-algebras, we shall freely use the notation introduced in our previous papers (see e.g:[4]). In particular for any C*-algebra A, $M(A)$ will denote the multiplier algebra of A. The set of all morphisms acting from A into B (A and B are C*-algebras) is denoted by $\mathrm{Mor}(A, B)$. We recall that $\varphi \in \mathrm{Mor}(A, B)$ if φ is a *-homomorphism mapping A into $M(B)$ such that the set $\varphi(A)B$ is norm dense in B. It is known that any $\varphi \in \mathrm{Mor}(A, B)$ admits unique extension to a unital *-homomorphism mapping $M(A)$ into $M(B)$. The unit of $M(A)$ and the identity morphism acting on A will be denoted by I_A and id_A respectively (in general for any set Λ, id_Λ is the identity map acting on Λ). The subscript may be omitted if the C*-algebra A is determined by the context. We shall also use unbounded elements affiliated with C*-algebras. Affiliation relation will be denoted by η. Morphisms can act on affiliated elements: if $a \,\eta\, A$ and $\varphi \in \mathrm{Mor}(A, B)$ then $\varphi(a)$ is a well defined element affiliated with B.

For any locally compact topological space Λ, $C_\infty(\Lambda)$ will denote the C*-algebra of all continuous functions vanishing at infinity on Λ. Then $M(C_\infty(\Lambda)) = C_{\mathrm{bounded}}(\Lambda)$ is the algebra of all continuous bounded functions on Λ. Elements affiliated with $C_\infty(\Lambda)$ are continuous functions on Λ.

The algebra of all compact operators acting on a Hilbert space H will be denoted by $CB(H)$. $M(CB(H)$ coincides with the algebra B(H) of all bounded operators acting on H. Elements affiliated with $CB(H)$ are closed operators acting on H.

Let A be a C*-algebra, G be a locally compact topological group, $e \in \mathrm{Mor}(C_\infty(G), \mathbf{C})$ and $\Delta_G \in \mathrm{Mor}(C_\infty(G), C_\infty(G) \otimes C_\infty(G))$ be the counit and comultiplication related to the group structure of G. We recall that an action of G on A is by definition a morphism $\alpha \in \mathrm{Mor}(A, A \otimes C_\infty(G))$ such that

$$(\mathrm{id} \otimes e)\alpha = \mathrm{id}_A \tag{0.1}$$

and the diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad\alpha\quad} & A \otimes C_\infty(G) \\ \alpha \Big\downarrow & & \Big\downarrow \alpha \otimes \mathrm{id} \\ A \otimes C_\infty(G) & \xrightarrow[\mathrm{id} \otimes \Delta_G]{} & A \otimes C_\infty(G) \otimes C_\infty(G) \end{array}$$

is commutative.

Denoting by $\chi_g \in \mathrm{Mor}(C_\infty(G), \mathbf{C})$ the evaluation functional ($\chi_g(f) = f(g)$ for any $f \in C_\infty(G)$ and $g \in G$) and setting

$$\alpha_g = (\mathrm{id} \otimes \chi_g)\alpha$$

we obtain a family $(\alpha_g)_{g \in G}$ of automorphisms of A such that $\alpha_{\text{neutral element}} = \mathrm{id}_G$, $\alpha_g \alpha_{g'} = \alpha_{gg'}$ for any $g, g' \in G$ and for any $a \in A$, $\alpha_g(a)$ is strictly continuous[1] with respect to g. Any family of automorphisms of A obeying the above properties can be obtained in this way.

In what follows the group $G = \mathbf{S}^1 \times \mathbf{Z}$ will play a distinguished role. Let $\alpha \in \mathrm{Mor}(A, A \otimes C_\infty(G))$ be an action of this group on a C*-algebra A. Then, for any $(\zeta, s) \in G$,

$$\alpha_{(\zeta,s)} = \beta_\zeta \gamma^s, \tag{0.2}$$

[1] In fact one can show that it is norm continuous.

where $\beta \in \mathrm{Mor}(A, A \otimes C(\mathbf{S}^1))$ is an action of $\mathbf{S}^1$ on A and γ is an automorphism of A commuting with the action β. The action β defines a $\mathbf{Z}$-grading of A:

$$A = \left\{ \sum_{k \in \mathbf{Z}}^{\oplus} A_k \right\}^{\text{norm closure}}, \tag{0.3}$$

where for any $k \in \mathbf{Z}$,

$$A_k = \{a \in A : \beta_\zeta(a) = \zeta^k a \text{ for all } \zeta \in \mathbf{S}^1\}$$

is the subspace of homogeneous elements of degree k. Clearly the decomposition (0 .3) is γ-invariant.

For any $a, b \in \mathbf{Z}$ and $(\zeta, s) \in G$ we put

$$f_{kl}(\zeta, s) = \zeta^k \delta_{ls}. \tag{0.4}$$

Then f_{kl} $(k, l \in \mathbf{Z})$ are continuous functions on G with compact support. Therefore $f_{kl} \in C_\infty(G)$. Functions f_{kl} $(k, l \in \mathbf{Z})$ are linearly independent and any element of $C_\infty(G)$ can be approximated by a finite linear combination of f_{kl} $(k, l \in \mathbf{Z})$. Using the Fourier series expansion and taking into account (0 .2) one can easily obtain

$$\alpha(a) = \sum_{rs \in \mathbf{Z}} \int_{\mathbf{S}^1} \beta_\zeta(\gamma^s(a)) \frac{\zeta^{-r} d\zeta}{2\pi i \zeta} \otimes f_{rs} \tag{0.5}$$

for any $a \in A$[2]. If a is a homogeneous element of degree k: $a \in A_k$, then the above formula becomes much simpler:

$$\alpha(a) = \sum_{s \in \mathbf{Z}} \gamma^s(a) \otimes f_{ks}. \tag{0.6}$$

1 Crossed products

Definition 1.1 *Let A, B, C be C^*-algebras, $j_1 \in \mathrm{Mor}(A, C)$, $j_2 \in \mathrm{Mor}(B, C)$ and*

$$j\left(\sum a_i \otimes b_i\right) = \sum j_1(a_i) j_2(b_i)$$

for any $a_1, a_2, \dots a_n \in A$ and $b_1, b_2, \dots b_n \in B$. Then $j : A \otimes_{\mathrm{alg}} B \to M(C)$ is a linear map. We say that (j_1, j_2, C) is a crossed product of A and B if:

1. *$j(A \otimes_{\mathrm{alg}} B)$ is a dense subset of C.*
2. *For any $x \in A \otimes_{\mathrm{alg}} B$,*

$$\left\{ j(x) = 0 \right\} \Longrightarrow \left\{ x = 0 \right\}. \tag{1.1}$$

Two crossed products (j_1, j_2, C) and (j'_1, j'_2, C') of A and B are said to be equivalent if there exists a C^*-isomorphism $\varphi : C \to C'$ such that $j'_1 = \varphi \circ j_1$ and $j'_2 = \varphi \circ j_2$

[2]It is well known that the Fourier series expansion converges uniformly only if the expanded function is smooth enough. For this reason, to have a good convergence of the series (0 .5) one has to impose a smoothness condition on the element a. It is sufficient to assume that $\beta_\zeta(a)$ is twicely differentiable with respect to $\zeta \in \mathbf{S}^1$. If this is the case then the series (0 .5) is norm converging. The reader should notice that the elements a with this property form a dense subset of A.

For the present paper the above definition is sufficient. We believe however that for the future applications it should be supplemented by a condition of topological nature. For example in all interesting cases the mapping j (considered as a densely defined mapping acting from Banach space $A \otimes B$ into Banach space C) is closeable and the implication (1.1) holds for any x belonging to the domain of the closure of j. The reader should notice that this condition depends on the choice of the C*-tensor product $A \otimes B$ (which may not be unique for non-nuclear A and B). In particular one may consider spacial and maximal crossed products. Another interesting condition says that the image $j(F \otimes B + A \otimes G)$ should be a norm-closed subspace of C for any finite-dimensional subspaces $F \subset A$ and $G \subset B$.

The reader should notice that the notion of crossed product introduced in Definition 1.1 generalizes that of C*-tensor product. A crossed product (j_1, j_2, C) of C*-algebras A and B is a tensor product if and only if $j_1(a)j_2(b) = j_2(b)j_1(a)$ for any $a \in A$ and $b \in B$. If this is a case then $C = A \otimes B$, $j_1(a) = a \otimes I_B$ and $j_2(b) = I_A \otimes b$ for any $a \in A$ and $b \in B$.

Remark 1.2 *Let A, B and D be C^*-algebras and (j_1, j_2, C) be a crossed product of A and B. Then any morphism $\varphi \in \mathrm{Mor}(C, D)$ is uniquely determined by $\varphi\circ j_1$ and $\varphi\circ j_1$.*

Indeed we have: $\varphi\left(j\left(\sum(a_i \otimes b_i)\right)\right) = \sum(\varphi\circ j_1)(a_i)(\varphi\circ j_2)(b_i)$.

We shall present an example of a crossed product that will play an essential role in the rest of the paper. Let $(e_{ab})_{a,b\in\mathbf{Z}}$ be the canonical basis in the Hilbert space $H = \ell^2(\mathbf{Z} \times \mathbf{Z})$ and v_1, v_2 be unitary and n_1, n_2 be selfadjoint operators acting on H such that

$$\left.\begin{array}{ll} v_1 e_{ab} = e_{a+1,b}, & n_1 e_{ab} = b e_{a,b}, \\ v_2 e_{ab} = e_{a,b-1}, & n_2 e_{ab} = a e_{a,b}. \end{array}\right\} \tag{1.2}$$

One can easily verify that (v_1, v_2), (v_1, n_1), (v_2, n_2) and (n_1, n_2) are pairs of strongly commuting operators and that

$$v_1^* n_2 v_1 = n_2 + I,$$

$$v_2 n_1 v_2^* = n_1 + 1.$$

The joint spectrum

$$\mathrm{Sp}\,(v_1, n_1) = \mathrm{Sp}\,(v_2, n_2) = \mathbf{S}^1 \times \mathbf{Z}$$

will be denoted by $G = \mathbf{S}^1 \times \mathbf{Z}$. Then G is a locally compact abelian selfdual group.

We shall use the functional calculus of strongly commuting normal operators. For any $f \in C_\infty(G)$ we set

$$\left.\begin{array}{l} i_1(f) = f(v_1, n_1), \\ i_2(f) = f(v_2, n_2). \end{array}\right\} \tag{1.3}$$

Clearly $i_1(f), i_2(f) \in B(H)$ and $i_1, i_2 \in \mathrm{Mor}(C_\infty(G), CB(H))$.

Let $f_{kl} \in C_\infty(G)$ $(k, l \in \mathbf{Z})$ be functions introduced by (0 .4). One can easily verify that

$$i_1(f_{kl}) e_{ab} = \delta_{lb} e_{a+k,b},$$

$$i_2(f_{kl})e_{ab} = \delta_{la}e_{a,b-k}$$

for any integers k, l, a, b. Combining these two results we get:

$$i_1(f_{kl})i_2(f_{rs}) = |e_{s+k,l})(e_{s,l+r}|. \tag{1.4}$$

Taking the hermitian conjugation of the both sides and replacing k and r by $-k$ and $-r$ respectively we obtain:

$$i_2(f_{rs})i_1(f_{kl}) = |e_{s,l-r})(e_{s-k,l}|.$$

Combining the last two formulae we get:

$$i_2(f_{rs})i_1(f_{kl}) = i_1(f_{k,l-r})i_2(f_{r,s-k}) \tag{1.5}$$

for all $k, l, r, s \in \mathbf{Z}$.

Let $f, g \in C_\infty(G)$. Using the Fourier series decomposition to write f and g as a linear combination of f_{kl} $(k, l \in \mathbf{Z})$ and applying (1.4) we get

$$(e_{ab}|i_1(f)i_2(g)|e_{cd}) = \int_{\mathbf{S}^1}\int_{\mathbf{S}^1} \zeta_1^{c-a}\zeta_2^{b-d} f(\zeta_1, b)g(\zeta_2, c)\frac{d\zeta_1}{2\pi i\zeta_1}\frac{d\zeta_2}{2\pi i\zeta_2} \tag{1.6}$$

For any $f_1, f_2, \ldots f_n, g_1, g_2, \ldots g_n \in C_\infty(G)$ we set

$$i(\sum f_k \otimes g_k) = \sum i_1(f_k)i_2(g_k) \tag{1.7}$$

Then $i : C_\infty(G) \otimes_{\rm alg} C_\infty(G) \to B(H)$ is a linear map. By virtue of (1.4), the image of this map is a dense subset of $CB(H)$. Identifying $C_\infty(G) \otimes_{\rm alg} C_\infty(G)$ with a subspace of $C_\infty(G \times G)$ and using formula (1.6) we obtain

$$(e_{ab}|i(h)|e_{cd}) = \int_{\mathbf{S}^1}\int_{\mathbf{S}^1} \zeta_1^{c-a}\zeta_2^{b-d} h(\zeta_1, b; \zeta_2, c)\frac{d\zeta_1}{2\pi i\zeta_1}\frac{d\zeta_2}{2\pi i\zeta_2} \tag{1.8}$$

for any $h \in C_\infty(G) \otimes_{\rm alg} C_\infty(G)$ and any integers a, b, c, d. Assume for the moment that $i(h) = 0$. Then the integral on the right hand side of (1.8) vanishes for all integers a, b, c, d and using the Fourier series expansion we see that $h(\zeta_1, b; \zeta_2, c) = 0$ for any $(\zeta_1, b; \zeta_2, c) \in G \times G$. It shows that $\ker(i) = \{0\}$. This way we proved that $(i_1, i_2, CB(H))$ is a crossed product of $C_\infty(G)$ by itself in the sense of Definition 1.1

Remark: Using the formula (1.8) in a more sophisticated way one can show that the map i considered as a densely defined linear mapping acting from Banach space $C_\infty(G)\otimes C_\infty(G) = C_\infty(G\times G)$ into $CB(H)$ is closeable and that the kernel of its closure is trivial.

It turns out that the crossed product of $C_\infty(G)$ by itself described above may be used as a pattern for a more general construction. Let A and B be C*-algebras endowed with actions of the group G. We shall use the notation introduced in Section 0. In particular the actions of G will be denoted by the same letter α and β and γ will be related with α by (0 .2). In the case of possible misunderstanding we shall use subscripts writing e.g: α_A or α_B instead of α. By σ (or more precisely $\sigma_{BA} \in \mathrm{Mor}(B \otimes A, A \otimes B)$) we denote the flip automorphism: $\sigma(b \otimes a) = a \otimes b$.

For any $a \in A$ and $b \in B$ we set

$$\left.\begin{aligned} j_1(a) &= (\sigma_{AB} \otimes i_1)(I_B \otimes \alpha_A(a)), \\ j_2(b) &= I_A \otimes (\mathrm{id}_B \otimes i_2)\alpha_B(b). \end{aligned}\right\} \tag{1.9}$$

Clearly $j_1 \in \mathrm{Mor}(A, A \otimes B \otimes CB(H))$ and $j_2 \in \mathrm{Mor}(B, A \otimes B \otimes CB(H))$. If a and b are homogeneous elements: $a \in A_k$ and $b \in B_l$, then using (0 .6) we get:

$$\begin{aligned} j_1(a) &= \sum_{s \in \mathbf{Z}} \gamma_A^s(a) \otimes I_B \otimes i_1(f_{ks}), \\ j_2(b) &= \sum_{r \in \mathbf{Z}} I_A \otimes \gamma_B^r(b) \otimes i_2(f_{lr}). \end{aligned}$$

Taking into account (1.5) we compute:

$$\begin{aligned} j_2(b)j_1(a) &= \sum_{s,r \in \mathbf{Z}} \gamma^s(a) \otimes \gamma^r(b) \otimes i_2(f_{lr})i_1(f_{ks}) \\ &= \sum_{s,r \in \mathbf{Z}} \gamma^s(a) \otimes \gamma^r(b) \otimes i_1(f_{k,s-l})i_2(f_{l,r-k}) \\ &= \sum_{s,r \in \mathbf{Z}} \gamma^{s+l}(a) \otimes \gamma^{r+k}(b) \otimes i_1(f_{k,s})i_2(f_{l,r}) \quad = j_1(\gamma^l(a))j_2(\gamma^k(b)) \end{aligned}$$

This way we showed that for any homogeneous elements $a \in A$ and $b \in B$ we have:

$$j_2(b)j_1(a) = j_1(\gamma_A^l(a))j_2(\gamma_B^k(b)), \tag{1.10}$$

where k and l are degrees of a and b respectively.

Let $j : A \otimes_{\mathrm{alg}} B \longrightarrow M(A \otimes B \otimes CB(H))$ be a linear map such that

$$j(\sum a_i \otimes b_i) = \sum j_1(a_i)j_2(b_i) \tag{1.11}$$

for any $a_1, a_2, \ldots a_n \in A$ and $b_1, b_2, \ldots b_n \in B$. We denote by $A \otimes_c B$ the norm closure of the image of j. It follows immediately from (1.10) that the image of j contains a dense subset closed under products and hermitian conjugation. Therefore $A \otimes_c B$ is a C*-algebra. Let $a \in A$. It follows immediately from the definition, that $j_1(a)$ and $j_1(a)^* = j_1(a^*)$ are left multipliers of $A \otimes_c B$. Therefore $j_1(a)$ is a multiplier of $A \otimes_c B$ and $j_1(A) \subset M(A \otimes_c B)$. Moreover, remembering that $A \cdot A$ is dense in A for any C*-algebra A, one can easily show that $j_1(A)(A \otimes_c B)$ is dense in $A \otimes_c B$. It means that $j_1 \in \mathrm{Mor}(A, A \otimes_c B)$. In the similar way one can prove that $j_2 \in \mathrm{Mor}(B, A \otimes_c B)$.

Combining the definitions (1.11), (1.9) and (1.7) we see that

$$j = (\mathrm{id}_{A \otimes B} \otimes i)(\mathrm{id}_A \otimes \sigma \otimes \mathrm{id}_{C_\infty(G)})(\alpha_A \otimes \alpha_B),$$

where the flip $\sigma = \sigma_{C_\infty(G),B}$. Due to (0 .1), the morphisms α_A and α_B are embedings. We already showed that $\ker(i) = \{0\}$. Therefore $\ker(j) = \{0\}$. It means, that $(j_1, j_2, A \otimes_c B)$ is a crossed product of A and B.

If the action of G on one of the algebras A or B is trivial, then $A \otimes_{\mathcal{C}} B$ coincides with $A \otimes B$. Indeed if for example the action of G on A is trivial, then all elements of A are of degree $k = 0$, $\gamma_A = \mathrm{id}_A$ and formula (1.10) shows that $j^2(b)j^1(a) = j^1(a)j^2(b)$ for any $a \in A$ and $b \in B$.

The case, when one of the algebras A or B is replaced by $C_\infty(G)$ (with the action of G given by $\alpha_{C_\infty(G)} = \Delta_G$) is also interesting.

For any $(\zeta, s) \in G$ we put

$$\left.\begin{aligned} v(\zeta, s) &= \zeta, \\ n(\zeta, s) &= s. \end{aligned}\right\} \qquad (1.12)$$

Then v and n are continuous functions on G. It means that v and n are affiliated with $C_\infty(G)$: $v, n \,\eta\, C_\infty(G)$. Elements v, n generate $C_\infty(G)$ in the sense explained in [4]. The comultiplication Δ_G acts on these generators in the following way:

$$\Delta_G(v) = v \otimes v,$$

$$\Delta_G(n) = n \otimes I + I \otimes n.$$

According to (1.3) we have:

$$\begin{aligned} i_1(v) &= v_1, & i_1(n) &= n_1, \\ i_2(v) &= v_2, & i_2(n) &= n_2. \end{aligned}$$

Using these formulae one can show, that the crossed products $(i_1, i_2, CB(H))$ and $(j_1, j_2, C_\infty(G) \otimes_{\mathcal{C}} C_\infty(G))$ of $C_\infty(G)$ by itself are equivalent.

2 Braided categories of C*-algebras

In this section we shall use to some extend the language of the theory of categories. In particular the notions of object, morphism, functor and natural mapping will be used. The class of objects of a category $\mathcal{C}$ will be denoted by $\mathrm{Ob}_{\mathcal{C}}$ and the set of morphisms of $\mathcal{C}$ acting from A into B ($A, B \in \mathrm{Ob}_{\mathcal{C}}$) will be denoted by $\mathrm{Mor}_{\mathcal{C}}(A, B)$. In the following definition Proj_1 and Proj_2 denote the canonical projections acting from $\mathcal{C} \times \mathcal{C}$ onto $\mathcal{C}$. Clearly Proj_1 and Proj_2 are covariant functors.

Definition 2.1 *Let $(\mathcal{C}, \otimes_{\mathcal{C}}, j_1, j_2)$ be a quadruple consisting of a category $\mathcal{C}$, a covariant functor $\otimes_{\mathcal{C}}$ acting from $\mathcal{C} \times \mathcal{C}$ into $\mathcal{C}$ and natural mappings j_1 acting from Proj_1 into $\otimes_{\mathcal{C}}$ and j_2 acting from Proj_2 into $\otimes_{\mathcal{C}}$. We say that $(\mathcal{C}, \otimes_{\mathcal{C}}, j_1, j_2)$ is a monoidal category if*

I. *For any $A, B, C \in \mathrm{Ob}_{\mathcal{C}}$ and any $\varphi, \varphi' \in \mathrm{Mor}_{\mathcal{C}}(A \otimes_{\mathcal{C}} B, C)$*

$$\left\{\begin{aligned} \varphi \circ j_1(A, B) &= \varphi' \circ j_1(A, B) \\ \varphi \circ j_2(A, B) &= \varphi' \circ j_2(A, B) \end{aligned}\right\} \Longrightarrow \left\{ \varphi = \varphi' \right\}$$

II. $\otimes_c$ *is associative, i.e: for any* $A, B, C \in \mathrm{Ob}_c$*, there exists an isomorphism* $\Psi_{ABC} \in \mathrm{Mor}_c((A \otimes_c B) \otimes_c C, A \otimes_c (B \otimes_c C))$ *such that*

$$\left.\begin{aligned} j_1(A, B \otimes_c C) &= \Psi_{ABC} \circ j_1(A \otimes_c B, C) \circ j_1(A, B), \\ j_2(A, B \otimes_c C) \circ j_1(B, C) &= \Psi_{ABC} \circ j_1(A \otimes_c B, C) \circ j_2(A, B), \\ j_2(A, B \otimes_c C) \circ j_2(B, C) &= \Psi_{ABC} \circ j_2(A \otimes_c B, C). \end{aligned}\right\} \tag{2.1}$$

A few words of explanation. The above definition means, that $\otimes_c$ is a binary operation acting on objects and morphisms of $\mathcal{C}$. For any $A, B \in \mathrm{Ob}_c$, $j_1(A, B)$ is a morphism acting from A into $A \otimes_c B$: $j_1(A, B) \in \mathrm{Mor}_c(A, A \otimes_c B)$. Similarly $j_2(A, B) \in \mathrm{Mor}_c(B, A \otimes_c B)$. Moreover for any $\varphi \in \mathrm{Mor}_c(A, A')$ and $\psi \in \mathrm{Mor}_c(B, B')$ the diagrams

$$\begin{array}{ccc} A & \xrightarrow{\;j_1(A,B)\;} & A \otimes_c B \\ \varphi \big\downarrow & & \big\downarrow \varphi \otimes_c \psi \\ A' & \xrightarrow[\;j_1(A',B')\;]{} & A' \otimes_c B' \end{array} , \qquad \begin{array}{ccc} B & \xrightarrow{\;j_2(A,B)\;} & A \otimes_c B \\ \psi \big\downarrow & & \big\downarrow \varphi \otimes_c \psi \\ B' & \xrightarrow[\;j_2(A',B')\;]{} & A' \otimes_c B' \end{array}$$

are commutative.

Condition I says that $A \otimes_c B$ is in a certain sense generated by $j_1(A)$ and $j_2(B)$. It implies that the isomorphism Ψ_{ABC} entering Condition II is unique. Due to Condition II we may identify $(A \otimes_c B) \otimes_c C$ with $A \otimes_c (B \otimes_c C)$ by setting $\Psi_{ABC} = \mathrm{id}$. With this identification, $\otimes_c$ becomes associative also on the level of morphisms. This fact follows easily from condition I. In what follows, writing the multiple $\otimes_c$ product we shall omit brackets. For any $A, B, C \in \mathrm{Ob}_c$ we set:

$$\begin{aligned} j_1(A, B, C) &= j_1(A, B \otimes_c C) &&= j_1(A \otimes_c B, C) \circ j_1(A, B), \\ j_2(A, B, C) &= j_2(A, B \otimes_c C) \circ j_1(B, C) &&= j_1(A \otimes_c B, C) \circ j_2(A, B), \\ j_3(A, B, C) &= j_2(A, B \otimes_c C) \circ j_2(B, C) &&= j_2(A \otimes_c B, C). \end{aligned}$$

Clearly $j_1(A, B, C)$ ($j_2(A, B, C)$, $j_3(A, B, C)$ respectively) is a morphism acting from A (B, C respectively) into $A \otimes_c B \otimes_c C$. In the similar way for any $A_1, A_2, \ldots, A_n \in \mathrm{Ob}_c$ one may introduce morphisms $j_k(A_1, A_2, \ldots, A_n) \in \mathrm{Mor}_c(A_k, A_1 \otimes_c A_2 \otimes_c \ldots \otimes_c A_n)$ ($k = 1, 2, \ldots, n$). These morphisms are called canonical.

Definition 2.2 *Let* $(\mathcal{C}, \otimes_c, j_1, j_2)$ *be a monoidal category such that objects of* $\mathcal{C}$ *are* C^**-algebras endowed with an additional structure and for any* $A, B \in \mathrm{Ob}_c$*,* $\mathrm{Mor}_c(A, B)$ *is a subset of* $\mathrm{Mor}(A, B)$ *consisting of all morphisms preserving this additional structure. We say that* $(\mathcal{C}, \otimes_c, j_1, j_2)$ *is a braided category of* C^**-algebras, if for any* $A, B \in \mathrm{Ob}_c$*,* $(j_1(A, B), j_2(A, B), A \otimes_c B)$ *is a crossed product of* A *and* B*.*

The reader should notice that for braided categories of C^*-algebras, Condition I of Definition 2.1 is automatically satisfied (cf Remark 1.2).

The crossed product of C^*-algebras endowed with actions of $G = \mathbf{S}^1 \times \mathbf{Z}$ group, introduced in the previous section, gives rise to an interesting braided category $(\mathcal{C}, \otimes_{\mathcal{C}}, j_1, j_2)$ of C^*-algebras. By definition objects of $\mathcal{C}$ are C^*-algebras endowed with an action of the group G. Morphisms of the category $\mathcal{C}$ are C^*-morphisms intertwining the actions of G: For any $A, B \in \mathrm{Ob}_{\mathcal{C}}$,

$$\mathrm{Mor}_{\mathcal{C}}(A, B) = \left\{\varphi \in \mathrm{Mor}(A, B) : \alpha_B \circ \varphi = (\varphi \otimes \mathrm{id}_{C_\infty(G)}) \circ \alpha_A\right\}$$

In other words, an element $\varphi \in \mathrm{Mor}(A, B)$ is a $\mathcal{C}$-morphism if and only if

$$\alpha_{Bg} \circ \varphi = \varphi \circ \alpha_{Ag}$$

for any $g \in G$. The reader should notice that $\alpha_{Ag} \in \mathrm{Mor}_{\mathcal{C}}(A, A)$ for any $A \in \mathrm{Ob}_{\mathcal{C}}$ and $g \in G$. This fact follows immediately from the abelianess of G.

The $\otimes_{\mathcal{C}}$-product of $\mathcal{C}$-objects and the families of morphism $j_1(A, B)$ and $j_2(A, B)$ $(A, B \in \mathrm{Ob}_{\mathcal{C}})$ are introduced in the previous section. At the moment, the C^*-algebras $A \otimes_{\mathcal{C}} B$ $(A, B \in \mathrm{Ob}_{\mathcal{C}})$ are not endowed with an action of G and consequently $j_1(A, B)$ and $j_2(A, B)$ are not $\mathcal{C}$-morphisms yet. To achieve the construction of $(\mathcal{C}, \otimes_{\mathcal{C}}, j_1, j_2)$ we have to introduce a $\otimes_{\mathcal{C}}$-product of morphisms and a natural action of G on all $A \otimes_{\mathcal{C}} B$ $(A, B \in \mathrm{Ob}_{\mathcal{C}})$ and then to show that the $\otimes_{\mathcal{C}}$-product of morphisms, $j_1(A, B)$ and $j_2(A, B)$ are $\mathcal{C}$-morphisms. We start with the $\otimes_{\mathcal{C}}$-product of morphisms.

Proposition 2.3 *Let $A, A', B, B' \in \mathrm{Ob}_{\mathcal{C}}$, $\varphi \in \mathrm{Mor}_{\mathcal{C}}(A, A')$ and $\psi \in \mathrm{Mor}_{\mathcal{C}}(B, B')$. Then there exists unique $\varphi \otimes_{\mathcal{C}} \psi \in \mathrm{Mor}(A \otimes_{\mathcal{C}} B, A' \otimes_{\mathcal{C}} B')$ such that*

$$\left.\begin{aligned} (\varphi \otimes_{\mathcal{C}} \psi) \circ j_1(A, B) &= j_1(A', B') \circ \varphi, \\ (\varphi \otimes_{\mathcal{C}} \psi) \circ j_2(A, B) &= j_2(A', B') \circ \psi. \end{aligned}\right\} \tag{2.2}$$

If moreover $A'', B'' \in \mathrm{Ob}_{\mathcal{C}}$, $\varphi' \in \mathrm{Mor}_{\mathcal{C}}(A', A'')$ and $\psi' \in \mathrm{Mor}_{\mathcal{C}}(B', B'')$, then

$$(\varphi' \otimes_{\mathcal{C}} \psi') \circ (\varphi \otimes_{\mathcal{C}} \psi) = \varphi' \varphi \otimes_{\mathcal{C}} \psi' \psi \tag{2.3}$$

Proof: According to Section 2, $A \otimes_{\mathcal{C}} B$ and $A' \otimes_{\mathcal{C}} B'$ are subalgebras of $A \otimes B \otimes CB(H)$ and $A' \otimes B' \otimes CB(H)$ respectively. Let $\varphi \in \mathrm{Mor}_{\mathcal{C}}(A, A')$ and $\psi \in \mathrm{Mor}_{\mathcal{C}}(B, B')$. Using definitions (1.9) and remembering that φ (ψ respectively) intertwines the actions of G on A and A' (B and B' respectively) one can easily verify that

$$\left.\begin{aligned} (\varphi \otimes \psi \otimes \mathrm{id}_{CB(H)}) \circ j_1(A, B) &= j_1(A', B') \circ \varphi, \\ (\varphi \otimes \psi \otimes \mathrm{id}_{CB(H)}) \circ j_2(A, B) &= j_2(A', B') \circ \psi. \end{aligned}\right\} \tag{2.4}$$

It is not difficult to see that these relations imply that $\varphi \otimes \psi \otimes \mathrm{id}_{CB(H)}$ maps $A \otimes_{\mathcal{C}} B$ into $M(A' \otimes_{\mathcal{C}} B')$ in a non-degenerate way. Therefore denoting by $\varphi \otimes_{\mathcal{C}} \psi$ the restriction of $\varphi \otimes \psi \otimes \mathrm{id}_{CB(H)}$ to $A \otimes_{\mathcal{C}} B$ we have: $\varphi \otimes_{\mathcal{C}} \psi \in \mathrm{Mor}(A \otimes_{\mathcal{C}} B, A' \otimes_{\mathcal{C}} B')$. Formulae (2.2) follows immediately from (2.4).

By virtue of (2.2) we have:

$$\begin{aligned}(\varphi'\otimes_{\mathcal{C}}\psi')\circ(\varphi\otimes_{\mathcal{C}}\psi)\circ j_1(A,B) &= (\varphi'\otimes_{\mathcal{C}}\psi')\circ j_1(A',B')\circ\varphi\\ &= j_1(A'',B'')\circ\varphi'\circ\varphi = (\varphi'\varphi\otimes_{\mathcal{C}}\psi'\psi)\circ j_1(A,B),\\ (\varphi'\otimes_{\mathcal{C}}\psi')\circ(\varphi\otimes_{\mathcal{C}}\psi)\circ j_2(A,B) &= (\varphi'\otimes_{\mathcal{C}}\psi')\circ j_2(A',B')\circ\psi\\ &= j_2(A'',B'')\circ\psi'\circ\psi = (\varphi'\varphi\otimes_{\mathcal{C}}\psi'\psi)\circ j_2(A,B)\end{aligned}$$

and (2.3) follows (cf Remark 1.2).

Q.E.D.

Proposition 2.4 *For any $A, B \in \mathrm{Ob}_{\mathcal{C}}$ there exists the unique action α_C of G on the crossed product $C = A\otimes_{\mathcal{C}} B$ such that*

$$\alpha_{Cg} = \alpha_{Ag}\otimes_{\mathcal{C}}\alpha_{Bg} \tag{2.5}$$

for any $g\in G$. This way, for any $A, B\in \mathrm{Ob}_{\mathcal{C}}$, the crossed product $A\otimes_{\mathcal{C}} B$ becomes an object of the category $\mathcal{C}$.

Proof: It sufficient to show that the right hand side of (2.5) is multiplicative (this fact follows immediately from (2.3)) and pointwise strictly continuous with respect to g. The later follows from the pointwise strict continuity of α_{Ag} and α_{Bg}. Indeed morphisms are strictly continuous and $A\otimes_{\mathcal{C}} B$ is generated by the images of $j_1(A,B)$ and $j_2(A,B)$.

Q.E.D.

Let $A, A', B, B' \in \mathrm{Ob}_{\mathcal{C}}$, $\varphi\in \mathrm{Mor}_{\mathcal{C}}(A,A')$ and $\psi\in\mathrm{Mor}_{\mathcal{C}}(B,B')$. Then for any $g\in G$ we have:

$$\begin{aligned}(\alpha_{A'g}\otimes_{\mathcal{C}}\alpha_{B'g})\circ(\varphi\otimes_{\mathcal{C}}\psi) &= \alpha_{A'g}\varphi\otimes_{\mathcal{C}}\alpha_{B'g}\psi\\ &= \varphi\alpha_{Ag}\otimes_{\mathcal{C}}\psi\alpha_{Bg} = (\varphi\otimes_{\mathcal{C}}\psi)\circ(\alpha_{Ag}\otimes_{\mathcal{C}}\alpha_{Bg}).\end{aligned}$$

It shows that $\varphi\otimes_{\mathcal{C}}\psi$ intertwins the actions of G on $A\otimes_{\mathcal{C}} B$ and $A'\otimes_{\mathcal{C}} B'$. Therefore $\varphi\otimes_{\mathcal{C}}\psi\in\mathrm{Mor}_{\mathcal{C}}(A\otimes_{\mathcal{C}} B, A'\otimes_{\mathcal{C}} B')$.

Inserting in (2.2), $A'=A$, $B'=B$, $\varphi=\alpha_{Ag}$ and $\psi=\alpha_{Bg}$ we see that $j_1(A,B)$ ($j_1(A,B)$ respectively) intertwins the actions of G on A (B respectively) and $A\otimes_{\mathcal{C}} B$. It means that $j_1(A,B)\in\mathrm{Mor}_{\mathcal{C}}(A, A\otimes_{\mathcal{C}} B)$ and $j_2(A,B)\in\mathrm{Mor}_{\mathcal{C}}(B, A\otimes_{\mathcal{C}} B)$. Now, formulae (2.2) show that j_1 and j_2 are natural mappings acting from Proj_1 and Proj_2 into $\otimes_{\mathcal{C}}$.

We have to show the associativity of $\otimes_{\mathcal{C}}$. The proof will be highly computational. Let $(e_{kl})_{k,l\in\mathbf{Z}}$ be the orthonormal basis considered in Section 1 and W be a unitary operator acting on $H\otimes H$ such that

$$We_{kl}\otimes e_{rs} = e_{r-k,s}\otimes e_{k,l+s}$$

By simple computations one can verify, that

$$\left.\begin{aligned} W(v_1\otimes v_1) &= (I_{CB(H)}\otimes v_1)W,\\ W(v_2\otimes v_1) &= (v_1\otimes v_2)W,\\ W(I_{CB(H)}\otimes v_2) &= (v_2\otimes v_2)W.\end{aligned}\right\} \tag{2.6}$$

and

$$\left.\begin{aligned} W(n_1 \otimes I_{CB(H)} + I_{CB(H)} \otimes n_1) &= (I_{CB(H)} \otimes n_1)W, \\ W(n_2 \otimes I_{CB(H)} + I_{CB(H)} \otimes n_1) &= (n_1 \otimes I_{CB(H)} + I_{CB(H)} \otimes n_2)W, \\ W(I_{CB(H)} \otimes n_2) &= (n_2 \otimes I_{CB(H)} + I_{CB(H)} \otimes n_2)W \end{aligned}\right\} \tag{2.7}$$

For any bounded operator Q acting on $H \otimes H$ we set

$$\psi(Q) = WQW^*$$

Then ψ is an automorphism belonging to $\mathrm{Mor}(CB(H) \otimes CB(H), CB(H) \otimes CB(H))$. We claim that

$$\left.\begin{aligned} \psi((i_1 \otimes i_1)\Delta_G(f)) &= I_{CB(H)} \otimes i_1(f), \\ \psi((i_2 \otimes i_1)\Delta_G(f)) &= (i_1 \otimes i_2)\Delta_G(f), \\ \psi(I_{CB(H)} \otimes i_2(f)) &= (i_2 \otimes i_2)\Delta_G(f) \end{aligned}\right\} \tag{2.8}$$

for any $f \in C_\infty(G)$. Indeed for $f = v$ ($f = n$ respectively), (2.8) coincides with (2.6) ((2.7) respectively). Remembering that $C_\infty(G)$ is generated by v and n we obtain (2.8) in full generality.

Let $A, B, C \in \mathrm{Ob}_{\mathcal{C}}$. According to the definitions of Section 1,

$$(A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \subset M(A \otimes B \otimes CB(H) \otimes C \otimes CB(H))$$

$$A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C) \subset M(A \otimes B \otimes C \otimes CB(H) \otimes CB(H))$$

We set:

$$\Psi = \mathrm{id}_{A \otimes B} \otimes \left[(\mathrm{id}_C \otimes \psi) \circ (\sigma \otimes \mathrm{id}_{CB(H)}) \right],$$

where the flip $\sigma = \sigma_{CB(H),C}$. Then Ψ is an isomorphism acting from $A \otimes B \otimes CB(H) \otimes C \otimes CB(H)$ onto $A \otimes B \otimes C \otimes CB(H) \otimes CB(H)$. Now, a moment of reflection shows that formulae (2.1) are equivalent to (2.8). Moreover, remembering that for any C*-algebras A and B, $j_1(A)j_2(B)$ is dense in $A \otimes_{\mathcal{C}} B$ and using (2.1) one can easily show that the restriction of Ψ to $(A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C$ is an isomorphism acting from $(A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C$ onto $A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C)$. This way the associativity of $\otimes_{\mathcal{C}}$ is proved and the construction of the braided category of C*-algebras $(\mathcal{C}, \otimes_{\mathcal{C}}, j_1, j_2)$ is completed.

3 Braided group structure on $\overline{\mathbf{C}}^q$

Let q be a real number such that $0 < q < 1$ and

$$\overline{\mathbf{C}}^q = \left\{ z \in \mathbf{C} : z = 0 \text{ or } |z| \in q^{\mathbf{Z}} \right\}. \tag{3.1}$$

Then $\overline{\mathbf{C}}^q$ is a locally compact space (for it is a closed subset of $\mathbf{C}$) and the group $G = \mathbf{S}^1 \times \mathbf{Z}$ acts on it in a natural way: $\mathbf{S}^1$ acts by rotations and $\mathbf{Z}$ by homotheties with factors being integer powers of q.

Let $\lambda = \mathrm{id}_{\overline{\mathbf{C}}^q}$. Then λ is a continuous complex valued continuous function on $\overline{\mathbf{C}}^q$: $\lambda \in C(\overline{\mathbf{C}}^q)$. Consequently λ is affiliated with the C*-algebra $A = C_\infty(\overline{\mathbf{C}}^q)$: $\lambda \,\eta\, A$. The action of G on $\overline{\mathbf{C}}^q$ described above gives rise to an action α_A of G on A. We have:

$$\alpha_A(\lambda) = \lambda \otimes vq^n. \tag{3.2}$$

where v and n are elements affiliated with $C_\infty(G)$ introduced by (1.12). This way A becomes an object of the braided category of C*-algebras $\mathcal{C}$ introduced in the previous section. We shall use the following shorthand notation for canonical morphisms: instead of $j_1(A,A)$, $j_2(A,A)$, $j_1(A,A,A)$, $j_2(A,A,A)$ and $j_3(A,A,A)$ we shall write j_1, j_2, $\bar{\jmath}_1$, $\bar{\jmath}_2$ and $\bar{\jmath}_3$ respectively: $j_1, j_2 \in \mathrm{Mor}_{\mathcal{C}}(A, A \otimes_{\mathcal{C}} A)$ and $\bar{\jmath}_1, \bar{\jmath}_2, \bar{\jmath}_3 \in \mathrm{Mor}_{\mathcal{C}}(A, A \otimes_{\mathcal{C}} A \otimes_{\mathcal{C}} A)$.

Let $R = j_1(\lambda)$ and $S = j_2(\lambda)$. Then R, S are normal elements affiliated with $A \otimes_{\mathcal{C}} A$ and their spectra coincide with $\overline{\mathbf{C}}^q$. According to (3.2), λ is a homogeneous element of degree 1 and $\gamma_A(\lambda) = q\lambda$. Therefore λ^* is a homogeneous element of degree -1 and $\gamma_A(\lambda^*) = q\lambda^*$and using (1.10) we obtain:

$$\left.\begin{aligned} SR &= q^2 RS, \\ SR^* &= R^* S. \end{aligned}\right\} \tag{3.3}$$

More detailed inspection shows that elements R and S satisfy the above relations in the strong sense explained in [3, Theorem 2.4.3]. In what follows, $\dot{+}$ will denote the closure of the sum of unbounded elements.

Theorem 3.1 *With the notation introduced above we have:*

1. The sum $j_1(\lambda)\dot{+}j_2(\lambda)$ is a normal element affiliated with $A \otimes_{\mathcal{C}} A$, the spectrum of $j_1(\lambda)\dot{+}j_2(\lambda)$ coincides with $\overline{\mathbf{C}}^q$.

2. There exists unique morphism $\Delta \in \mathrm{Mor}_{\mathcal{C}}(A, A \otimes_{\mathcal{C}} A)$ such that

$$\Delta(\lambda) = j_1(\lambda)\dot{+}j_2(\lambda).$$

3. The morphism Δ is coassociative:

$$(\Delta \otimes_{\mathcal{C}} \mathrm{id}_A)\circ\Delta = (\mathrm{id}_A \otimes_{\mathcal{C}} \Delta)\circ\Delta.$$

In a very loose language, this theorem means that $\overline{\mathbf{C}}^q$ equipped with the arithmetic sum '+' is a braided group.

Proof: Statement 1 is contained in [2, Theorems 2.1, 2.2 and 5.1]. Statement 2 follows immediately from Statement 1. To prove Statement 3, one has to show that $(\bar{\jmath}_1(\lambda)\dot{+}\bar{\jmath}_2(\lambda))\dot{+}\bar{\jmath}_3(\lambda) = \bar{\jmath}_1(\lambda)\dot{+}(\bar{\jmath}_2(\lambda)\dot{+}\bar{\jmath}_3(\lambda))$. To this end one may use [2, Lemma 3.2] in the way, it was used in the prove of [2, Theorem 3.1].

Q.E.D.

In the following we shall use a special function defined on $\overline{\mathbf{C}}^q$. For any $z \in \overline{\mathbf{C}}^q$ we set:

$$F_q(z) = \begin{cases} -1 & \text{for } z = -1, -q^{-2}, -q^{-4}, -q^{-6}, \dots \\ \displaystyle\prod_{k=0}^{\infty} \frac{1 + q^{2k}\bar{z}}{1 + q^{2k} z} & \text{otherwise.} \end{cases}$$

Then F_q is a continuous function on $\overline{\mathbf{C}}^q$ with values in $\mathbf{S}^1$. It turns out that for any $z \neq -1, -q^{-2}, -q^{-4}, -q^{-6}, \ldots$,

$$F_q(z) = \frac{\exp_{1/q}\left(\dfrac{\overline{z}}{1-q^2}\right)}{\exp_{1/q}\left(\dfrac{z}{1-q^2}\right)},$$

where for any complex $\zeta \in \mathbf{C}$,

$$\exp_{1/q}(\zeta) = \sum_{k=0}^{\infty} \frac{\zeta^k}{\mathrm{Fact}_{1/q}(k)}, \tag{3.4}$$

where for any nonnegative integer k,

$$\begin{aligned} \mathrm{Fact}_{1/q}(k) &= \prod_{n=1}^{k} \frac{1-q^{-2n}}{1-q^{-2}} \\ &= \sum_{\mathrm{perm}(k)} q^{-2(\text{number of inversions})}, \end{aligned}$$

where the summation runs over all permutations of k elements.

Let $\chi \in M(A)$. We say that χ a character of the braided group $\overline{\mathbf{C}}^q$, if $\Delta(\chi) = j_1(\chi)j_2(\chi)$. Let K be a Hilbert space. We say that u is a unitary representation of the braided group $\overline{\mathbf{C}}^q$ if u is a unitary element of $M(CB(K)\otimes A)$ such that $(\mathrm{id}_{CB(K)}\otimes\Delta)u = (\mathrm{id}_{CB(K)} \otimes j_1)u(\mathrm{id}_{CB(K)} \otimes j_2)u$. The main facts concerning the braided group $\overline{\mathbf{C}}^q$ are contained in the following two theorems:

Theorem 3.2 *Let $z \in \overline{\mathbf{C}}^q$ and*

$$\chi_z = F_q(\overline{z}\lambda).$$

Then χ_z is a character of the braided group $\overline{\mathbf{C}}^q$. Any invertible character of $\overline{\mathbf{C}}^q$ is of this form, with the parameter z uniquely determined.

Theorem 3.3 *Let K be a Hilbert space, $dE(z)$ be a spectral measure defined on $\overline{\mathbf{C}}^q$ with values in the set of orthogonal projection acting on K and*

$$u = \int^{\oplus} dE(z) \otimes F_q(\overline{z}\lambda),$$

where the direct integral is taken over $\overline{\mathbf{C}}^q$. Then u is a unitary representation of the braided group $\overline{\mathbf{C}}^q$. Any unitary representation of $\overline{\mathbf{C}}^q$ is of this form, with the spectral measure $dE(z)$ uniquely determined.

Theorems 3.2 and 3.3 follow immediately from [2, Theorems 3.1 and 4.1] and [2, Theorems 4.2] respectively. Theorem 3.3 may be rephrased in the following way:

Theorem 3.4 *Let U be the unitary element of $M(A \otimes A)$ introduced by the formula:*

$$U = F_q(\lambda^* \otimes \lambda). \tag{3.5}$$

Then, for any nondegenerate representation π of A acting on a Hilbert space K, $(\pi \otimes \mathrm{id}_A)U$ is a unitary representation of the braided group $\overline{\mathbf{C}}^q$. Any unitary representation of $\overline{\mathbf{C}}^q$ is of this form, with the representation π uniquely determined.

It shows that the braided group $\overline{\mathbf{C}}^q$ is selfdual: the Pontryagin dual of $\overline{\mathbf{C}}^q$ coincides with $\overline{\mathbf{C}}^q$. The universal bicharacter establishing this duality is given by (3.5).

Remark 3.5 Inserting in Theorem 3.2 $z = 1$ we see that $F_q(\lambda)$ is a character of $\overline{\mathbf{C}}^q$. It means that $F_q(j_1(\lambda)\dot{+}j_2(\lambda)) = F_q(j_1(\lambda))F_q(j_2(\lambda))$ and

$$F_q(R\dot{+}S) = F_q(R)F_q(S). \tag{3.6}$$

One may try to prove this equality in purely computational manner. Taking into account the first of the relations (3.3) and using the power series expansion (3.4) we obtain:

$$\exp_{1/q}\left(\frac{R+S}{1-q^2}\right) = \exp_{1/q}\left(\frac{S}{1-q^2}\right)\exp_{1/q}\left(\frac{R}{1-q^2}\right).$$

Therefore

$$\exp_{1/q}\left(\frac{R^*+S^*}{1-q^2}\right) = \exp_{1/q}\left(\frac{R^*}{1-q^2}\right)\exp_{1/q}\left(\frac{S^*}{1-q^2}\right),$$

$$\exp_{1/q}\left(\frac{R+S}{1-q^2}\right)^{-1} = \exp_{1/q}\left(\frac{R}{1-q^2}\right)^{-1}\exp_{1/q}\left(\frac{S}{1-q^2}\right)^{-1}$$

and

$$\begin{aligned} F_q(R+S) &= \exp_{1/q}\left(\frac{R^*+S^*}{1-q^2}\right)\exp_{1/q}\left(\frac{R+S}{1-q^2}\right)^{-1} \\ &= \exp_{1/q}\left(\frac{R^*}{1-q^2}\right)\exp_{1/q}\left(\frac{S^*}{1-q^2}\right)\exp_{1/q}\left(\frac{R}{1-q^2}\right)^{-1}\exp_{1/q}\left(\frac{S}{1-q^2}\right)^{-1} \\ &= \exp_{1/q}\left(\frac{R^*}{1-q^2}\right)\exp_{1/q}\left(\frac{R}{1-q^2}\right)^{-1}\exp_{1/q}\left(\frac{S^*}{1-q^2}\right)\exp_{1/q}\left(\frac{S}{1-q^2}\right)^{-1} \\ &= F_q(R)F_q(S), \end{aligned}$$

where in the last but one step we used the second relation (3.3).

However, these computations are not correct: there is no way to control the convergence of the power series used above. In fact the situation is even worse: Using the same method one could show that $F_q(zR\dot{+}zS) = F_q(zR)F_q(zS)$ for any $z \in \mathbf{C}$. On the other hand by virtue of Theorem 3.2, this relation holds only for $z \in \overline{\mathbf{C}}^q$. The rigorous prove of (3.6) is given in [2, Proof of Theorem 3.1].

Remark 3.6 One may try to remove the radius quantization introduced in (3.1) and replace $\overline{\mathbf{C}}^q$ by a larger G-invariant closed subset $\Lambda \subset \mathbf{C}$. However in this case the spectra of R and S coincide with Λ and Condition 3 of [2, Theorem 2.1] is not satisfied. Consequently, $j_1(\lambda)\dot{+}j_2(\lambda)$ is no longer normal and the morphism $\Delta \in \mathrm{Mor}_c(C_\infty(\mathbf{C}), C_\infty(\mathbf{C}) \otimes_c C_\infty(\mathbf{C}))$ does not exist.

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