A QUANTUM GL(2,C) GROUP AT ROOTS OF UNITY *

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Abstract

We construct a quantum deformation of $GL(2, \mathbf{C})$ corresponding to the deformation parameter $q = e^{\frac{2\pi i}{N}}$ where N is an even natural number. Hopf *-algebra, Hilbert space and C^*-algebra levels are considered. The C^*-algebra A, which may be interpreted as *the algebra of all continuous functions on the group vanishing at infinity*, is generated (in the sense of [12]) by five elements α, β, γ, δ and $\det^{-1}$ satisfying certain commutation relations completed by hermiticity conditions. The group structure is encoded by a comultiplication $\Phi \in \mathrm{Mor}(A, A \otimes A)$ acting on generators in the standard way. On the Hopf algebra level our deformation corresponds to a one-parameter family $GL_{q^2,1}(2, \mathbf{C})$ of the standard two-parameter deformations $GL_{p,q}(2, \mathbf{C})$.

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0 Introduction

The theory of compact quantum groups is now well established. Different approaches are more or less equivalent and as in the classical case a complete theory may be developed starting from a commonly accepted basic notions [10],[16]. We also have a rich set of examples containing standard deformations of classical simple compact groups.

On the contrary, the situation of non-compact groups is rather unsatisfactory. In this case there is an essential discrepancy between Hopf and C^*-algebra levels. It seems that there is no common agreement as to the definition of a non-compact quantum group though there are some promising approaches [1], [17], [7], [4]. A list of examples is rather short. Known examples as a rule concern groups of low dimension and for the most part the standard deformation parameter q is real.

On the other hand on the Hopf algebra level the values of q being a roots of unity are of special interest. Despite great effort (e.g.cf.[9]) these values of deformation parameter seemed to be inaccesible for the C^*-algebra approach. Only recently the deformations of the groups of affine transformations of $\mathbf{R}$ and $\mathbf{C}$ corresponding to $q = e^{\frac{2\pi i}{N}}$ were discovered [14], [18]. It is an interesting challange to construct quantum $SL(2, \mathbf{C})$ at roots of unity. The present paper is a step in this direction. We shall define a quantum deformation of $GL(2, \mathbf{C})$ with q being a root of unity. All three levels (Hopf *-algebra, Hilbert space and C^*) are considered. We belive that the gathered experience will be useful in future work with $SL(2, \mathbf{C})$ group.

A quantum deformation of $GL(2, \mathbf{C})$ on Hopf algebra level has been known for a long time [8]. It is a two parameter family $GL_{p,q}(2)$ were p and q are non-zero complex numbers. This deformation for special values of parameters, $p = e^{\frac{4\pi i}{N}}$ and $q = 1$ is a starting point for the description of quantum deformation of $GL(2, \mathbf{C})$ on the C^*-algebra level.

The paper is arranged as follows. In Section 1 we introduce Hopf *-algebra $\mathcal{A}$ being a quantum deformation of the polynomial algebra on $GL(2, \mathbf{C})$. It is generated by five elements α, β, γ, δ and $\det^{-1}$ satisfying certain commutation relations supplemented by hermiticity conditions. The relations determining Hopf - algebra structure coincide with that for a Hopf - algebra structure of the quantum deformation $GL_{p,q}(2)$ described in [8], at special values of parameters $(p, q) = (q^2, 1)$. In particular det is the quantum determinant. A *-algebra structure is imposed by "complexification" procedure and determines the other commutation relations. Now hermiticity conditions state that $\frac{N}{2}$-power of any generator is a hermitian element which complete the structure of $\mathcal{A}$. At this point we would like to stress the role of the hermiticity conditions: they will guarantee the existence of a comultiplication on the C^*-algebra level.

In Section 2 we investigate Hilbert space representations of the considered relations. Since unbounded operators are involved one has to assign a proper operator meaning to the relations. Inspecting the anatomy of the relations at first we conclude that det is an invertible operator and α, β, γ, δ and $\det^{-1}$ are normal operators such that their spectra are localized in a set $\overline{\Gamma} \subset \mathbf{C}$ due to hermiticity conditions (cf. (2.1)). Moreover, it turns out, in contrast to the classical case, that α and δ are invertible, too. Next we have to give a precise meaning to the relations of the form

$$XY = q^2 YX \qquad \text{and} \qquad XY^* = Y^* X$$

where X and Y are normal operators such that $\operatorname{Sp} X,\ \operatorname{Sp} Y \subset \overline{\Gamma}$. It turns out that relations of this form were investigated in [14] under additional assumption that X and Y were invertible. The notion of a q^2-pair (cf. Definition 2.1) extends the idea of [14] to our more degenerate case. q^2-pairs and their properties play a crucial role in our paper. In consequence the precise operator meaning of the commutation relations and hermiticity conditions is established by the notion of a G-matrix (cf. Definition 2.5). The main result of the Section shows that a tensor product of two G-matrices is a G-matrix again. This is used in the next Section to prove the existence of a comultiplication on the C^*-level.

Section 4 is devoted to the construction of our quantum group on the C^*-algebra level. We are looking for the universal C^*-algebra A, *"the algebra of all continuous functions on the group vanishing at infinity"*, generated by α, β, γ, δ and $\det^{-1}$ in the sense of [12]. It turns out that A is a crossed product C^*-algebra, $A = C_\infty(\overline{\Gamma} \times \overline{\Gamma}) \times_{\sigma\times\sigma} (\Gamma \times \Gamma)$ where σ is a natural action of the group Γ on $\overline{\Gamma}$. Then we show the existence of a comultiplication $\Phi \in \mathrm{Mor}(A, A \otimes A)$ acting on generators in the standard way.

Our investigations show that our quantum group $GL_{q^2,1}(2, C)$ is related to the quantum '$az + b$' group. In fact it is the class of quantum groups which appear in the double group construction applied to '$az + b$'. This will be presented in a forthcoming paper.

The notation used in the paper is explained in [11], [12] and [14]. In principle all C^*-algebras considered in the paper are separable. Non-separable C^*-algebras appear only as multiplier algebras. For any Hilbert space H, $C^*(H)$ denotes the set of all nondegenerate separable C^*-subalgebras of $B(H)$. Nondegeneracy for an algebra A means that $A(H)$ is a dense subset of H. We remind that for any C^*-algebras A and B, $M(A)$ denotes the multiplier algebra of A and $\phi \in \mathrm{Mor}(A, B)$ means that $\phi : A \to M(B)$ is a *-homomorphism such that $\phi(A)B$ is dense in B. C^*-affiliation relation is denoted by "η". Elements affiliated with the C^*-algebra may by treated as "unbounded multipliers" or "continuous functions" on a quantum space but the reader should be warned that in general the set of all elements affiliated with the C^*-algebra is not an algebra or even a vector space.

1 Hopf *-algebra level

In this paper we consider a subfamily $GL_{q^2,1}(2, \mathbf{C})$, where q is a root of unity, of the standard 2-parameter quantum deformations of $GL(2, \mathbf{C})$ group.

The *-algebra $\mathcal{A}_o$ of commutative polynomials on $GL(2, \mathbf{C})$ is generated by four normal elements α, β, γ, δ subject to the relation

$$\det := \alpha\delta - \gamma\beta \quad \text{is invertible.}$$

The group structure of $GL(2, \mathbf{C})$ is encoded by the comultiplication Φ defined as the unique *-algebra homomorphism from $\mathcal{A}_o$ into $\mathcal{A}_o \otimes \mathcal{A}_o$ such that

$$\begin{aligned} \Phi(\alpha) &= \alpha \otimes \alpha + \beta \otimes \gamma, \quad \Phi(\beta) = \alpha \otimes \beta + \beta \otimes \delta \\ \Phi(\gamma) &= \gamma \otimes \alpha + \delta \otimes \gamma, \quad \Phi(\delta) = \gamma \otimes \beta + \delta \otimes \delta. \end{aligned} \tag{1.1}$$

It is clear that $(\mathcal{A}_o, \Phi)$ is a Hopf *-algebra. In particular counit and coinverse κ are given by the formulae

$$
\begin{aligned}
&e(\alpha) = 1 = e(\delta), &&e(\beta) = 0 = e(\gamma)\\
&\kappa(\alpha) = \det{}^{-1}\delta, &&\kappa(\beta) = -\det{}^{-1}\beta\\
&\kappa(\gamma) = -\det{}^{-1}\gamma, &&\kappa(\delta) = \det{}^{-1}\alpha.
\end{aligned}
\tag{1.2}
$$

To perform quantum deformation of $GL(2,\mathbf{C})$ we fix a deformation parameter q. In what follows q is a root of unity of the form

$$
q = e^{\frac{2\pi}{N}i} \qquad\qquad N = 6, 8, 10, ... \tag{1.3}
$$

The quantum $GL(2,\mathbf{C})$ group on the level of Hopf *-algebra can be easily described.

Let α, β, γ, δ be elements subject to the following relations

$$
\left.
\begin{aligned}
&\alpha\beta = q^2\beta\alpha, &&\alpha\gamma = \gamma\alpha\\
&\gamma\delta = q^2\delta\gamma, &&\beta\delta = \delta\beta\\
&\gamma\beta = q^2\beta\gamma &&\\
&\alpha\delta - \delta\alpha = (q^2-1)\beta\gamma. &&
\end{aligned}
\right\}
\tag{1.4}
$$

We shall use the quantum determinant

$$
\det := \alpha\delta - \gamma\beta. \tag{1.5}
$$

Clearly by (1.4) we have also

$$
\det = \delta\alpha - \beta\gamma \tag{1.6}
$$

and

$$
\alpha\det = \det\alpha, \qquad \delta\det = \det\delta, \qquad \beta\det = q^{-2}\det\beta, \qquad \gamma\det = q^2\det\gamma. \tag{1.7}
$$

We shall assume that det is invertible. Then

$$
\det{}^{-1}(\alpha\delta - \gamma\beta) = (\alpha\delta - \gamma\beta)\det{}^{-1} = I \tag{1.8}
$$

By definition the algebra $\mathcal{A}_{hol}$ is generated by $\det^{-1}$ and four elements α, β, γ, δ satisfying (1.4) and (1.8).

We shall show that $\alpha^{\frac{N}{2}}$, $\beta^{\frac{N}{2}}$, $\gamma^{\frac{N}{2}}$ and $\delta^{\frac{N}{2}}$ belong to the center of the algebra $\mathcal{A}_{hol}$. To this end it is enough to verify that all 16 commutators of these operators with α, β, γ and δ vanish. Only computation of $\left[\alpha^{\frac{N}{2}}, \delta\right]$ and $\left[\delta^{\frac{N}{2}}, \alpha\right]$ is a little more complicated:

$$
\begin{aligned}
\left[\alpha^{\frac{N}{2}}, \delta\right] &= \sum_{j=0}^{\frac{N}{2}-1} \alpha^j\,[\alpha,\delta]\,\alpha^{\frac{N}{2}-j-1}\\
&= (q^2-1)\sum_{j=0}^{\frac{N}{2}-1}\alpha^j\beta\gamma\alpha^{\frac{N}{2}-j-1} = (q^2-1)\sum_{j=0}^{\frac{N}{2}-1} q^{2j}\beta\gamma\alpha^{\frac{N}{2}-1}\\
&= (q^N-1)\beta\gamma\alpha^{\frac{N}{2}-1} = 0
\end{aligned}
$$

and the similar calculation shows that $\left[\delta^{\frac{N}{2}}, \alpha\right] = 0$.

By (1.7) det does not belong to the center of $\mathcal{A}_{hol}$ but it is clear that $\det^{\frac{N}{2}}$ is a central element of $\mathcal{A}_{hol}$. Moreover one can find useful formulae relating $\det^{\frac{N}{2}}$ with other central elements. Extend for a moment $\mathcal{A}_{hol}$ by adjoing an inverse of δ. Then (cf. (1.5))

$$\alpha = \det \delta^{-1} + \gamma\beta\delta^{-1}.$$

Now let us observe that $(\det \delta^{-1})(\gamma\beta\delta^{-1}) = q^2(\gamma\beta\delta^{-1})(\det \delta^{-1})$. Therefore using q-deformed binomial theorem (cf. e.g [3, Lemma 1.3.1]) one can easily check that

$$\alpha^{\frac{N}{2}} = (\det \delta^{-1})^{\frac{N}{2}} + (\gamma\beta\delta^{-1})^{\frac{N}{2}}.$$

(deformed binomial coefficients $\left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right]_{q^2}$ vanish for $n = \frac{N}{2}$ and $0 < j < \frac{N}{2}$). Remembering that det and δ commute, γ commutes with $\beta\delta^{-1}$ and β commutes with δ we get

$$\alpha^{\frac{N}{2}} = \det^{\frac{N}{2}}\delta^{-\frac{N}{2}} + \gamma^{\frac{N}{2}}\beta^{\frac{N}{2}}\delta^{-\frac{N}{2}} = (\det^{\frac{N}{2}} + \gamma^{\frac{N}{2}}\beta^{\frac{N}{2}})\delta^{-\frac{N}{2}}.$$

Therefore we obtain the formula

$$\det^{\frac{N}{2}} \quad = (\alpha\delta - \gamma\beta)^{\frac{N}{2}} = \alpha^{\frac{N}{2}}\delta^{\frac{N}{2}} - \gamma^{\frac{N}{2}}\beta^{\frac{N}{2}}. \tag{1.9}$$

In the same manner starting from (1.6) we obtain

$$\det^{\frac{N}{2}} \quad = (\delta\alpha - \beta\gamma)^{\frac{N}{2}} = \delta^{\frac{N}{2}}\alpha^{\frac{N}{2}} - \beta^{\frac{N}{2}}\gamma^{\frac{N}{2}}. \tag{1.10}$$

It turns out that formulae (1.9) and (1.10) are also valid within $\mathcal{A}_{hol}$. The rigorous proof of these equalities making no use of δ^{-1} is based on following lemma

Lemma 1.1
For any natural n :

$$\alpha^n\delta^n = \sum_{j=0}^{n} \left[\begin{array}{c} n \\ j \end{array}\right]_{q^2} \gamma^j\beta^j\det^{n-j} \tag{1.11}$$

$$\delta^n\alpha^n = \sum_{j=0}^{n} \left[\begin{array}{c} n \\ j \end{array}\right]_{q^2} q^{2j(j-n)}\beta^j\gamma^j\det^{n-j}. \tag{1.12}$$

In particular for $n = \frac{N}{2}$ we obtain (1.9) and (1.10).
Proof. For $n = 1$ the formulae coincide with (1.5) and (1.6). For any natural k we have

$$\alpha^{k+1}\delta^{k+1} = \alpha^k\alpha\delta\delta^k = \alpha^k(\det + \gamma\beta)\delta^k = \alpha^k\delta^k(\det + q^{2k}\gamma\beta).$$

Assuming that (1.11) holds for $n = k$ and using the binomial coefficient identity

$$\left[\begin{array}{c} k+1 \\ j \end{array}\right]_{q^2} = \left[\begin{array}{c} k \\ j \end{array}\right]_{q^2} + q^{2(k+1-j)}\left[\begin{array}{c} k \\ j-1 \end{array}\right]_{q^2}$$

one can easily check show that (1.11) holds for $n = k+1$. This proves (1.11) for all natural n. In the similar way one can prove (1.12) using the identity

$$\left[\begin{array}{c} k+1 \\ j \end{array}\right]_{q^2} = q^{2j}\left[\begin{array}{c} k \\ j \end{array}\right]_{q^2} + \left[\begin{array}{c} k \\ j-1 \end{array}\right]_{q^2}.$$

□

It is known [8] that the transformation (1.1) respects relations (1.4). Therefore there exists a unique algebra homomorphism $\Phi : \mathcal{A}_{hol} \longrightarrow \mathcal{A}_{hol} \otimes \mathcal{A}_{hol}$ acting on generators α, β, γ and δ in the way described by (1.1). Endowing $\mathcal{A}_{hol}$ with counit e and coinverse κ described on generators by the same formula (1.2) as in the classical case one verifies that $(\mathcal{A}_{hol}, \Phi)$ is a Hopf-algebra. Simple computation shows that det is a character (group-like element): $\Phi(\det) = \det \otimes \det$, and $e(\det) = 1$, $\kappa(\det) = \det^{-1}$.

Since

$$
\begin{aligned}
(\alpha \otimes \alpha)(\beta \otimes \gamma) &= q^2(\beta \otimes \gamma)(\alpha \otimes \alpha), \\
(\alpha \otimes \beta)(\beta \otimes \delta) &= q^2(\beta \otimes \delta)(\alpha \otimes \beta), \\
(\gamma \otimes \alpha)(\delta \otimes \gamma) &= q^2(\delta \otimes \gamma)(\gamma \otimes \alpha), \\
(\gamma \otimes \beta)(\delta \otimes \delta) &= q^2(\delta \otimes \delta)(\gamma \otimes \beta),
\end{aligned}
$$

using the q-deformed binomial theorem one easily checks that

$$
\begin{aligned}
\Phi(\alpha^{\frac{N}{2}}) &= \alpha^{\frac{N}{2}} \otimes \alpha^{\frac{N}{2}} + \beta^{\frac{N}{2}} \otimes \gamma^{\frac{N}{2}}, \\
\Phi(\beta^{\frac{N}{2}}) &= \alpha^{\frac{N}{2}} \otimes \beta^{\frac{N}{2}} + \beta^{\frac{N}{2}} \otimes \delta^{\frac{N}{2}}, \\
\Phi(\gamma^{\frac{N}{2}}) &= \gamma^{\frac{N}{2}} \otimes \alpha^{\frac{N}{2}} + \delta^{\frac{N}{2}} \otimes \gamma^{\frac{N}{2}}, \\
\Phi(\delta^{\frac{N}{2}}) &= \gamma^{\frac{N}{2}} \otimes \beta^{\frac{N}{2}} + \delta^{\frac{N}{2}} \otimes \delta^{\frac{N}{2}}.
\end{aligned}
\tag{1.13}
$$

Now we "complexify" $\mathcal{A}_{hol}$.

Let $\mathcal{A}_o$ be the *-algebra generated by α, β, γ, δ and $\det^{-1}$ satisfying (1.4), (1.8) and

$$
cc' = c'c \quad \text{for any } c \in \{\alpha, \beta, \gamma, \delta, \} \text{ and } c' \in \{\alpha^*, \beta^*, \gamma^*, \delta^*\} \ . \tag{1.14}
$$

Then $\mathcal{A}_{hol}$ is a subalgebra of $\mathcal{A}_o$. $\mathcal{A}_o$ may be identified with $\mathcal{A}_{hol} \otimes \mathcal{A}_{hol}^*$. The multiplication map

$$
\mathcal{A}_{hol} \otimes \mathcal{A}_{hol}^* \ni a \otimes b^* \longrightarrow ab^* \in \mathcal{A}_o.
$$

is an algebra isomorphism. $\alpha, \beta, \gamma, \delta$ and det are normal elements of $\mathcal{A}_o$ and $\alpha^{\frac{N}{2}}$, $\beta^{\frac{N}{2}}$, $\gamma^{\frac{N}{2}}$, $\delta^{\frac{N}{2}}$ and $\det^{\frac{N}{2}}$ are in the center of $\mathcal{A}_o$.

We already introduced the comultiplication $\Phi : \mathcal{A}_{hol} \longrightarrow \mathcal{A}_{hol} \otimes \mathcal{A}_{hol}$. Using the formula $\Phi(ab^*) = \Phi(a)\Phi(b)^*$ we extend Φ to $\mathcal{A}_o$. Then one can easily show that $(\mathcal{A}_o, \Phi)$ is a Hopf *-algebra.

Many authors working with quantum groups at roots of unity assume additional relations on generators of the algebra. For $GL_{q^2,1}(2, \mathbf{C})$ one may assume that $\alpha^{\frac{N}{2}} = \delta^{\frac{N}{2}} = I$ and $\beta^{\frac{N}{2}} = \gamma^{\frac{N}{2}} = 0$. This procedure leads to a finite-dimensional Hopf algebra (e.g.[6]). We shall make a half-step in this direction assuming that the elements $\alpha^{\frac{N}{2}}$, $\beta^{\frac{N}{2}}$, $\gamma^{\frac{N}{2}}$ and $\delta^{\frac{N}{2}}$ are hermitian:

$$
\begin{aligned}
(\alpha^{\frac{N}{2}})^* &= \alpha^{\frac{N}{2}}, \quad (\beta^{\frac{N}{2}})^* = \beta^{\frac{N}{2}}, \\
(\gamma^{\frac{N}{2}})^* &= \gamma^{\frac{N}{2}}, \quad (\delta^{\frac{N}{2}})^* = \delta^{\frac{N}{2}}.
\end{aligned}
\tag{1.15}
$$

The reader should notice that $\alpha^{\frac{N}{2}}$ and $(\alpha^{\frac{N}{2}})^*$ have the same commutation relations with all generators of $\mathcal{A}_o$ (both elements are in the center of $\mathcal{A}_o$). Therefore the relation $(\alpha^{\frac{N}{2}})^* = \alpha^{\frac{N}{2}}$ is compatible with the algebraic structure of $\mathcal{A}_o$. The same holds for remaining relations (1.15). Taking into account (1.9) and (1.10) we obtain

$$(\det{}^{\frac{N}{2}})^* = \det{}^{\frac{N}{2}}. \tag{1.16}$$

According to (1.13) the hermiticity of $\alpha^{\frac{N}{2}}$, $\beta^{\frac{N}{2}}$, $\gamma^{\frac{N}{2}}$ and $\delta^{\frac{N}{2}}$ implies the hermiticity of $\Phi(\alpha^{\frac{N}{2}})$, $\Phi(\beta^{\frac{N}{2}})$, $\Phi(\gamma^{\frac{N}{2}})$ and $\Phi(\delta^{\frac{N}{2}})$. Therefore the relations (1.15) are compatible with the coalgebra structure of $\mathcal{A}_o$. More precisely we have

Proposition 1.2
Let $\mathcal{A}$ be a *-algebra generated by four normal elements α, β, γ, δ and $\det^{-1}$ satisfying (1.4), (1.8), (1.14) and additional hermiticity conditions (1.15). Then

1. *There exists a unique *-algebra homomorphism*

$$\Phi : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$$

such that

$$\begin{pmatrix} \Phi(\alpha), & \Phi(\beta) \\ \Phi(\gamma), & \Phi(\delta) \end{pmatrix} = \begin{pmatrix} \alpha\otimes\alpha + \beta\otimes\gamma, & \alpha\otimes\beta + \beta\otimes\delta \\ \gamma\otimes\alpha + \delta\otimes\gamma, & \gamma\otimes\beta + \delta\otimes\delta \end{pmatrix} \tag{1.17}$$

2. *There exists a unique *-character*

$$e : \mathcal{A} \longrightarrow \mathbf{C}$$

such that

$$\begin{pmatrix} e(\alpha), & e(\beta) \\ e(\gamma), & e(\delta) \end{pmatrix} = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}.$$

3. *There exists a unique linear antimultiplicative mapping*

$$\kappa : \mathcal{A} \longrightarrow \mathcal{A}$$

such that

$$\begin{pmatrix} \kappa(\alpha), & \kappa(\beta) \\ \kappa(\gamma), & \kappa(\delta) \end{pmatrix} = \begin{pmatrix} \det^{-1}\delta, & -\det^{-1}\beta \\ -\det^{-1}\gamma, & \det^{-1}\alpha \end{pmatrix},$$

$$\begin{pmatrix} \kappa(\alpha^*), & \kappa(\beta^*) \\ \kappa(\gamma^*), & \kappa(\delta^*) \end{pmatrix} = \begin{pmatrix} (\det^*)^{-1}\delta^*, & -(\det^*)^{-1}\beta^* \\ -(\det^*)^{-1}\gamma^*, & (\det^*)^{-1}\alpha^* \end{pmatrix}.$$

4. *$(\mathcal{A}, \Phi)$ is a Hopf *-algebra.*

In particular

$$u = \begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix} \tag{1.18}$$

is a two dimensional corepresentation of $(\mathcal{A}, \Phi)$.

The above Proposition justifies the following definition

Definition 1.3 *The Hopf *-algebra $(\mathcal{A}, \Phi)$ described in Proposition 1.2 is called the algebra of polynomials on quantum $GL(2, \mathbf{C})$.*

Therefore u (cf. (1.18)) is a two dimensional representation of G, the quantum $GL(2, \mathbf{C})$-group.

2 Hilbert space level and q^2-pairs.

In this Section we consider quantum group G on the Hilbert space level. In particular we would like to represent generators of the Hopf *-algebra $\mathcal{A}$ by operators acting on a Hilbert space H. Inspecting algebraic formulae involving the generators we see that in general α, β, γ and δ should be treated as unbounded operators. Therefore one is forced to give a more precise meaning to considered formulae.

Formula (1.4) suggests that α, β, γ and δ are normal operators and by (1.5) we expect that det is normal as well. Then formulae (1.15) and (1.16) state that $\alpha^{\frac{N}{2}}$, $\beta^{\frac{N}{2}}$, $\gamma^{\frac{N}{2}}$, $\delta^{\frac{N}{2}}$ and $\det^{\frac{N}{2}}$ are selfadjoint operators and give rise to spectral conditions localizing spectra of α, β, γ, δ and det . To formulate these conditions we follow the notation introduced in [14]. Let

$$\begin{gathered}\mathbf{R}_+ := \{\, r \in \mathbf{R} : r > 0 \,\}, \\ \Gamma := \left\{ z \in \mathbf{C} \setminus \{0\} :\ \operatorname{Phase} z \in q^{\mathbf{Z}} \right\} = \bigcup_{k=0}^{N-1} q^k \mathbf{R}_+, \\ \overline{\Gamma} = \Gamma \cup \{0\}.\end{gathered} \tag{2.1}$$

Clearly Γ is a multiplicative subgroup of $\mathbf{C} \setminus \{0\}$ and $\overline{\Gamma}$ is the closure of Γ in $\mathbf{C}$. Therefore formulae (1.15) and (1.16) imply that

$$\operatorname{Sp} \alpha,\ \operatorname{Sp} \beta,\ \operatorname{Sp} \gamma,\ \operatorname{Sp} \delta,\ \operatorname{Sp} \det \subset \overline{\Gamma}. \tag{2.2}$$

Now we shall analyze relations involved in formulae (1.4) and (1.7). The meaning of the relations describing the commuting pairs of operators seems to be clear. Due to (1.14) it is natural to assume that these are pairs of normal strongly commuting operators. Another part of relations describes pairs of normal operators commuting up to the q^2 factor. To be more precise they involve pairs (Y, X) of normal operators such that

$$XY = q^2 YX \quad \text{and} \quad XY^* = Y^* X. \tag{2.3}$$

Such pairs will be called q^2-pairs on H. Since this is the crucial notion for the rest of the paper we discuss it in more detail. We shall follow the idea of [14].

For any z, $z', \in \Gamma$ we set (cf. [14, Section 1])

$$\chi(z, z') = q^{kk'} e^{\frac{N}{2\pi i}(\log r)(\log r')} \tag{2.4}$$

where k, k' are integers and r, r' are strictly positive real numbers such that $z = q^k r$ and $z' = q^{k'} r'$. Then $\chi : \Gamma \times \Gamma \longrightarrow S^1$ and χ is a symmetric bicharacter on Γ. One can easily verify (cf. [14, (1.2) and (1.3)] that

$$\chi(z,z') = \frac{\omega(zz')}{\omega(z)\omega(z')} \tag{2.5}$$

where

$$\omega(z) = e^{\frac{\pi i}{N}k^2} e^{\frac{N}{4\pi i}(\log r)^2} \tag{2.6}$$

for any $z \in \Gamma,\ z = q^k r$.

For $z' \in \Gamma$ and a normal operator X acting on a Hilbert space H we may use the functional calculus to introduce $\chi(X,z')$. Clearly $\chi(X,z')$ is well defined whenever $\operatorname{Sp} X \subset \overline{\Gamma}$ and X is invertible. Then $\chi(X,z')$ is a unitary operator acting on H. To be able to deal with non-invertible operator X we extend χ to $\overline{\Gamma}$. For any $z, z' \in \overline{\Gamma}$ we set

$$\tilde{\chi}(z,z') = \begin{cases} \chi(z,z') & \text{for } z,z' \in \Gamma \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

Then $\tilde{\chi} : \overline{\Gamma} \times \overline{\Gamma} \longrightarrow S^1 \cup \{0\},\ \tilde{\chi}$ is symmetric and

$$\tilde{\chi}(zz',z'') = \tilde{\chi}(z,z'')\tilde{\chi}(z',z'') \tag{2.8}$$

for any $z,z',z'' \in \overline{\Gamma}$.

For $z' \in \overline{\Gamma}$ and any normal operator X acting on a Hilbert space H such that $\operatorname{Sp} X \subset \overline{\Gamma}$ the operator $\tilde{\chi}(X,z')$ may be zero operator or a unitary one. In general $\tilde{\chi}(X,z')$ is a direct sum of both of them. Moreover

$$\tilde{\chi}(zX,z') = \tilde{\chi}(z,z')\tilde{\chi}(X,z') \tag{2.9}$$

for any $z,z' \in \overline{\Gamma}$.

Now we have

Definition 2.1 *A pair (Y,X) of normal operators acting on a Hilbert space H is a q^2-pair on H if*

1. $\operatorname{Sp} X \subset \overline{\Gamma},\ \operatorname{Sp} Y \subset \overline{\Gamma}$;

2. $$\tilde{\chi}(X,z)\tilde{\chi}(Y,z') = \chi(z,z')\tilde{\chi}(Y,z')\tilde{\chi}(X,z) \tag{2.10}$$

   for any $z,z' \in \Gamma$.

The set of all q^2-pairs on H will be denoted by $\overline{\mathcal{D}}_H$.

Remark. Let us note that if (Y,X) is a q^2-pair and X is invertible then the relation (2.10) can be written in the form (cf. [14, (2.2)])

$$\chi(X,z)Y\chi(X,z)^* = zY \tag{2.11}$$

for any $z \in \Gamma$.

We list four special types of q^2-pairs. At first clearly $(Y, X) = (0, 0)$ is a q^2-pair. Next for any normal invertible operator Z such that $\operatorname{Sp} Z \subset \overline{\Gamma}$, $(0, Z)$ and $(Z, 0)$ are q^2-pairs. This is second and third type. In some sense these cases are degenerate. Finally assuming that both X and Y are invertible we get the non-degenerate fourth type. It is not difficult to show that in general any q^2-pair is a direct sum of the special ones listed above.

The non-degenerate q^2-pairs are more interesting. They were considered in [14] (cf. [14, (2.1)]). It was shown that non-degenerate q^2-pair is unique up to the multiplicity. Combining this with the previous remark we obtain a complete description of q^2-pairs.

Many results concerning non-degenerate q^2-pairs obtained in [14] can be extended to the q^2-pairs after obvious modifications. At first we state the result concerning a relationship between the commutation relation of the form (2.10) with those of (2.3). It shows that formula (2.10) is a Weyl form of (2.3). As in [14] we shall call (2.10) the Weyl pentagonal relation.

We have (cf. [14, Proposition 2.1])

Proposition 2.2
Let H be a Hilbert space and $(Y, X) \in \overline{\mathcal{D}}_H$. Then the compositions $X \circ Y$, $Y \circ X$, $X \circ Y^*$, $Y^* \circ X$ and $Y^* \circ X^*$ are densely defined closeable operators and denoting by XY, YX, XY^*, Y^*X and Y^*X^* their closures we have

$$XY = q^2 YX, \qquad XY^* = Y^*X, \qquad (XY)^* = Y^*X^*.$$

Moreover XY is a normal operator, $\operatorname{Sp} XY \subset \overline{\Gamma}$, *and*

$$\tilde{\chi}(XY, z) = \omega(z)\chi(q, z)\tilde{\chi}(Y, z)\tilde{\chi}(X, z) \tag{2.12}$$

for any $z \in \Gamma$. In particular XY is invertible iff (Y, X) is non-degenerate.

As a straightforward generalization of [14, Proposition 2.2] we obtain

Proposition 2.3
Let $(Y, X) \in \overline{\mathcal{D}}_H$. Then (Y^*, X^*), (Y, XY) and (YX, X) are q^2-pairs on H. Moreover

1. *If X is invertible then $(X^{-1}, Y) \in \overline{\mathcal{D}}_H$.*
2. *If Y is invertible then $(X, Y^{-1}) \in \overline{\mathcal{D}}_H$.*

We also have

Proposition 2.4
Let $(Y, X) \in \overline{\mathcal{D}}_H$ and $(Z, X) \in \overline{\mathcal{D}}_H$. Then

1. $$\left(\begin{array}{c} Z \text{ and } Y \\ \text{strongly commute} \end{array} \right) \Longrightarrow \left(\begin{array}{c} (Z, XY) \in \overline{\mathcal{D}}_H \\ \text{and} \\ (Z, YX) \in \overline{\mathcal{D}}_H \end{array} \right).$$

2. $\Big(\ (Y,Z) \in \overline{\mathcal{D}}_H \ \Big) \Longrightarrow \Big(\ Z$ *strongly commutes with* XY *and* $YX \ \Big)$.

Proof. At first let us note that if U and V are normal operators acting on H such that $\mathrm{Sp}\, U \in \overline{\Gamma}$ and $\mathrm{Sp}\, V \in \overline{\Gamma}$ then U and V strongly commute if and only if

$$\tilde{\chi}(U,z)\tilde{\chi}(V,z') = \tilde{\chi}(V,z')\tilde{\chi}(U,z)$$

for any $z, z' \in \Gamma$.

By assumptions X, Y and Z are normal operators with their spectra localized in $\overline{\Gamma}$. Moreover also XY and YX are normal operators and $\mathrm{Sp}\, XY$, $\mathrm{Sp}\, YX$ are contained in $\overline{\Gamma}$ by Proposition 2.2. Since $XY = q^2 YX$ we shall consider the case of XY only. We have to compare $\tilde{\chi}(XY,z)\tilde{\chi}(Z,z')$ and $\tilde{\chi}(Z,z')\tilde{\chi}(XY,z)$ for any z and $z' \in \Gamma$. According to (2.12)

$$\tilde{\chi}(XY,z)\tilde{\chi}(Z,z') = \omega(z)\chi(q,z)\tilde{\chi}(Y,z)\tilde{\chi}(X,z)\tilde{\chi}(Z,z'). \tag{2.13}$$

We have assumed that $\tilde{\chi}(X,z)$ and $\tilde{\chi}(Z,z')$ satisfy the Weyl pentagonal relation (2.10). To prove Statement 1 we assume in addition that $\tilde{\chi}(Z,z')$ and $\tilde{\chi}(Y,z)$ commute. Therefore

$$\tilde{\chi}(XY,z)\tilde{\chi}(Z,z') = \chi(z,z')\tilde{\chi}(Z,z')\tilde{\chi}(XY,z)$$

and (Z, XY) is a q^2-pair.

To prove Statement 2 we assume in addition that $\tilde{\chi}(Z,z')$ and $\tilde{\chi}(Y,z)$ satisfy Weyl pentagonal relation (2.10). Now (2.13) leads to

$$\tilde{\chi}(XY,z)\tilde{\chi}(Z,z') = \chi(z',z)\chi(z',z)^{-1}\tilde{\chi}(Z,z')\tilde{\chi}(XY,z) = \tilde{\chi}(Z,z')\tilde{\chi}(XY,z).$$

Therefore Z and XY strongly commute. This proves Statement 2.

□

In what follows we shall assume that the pairs of operators in formulae (1.4) and (1.7) commuting up to q^2 factor are q^2-pairs in the sense of Definition 2.1.

In particular (β,γ), $(\det,\gamma)$ and $(\beta,\det)$ are q^2-pairs. Therefore $\mathrm{Sp}\,\gamma\beta \subset \overline{\Gamma}$ and the operator det strongly commutes with $\gamma\beta$ and $\beta\gamma$ by the second part of Proposition 2.4. Therefore the sums $\det + \gamma\beta$ and $\det^* + (\beta\gamma)^*$ are well defined and give rise to normal operators on H. On the other hand (δ,γ) is q^2-pair and δ strongly commutes with β therefore $(\delta,\gamma\beta)$ and $(\delta,\beta\gamma)$ are q^2-pairs by the first part of Proposition 2.4. Then $(\delta^*,(\beta\gamma)^*)$ is a q^2-pair due to Proposition 2.3. In particular the space $\ker\delta$ which coincides with $\ker\delta^*$ is invariant under the action of $\gamma\beta$ and $(\beta\gamma)^*$. The same statement holds for det and $\det^*$ since δ and det strongly commute. Therefore $\ker\delta$ is an invariant subspace for $\det + \gamma\beta$ and $\det^* + (\beta\gamma)^*$. Now formulae (cf. (1.5) and (1.6))

$$\alpha\delta = \det + \gamma\beta, \qquad \alpha^*\delta^* = \det^* + (\beta\gamma)^*$$

indicate that

$$\det + \gamma\beta = 0, \quad \text{and} \quad \det^* + (\beta\gamma)^* = 0$$

on $\ker\delta \subset (\ker\alpha\delta) \cap (\ker\alpha^*\delta^*)$. Therefore $\det + \gamma\beta = \det + \beta\gamma$. This means that $\beta\gamma = 0$ since $\gamma\beta = q^2\beta\gamma$ and in consequence $\det = 0$ on $\ker\delta$. Remembering that det is an invertible operator we conclude that $\ker\delta = \{0\}$, i.e. δ is an invertible operator. By a similar reasoning one can show that $\ker\alpha = \{0\}$ and conclude that also α is an invertible operator.

We shall consider matrices of the form

$$\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix} \tag{2.14}$$

where α, β, γ and δ are normal operators acting on a Hilbert space H. The above observations lead to the following definition.

Definition 2.5 *We say that (2.14) is a G-matrix whenever there exists a normal operator* det *such that*

1. $\mathrm{Sp}\,\beta,\ \mathrm{Sp}\,\gamma,\ \mathrm{Sp}\,\delta,\ \mathrm{Sp}\,\mathrm{det}\ \ \subset \overline{\Gamma}$;
2. δ *strongly commutes with* β *and* det;
3. $(\beta,\gamma),\ (\delta,\gamma),\ (\beta,\mathrm{det}),\ (\mathrm{det},\gamma)\ \in\ \overline{\mathcal{D}}_H$;
4. det *and* δ *are invertible operators;*
5. *If* $x \in D(\delta)\cap D(\gamma\beta)\cap D(\mathrm{det})$ *then* $\delta(x)\in D(\alpha)$ *and*
$$\alpha\delta(x) = \mathrm{det}(x) + \gamma\beta(x).$$

We shall explain the meaning of the last condition of the above Definition.

Let $y \in D(\delta^{-1})$ and $\delta^{-1}(y) \in D(\gamma\beta)\cap D(\mathrm{det})$. Then $\delta^{-1}(y)\in D(\delta)$ and $x := \delta^{-1}(y)$ satisfies the assumption of the last condition. Therefore

$$\alpha(y) = \mathrm{det}\delta^{-1}(y) + \gamma\beta\delta^{-1}(y).$$

This means that

$$\alpha \supset \mathrm{det}\circ\delta^{-1}\ +\ \gamma\beta\circ\delta^{-1}. \tag{2.15}$$

We know that $\mathrm{Sp}\,\delta\ \subset\ \overline{\Gamma}$. Therefore $\mathrm{Sp}\,\delta^{-1}\ \subset\ \overline{\Gamma}$. Since det and δ strongly commute then $\mathrm{Sp}\,(\mathrm{det}\,\delta^{-1}) \subset \overline{\Gamma}\cdot\overline{\Gamma} \subset \overline{\Gamma}$. As before $(\delta,\gamma\beta)$ is a q^2-pair. Then $(\gamma\beta,\delta^{-1})$ and $(\gamma\beta,\gamma\beta\delta^{-1})$ are q^2-pairs by Proposition 2.3. In particular $\mathrm{Sp}\,\gamma\beta\delta^{-1}\subset\overline{\Gamma}$. Now as in the proof of Proposition 2.4 using (2.13) one can check that $\tilde{\chi}(\mathrm{det}\,\delta^{-1},z)$ and $\tilde{\chi}(\gamma\beta\delta^{-1},z')$ satisfy the Weyl pentagonal relation (2.10) for any $z,z'\in\Gamma$. Therefore $(\gamma\beta\delta^{-1},\mathrm{det}\,\delta^{-1})$ is a q^2-pair. We claim that (2.15) implies that

$$\alpha \supset \mathrm{det}\,\delta^{-1}\ +\ \gamma\beta\delta^{-1}. \tag{2.16}$$

This will follow from a more general result which in our opinion is very important in itself.

Definition 2.6 *Let $\mathcal{T}$ be a set of closeable operators acting on a Hilbert space H and let D_o be a linear subset of H. We say that D_o is a core for $\mathcal{T}$ if*

1. $D_o \subset D(T)$ *for all* $T\in\mathcal{T}$;
2. *for any* $x\in H$ *there exists a sequence* $(x_n)_{n=1,2...}$ *of elements of* D_o *converging to* x *such that* $(Tx_n)_{n=1,2...}$ *converges to* $\overline{T}x$ *for all* $T\in\mathcal{T}$ *such that* $x\in D(\overline{T})$.

Let us point out two crutial features of this definition. First the vector x need not belong to the domain $\overline{T}$ for all $T \in \mathcal{T}$. Secondly, the approximating sequence $(x_n)_{n=1,2\ldots}$ is the same for all $T \in \mathcal{T}$ that may be taken into account.

Theorem 2.7
Let $T_1, T_2, \ldots T_p$ be normal operators with spectra contained in $\overline{\Gamma}$ and $\mathcal{T}$ be the set of all compositions of the form $T^{\#}_{i_1} \circ T^{\#}_{i_2} \circ \ldots \circ T^{\#}_{i_k}$ where $k \in \mathbf{N}$, $i_1, i_2, \ldots i_k = 1, 2, \ldots, p$ and $T^{\#}$ denotes either T or T^*. Assume that for each pair of indices $i, j = 1, 2, \ldots, p$, one of the following conditions hold:

1. *T_i strongly commutes with T_j,*
2. *(T_i, T_j) is a q^2-pair,*
3. *(T_j, T_i) is a q^2-pair.*

Then all operators in $\mathcal{T}$ are densely defined and closeable. Moreover there exists a core for $\mathcal{T}$.

Proof. For any $i = 1, 2, \ldots, p$ the decomposition

$$H = \ker T_i \oplus (\ker T_i)^{\perp}$$

is preserved by any T_j $(j = 1, 2, \ldots, p)$ since T_i and T_j strongly commute or form a q^2-pair. Therefore by passing eventually to a smaller number of variables we may assume that all operators T_i $(i = 1, 2, \ldots, p)$ are invertible. We shall show that further reduction is possible.

Writting normal invertible operator R in the polar decomposition form, $R = (\mathrm{Phase}\, R)|R|$ we conclude that $\mathrm{Phase}\, R$ is unitary, $|R|$ selfadjoint and strictly positive (positive and invertible) and $(\mathrm{Phase}\, R)|R| = |R|(\mathrm{Phase}\, R)$. Moreover if R and S are invertible operators then (R, S) is a q^2-pair if and only if (cf.[14, formula (2.4)])

$$(\mathrm{Phase}\, S)|R| = |R|(\mathrm{Phase}\, S), \qquad (\mathrm{Phase}\, R)|S| = |S|(\mathrm{Phase}\, R) \tag{2.17}$$

$$(\mathrm{Phase}\, S)(\mathrm{Phase}\, R) = q(\mathrm{Phase}\, R)(\mathrm{Phase}\, S)$$

and

$$|S|^{it}|R|^{it'} = e^{-\frac{2\pi i}{N}tt'}|R|^{it'}|S|^{it} \tag{2.18}$$

for any $t, t' \in \mathbf{R}$. Formally the last relation means that $|S|\,|R| = q|R|\,|S|$. Replacing $\frac{2\pi}{N}$ in formula (2.18) by some real number $\hbar > 0$ we obtain more general class known as Zakrzewski relations. They are denoted by $|R| \multimap |S|$ and were investigated in [15]. By [15, Definition 2.1] (2.18) means that $|R| \multimap |S|$ for

$$\hbar = \frac{2\pi}{N}. \tag{2.19}$$

Now writing any T_i, $(i = 1, 2, \ldots, p)$ in the polar decomposition form we see that for any pair of indices

$$(\mathrm{Phase}\, T_i)|T_j| = |T_j|(\mathrm{Phase}\, T_i)$$

and

$$(\ |T_i| \text{ and } |T_j| \text{ strongly commute }\ ) \quad \text{or} \quad (\ |T_i| \multimap |T_j|\ ) \quad \text{or} \quad (\ |T_j| \multimap |T_i|\ )\,. \tag{2.20}$$

Let $\mathcal{T}'$ be the set of all compositions of the form $|T_{i_1}|\circ|T_{i_2}|\circ ...\circ|T_{i_k}|$, where ($k \in \mathbf{N}$, and $i_1,\ i_2, ...i_k = 1, 2, ..., p$). By (2.17) elements $T \in \mathcal{T}$ admit decomposition $T = UT'$, where U is unitary and $T' \in \mathcal{T}'$. Therefore it is enough to prove that all operators in $\mathcal{T}'$ are densely defined and closeable and that there exists a core for $\mathcal{T}'$. Actually if all operators in $\mathcal{T}'$ are densely defined then they are closeable. Indeed an operator is closeable iff its adjoint is densely defined. On the other hand $\mathcal{T}'$ (as well as $\mathcal{T}$) is symmetric in the following sense: for any $T \in \mathcal{T}'$, the adjoint T^* is an extension of an operator belonging to $\mathcal{T}'$.

Let $\mathcal{F}$ be the family of all entire analytic functions ϕ on $\mathbf{C}$, such that

$$\sup_{x\in\mathbf{R}} |\phi(x + i\mu)|e^{\nu x} < +\infty \tag{2.21}$$

for all $\mu,\ \nu \in \mathbf{R}$. Denote by $\mathcal{R}$ the set of all operators of the form

$$R(\phi_1, \phi_2, ..., \phi_p) := \phi_1(\log|T_1|) \cdot \phi_2(\log|T_2|) \cdot ... \cdot \phi_p(\log|T_p|) \tag{2.22}$$

where $\phi_1, \phi_2, ..., \phi_p \in \mathcal{F}$. Clearly $\mathcal{R} \subset B(H)$.

For any $i, j = 1, 2, ...p$ and $\phi \in \mathcal{F}$ we set

$$\phi^{ij}(z) = \begin{cases} \phi(z) & \text{if} \quad |T_i| \text{ and } |T_j| \text{ strongly commute,} \\ \phi(z + i\hbar) & \text{if} \quad |T_i| \multimap |T_j|, \\ \phi(z - i\hbar) & \text{if} \quad |T_j| \multimap |T_i|. \end{cases} \tag{2.23}$$

and

$$\hat{\phi}(z) = e^z \phi(z).$$

Clearly $\phi^{ij},\ \hat{\phi} \in \mathcal{F}$.

Using Theorem 3.1 (2) of [15] we obtain

$$|T_i|\phi(\log|T_j|) \supset \phi^{ij}(\log|T_j|)|T_i|. \tag{2.24}$$

On the other hand obviously $\phi(\log|T_i|)H \subset D(|T_i|)$ and

$$|T_i|\phi(\log|T_i|) = \hat{\phi}(\log|T_i|). \tag{2.25}$$

By repeated use of (2.24) and (2.25) we see that $RH \subset D(|T_i|)$ and $|T_i|R \in \mathcal{R}$ for any $R \in \mathcal{R}$. Consequently

$$RH \subset D(T) \tag{2.26}$$

and $TR \in \mathcal{R}$ for all $T \in T'$ and $R \in \mathcal{R}$.

For any $\epsilon > 0$ we set $\phi_\epsilon(z) = e^{-\epsilon z^2}$. Then $\phi_\epsilon \in \mathcal{F}$. Let (cf. (2.22))

$$R_\epsilon := R(\phi_\epsilon, \phi_\epsilon, ..., \phi_\epsilon) \tag{2.27}$$

Then $\|R_\epsilon\| \leq 1$ and $s - \lim_{\epsilon\to 0+} R_\epsilon = I$. Therefore

$$D_o := \left(\bigcup_{\epsilon>0} R_\epsilon H\right)^{\text{linear span}}.$$

is a dense linear subset of H. By (2.26), $D_o \subset D(T)$ for all $T \in \mathcal{T}'$. In particular any operator in $\mathcal{T}'$ is densely defined. It is closeable since $\mathcal{T}'$ is symmetric.

Let $T \in \mathcal{T}'$. Using repeatedly (2.24) one can easily show that there exists $R^T_\epsilon \in \mathcal{R}$ such that

$$TR_\epsilon \supset R^T_\epsilon T.$$

This formula shows that

$$TR_\epsilon x = R^T_\epsilon \overline{T} x$$

for any $x \in D(T)$. Remembering that R^T_ϵ and TR_ϵ are bounded we see that the above equality holds for all $x \in D(\overline{T})$. Therefore

$$TR_\epsilon \supset R^T_\epsilon \overline{T}. \tag{2.28}$$

Inspecting im more detail the dependence of R^T_ϵ on ϵ we see that

$$R^T_\epsilon = R(\phi^1_\epsilon, \phi^2_\epsilon, ..., \phi^p_\epsilon)$$

where

$$\phi^k_\epsilon(z) = e^{-\epsilon(z+in_k\hbar)^2}$$

where n_k is an integer depending on T and k but independent of ϵ. Due to this property R^T_ϵ converges strongly to I when $\epsilon \to 0$. In particular

$$\lim_{n\to+\infty} R^T_{\frac{1}{n}} x = x \tag{2.29}$$

for any $x \in H$.

For any $x \in H$ we set

$$x_n := R_{\frac{1}{n}} x \in D_o.$$

Then $x_n \to x$ when $n \to +\infty$. Let T be an element of $\mathcal{T}'$ such that $x \in D(\overline{T})$. Formula (2.28) shows that

$$Tx_n = R^T_{\frac{1}{n}} \overline{T} x$$

and $Tx_n \to \overline{T}x$ by (2.29). This shows that D_o is a core for $\mathcal{T}'$. □

Now to prove (2.16) we note that operators $T_1 := \det$, $T_2 := \delta^{-1}$, $T_3 := \gamma\beta$ satisfy the assumptions of the above Theorem. If $x \in D(\det\delta^{-1} + \gamma\beta\delta^{-1})$ then $x \in D(\det\delta^{-1})$, $x \in D(\gamma\beta\delta^{-1})$ and there exists a sequence $(x_n)_{n=1,2...}$ in the core such that $x = \lim x_n$, $\gamma\beta\delta^{-1}(x_n) \longrightarrow (\gamma\beta\delta^{-1})(x)$ and $\det\delta^{-1}(x_n) \longrightarrow (\det\delta^{-1})(x)$. Therefore by (2.15)

$$\alpha(x_n) = \det\delta^{-1}(x_n) + (\gamma\beta)\delta^{-1}(x_n) \longrightarrow (\det\delta^{-1})(x) + (\gamma\beta\delta^{-1})(x) = (\det\delta^{-1} + \gamma\beta\delta^{-1})(x).$$

Remembering that α is closed we get $\alpha(x) = (\det\delta^{-1} + \gamma\beta\delta^{-1})(x)$. The proof of (2.16) is done.

For normal operators X and Y their sum $X+Y$ in general does not behave well. However the case when (Y, X) is a q^2-pair is more regular. The corresponding result involves a special function F_N introduced in [14, Section 1]. For the precise and rather complicated formula defining F_N and fundamental properties of F_N we refer to [14]. We mention only basic results beeing a simple extension of Theorem 2.4, Corollary 2.5 and Theorem 2.6 of [14]. To this end we recall that F_N is a continuous function on $\overline{\Gamma}$, $F_N : \overline{\Gamma} \longrightarrow S^1$ and $F_N(0) = 1$. Therefore $F_N(V)$ makes sense for any normal operator V such that $\mathrm{Sp}\, V \subset \overline{\Gamma}$ and defines then a unitary operator.

Proposition 2.8
Let $(Y,X) \in \overline{\mathcal{D}}_H$. *Then*

1. $X + X{\circ}Y$ *and* $X{\circ}Y + Y$ *are densely defined closeable operators and their closures*
$$X \dot{+} XY = F_N(Y)^* X F_N(Y) \tag{2.30}$$
$$XY \dot{+} Y = F_N(X) Y F_N(X)^*. \tag{2.31}$$
In particular $X \dot{+} XY$ *and* $XY \dot{+} Y$ *are normal operators and* $\mathrm{Sp}\,(X \dot{+} XY)$, $\mathrm{Sp}\,(XY \dot{+} Y) \subset \overline{\Gamma}$.

2. $X + Y$ *is a densely defined closeable operator, its closure* $X \dot{+} Y$ *is normal operator with* $\mathrm{Sp}\,(X \dot{+} Y) \subset \overline{\Gamma}$. *Moreover*
$$X \dot{+} Y = \begin{cases} F_N(X^{-1}Y)^* X F_N(X^{-1}Y) & \text{if } X \text{ is invertible} \\ F_N(XY^{-1}) Y F_N(XY^{-1})^* & \text{if } Y \text{ is invertible.} \end{cases} \tag{2.32}$$
In particular $X \dot{+} Y$ *is an invertible operator if* X *or* Y *is invertible.*

3. $$F_N(Y \dot{+} X) = F_N(Y) F_N(X). \tag{2.33}$$

Let us remark that F_N is a quantum exponential function. This name is justified by the formula (2.33).

Combining (2.30) and (2.32) with (2.33) we get pentagonal relations

Corollary 2.9
Let $(Y,X) \in \overline{\mathcal{D}}_H$. *Then*
$$F_N(XY) \;=\; F_N(Y)^* F_N(X) F_N(Y) F_N(X)^* \tag{2.34}$$
and
$$F_N(Y)F_N(X) = \begin{cases} F_N(X^{-1}Y)^* F_N(X) F_N(X^{-1}Y) & \text{if } X \text{ is invertible} \\ F_N(XY^{-1}) F_N(Y) F_N(XY^{-1})^* & \text{if } Y \text{ is invertible.} \end{cases} \tag{2.35}$$

To continue the discussion of the last condition of Definition 2.5 we put $X = \det\delta^{-1}$ and $Y = \gamma\beta\delta^{-1}$ in the second part of Proposition 2.8 and conclude that $\det\delta^{-1} + \gamma\beta\delta^{-1}$ is densely defined closeable operator on H whose closure $\det\delta^{-1} \dot{+} \gamma\beta\delta^{-1}$ is a normal operator with $\mathrm{Sp}\,(\det\delta^{-1} \dot{+} \gamma\beta\delta^{-1}) \subset \overline{\Gamma}$. Since a normal operator admits no proper normal extension then the last condition of Definition 2.5 implies that in fact a stronger relation holds:
$$\alpha = \det\delta^{-1} \dot{+} \gamma\beta\delta^{-1}. \tag{2.36}$$
In particular $\mathrm{Sp}\,\alpha \subset \overline{\Gamma}$.

Definition 2.5 assumes the existence of operator det. Now we investigate the problem of uniquness. We shall show that there exists at most one such det. Let us note that due to the last condition of Definition 2.5 the operator det is defined on the domain
$$D_{\det} := \{\, x \in H : x \in\; D(\delta) \cap D(\gamma\beta) \cap D(\det) \,\} \tag{2.37}$$

since $\delta(D_{\det}) \subset D(\alpha)$ and

$$\det(x) = \alpha\delta(x) - \gamma\beta(x) \tag{2.38}$$

for any $x \in D_{\det}$. Clearly $D_{\det}$ is a core for det, but $D_{\det}$ depends on det. Therefore the uniqueness of det is not a straightforward consequence of (2.38). Nevertheless we can find a domain which is a core for det, does not depend on det and such that (2.38) is satisfied.

At first let us note that due to the last condition of Definition 2.5

$$D_{\det} \subset D_\alpha := \{\, x \in H :\, x \in D(\delta) \cap D(\gamma\beta) \text{ and } \delta(x) \in D(\alpha)\,\}. \tag{2.39}$$

We shall check that D_α has desirable properties. Clearly D_α does not depend on det. By (2.39) D_α will be a core for det if we prove that $D_\alpha \subset D(\det)$. We have

Lemma 2.10
Let $(Y,\, X)$ be a q^2-pair. Then $D(Y) \cap D(Y \dot{+} X) \subset D(X)$.

Proof. Let $x \in D(Y) \cap D(Y \dot{+} X)$. Then for any $y \in D(Y) \cap D(X)$

$$(y|(Y \dot{+} X)x) = (Y^* y + X^* y|x)$$

(remember that $Y \dot{+} X$ is the closure of $Y + X$). On the other hand $(y|Yx) = (Y^* y|x)$ and

$$(y|(Y \dot{+} X)x - Yx) = (X^* y|x).$$

Since $D(Y) \cap D(X)$ is a core for X^* we get that $x \in D(X)$.

□

Applying the above Lemma to the q^2-pair $(\gamma\beta\delta^{-1},\, \det\delta^{-1})$ we find (cf. (2.36)) that

$$\{\, y \in H : y \in D(\delta^{-1}) \cap D(\alpha) \text{ and } \delta^{-1}(y) \in D(\gamma\beta)\ \} \subset D(\det\delta^{-1}).$$

In particular if $y \in D(\delta^{-1}) \cap D(\alpha)$ and $\delta^{-1}(y) \in D(\gamma\beta)$ then $\delta^{-1}(y) \in D(\det\delta^{-1} \circ \delta) \subset D(\det)$. Therefore

$$\begin{aligned} D_\alpha &= \{\, x \in H :\, x \in D(\delta) \cap D(\gamma\beta) \text{ and } \delta(x) \in D(\alpha)\,\} \\ &= \{\, \delta^{-1}(y) :\, y \in D(\delta^{-1}) \cap D(\alpha) \text{ and } \delta^{-1}(y) \in D(\gamma\beta)\ \} \subset D(\det). \end{aligned}$$

Now (2.38) is satisfied (cf. (2.36)) for any $x \in D_\alpha$. The uniqueness of det is proved.

Remembering that det and δ strongly commute and that

$$\gamma\beta\delta^{-1} = q^{-2}\delta^{-1}\gamma\beta = \delta^{-1}\beta\gamma$$

we also get

$$\alpha = \delta^{-1}\det \dot{+}\, \delta^{-1}\beta\gamma.$$

Since $\det\delta^{-1}$ is invertible, by the first formula of (2.32) we obtain

$$\alpha = F_N(Z)^* (\det\delta^{-1}) F_N(Z) \tag{2.40}$$

where $Z := (\det^{-1}\delta)(\gamma\beta\delta^{-1})$. Therefore α is an invertible operator. Since $(\gamma\beta, \delta^{-1})$ is a q^2-pair,

$$Z = \det^{-1}\beta\gamma. \tag{2.41}$$

Clearly $\operatorname{Sp} Z \subset \overline{\Gamma}$ and using the standard argument we verify that the operators $\det, \gamma$ and β strongly commute with Z. Therefore $\det, \gamma$ and β commute with the unitary operator $F_N(Z)$. Taking into account that det and γ strongly commute with $\det\delta^{-1}$ and $(\beta, \det\delta^{-1})$ is a q^2-pair we see, due to formula (2.40), that α strongly commutes with operators det and γ. Moreover $(\beta,\ \alpha)$ is a q^2-pair. We summarize our discussion in

Proposition 2.11
Let (2.14) be G-matrix and det *be the operator appearing in Definition 2.5. Then*

1. $\operatorname{Sp}\alpha \subset \overline{\Gamma}$;
2. α *is an invertible operator;*
3. α *strongly commutes with* γ *and* det;
4. (β, α) *is a* q^2*-pair.*
5. *The set* $D_\alpha := \{\, x \in D(\delta) \cap D(\gamma\beta) : \delta(x) \in D(\alpha) \,\}$ *is a core for* det *and*

$$\det(x) = \alpha\delta(x) - \gamma\beta(x)$$

for any $x \in D_\alpha$.

Definition 2.5 and Proposition 2.11 give a precise operator meaning to the relations (1.4), (1.5), (1.6), (1.7), (1.14) and (1.15) on the Hilbert space level.

Operator α plays a special role in Definition 2.5. But let us note that we can give a definition of a G-matrix in which δ plays the role of α. As we know α is an invertible operator. Therefore formula (2.40) implies

$$\alpha^{-1} = F_N(Z)^*(\det^{-1}\delta)F_N(Z)$$

and

$$\delta = F_N(Z)(\det\alpha^{-1})F_N(Z)^* \tag{2.42}$$

since det and Z strongly commute. Now $(\det\alpha^{-1},\ \beta\gamma\alpha^{-1})$ is a q^2-pair, $(\beta\gamma\alpha^{-1})(\det\alpha^{-1})^{-1} = Z$ and by the second formula (2.32)

$$\delta = \det\alpha^{-1} \dot{+}\, \beta\gamma\alpha^{-1}. \tag{2.43}$$

Now one can easily prove that

Proposition 2.12

1. *If* $y \in D(\alpha) \cap D(\beta\gamma) \cap D(\det)$ *then* $\alpha(y) \in D(\delta)$ *and*

$$\delta\alpha(y) = \det(y) + \beta\gamma(y).$$

2. *The set* $D_\delta := \{\, y \in D(\delta) \cap D(\beta\gamma) : \alpha(y) \in D(\delta) \,\}$ *is a core for* det *and*

$$\det(y) = \delta\alpha(y) - \beta\gamma(y)$$

for any $y \in D_\delta$.

Now the above mentioned alternative definition of a G-matrix is the following

Definition 2.13 *We say that (2.14) is a G-matrix whenever there exists a normal operator* det *such that*

1. $\mathrm{Sp}\,\beta,\ \mathrm{Sp}\,\gamma,\ \mathrm{Sp}\,\alpha,\ \mathrm{Sp}\det \quad \subset \overline{\Gamma}$;
2. α *strongly commutes with* γ *and* det;
3. $(\beta,\, \gamma),\ (\beta,\, \alpha),\ (\beta,\, \det),\ (\det,\, \gamma)\ \in\ \overline{\mathcal{D}}_H$;
4. det *and* α *are invertible operators;*
5. *If* $x \in D(\alpha) \cap D(\gamma\beta) \cap D(\det)$ *then* $\alpha(x) \in D(\delta)$ *and*

$$\delta\alpha(x) = \det(x) + \beta\gamma(x).$$

Due to Proposition 2.11 and Proposition 2.12 we see that both definitions of a G-matrix are equivalent.

Proposition 2.11 shows that the operator α is completely determined by (2.11). Therefore G-matrices are in fact parametrized by elements β, γ, δ, det satisfying conditions $1-4$ of Definition 2.5. More precisely we have

Proposition 2.14 *Let (2.14) be a G-matrix (in the sense of Definition 2.5) on the Hilbert space* H *and let* $C \in C^*(H)$*. Then*

$$(\,\alpha,\, \beta,\, \gamma,\, \delta,\, \det^{-1} \eta\, C\,) \Longleftrightarrow (\,\beta,\, \gamma,\, \delta,\, \delta^{-1},\, \det,\, \det^{-1} \eta\, C\,) \tag{2.44}$$

Proof. The proof is based on the following lemma:

Lemma 2.15
Let X *and* Y *be normal operators on a Hilbert space* H *and let* $C \in C^*(H)$*. Assume that* $X\,\eta\, C$ *and* $Y\,\eta\, C$*. Then*

1. $\Big(\ X \text{ and } Y \text{ strongly commute}\ \Big) \Longrightarrow \Big(\ XY\,\eta\, C\ \Big)$.
2. $\Big(\ (Y, X) \in \overline{\mathcal{D}}_H\ \Big) \Longrightarrow \Big(\ (X \dot{+} Y)\,\eta\, C \text{ and } XY\,\eta\, C\ \Big)$.

Proof of the Lemma will be presented later. Now let us assume that α, β, γ, δ, $\det^{-1} \eta\, C$. Then due to (2.40)

$$\det\delta^{-1} = F_N(Z)\alpha F_N(Z)^*. \tag{2.45}$$

where $Z = \det^{-1}\beta\gamma$. By above Lemma $Z\,\eta\, C$ since $\beta\gamma\,\eta\, C$ ($(\beta,\, \gamma)$ is a q^2-pair) and $\beta\gamma$ strongly commutes with $\det^{-1}$. Now $F_N(Z) \in M(C)$ as F_N is a continuous function on $\mathrm{Sp}\, Z$.

Therefore $\det\delta^{-1}\,\eta\,C$. As we know operators det and δ strongly commute and $\det = (\det\delta^{-1})\delta$, $\delta^{-1} = \det^{-1}(\det\delta^{-1})$. Therefore $\det\eta\,C$ and $\delta^{-1}\,\eta\,C$.

Conversely assume that β, γ, δ, δ^{-1}, det, $\det^{-1}\eta\,C$. Then $F_N(Z) \in M(C)$ as before and $\det\delta^{-1}\,\eta\,C$. Therefore by formula (2.40) $\alpha\,\eta\,C$. (The same conclusion follows also from (2.36) and above Lemma since $(\gamma\beta\delta^{-1}, \det\delta^{-1})$ is a q^2-pair which consists of elements affiliated with C.)

□

Proof of Lemma2.15. It is known that if X and Y are strongly commuting operators affiliated with a C^*-algebra C then $f(X,Y)\,\eta\,C$ for any continuous function f on $\mathrm{Sp}\,X \times \mathrm{Sp}\,Y$ and the Statement 1 follows.

We prove Statement 2. By [14, Theorem 5.1] a normal operator T with $\mathrm{Sp}\,T \subset \overline{\Gamma}$ is affiliated with $C \in C^*(H)$ iff $F_N(zT) \in M(C)$ for any $z \in \overline{\Gamma}$ and the mapping

$$\overline{\Gamma} \ni z \longrightarrow F_N(zT) \in M(C)$$

is continuous provided $M(C)$ is considered with the strict topology. By assumption (Y, X) is a q^2-pair and $X\,\eta\,C$, $Y\,\eta\,C$. Therefore $\mathrm{Sp}\,X \subset \overline{\Gamma}$, $\mathrm{Sp}\,Y \subset \overline{\Gamma}$ and the mappings

$$\begin{aligned} \overline{\Gamma} \ni z &\longrightarrow F_N(zX) \in M(C) \\ \overline{\Gamma} \ni z &\longrightarrow F_N(zY) \in M(C) \end{aligned} \tag{2.46}$$

are continuous in strict topology on $M(C)$. We know that $X\dot{+}Y$ and XY are normal operators and $\mathrm{Sp}\,(X\dot{+}Y) \subset \overline{\Gamma}$, $\mathrm{Sp}\,XY \subset \overline{\Gamma}$ (cf. Statement 2 of Proposition 2.8 and Proposition 2.2). Therefore by (2.33) and (2.34) $F_N(z(X\dot{+}Y)) = F_N(zY)F_N(zX) \in M(C)$ and $F_N(zXY) = F_N(Y)^*F_N(zX)F_N(Y)F_N(zX)^* \in M(C)$ for any $z \in \overline{\Gamma}$ and the mappings

$$\overline{\Gamma} \ni z \longrightarrow F_N(zY)F_N(zX) \in M(C)$$

$$\overline{\Gamma} \ni z \longrightarrow F_N(Y)^*F_N(zX)F_N(Y)F_N(zX)^* \in M(C)$$

are continuous in strict topology on $M(C)$ due to the continuity of (2.46) (note that multiplication map restricted to bounded subsets of $M(C)$ and *-operation are continuous in strict topology (cf. [11, p.489])). This shows that $(X\dot{+}Y)\,\eta\,C$ and $XY\,\eta\,C$.

□

Remark. Let us note that parametrization of G-matrices based on elements α, β, γ, det appearing in Definition 2.13 can be used as well. Then instead of (2.44) we obtain that

$$(\,\alpha,\,\beta,\,\gamma,\,\delta,\,\det^{-1}\eta\,C\,) \Longleftrightarrow (\,\beta,\,\gamma,\,\alpha,\,\alpha^{-1},\,\det,\,\det^{-1}\eta\,C\,). \tag{2.47}$$

Now we consider tensor products. Let H_1 and H_2 be Hilbert spaces and T_1, T_2 be normal operators on H_1 and H_2 respectively. Then $T = T_1 \otimes T_2$ is a normal operator on $H = H_1 \otimes H_2$ and $D(T_1) \otimes_{alg} D(T_2)$ is a core for T. Moreover $\mathrm{Sp}\,T \subset \overline{(\mathrm{Sp}\,T_1)(\mathrm{Sp}\,T_2)}^{\text{closure}}$. In particular if $\mathrm{Sp}\,T_1$, $\mathrm{Sp}\,T_2 \subset \overline{\Gamma}$ then $\mathrm{Sp}\,(T_1 \otimes T_2) \subset \overline{\Gamma}$ and (cf. (2.8))

$$\tilde{\chi}(T_1 \otimes T_2, z) = \tilde{\chi}(T_1, z) \otimes \tilde{\chi}(T_2, z) \tag{2.48}$$

for any $z \in \Gamma$. Using this and (2.10) one easily obtains

Lemma 2.16
Let $(Y_1, X_1) \in \overline{\mathcal{D}}_{H_1}$ and X_2, Y_2 be normal operators on H_2. Then

1. $$\left(\begin{array}{c} \operatorname{Sp} X_2 \subset \overline{\Gamma},\ \operatorname{Sp} Y_2 \subset \overline{\Gamma} \\ \textit{and} \\ X_2 \textit{ strongly commutes with } Y_2 \end{array}\right) \Longrightarrow \Big((Y_1 \otimes Y_2, X_1 \otimes X_2) \in \overline{\mathcal{D}}_{H_1 \otimes H_2} \Big).$$

2. $\Big((Y_2, X_2) \in \overline{\mathcal{D}}_{H_2} \Big) \Longrightarrow ($ $X_1 \otimes Y_2$ and $Y_1 \otimes X_2$ strongly commute $)$.

We state the main result concerning the tensor products of G-matrices.

Theorem 2.17
Let

$$u_1 = \begin{pmatrix} \alpha_1, & \beta_1 \\ \gamma_1, & \delta_1 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} \alpha_2, & \beta_2 \\ \gamma_2, & \delta_2 \end{pmatrix} \tag{2.49}$$

be G-matrices on Hilbert spaces H_1 and H_2 respectively and

$$\tilde{u} = \begin{pmatrix} \tilde{\alpha}, & \tilde{\beta} \\ \tilde{\gamma}, & \tilde{\delta} \end{pmatrix} = u_1 \textcircled{\scriptsize T} u_2 := \begin{pmatrix} \alpha_1 \otimes \alpha_2 \dotplus \beta_1 \otimes \gamma_2, & \alpha_1 \otimes \beta_2 \dotplus \beta_1 \otimes \delta_2 \\ \gamma_1 \otimes \alpha_2 \dotplus \delta_1 \otimes \gamma_2, & \gamma_1 \otimes \beta_2 \dotplus \delta_1 \otimes \delta_2 \end{pmatrix}. \tag{2.50}$$

Then $\tilde{u}$ is a G-matrix on $H_1 \otimes H_2$. Moreover

$$\tilde{\det} = \det_1 \otimes \det_2$$

where $\tilde{\det}$, $\det_1$ *and* $\det_2$ *are the corresponding quantum determinants for $\tilde{u}$, u_1 and u_2 respectively.*

Remark. The reader should notice that $(\beta_1 \otimes \gamma_2,\ \alpha_1 \otimes \alpha_2)$, $(\beta_1 \otimes \delta_2,\ \alpha_1 \otimes \beta_2)$, $(\delta_1 \otimes \gamma_2,\ \gamma_1 \otimes \alpha_2)$ and $(\delta_1 \otimes \delta_2,\ \gamma_1 \otimes \beta_2)$ are q^2-pairs on $H = H_1 \otimes H_2$. This follows from Definition 2.5, Proposition 2.11 and Lemma 2.16. Therefore the matrix elements of $\tilde{u}$ are well defined. They are normal operators on H with their spectra contained in $\overline{\Gamma}$ (cf. Proposition 2.32, Statement 2).

Proof. Let us note that the notion of G-matrix on H is invariant under unitary transformation of H. Therefore we may replace $\tilde{u}$ by a simpler unitarily equivalent matrix $\tilde{u}_U$ for a suitable unitary operator U on H :

$$\tilde{u} = (\mathrm{id} \otimes U)\tilde{u}_U(\mathrm{id} \otimes U^*)$$

where

$$\tilde{u}_U = \begin{pmatrix} \tilde{\alpha}_U, & \tilde{\beta}_U \\ \tilde{\gamma}_U, & \tilde{\delta}_U \end{pmatrix}. \tag{2.51}$$

Since $\delta_1 \otimes \delta_2$ is an invertible operator on H $\tilde{\delta}$ is invertible and we shall show that there exists a unitary operator U commuting with $\tilde{\det}$ such that $\tilde{\delta}_U = \delta_1 \otimes \delta_2$ and $\tilde{u}_U$ is G-matrix with the same determinant $\tilde{\det}_U = \tilde{\det}$. At first let us note that by Statement 2 of Proposition 2.8

$$\tilde{\delta} = \gamma_1 \otimes \beta_2 \dotplus \delta_1 \otimes \delta_2 = F_N(\gamma_1 \delta_1^{-1} \otimes \beta_2 \delta_1^{-1})(\delta_1 \otimes \delta_2) F_N(\gamma_1 \delta_1^{-1} \otimes \beta_2 \delta_1^{-1})^*. \tag{2.52}$$

Set

$$Y = \gamma_1\delta_1^{-1} \otimes \beta_2\delta_1^{-1}. \tag{2.53}$$

Then

$$\tilde{u} = (\mathrm{id} \otimes F_N(Y)) \begin{pmatrix} F_N(Y)^*\tilde{\alpha}F_N(Y), & F_N(Y)^*\tilde{\beta}F_N(Y) \\ F_N(Y)^*\tilde{\gamma}F_N(Y), & \delta_1 \otimes \delta_2 \end{pmatrix} (\mathrm{id} \otimes F_N(Y))^* \tag{2.54}$$

and

$$\tilde{\det} = F_N(Y)\tilde{\det}F_N(Y)^*$$

since Y and $\tilde{\det}$ strongly commute. Due to formula (2.40)

$$\begin{aligned} \alpha_1 \otimes I_2 &= F_N(Z_1)^*(\det{}_1\, \delta_1^{-1} \otimes I_2)F_N(Z_1) \\ I_1 \otimes \alpha_2 &= F_N(Z_2)^*(I_1 \otimes \det{}_2\, \delta_2^{-1})F_N(Z_2) \end{aligned} \tag{2.55}$$

where following (2.41) we denoted

$$Z_1 = \det{}_1^{-1}\beta_1\gamma_1 \otimes I_2 \quad \text{and} \quad Z_2 = I_1 \otimes \det{}_2^{-1}\beta_2\gamma_2. \tag{2.56}$$

Note that Z_1, Z_2 strongly commute and $(Z_1, Y) \in \overline{\mathcal{D}}_H$, $(Z_2, Y) \in \overline{\mathcal{D}}_H$. Since $\beta_1 \otimes \gamma_2$ strongly commutes with Z_1 and Z_2, $\beta_1 \otimes \delta_2$ strongly commutes with Z_1 and $\delta_1 \otimes \gamma_2$ strongly commutes with Z_2 we obtain

$$\begin{aligned} \tilde{\alpha} &= F_N(Z_1)^*F_N(Z_2)^* \left[\det{}_1\delta_1^{-1} \otimes \det{}_2\delta_2^{-1} \dot{+} \beta_1 \otimes \gamma_2\right] F_N(Z_2)F_N(Z_1) \\ \tilde{\beta} &= F_N(Z_1)^* \left[\det{}_1\delta_1^{-1} \otimes \beta_2 \dot{+} \beta_1 \otimes \delta_2\right] F_N(Z_1) \\ \tilde{\gamma} &= F_N(Z_2)^* \left[\gamma_1 \otimes \det{}_2\delta_2^{-1} \dot{+} \delta_1 \otimes \gamma_2\right] F_N(Z_2). \end{aligned} \tag{2.57}$$

Further discussion will depend on invertibility of $\beta_1 \otimes \delta_2$, $\delta_1 \otimes \gamma_2$, i.e on invertibility of β_1 and γ_2 since δ_1, δ_2 are invertible. Let us note that if (2.14) is a G-matrix then operators β and γ either strongly commute or form q^2-pair with any of the operators α, β, γ, δ and det (cf. Definition 2.5 and Proposition 2.11). In particular $\ker\beta$, $\ker\gamma$ and their orthogonal complements are invariant subspaces under the action of all operators. Therefore the decomposition

$$H = \left(\ker\beta_1^{\perp} \otimes \ker\gamma_2^{\perp}\right) \oplus \left(\ker\beta_1 \otimes \ker\gamma_2^{\perp}\right) \oplus \left(\ker\beta_1^{\perp} \otimes \ker\gamma_2\right) \oplus (\ker\beta_1 \otimes \ker\gamma_2) \tag{2.58}$$

is respected by $\tilde{\det}$ and the matrix elements of $\tilde{u}$. In particular it is enough to consider each of the above direct summands separately.

Case 1. β_1 and γ_2 are invertible operators on H_1 and H_2 respectively.
In particular in this case $\beta_1 \otimes \gamma_2$, $\beta_1 \otimes \delta_2$ and $\delta_1 \otimes \gamma_2$ are invertible operators on H. Since $(\beta_1 \otimes \gamma_2, \det_1\delta_1^{-1} \otimes \det_2\delta_2^{-1})$, $(\beta_1 \otimes \delta_2, \det_1\delta_1^{-1} \otimes \beta_2)$ and $(\delta_1 \otimes \gamma_2, \gamma_1 \otimes \det_2\delta_2^{-1})$ are q^2-pairs on H, by Statement 2 of Proposition 2.8

$$\begin{aligned} \det{}_1\delta_1^{-1} \otimes \det{}_2\delta_2^{-1} \dot{+} \beta_1 \otimes \gamma_2 &= F_N(R)(\beta_1 \otimes \gamma_2)F_N(R)^* \\ \det{}_1\delta_1^{-1} \otimes \beta_2 \dot{+} \beta_1 \otimes \delta_2 &= F_N(X_1)(\beta_1 \otimes \delta_2)F_N(X_1)^* \\ \gamma_1 \otimes \det{}_2\delta_2^{-1} \dot{+} \delta_1 \otimes \gamma_2 &= F_N(X_2)(\delta_1 \otimes \gamma_2)F_N(X_2)^* \end{aligned} \tag{2.59}$$

where

$$\begin{aligned} X_1 &= \det{}_1\delta_1^{-1}\beta_1^{-1}\otimes\beta_2\delta_2^{-1}, \\ X_2 &= \gamma_1\delta_1^{-1}\otimes\det{}_2\delta_2^{-1}\gamma_2^{-1} \end{aligned} \tag{2.60}$$

and

$$R = \det{}_1\delta_1^{-1}\beta_1^{-1}\otimes\det{}_2\delta_2^{-1}\gamma_2^{-1}. \tag{2.61}$$

One can easily check that X_1 and X_2 are invertible and strongly commute, $(Y, X_1) \in \overline{\mathcal{D}}_H$, $(Y, X_2) \in \overline{\mathcal{D}}_H$ and $Z_1 = X_1^{-1}Y$, $Z_2 = X_2^{-1}Y$. Moreover Y is invertible and

$$R = X_1Y^{-1}X_2 = X_2Y^{-1}X_1.$$

Set

$$V = F_N(Y)^*F_N(X_1^{-1}Y)^*F_N(X_2^{-1}Y)^*F_N(X_2Y^{-1}X_1). \tag{2.62}$$

Then using (2.57) and (2.59) matrix elements appearing in (2.54) can be written in the form

$$F_N(Y)^*\tilde{\alpha}F_N(Y) = V(\beta_1\otimes\gamma_2)V^*, \tag{2.63}$$

$$F_N(Y)^*\tilde{\beta}F_N(Y) = F_N(Y)^*F_N(X_1^{-1}Y)^*F_N(X_1)(\beta_1\otimes\delta_2)F_N(X_1)^*F_N(X_1^{-1}Y)F_N(Y), \tag{2.64}$$

$$F_N(Y)^*\tilde{\gamma}F_N(Y) = F_N(Y)^*F_N(X_2^{-1}Y)^*F_N(X_2)(\delta_1\otimes\gamma_2)F_N(X_2)^*F_N(X_2^{-1}Y)F_N(Y). \tag{2.65}$$

Now by (2.35) we get

$$F_N(Y)^*F_N(X_i^{-1}Y)^*F_N(X_i) = F_N(X_i)F_N(X_i^{-1}Y)^* \tag{2.66}$$

for $i = 1, 2$. Since $\beta_1\otimes\delta_2$ strongly commutes with $X_1^{-1}Y = Z_1$ and $\delta_1\otimes\gamma_2$ strongly commutes with $X_2^{-1}Y = Z_2$ (cf. (2.64) and (2.65)) it follows that

$$F_N(Y)^*\tilde{\beta}F_N(Y) = F_N(X_1)(\beta_1\otimes\delta_2)F_N(X_1)^* \tag{2.67}$$

and

$$F_N(Y)^*\tilde{\gamma}F_N(Y) = F_N(X_2)(\delta_1\otimes\gamma_2)F_N(X_2)^*. \tag{2.68}$$

On the other hand applying successively (2.35) to q^2-pairs $(X_1,\, X_1Y^{-1}X_2)$, (Y, X_1) and $(X_2,\, X_2Y^{-1}X_1)$ in equlity (2.62) we obtain

$$\begin{aligned} V &= F_N(Y)^*F_N(X_1^{-1}Y)^*F_N(X_1)F_N(X_2Y^{-1}X_1)F_N(X_2^{-1}Y)^* \\ &= F_N(X_1)F_N(X_1^{-1}Y)^*F_N(X_1Y^{-1}X_2)F_N(X_2^{-1}Y)^* \\ &= F_N(X_1)F_N(X_2)F_N(X_1Y^{-1}X_2)F_N(X_1^{-1}Y)^*F_N(X_2^{-1}Y)^* \\ &= F_N(X_1)F_N(X_2)F_N(R)F_N(Z_1)^*F_N(Z_2)^*. \end{aligned}$$

Now $\beta_1\otimes\gamma_2$ commutes with Z_1 and Z_2 and (cf. (2.63))

$$F_N(Y)^*\tilde{\alpha}F_N(Y) = F_N(X_1)F_N(X_2)F_N(R)(\beta_1\otimes\gamma_2)F_N(R)^*F_N(X_2)^*F_N(X_1)^*.$$

Moreover X_1 strongly commutes with $\delta_1\otimes\gamma_2$, X_2 strongly commutes with $\beta_1\otimes\delta_2$ and X_1, X_2 strongly commute with $\det_1\otimes\det_2$ and $\delta_1\otimes\delta_2$. Set

$$U = F_N(Y)F_N(X_1)F_N(X_2).$$

Therefore in this case the matrix $\tilde{u}_U$ (cf. (2.51) and (2.59)) is of the form

$$\tilde{u}_U = \begin{pmatrix} \det_1\delta_1^{-1} \otimes \det_2\delta_2^{-1} \dot{+} \beta_1 \otimes \gamma_2, & \beta_1 \otimes \delta_2 \\ \delta_1 \otimes \gamma_2, & \delta_1 \otimes \delta_2 \end{pmatrix} \tag{2.69}$$

and

$$\tilde{\det}_U = U\tilde{\det}U^* = \tilde{\det}.$$

Now conditions 1-4 of Definition 2.5 are easily verified due to Lemma 2.16. To verify the fifth condition it is enough to check that (cf. (2.36))

$$\tilde{\alpha}_U = \tilde{\det}_U\, \tilde{\delta}_U^{-1} \dot{+} \tilde{\gamma}_U \tilde{\beta}_U \tilde{\delta}_U^{-1}. \tag{2.70}$$

It is obvious since $\tilde{\det}_U\, \tilde{\delta}_U^{-1} = \det_1\delta_1^{-1} \otimes \det_2\delta_2^{-1}$ and

$$\tilde{\gamma}_U \tilde{\beta}_U \tilde{\delta}_U^{-1} = (\delta_1 \otimes \gamma_2)(\beta_1 \otimes \delta_2)(\delta_1^{-1} \otimes \delta_2^{-1}) = \beta_1 \otimes \gamma_2.$$

Case 2. $\beta_1 = 0$ and γ_2 is an invertible operator on H_2.
In this case $\delta_1 \otimes \gamma_2$ is an invertible operator on H as before and $\alpha_1 = \det_1\delta_1^{-1}$. Now $\det_1\delta_1^{-1} \otimes \beta_2$ and Y (cf. (2.53)) commute and (2.54) takes the form

$$\tilde{u} = (\mathrm{id} \otimes F_N(Y)) \begin{pmatrix} F_N(Y)^*(\det_1\delta_1^{-1} \otimes \tilde{\alpha}_2)F_N(Y), & \det_1\delta_1^{-1} \otimes \beta_2 \\ F_N(Y)^* \tilde{\gamma} F_N(Y), & \delta_1 \otimes \delta_2 \end{pmatrix} (\mathrm{id} \otimes F_N(Y))^*. \tag{2.71}$$

Note that $Z_1 = 0$ and $F_N(Z_1) = I$ but $Z_2 = X_2^{-1}Y$ is invertible as in the previous case. Therefore formula (2.68) is still valid and X_2 commutes with $\det_1\delta_1^{-1} \otimes \beta_2$. Set

$$U = F_N(Y)F_N(X_2).$$

Then $\tilde{\alpha}_U = U^*(\det_1\delta_1^{-1} \otimes \tilde{\alpha}_2)U$. Now using (2.55) and (2.66) for $i = 2$ we get

$$F_N(X_2)^* F_N(Y)^* F_N(X_2^{-1}Y)^* = F_N(X_2^{-1}Y)^* F_N(X_2)^*$$

and

$$\begin{aligned} \tilde{\alpha}_U &= U^* F_N(X_2^{-1}Y)^*(\det_1\delta_1^{-1} \otimes \det_2\delta_2^{-1}) F_N(X_2^{-1}Y)U \\ &= F_N(Z_2)^*(\det_1\delta_1^{-1} \otimes \det_2\delta_2^{-1})F_N(Z_2) = \det_1\delta_1^{-1} \otimes \tilde{\alpha}_2 \end{aligned}$$

since X_2 and $\det_1\delta_1^{-1} \otimes \det_2\delta_2^{-1}$ commute. Therefore in this case

$$\tilde{u}_U = \begin{pmatrix} \det_1\delta_1^{-1} \otimes \tilde{\alpha}_2, & \det_1\delta_1^{-1} \otimes \beta_2 \\ \delta_1 \otimes \gamma_2, & \delta_1 \otimes \delta_2 \end{pmatrix} \tag{2.72}$$

and $\tilde{\det}_U = U\tilde{\det}U^* = \tilde{\det}$. The verification of conditions 1-4 of Definition 2.5 is as easy as before. Moreover we have $\tilde{\gamma}_U \tilde{\beta}_U \tilde{\delta}_U^{-1} = \det_1\delta_1^{-1} \otimes \gamma_2\beta_2\delta_2^{-1}$. Therefore remembering that $(\det_1\delta_1^{-1} \otimes \gamma_2\beta_2\delta_2^{-1}, \det_1\delta_1^{-1} \otimes \det_2\delta_2^{-1})$ is a q^2-pair and $\det_1\delta_1^{-1} \otimes \det_2\delta_2^{-1}$ is invertible we get (cf. (2.32))

$$\begin{aligned} \tilde{\det}_U\, \tilde{\delta}_U^{-1} \dot{+} \tilde{\gamma}_U \tilde{\beta}_U \tilde{\delta}_U^{-1} &= (\det_1\delta_1^{-1} \otimes \det_2\delta_2^{-1}) \dot{+} (\det_1\delta_1^{-1} \otimes \gamma_2\beta_2\delta_2^{-1}) \\ &= F_N(Z_2)^*(\det_1\delta_1^{-1} \otimes \det_2\delta_2^{-1})F_N(Z_2) = \det_1\delta_1^{-1} \otimes \tilde{\alpha}_2. \end{aligned} \tag{2.73}$$

This proves the last condition of Definition 2.5.

Case 3. β_1 is an invertible operator on H_1 and $\gamma_2 = 0$.
This case is complementary to the previous one. Now $\beta_1 \otimes \delta_2$ is an invertible operators on H as in Case 1, $\alpha_2 = \mathrm{det}_2\delta_2^{-1}$ and $\gamma_1 \otimes \mathrm{det}_2\delta_2^{-1}$ strongly commutes with Y. Therefore (2.54) takes the form

$$\tilde{u} = (\mathrm{id} \otimes F_N(Y)) \begin{pmatrix} F_N(Y)^*(\tilde{\alpha}_1 \otimes \mathrm{det}_2\delta_2^{-1})F_N(Y), & F_N(Y)^*\tilde{\beta}F_N(Y) \\ \gamma_1 \otimes \mathrm{det}_2\delta_2^{-1}, & \delta_1 \otimes \delta_2 \end{pmatrix} (\mathrm{id} \otimes F_N(Y))^*. \tag{2.74}$$

Since $Z_2 = 0$, $F_N(Z_2) = I$ and $Z_1 = X_1^{-1}Y$ is invertible formula (2.67) is still valid and X_1 commutes with $\gamma_1 \otimes \mathrm{det}_2\delta_2^{-1}$. Let

$$U = F_N(Y)F_N(X_1).$$

Then $\tilde{\alpha}_U = U^*(\tilde{\alpha}_1 \otimes \mathrm{det}_2\delta_2^{-1})U$. Now in a similar way as in Case 2 we get

$$\begin{aligned} \tilde{\alpha}_U \quad &= U^*F_N(X_1^{-1}Y)^*(\mathrm{det}_1\delta_1^{-1} \otimes \mathrm{det}_2\delta_2^{-1})F_N(X_1^{-1}Y)U \\ &= F_N(Z_1)^*(\mathrm{det}_1\delta_1^{-1} \otimes \mathrm{det}_2\delta_2^{-1})F_N(Z_1) = \tilde{\alpha}_1 \otimes \mathrm{det}_2\delta_2^{-1} \end{aligned}$$

due to commutation of X_1 and $\mathrm{det}_1\delta_1^{-1} \otimes \mathrm{det}_2\delta_2^{-1}$. Therefore

$$\tilde{u}_U = \begin{pmatrix} \tilde{\alpha}_1 \otimes \mathrm{det}_2\delta_2^{-1}, & \beta_1 \otimes \delta_2, \\ \gamma_1 \otimes \mathrm{det}_2\delta_2^{-1}, & \delta_1 \otimes \delta_2 \end{pmatrix} \tag{2.75}$$

and $\tilde{\mathrm{det}}_U = U\tilde{\mathrm{det}}U^* = \tilde{\mathrm{det}}$. As in Case 2 one proves that $\tilde{u}_U$ is a G-matrix.
At the end we consider the simplest case.

Case 4. $\beta_1 = 0$ and $\gamma_2 = 0$.
Now $\alpha_1 = \mathrm{det}_1\delta_1^{-1}, \alpha_2 = \mathrm{det}_2\delta_2^{-1}$ and the operators $\mathrm{det}_1\delta_1^{-1} \otimes \beta_2$, $\gamma_1 \otimes \mathrm{det}_2\delta_2^{-1}$ strongly commute with Y. Therefore we set $U = F_N(Y)$ and then (cf. (2.54))

$$\tilde{u}_U = \begin{pmatrix} F_N(Y)^*(\mathrm{det}_1\delta_1^{-1} \otimes \mathrm{det}_2\delta_2^{-1})F_N(Y), & \mathrm{det}_1\delta_1^{-1} \otimes \beta_2 \\ \gamma_1 \otimes \mathrm{det}_2\delta_2^{-1}, & \delta_1 \otimes \delta_2 \end{pmatrix} \tag{2.76}$$

As in the previous cases it is not difficult to verify conditions 1-4 of Definition 2.5. Now

$$\begin{aligned} \tilde{\mathrm{det}}_U\, \tilde{\delta}_U^{-1} \dot{+} \tilde{\gamma}_U\tilde{\beta}_U\tilde{\delta}_U^{-1} \quad &= \quad (\mathrm{det}_1\delta_1^{-1} \otimes \mathrm{det}_2\delta_2^{-1})\dot{+}(\gamma_1\mathrm{det}_1\delta_1^{-2} \otimes \mathrm{det}_2\beta_2\delta_2^{-2}) \\ &= \quad F_N(Y)^*(\mathrm{det}_1\delta_1^{-1} \otimes \mathrm{det}_2\delta_2^{-1})F_N(Y) \end{aligned} \tag{2.77}$$

due to (2.32). This shows that $\tilde{u}_U$ is a G-matrix.

□

3 C^*-algebra level

In this section we construct the universal C^*-algebra A related to G-matrices. It is generated by α, β, γ, δ and $\det^{-1}$ and corresponds to "the algebra of continuous functions vanishing at infinity" on quantum $GL_{q^2,1}(2,\mathbf{C})$ group. Then a group structure is introduced by a comultiplication $\Phi \in \mathrm{Mor}(A, A\otimes A)$.

We start with a proposition which easily follows from Lemma 2.15 and Proposition 2.14.

Proposition 3.1 *Let (2.14) be a G-matrix (in the sense of Definition 2.5) on a Hilbert space H and let $C \in C^*(H)$. Then*

$$(\alpha,\, \beta,\, \gamma,\, \delta,\, \det^{-1} \eta\, C\,) \iff (\,\beta\delta^{-1},\, \gamma,\, \delta,\, \delta^{-1},\, \det\delta^{-1},\, \det^{-1}\delta\,\eta\, C\,) \tag{3.1}$$

This shows that $\beta\delta^{-1}$, γ, δ, δ^{-1}, $\det\delta^{-1}$ and $\det^{-1}\delta$ also parametrize G-matrices. Clearly

$$\left.\begin{array}{c}(\gamma,\, \delta^{-1}) \in \overline{\mathcal{D}}_H,\ (\beta\delta^{-1},\, \det\delta^{-1}) \in \overline{\mathcal{D}}_H \text{ and } \delta^{-1},\ \det\delta^{-1} \text{ are invertible,}\\ \gamma \text{ and } \delta^{-1} \text{ strongly commute with } \beta\delta^{-1} \text{ and } \det\delta^{-1}.\end{array}\right\} \tag{3.2}$$

Therefore A should be a tensor product of two copies of the C^*-algebra generated by a q^2-pair (Y, X) where X is invertible. This algebra was considered in [14].

Set

$$\sigma_z(f)(z') := f(zz')$$

for any $f \in C_\infty(\overline{\Gamma})$ and $z \in \Gamma$. Then $(C_\infty(\overline{\Gamma}), (\sigma_z)_{z\in\Gamma})$ is a C^*-dynamical system. Denote by B the corresponding C^*-crossed product algebra [5]. Let $Y \in C(\overline{\Gamma})$,

$$Y(z) = z$$

for $z \in \overline{\Gamma}$, $U = (U_z)_{z\in\Gamma}$ be one parameter group of unitaries in $M(B)$ implementing the action σ of Γ on $C_\infty(\overline{\Gamma})$,

$$U_z f U_z^* = \sigma_z(f)$$

for all $f \in C_\infty(\overline{\Gamma})$ and $z \in \Gamma$ and let X be the analytic generator related to U by the formula

$$U_z = \chi(X, z)$$

for all $z \in \Gamma$. It is known (cf. [14, Proposition 4.1]) that $Y\,\eta\, B$, $X\,\eta\, B$, X is invertible and $X^{-1}\,\eta\, B$. Moreover (Y, X) is a q^2-pair and B is the universal C^*-algebra generated in the sense of [12] by (Y, X). Therefore let

$$A := B \otimes B = C_\infty(\overline{\Gamma}\times\overline{\Gamma}) \times_{\sigma\times\sigma} (\Gamma\times\Gamma). \tag{3.3}$$

Now by definition

$$(\gamma,\, \delta^{-1}) = (Y\otimes I, X\otimes I) \ \text{ and } \ (\beta\delta^{-1},\, \det\delta^{-1}) = (I\otimes Y, I\otimes X). \tag{3.4}$$

Remembering that tensor product of two elements affiliated with C^*-algebras is element afilliated with the tensor product of the corresponding algebras we see that

$$\gamma,\ \delta^{-1},\ \delta,\ \beta\delta^{-1},\ \det\delta^{-1}, \det{}^{-1}\delta\ \eta\ A.$$

Let

$$\begin{aligned} \alpha &= I\otimes X\dot{+}Y\otimes Y, \qquad & \beta &= X^{-1}\otimes Y \\ \gamma &= Y\otimes I, & \delta &= X^{-1}\otimes I \end{aligned} \tag{3.5}$$

and

$$\det = X^{-1}\otimes X. \tag{3.6}$$

Taking into account Proposition 3.1 we get by universality of the algebra B

Proposition 3.2
The C^*-algebra A is generated by five affiliated elements α, β, γ, δ, $\det^{-1}\eta\, A$.

Moreover the matrix (2.14) where matrix elements and det are given by (3.5) and (3.6) is a universal G-matrix. More precisely due to Proposition 4.2 of [14] one gets

Proposition 3.3
1.Let π be a representation of A on a Hilbert space H and

$$\begin{aligned} \alpha_o &= \pi(\alpha), \qquad & \beta_o &= \pi(\beta) \\ \gamma_o &= \pi(\gamma), & \delta_o &= \pi(\delta). \end{aligned} \tag{3.7}$$

Then

$$\begin{pmatrix} \alpha_o, & \beta_o \\ \gamma_o, & \delta_o \end{pmatrix} \tag{3.8}$$

is a G-matrix (in the sense of Definition 2.5) on the Hilbert space H. In this case $\det_o = \pi(\det)$.

2.Conversely if (3.8) is a G-matrix on a Hilbert space H then there exists unique representation π of A on H such that (3.7) holds. Moreover if $C\in C^*(H)$ then

$$(\,\alpha_o,\ \beta_o,\ \gamma_o,\ \delta_o,\ \det{}_o^{-1}\ \eta\ C\,) \Longrightarrow (\,\pi\in \mathrm{Mor}(A,C)\,).$$

Now we have

Theorem 3.4
Let A be the C^*-algebra (3.3) and α, β, γ and δ the universal elements (3.5). Then there exists a unique $\Phi\in\mathrm{Mor}(A, A\otimes A)$ such that

$$\begin{aligned} \Phi(\alpha) &= \alpha\otimes\alpha\dot{+}\beta\otimes\gamma, \qquad & \Phi(\beta) &= \alpha\otimes\beta\dot{+}\beta\otimes\delta \\ \Phi(\gamma) &= \gamma\otimes\alpha\dot{+}\delta\otimes\gamma, & \Phi(\delta) &= \gamma\otimes\beta\dot{+}\delta\otimes\delta. \end{aligned} \tag{3.10}$$

Proof. We know (cf. Theorem 2.17) that matrix

$$\begin{pmatrix} \alpha\otimes\alpha\dot{+}\beta\otimes\gamma, & \alpha\otimes\beta\dot{+}\beta\otimes\delta \\ \gamma\otimes\alpha\dot{+}\delta\otimes\gamma, & \gamma\otimes\beta\dot{+}\delta\otimes\delta \end{pmatrix}$$

is a G-matrix (with determinant $\det\otimes\det$). Therefore existence and uniqueness of Φ follows from Lemma 2.15 and Statement 2 of Proposition 3.3. □

It remains to prove that Φ is coassociative. To this end we need two lemmas.

Lemma 3.5
Let T_1 and T_2 be normal operators acting on a Hilbert space H such that $\operatorname{Sp} T_1 \subset \overline{\Gamma}$ and $\operatorname{Sp} T_2 \subset \overline{\Gamma}$. Then

$$\begin{pmatrix} F_N(zT_1) = F_N(zT_2) \\ \text{for any } z\in\Gamma \end{pmatrix} \Longleftrightarrow (\ T_1 = T_2\ ). \tag{3.11}$$

Proof. Let us note that above result can be deduced from the proof of Theorem 5.1 in [14]. Here we present an independent proof of it.

Let (R,S) be a non-degenerate q^2-pair on a Hilbert space K, i.e. R and S are invertible. Then we set

$$Y_1 := R\otimes T_1,\quad Y_2 := R\otimes T_2 \quad\text{and } X := S\otimes I.$$

Clearly they are normal operators acting on $K\otimes H$ and (Y_1,X), (Y_2,X) are q^2-pairs. Since X is invertible, by (2.30)

$$X\dot{+}XY_i = F_N(Y_i)^* X F_N(Y_i) \tag{3.12}$$

for $i=1,2$. Assume that $F_N(zT_1)=F_N(zT_2)$ for any $z\in\Gamma$. By the spectral theorem

$$F_N(Y_i) = \int_{\overline{\Gamma}} dE_R(z)\otimes F_N(zT_i)$$

$(i=1,2)$, where E_R is the spectral measure associated with the normal operator R. Therefore $F_N(Y_1)=F_N(Y_2)$. Now (3.12) leads to $X\dot{+}XY_1 = X\dot{+}XY_2$ i.e.

$$SR\otimes T_1\dot{+}S\otimes I = SR\otimes T_2\dot{+}S\otimes I$$

and

$$(SR\otimes T_1 + S\otimes I)^* = (SR\otimes T_2 + S\otimes I)^*\,. \tag{3.13}$$

Choose $y,y'\in D(SR)\cap D(S)$ such that $(y'|SRy)\neq 0$. Then using (3.13) we get for any $x_1\in D(T_1)$ and $x_2\in D(T_2)$

$$\begin{aligned} ((SR)^*y'|y)(T_2^*x_2|x_1) + (S^*y'|y)(x_2|x_1) &= ((SR\otimes T_2 + S\otimes I)^*(y'\otimes x_2)|y\otimes x_1) \\ &= (y'\otimes x_2|(SR\otimes T_1 + S\otimes I)(y\otimes x_1)) \\ &= (y'|SRy)(x_2|T_1x_1) + (y'|Sy)(x_2|x_1). \end{aligned}$$

Therefore

$$((T_2)^*x_2|x_1) = (x_2|T_1x_1).$$

This means that $T_1 \subset T_2^{**} = T_2$ and $T_1 = T_2$ (normal operators have no proper normal extensions). This ends the proof since the opposite implication is obvious.

□

Lemma 3.6
Let $(Y,X) \in \overline{\mathcal{D}}_H$, $(R,S) \in \overline{\mathcal{D}}_H$, $(S,X) \in \overline{\mathcal{D}}_H$, $(R,Y) \in \overline{\mathcal{D}}_H$ *and* Y *strongly commutes with* S. *Assume that* $((S \dot{+} R), (X \dot{+} Y)) \in \overline{\mathcal{D}}_H$ *and* $((Y \dot{+} R), (X \dot{+} S)) \in \overline{\mathcal{D}}_H$. *Then*

$$(X \dot{+} Y) \dot{+} (S \dot{+} R) = (X \dot{+} S) \dot{+} (Y \dot{+} R). \tag{3.14}$$

In particular if $(Y,X) \in \overline{\mathcal{D}}_H$, $(Z,Y) \in \overline{\mathcal{D}}_H$ *and* $(Z, (X \dot{+} Y)) \in \overline{\mathcal{D}}_H$, $((Y \dot{+} Z), X) \in \overline{\mathcal{D}}_H$ *then*

$$(X \dot{+} Y) \dot{+} Z = X \dot{+} (Y \dot{+} Z). \tag{3.15}$$

Proof. Using (2.33) we obtain for any $z \in \Gamma$

$$F_N(z[(X \dot{+} Y) \dot{+} (S \dot{+} R)]) = F_N(z(S \dot{+} R))F_N(z(X \dot{+} Y)) = F_N(zR)F_N(zS)F_N(zY)F_N(zX)$$

and

$$F_N(z[(X \dot{+} S) \dot{+} (Y \dot{+} R)]) = F_N(z(Y \dot{+} R))F_N(z(X \dot{+} S)) = F_N(zR)F_N(zY)F_N(zS)F_N(zX).$$

Now both expressions coincide since $F_N(zY)$ commutes with $F_N(zS)$ and (3.14) follows by Lemma 3.5. Taking $Z = R$ and $S = 0$ we obtain (3.15).

□

Proposition 3.7
Let $\Phi \in \mathrm{Mor}(A, A \otimes A)$ *be the morphism described in Theorem 3.9. Then* Φ *is coassociative,*

$$(\Phi \otimes \mathrm{id})\Phi = (\mathrm{id} \otimes \Phi)\Phi \tag{3.16}$$

.

Proof. Clearly $(\Phi \otimes \mathrm{id})\Phi$, $(\mathrm{id} \otimes \Phi)\Phi \in \mathrm{Mor}(A, A \otimes A \otimes A)$. Therefore to prove equlity (3.16) it is enough check that the morphisms coincide on the set of generators of A. They take the same value on det equal to $\det \otimes \det \otimes \det$. Now (cf. (3.10))

$$(\mathrm{id} \otimes \Phi \otimes \mathrm{id})(\mathrm{id} \otimes \Phi) = \begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix} = \begin{pmatrix} \Phi(\alpha) \otimes \alpha \dot{+} \Phi(\beta) \otimes \gamma, & \Phi(\alpha) \otimes \beta \dot{+} \Phi(\beta) \otimes \delta \\ \Phi(\gamma) \otimes \alpha \dot{+} \Phi(\delta) \otimes \gamma, & \Phi(\gamma) \otimes \beta \dot{+} \Phi(\delta) \otimes \delta \end{pmatrix} \tag{3.17}$$

and

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Phi)(\mathrm{id} \otimes \Phi) \begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix} = \begin{pmatrix} \alpha \otimes \Phi(\alpha) \dot{+} \beta \otimes \Phi(\gamma), & \alpha \otimes \Phi(\beta) \dot{+} \beta \otimes \Phi(\delta) \\ \gamma \otimes \Phi(\alpha) \dot{+} \delta \otimes \Phi(\gamma), & \gamma \otimes \Phi(\beta) \dot{+} \delta \otimes \Phi(\delta) \end{pmatrix}. \tag{3.18}$$

We have to show that corresponding matrix elements of (3.17) and (3.18) coincide. Remembering that Φ is morphism we know that $(\Phi(\beta) \otimes \gamma, \Phi(\alpha) \otimes \alpha)$ and $(\beta \otimes \Phi(\gamma), \alpha \otimes \Phi(\alpha))$ are q^2-pairs (cf. Definition 2.5 and Propostion 2.11). Moreover

$$\begin{aligned} \Phi(\alpha) \otimes \alpha \dot{+} \Phi(\beta) \otimes \gamma &= (\alpha \otimes \alpha \dot{+} \beta \otimes \gamma) \otimes \alpha \dot{+} (\alpha \otimes \beta \dot{+} \beta \otimes \delta) \otimes \gamma \\ &= (\alpha \otimes \alpha \otimes \alpha \dot{+} \beta \otimes \gamma \otimes \alpha) \dot{+} (\alpha \otimes \beta \otimes \gamma \dot{+} \beta \otimes \delta \otimes \gamma) \end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\alpha\otimes\Phi(\alpha)\dot{+}\beta\otimes\Phi(\gamma) &= \alpha\otimes(\alpha\otimes\alpha\dot{+}\beta\otimes\gamma)\dot{+}\beta\otimes(\gamma\otimes\alpha\dot{+}\delta\otimes\gamma)\\ &= (\alpha\otimes\alpha\otimes\alpha\dot{+}\alpha\otimes\beta\otimes\gamma)\dot{+}(\beta\otimes\gamma\otimes\alpha\dot{+}\beta\otimes\delta\otimes\gamma).\end{aligned} \tag{3.20}$$

Set in Lemma 3.6

$$X=\alpha\otimes\alpha\otimes\alpha, \qquad Y=\beta\otimes\gamma\otimes\alpha$$

$$S=\alpha\otimes\beta\otimes\gamma, \qquad R=\beta\otimes\delta\otimes\gamma.$$

Then one easily checks that all assumption are satisfied. Therefore the last expressions in (3.19) and (3.20) are equal. This proves that

$$(\Phi\otimes\mathrm{id})\Phi(\alpha)=(\mathrm{id}\otimes\Phi)\Phi(\alpha).$$

In the case of β, γ and δ one proceeds in the same manner. Simple calculations are left to the reader.

□

The pair (A,Φ) describes the quantum group G on the C^*-algebra level.

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