C*-algebras generated by unbounded elements

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Abstract

The main aim of this paper is to provide a proper mathematical framework for the theory of topological non-compact quantum groups, where we have to deal with non-unital C*-algebras. The basic concepts and results related to the affiliation relation in the C*-algebra theory are recalled. In particular natural topologies on the set of affiliated elements and on the set of morphisms are considered. The notion of a C*-algebra generated by a finite sequence of unbounded elements is introduced and investigated. It is generalized to include continuous quantum families of generators. An essential part of the duality theory for C*-algebras is presented including complete proofs of many theorems announced in [17]. The results are used to develop a presentation method of introducing non-unital C*-algebras. Numerous examples related mainly to the quantum group theory are presented.

0 Introduction

In the theory of quantum groups we often deal with unital algebras introduced in terms of *generators and relations*. Generators are distinguished elements of the algebra being introduced and relations are the algebraic equalities imposed on the generators. The list of generators and relations is called a *presentation* of the algebra. It is always assumed that the algebra coincides with the set of all algebraic combinations of generators and that the set of relations is complete: any algebraic equality satisfied by generators must be a logical consequence of the relations. Given a presentation, the corresponding algebra is the quotient

$$A = A_{\mathrm{free}}/J, \tag{0. 1}$$

where A_{free} is the algebra of all (noncommutatve) polynomials (with complex coefficients) of free variables $t_1, t_2, \dots t_N$ (N is the number of generators; dealing with *-algebras one has to double the number of variables including $t_1{}^*, t_2{}^*, \dots t_N{}^*$) and J is

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the ideal of $A_{\rm free}$ generated by the relations. There is an alternative way of introducing J.

Let φ be a homomorphism of $A_{\rm free}$ into an algebra B. We say that φ is compatible with the relations if $\varphi(t_1), \varphi(t_2), \dots, \varphi(t_N)$ satisfy the relations in B. It turns out that J is the intersection of kernels of all homomorphisms compatible with the relations.

With no essential changes, the presentation method works for unital C*-algebras. In this case there exist an universal object $B = B(H)$ (the algebra of all bounded operators acting on a separable infinite-dimensional Hilbert space H) that may serve as a target algebra for homomorphisms φ. The relations must imply an uniform estimate of the norm of generators: there exist numbers M_i $(i = 1, 2, \dots N)$ such that

$$\|T_i\| \leq M_i \tag{0. 2}$$

for any N-tuples $(T_1, T_2, \dots T_N)$ of bounded operators acting on H satisfying the relations. For any $w \in A_{\rm free}$ we set

$$\|w\| = \sup \|\varphi(w)\|, \tag{0. 3}$$

where supremum is taken over all *-algebra homomorphisms $\varphi : A_{\rm free} \to B(H)$ compatible with the relations. Due to (0. 2), $\|w\| < \infty$ for any $w \in A_{\rm free}$. Clearly $\|\cdot\|$ introduced by (0. 3) is a C*-semonorm on $A_{\rm free}$. Moreover

$$J = \{w \in A_{\rm free} : \|w\| = 0\}\,.$$

The definition of the algebra corresponding to the considered presentation should include the completion procedure. Instead of (0. 1) we have:

$$A = \{A_{\rm free}/J\}^{\rm completion}, \tag{0. 4}$$

where completion is taken with respect to the C*-norm on $A_{\rm free}/J$ induced by the seminorm (0. 3). Clearly (0. 4) is a unital C*-algebra.

A theory of unital C*-algebras generated by a finite number of elements is developed in [9]. The main results of this paper says, that such algebras are isomorphic to the algebras of all continuous operator functions on compact domains. In the present paper we shall generalize this result to include non-unital C*-algebras and non-compact domains.

The need of such a generalization comes from the theory of non-compact quantum groups. Following the theory of compact groups we would like to say that the C*-algebra of all *continuous functions vanishing at infinity on a group* is generated by matrix elements of a fundamental representations. However in the non-compact case the fundamental representation is not unitary and its matrix elements are not bounded. It means that we have to develop a framework enabling us to consider C*-algebras generated by unbounded elements.

The first step in this direction is already done. Unbounded generators can not belong to the algebra. Therefore one has to replace the membership relation '$\in$' by a weaker one denoted by '$\,\eta\,$'. If $T \,\eta\, A$ then one says that T is an element affiliated with the C*-algebra A. The affiliation relation in the theory of C*-algebras was introduced by S. Baaj and P. Jungl in [3].

In the present paper we use this relation to introduce the concept of a C*-algebra generated by a number of (unbounded) elements affiliated with it. One has to stress at this moment some peculiar features of this concept. The sentence

$$\boxed{\begin{array}{c}A \text{ is generated by ele-}\\ \text{ments } T_1, T_2, \dots T_N.\end{array}} \tag{0. 5}$$

describes a definite relation between a C*-algebra A and elements $T_1, T_2, \dots T_N \,\eta\, A$. (0. 5) can not be used as a definition of A. The algebra A and the affiliated elements $T_1, T_2, \dots T_N$ must be known in advance. Whenever in this paper we write:

$$\boxed{\begin{array}{c}\text{Let } A \text{ be a C}^*\text{-algebra ge-}\\ \text{nerated by } T_1, T_2, \dots T_N.\end{array}},$$

it means:

$$\boxed{\begin{array}{c}\text{Let } A \text{ be a C}^*\text{-algebra and } T_1, T_2, \dots T_N \,\eta\, A.\\ \text{Assume that } A \text{ is generated by } T_1, T_2, \dots T_N.\end{array}}.$$

If B is a C*-algebra and $T_1, T_2, \dots T_N \,\eta\, B$, then for some non-degenerate subalgebra A of (the multiplier algebra of) B, the elements $T_1, T_2, \dots T_N$ are affiliated with A and the statement (0. 5) is meaningful. However no effective procedure producing A generated by $T_1, T_2, \dots T_N$ is known. What is more, even the existence of A is not guaranteed.

The paper is organized in the following way. Section 1 contains necessary material concerning the theory of non-unital C*-algebras. Throughout the paper H is an infinite-dimensional separable Hilbert space, $B(H)$ is the algebra of all bounded linear operators acting on H and $CB(H)$ is the subalgebra composed of all compact operators. To make the paper selfconsistent we formulate the definitions of multiplier algebra, z-transform, affiliated elements and morphisms. Introducing these notions we shall assume that the considered C*-algebras are concrete i.e. embedded into $B(H)$. This approach is more suitable for our purposes. One should stress however that the definitions (being equivalent to the ones given in previous papers [10, 16, 3, 20, 23]) are essentially independent of the choice of the embeding. The only new result contained in Section 1 says that any separable C*-algebra is uniquely determined by its multiplier algebra. An important technical tool used in this Section is the relation of strict inequality defined for selfadjoint elements of multiplier algebras.

In Section 2 we investigate natural topologies on the multiplier algebra $M(A)$ of any C*-algebra A, on the set A^η of all elements affiliated with A and on the set $\mathrm{Mor}(A, B)$ of all morphisms from A into a C*-algebra B. The topology on $M(A)$ is introduced in such a way that for any locally compact space Λ, the set of all continuous bounded mappings from Λ into $M(A)$ may be identified with $M(C_\infty(\Lambda) \otimes A)$, where $C_\infty(\Lambda)$ denotes the C*-algebra of all continuous functions vanishing at infinity on Λ. Similarly, the topologies on A^η and $\mathrm{Mor}(A, B)$ are introduced in such a way that the sets of all continuous mappings from Λ into A^η and $\mathrm{Mor}(A, B)$ respectively may be identified with $(C_\infty(\Lambda) \otimes A)^\eta$ and $\mathrm{Mor}(A, C_\infty(\Lambda) \otimes B)$ respectively. These topologies will play a fundamental role in Sections 5, 6 and 7.

Sections 3 and 4 contain the basic concepts of the paper. In Section 3 we give the precise meanning to the sentence (0. 5). In Section 4 the sequence of generators $T_1, T_2, \dots T_N \,\eta\, A$ is replaced by an element $T \,\eta\, C \otimes A$, where C is a C*-algebra. Due to this generalization we may consider in particular C*-algebras generated by continuous families of affiliated elements. This case corresponds to commutative C. In the theory of quantum groups the C*-algebra of *all continuous functions vanishing at infinty* on a matrix quantum group is generated by the fundamental representation of the group. In this case $C = M_N(\mathbf{C})$. The definition of a C*-algebra A generated by a sequence $T_1, T_2, \dots T_N \,\eta\, A$ (an element $T \,\eta\, C \otimes A$ respectively) formulated in Section 3 (Section 4 respectively) contains a very strong condition that is not easy to verify directly in most cases. To deal with this problem we shall give a few criteria (sufficient conditions) that are useful in concrete cases. On the other hand, if one knows that a C*- algebra is generated by $T_1, T_2, \dots T_N \,\eta\, A$ ($T \,\eta\, C \otimes A$ respectively), then a number of interesting conclusions follows easily. The Sections contain as many as twenty examples coming mainly from the theory of quantum groups.

Section 5 is devoted to the theory of topological W*-categories. It contains the proofs of many theorems announced in [16]. We shall prove that elements affiliated with a C*-algebra A may be identified with continuous operator functions on the W*-category $\mathrm{Rep}\,(A, H) = \mathrm{Mor}(A, CB(H))$ of all non-degenerate representations of A (cf (5. 15)). Moreover for any C*-algebras A and B, the set $\mathrm{Mor}(A, B)$ is in one to one natural correspondence with the set of all W*-category morphisms from $\mathrm{Rep}\,(B, H)$ into $\mathrm{Rep}\,(A, H)$ (cf Theorem 5.6).

The last result is used in Section 6 to prove the main theorem of this paper. It says that an element $T \,\eta\, C \otimes A$ generates a C*-algebra A if and only if the evaluation map: $\pi \longmapsto (\mathrm{id} \otimes \pi)T$ is an homeomorphism of $\mathrm{Rep}\,(A, H)$ onto a closed subset of $(C \otimes CB(H))^\eta$. This result is the cornerstone of the presentation theory of C*-algebras developed in Section 7. A large number of examples is given.

We belive that the concepts introduced in this paper will play a crucial role in the future general theory of non-compact quantum groups. In particular these concepts allow us to reformulate Conditions 1 and 2 of the Definition 1.1 of [19] to cover a non-compact matrix quantum group case. Instead of Condition 1 we postulate that the algebra A of all *continuous functions vanishing at infinity* on the group is generated by the fundamental representation $u \,\eta\, M_N \otimes A$. In Condition 2 we demand that the comultiplication $\Phi \in \mathrm{Mor}(A, A \otimes A)$. At the moment we do not know, how to reformulate Condition 3 concerning the coinverse (antipode) map.

1 C*-algebras, affiliated elements and morphisms

In this section we recall the main concepts of the theory of non-unital C*-algebras introduced in [3, 16, 20]. To this end it is convenient to assume that we deal with concrete

C*-algebras. The definitions given below use particular embeddings of C*-algebras into $B(H)$. However the notions introduced in this way are essentially independent of the choice of the embeddings. We shall also assume that all C*-algebras considered in this paper are separable. The only exceptions are the multiplier algebras; according to Proposition 1.1, in the non-compact case, they are never separable.

If A is a commutative C*-algebra then there exists a locally compact space Λ such that $A = C_\infty(\Lambda)$. The space Λ is called a spectrum of A: $\Lambda = \mathrm{Sp}\, A$. If A is noncommutative then $\mathrm{Sp}\, A$ will denote the quantum space (pseudospace) corresponding to A [16].

For any separable Hilbert space H, $C^*(H)$ will denote the set of all separable nondegenerate C*-subalgebras of $B(H)$ (a C*-subalgebra $A \subset B(H)$ is nondegenerate if AH is dense in H). For any $A \in C^*(H)$, the embedding $A \hookrightarrow B(H)$ will be denoted by i_A.

Let H be a separable Hilbert space and $A \in C^*(H)$. An operator $a \in B(H)$ is called a multiplier of A if aA and Aa are contained in A. The set of all multipliers of A is denoted by $M(A)$:

$$M(A) = \left\{ a \in B(H) : \begin{array}{c} ab,\ ba \in A \\ \text{for any } b \in A \end{array} \right\}$$

Clearly $M(A)$ is a C*-subalgebra of $B(H)$ and $I_{B(H)} \in M(A)$. A is an ideal in $M(A)$. If $a \in M(A)$ and $ab = 0$ for all $b \in A$ then $a = 0$. We say that A is an essential ideal of $M(A)$. In fact $M(A)$ is the largest C*-algebra containing A as an essential ideal (cf [10, Proposition 3.12.8]).

If $CB(H) \in C^*(H)$ is the algebra of all compact operators acting on H, then $M(CB(H)) = B(H)$. If A is commutative, then $M(A) = C_{\text{bounded}}(\mathrm{Sp}\, A)$.

Let A be a C*-algebra. An element $a \in M(A)$ is said to be strictly positive on $\mathrm{Sp}\, A$ if $0 \leq a$ and aA is dense in A. For any separable C*-algebra A there exists $a \in A$ strictly positive on $\mathrm{Sp}\, A$. To construct such an element it is sufficient to choose a denumerable dense subset $(a_i)_{i \in \mathbf{N}}$ of the unit ball of A and set

$$a = \sum_{i=1}^{\infty} 2^{-i} {a_i}^* a_i. \tag{1. 1}$$

If an element $a \in A$ is strictly positive on $\mathrm{Sp}\, A$ and $\|a\| \leq 1$ then the sequence $\left(a^{1/n}\right)_{n=1,2,\ldots}$ is an approximate unit of A.

Let ω be a state (a positive linear functional of norm 1) on a C*-algebra A. According to [6, Proposition 2.11.7] there exists unique state on $M(A)$ that extends ω. It will be denoted by the same symbol ω. Let $a \in M(A)$. Using [6, Theorem 2.9.5] one can easily show that $a > 0$ if and only if $\omega(a) > 0$ for any pure state ω of A.

For any selfadjoint $a, b \in M(A)$ we say that $a < b$ on $\mathrm{Sp}\, A$ whenever $b - a$ is strictly positive on $\mathrm{Sp}\, A$. One can easily show that $a + b > 0$ for any $a > 0$ and $b \geq 0$. Therefore for any $a, b, c \in M(A)$ such that $a > b$ and $b \geq c$ we have $a > c$. The same result follows from $a \geq b$ and $b > c$. In particular the relation of strict inequality is transitive.

It turns out that A is uniquely determined by $M(A)$. We have[1]:

[1]the author is grateful to Alain Connes for setting a problem leading to this result

Proposition 1.1 *Let A be a separable C^*-algebra. Then*

$$A = \{a \in M(A) : aM(A) \text{ is separable}\}. \tag{1. 2}$$

The proof will be given latter.

For any closed densely defined operator T acting on a Hilbert space H we set

$$z_T = T(I + T^*T)^{-\frac{1}{2}} \tag{1. 3}$$

One should notice that $z_T \in B(H)$ and $\|z_T\| \leq 1$. The operator z_T is called the z-transform of T. It contains the full information about T:

$$T = z_T(I - {z_T}^* z_T)^{-\frac{1}{2}}$$

Clearly $\|T\| < \infty$ if and only if $\|z_T\| < 1$. Let $M < \infty$. Then (by the Weierstrasse approximation theorem) there exist sequences of polynomials v_n and w_n (with real coefficients) such that

$$\begin{aligned} T &= \text{norm-}\lim z_T v_n({z_T}^* z_T) \\ z_T &= \text{norm-}\lim T w_n(T^*T) \end{aligned} \tag{1. 4}$$

for any T with $\|T\| \leq M$. It shows that T and z_T belong to the same C*-algebra.

Let H be a separable Hilbert space, $A \in C^*(H)$ and T be a closed, densely defined operator acting on H. We say that T is affiliated with A and write $T \,\eta\, A$ if $z_T \in M(A)$ and ${z_T}^* z_T < I$ on $\operatorname{Sp} A$.

The set of all elements affiliated with A will be denoted by A^η. It is known that T^* and T^*T are affiliated with A for any $T \,\eta\, A$. Multipliers are the only bounded elements affiliated with A: $\{T \,\eta\, A : \|T\| < \infty\} = M(A)$. Any closed, densely defined operator acting on H is affiliated with the algebra $CB(H)$ of all compact operators on H. If A is commutative, then $A^\eta = C(\operatorname{Sp} A)$. A is unital if and only if $A^\eta = A$.

Let A be a C*-algebra and H be a separable Hilbert space. The symbol $\operatorname{Rep}(A, H)$ will denote the set of all non-degenerate representations of A in H. By definition $\pi \in \operatorname{Rep}(A, H)$ if and only if $\pi : A \longrightarrow B(H)$ is a *-algebra homomorphism such that $\pi(A)H$ is dense in H. If $A \in C^*(H)$ then $i_A \in \operatorname{Rep}(A, H)$.

Remembering that A is an ideal in $M(A)$ and using Proposition 2.10.4 of [6] we see that any $\pi \in \operatorname{Rep}(A, H)$ admits the unique extension to a unital *-algebra homomorphism[2] $\pi : M(A) \longrightarrow B(H)$. Clearly $\pi(z_T) = z_{\pi(T)}$ for any $T \in M(A)$. Using this property one can easily extend the action of π to elements affiliated with A: If $T \,\eta\, A$ then $z_T \in M(A)$ and there exists unique densely defined closed operator S acting on H such that $\pi(z_T) = z_S$. We say that S is the π-image of T and write $S = \pi(T)$.

Let A and B be C*-algebras. Assume that $B \in C^*(H)$. We say that π is a morphism from A into B if $\pi \in \operatorname{Rep}(A, H)$ and $\pi(A)B$ is a dense subset of B. The set of all morphisms from A into B will be denoted by $\operatorname{Mor}(A, B)$:

[2]denoted by the same symbol

$$\mathrm{Mor}(A,B)=\left\{\pi\in\mathrm{Rep}\,(A,H):\ \overline{\pi(A)B}=B\right\}\tag{1. 5}$$

The reader should notice that $\mathrm{Rep}\,(A,H)=\mathrm{Mor}(A,CB(H))$.

Let A, B be C*-algebras and $\pi\in\mathrm{Mor}(A,B)$. Then $\pi(T)\,\eta\,B$ for any $T\,\eta\,A$. In other words any morphism from A into B defines in a natural way a mapping from A^η into B^η.

Morphisms may be composed. Let A, B, C be C*-algebras, $\varphi\in\mathrm{Mor}(A,B)$ and $\psi\in\mathrm{Mor}(B,C)$. Then there exists unique composition $\psi\varphi\in\mathrm{Mor}(A,C)$ such that $(\psi\varphi)(T)=\psi(\varphi(T))$ for any $T\,\eta\,A$.

For any C*-algebra A there exists unique morphism $\mathrm{id}_A\in\mathrm{Mor}(A,A)$ such that $\mathrm{id}_A(T)=T$ for any $T\,\eta\,A$. Clearly $\pi\,\mathrm{id}_A=\mathrm{id}_B\pi=\pi$ for any $\pi\in\mathrm{Mor}(A,B)$. It means that we have constructed a category of C*-algebras: Objects of this category are C*-algebras and morphisms are introduced by (1. 5). The category of quantum spaces is by definition the category dual to the category of C*-algebras described above [16].

Let A and B be C*-algebras, $a\in A$ and $\pi\in\mathrm{Mor}(A,B)$. We shall show that π preserves the strict inequalities:

$$\left(\begin{array}{c}a>0\\ \text{on }\mathrm{Sp}\,A\end{array}\right)\Longrightarrow\left(\begin{array}{c}\pi(a)>0\\ \text{on }\mathrm{Sp}\,B\end{array}\right).\tag{1. 6}$$

Indeed, if $a>0$ on $\mathrm{Sp}\,A$, then aA is dense in A, $\pi(a)\pi(A)$ is dense in $\pi(A)$ and $\pi(a)B\supset\pi(a)\pi(A)B$ is dense in $\pi(A)B$ which in turn is dense in B (see (1. 5)). It shows that $\pi(a)>0$ on $\mathrm{Sp}\,B$ and (1. 6) holds.

Let A be a C*-algebra and a be an element of $M(A)$ such that $rI<a<sI$ on $\mathrm{Sp}\,A$ ($r,s\in\mathbf{R}$). Then $(a-rI)(sI-a)A$ is dense in A. For any $f\in C_\infty(]r,s[)$ we set $\pi(f)=f(a)$. Then π is a *-algebra homomorphism acting from $C_\infty(]r,s[)$ into $M(A)$. Clearly the function $]r,s[\ni\lambda\mapsto(\lambda-r)(s-\lambda)\in\mathbf{R}$ belongs to $C_\infty(]r,s[)$. Therefore $\pi(C_\infty(]r,s[))A\supset(a-rI)(sI-a)A$ is dense in A and

$$\pi\in\mathrm{Mor}(C_\infty(]r,s[),A).$$

One can easily show that $\mathrm{Sp}\,a\subset[r,s]$ and that the end points r and s are not eigenvalues of a. The extension of π to $C(]r,s[)=C_\infty(]r,s[)^\eta$ is given by the same formula: $\pi(f)=f(a)$. It shows that $f(a)\,\eta\,A$ for any $f\in C(]r,s[)$. In particular

$$f(a)\in M(A)\tag{1. 7}$$

for any $f\in C_{\mathrm{bounded}}(]r,s[)$. Moreover, using (1. 6) we get

$$\left(\begin{array}{c}rI<a<sI\text{ on }\mathrm{Sp}\,A\\ \text{and}\\ f\in C(]r,s[,]r',s'[)\end{array}\right)\Longrightarrow\left(\begin{array}{c}r'I<f(a)<s'I\\ \text{on }\mathrm{Sp}\,A\end{array}\right).\tag{1. 8}$$

Clearly in the above results the interval $]r,s[$ may be replaced by $]r,\infty[$ and $]-\infty,s[$.

Proof of Proposition 1.1**:**

Let B be the right hand side of (1. 2). Clearly, B is a two sided ideal in $M(A)$. If $a_n \in B$ and $a_n \longrightarrow a$ in norm, then $aM(A)$ is contained in the closure of separable set $\cup_n a_n M(A)$. Therefore $aM(A)$ is separable and $a \in B$. It shows that B is a closed ideal in $M(A)$. In particular (cf [6]) $a^* \in B$ for any $a \in B$.

For any $a \in A$, $aM(A) \subset A$ is separable. Therefore $A \subset B$. To prove the converse we have to show that the quotient C*-algebra $B/A = \{0\}$.

Assume that the latter does not hold. Let $\tilde{b}$ be a positive element of B/A of norm 1 and b be an element of B representing $\tilde{b}$. Positive elements admit positive lifts. Therefore b may be choosen in such a way that $b = b^*$ and $b \leq I$. Replacing if necessary b by $b - a$ (where a is introduced by (1. 1)) we may assume that $b < I$.

According to (1. 7), $f(b) \in M(A)$ for any $f \in C_{\text{bounded}}(]-\infty, 1[)$. We know that 1 is not an eigenvalue of b. Therefore

$$\|bf(b)\| = \sup\{|\lambda f(\lambda)| : \lambda \in \mathrm{Sp}\, b,\, \lambda \neq 1\} \tag{1. 9}$$

Since $b \in B$, $\{bf(b) : f \in C_{\text{bounded}}(]-\infty, 1[)\} \subset bM(A)$ is separable, so is $C_{\text{bounded}}(]-\infty, 1[)$ equipped with the seminorm given by the right hand side of (1. 9). It is not difficult to show that this is the case if and only if $\mathrm{Sp}\, b \subset]-\infty, 1[$. Therefore $1 \notin \mathrm{Sp}\, b$. Consequently $1 \notin \mathrm{Sp}\, \tilde{b}$ and $\|\tilde{b}\| < 1$. We obtained the contradiction with $\|\tilde{b}\| = 1$. It shows that B/A contains no positive element of norm 1. Therefore $B/A = \{0\}$, $A = B$ and (1. 2) follows.

Q.E.D.

To fix a notation we insert a few remarks concerning the tensor products. The tensor product of C*-algebras corresponds to the cartesian product of underlying quantum spaces. We shall exclusively use the minimal tensor product. Let $A \in C^*(H)$ and $B \in C^*(K)$ (H and K are Hilbert spaces). The norm closure of the linear span of all operators of the form $a \otimes b$, where $a \in A$ and $b \in B$, will be denoted by $A \otimes B$. Clearly $A \otimes B \in C^*(H \otimes K)$.

Let S and T be closed operators acting on the Hilbert spaces H and K. Then (cf [23]) there exists unique closed operator $S \otimes T$ acting on $H \otimes K$ such that the algebraic tensor product of the domains of S and T is a core for $S \otimes T$ and $(S \otimes T)(h \otimes k) = Sh \otimes Tk$ for any $h \in H$ and $k \in K$. If $S \,\eta\, A$ and $T \,\eta\, B$, then $S \otimes T$ is affiliated with $A \otimes B$ (see [23] for details)

Let $\varphi \in \mathrm{Mor}(A_1, A_2)$ and $\psi \in \mathrm{Mor}(B_1, B_2)$ (A_1, A_2, B_1 and B_2 are C*-algebras). Then there exists unique morphism $\varphi \otimes \psi \in \mathrm{Mor}(A_1 \otimes B_1, A_2 \otimes B_2)$ such that $(\varphi \otimes \psi)(S \otimes T) = \varphi(S) \otimes \psi(T)$.

Let $B \in C^*(H)$, where H is a Hilbert space. Then the embedding $i_B : B \hookrightarrow B(H)$ belongs to $\mathrm{Mor}(B, CB(H))$, $\mathrm{id} \otimes i_B \in \mathrm{Mor}(A \otimes B, A \otimes CB(H))$ and $(\mathrm{id} \otimes i_B)U \,\eta\, A \otimes CB(H)$ for any $U \,\eta\, A \otimes B$. In what follows we shall omit $(\mathrm{id} \otimes i_B)$ identifying in this way $(A \otimes B)^\eta$ with a subset of $(A \otimes CB(H))^\eta$:

$$(A \otimes B)^\eta \subset (A \otimes CB(H))^\eta$$

Let N be an integer. Throughout the paper, M_N will denote the C*-algebra of all $N \times N$ matrices with complex entries. If A is a C*-algebra, then $M_N \otimes A$ coincides with the C*-algebra $M_N(A)$ of all $N\times N$ matrices with entries belonging to A. Clearly $M(M_N\otimes A)=M_N\otimes M(A)$. We shall use the following simple

Proposition 1.2 *Let A be a C^*-algebra and $p,q,s\in M(A)$. Then*

$$\left(\begin{array}{c}\begin{pmatrix} p\ ,\ q\\ q^*\ ,\ s\end{pmatrix}>0\\ \textit{on}\ \mathrm{Sp}\,(M_2\otimes A)\end{array}\right)\Longleftrightarrow\left(\begin{array}{c} s>0\ \textit{and}\\ p-qs^{-1}q^*>0\\ \textit{on}\ \mathrm{Sp}\,A\end{array}\right).$$

The standard proof is left to the reader.

2 Natural topologies

The spaces $M(A)$, A^η, $\mathrm{Rep}\,(A,H)$ and $\mathrm{Mor}(A,B)$ (where A, B are C*-algebras and H is a Hilbert space) are endowed with natural topologies.

Let A be a C*-algebra. Strict topology on $M(A)$ is the weakest topology such that for all $a\in A$ the mappings

$$\begin{array}{c} M(A)\ni x\longmapsto ax\in A\\ M(A)\ni x\longmapsto xa\in A\end{array}$$

are continuous. The strict topology is weaker than the norm topology. A net $(x_\lambda)_{\lambda\in\Lambda}$ of elements of $M(A)$ strictly converges to 0 if and only if $\|ax_\lambda\|\to 0$ and $\|x_\lambda a\|\to 0$ for any $a\in A$. Let us notice that A is dense in $M(A)$. Indeed if $(e_\lambda)_{\lambda\in\Lambda}$ is an approximate unit in A, then for any $x\in M(A)$, $(e_\lambda x)_{\lambda\in\Lambda}$ is a net of elements of A converging strictly to x. The strict topology on $B(H)=M(CB(H)$ coincides with the *-strong operator topology.

$M(A)$ endowed with the strict topology is a locally convex topological vector space. Moreover the mapping

$$M(A)\ni x\longmapsto x^*\in M(A) \tag{2. 1}$$

is continuous.

Let $(x_\lambda)_{\lambda\in\Lambda}$ and $(y_\lambda)_{\lambda\in\Lambda}$ be bounded nets of elements of $M(A)$ strictly converging to x and y respectively. Using the estimates $\|x_\lambda y_\lambda a-xya\|\le\|x_\lambda\|\|y_\lambda a-ya\|+\|x_\lambda ya-xya\|$ and $\|ax_\lambda y_\lambda-axy\|\le\|ax_\lambda-ax\|\|y_\lambda\|+\|axy_\lambda-axy\|$ (where $a\in A$) we see that $x_\lambda y_\lambda$ strictly converges to xy. It shows that the multiplication map

$$Q\times Q\ni(x,y)\longmapsto xy\in M(A) \tag{2. 2}$$

where Q is a bounded subset of $M(A)$ is continuous.

Let $(a_\lambda)_{\lambda\in\Lambda}$ be a net of elements of A norm converging to $a\in A$ and $(x_\lambda)_{\lambda\in\Lambda}$ be a bounded net of elements of $M(A)$ strictly converging to $x\in M(A)$. Using the estimates $\|x_\lambda a_\lambda-xa\|\le\|x_\lambda\|\|a_\lambda-a\|+\|x_\lambda a-xa\|$ and $\|a_\lambda x_\lambda-ax\|\le\|a_\lambda-a\|\|x_\lambda\|+\|ax_\lambda-ax\|$ we see that $x_\lambda a_\lambda$ and $a_\lambda x_\lambda$ are norm converging to xa and ax respectively. It shows that the multiplication maps

$$
\begin{array}{rcl}
Q \times A \ni (x, a) & \longmapsto & xa \in A \\
A \times Q \ni (a, x) & \longmapsto & ax \in A
\end{array} \tag{2. 3}
$$

where Q is a bounded subset of $M(A)$ are continuous. It turns out that the z-transform is continuous. We have:

Proposition 2.1 *Let A be a C^*-algebra, $(a_n)_{n=1,2,\ldots}$ be a sequence of elements of $M(A)$ and $a \in M(A)$. Then the following two conditions are equivalent:*

1. *a_n strictly converge to a.*
2. *The sequence $(a_n)_{n=1,2,\ldots}$ is bounded and z_{a_n} strictly converge to z_a.*

Proof: The boundness of strictly converging sequences of multipliers follows immediately from [7, Chapter II, Section 1, Theorem 17]. Now, taking into account the continuity of (2. 2) and using (1. 4) we get the equivalence of Conditions 1 and 2.

Q.E.D.

In Section 4 we shall use the following version of the Stone - Weierstrass Theorem:

Proposition 2.2 *Let A be a C^*-algebra and Q be a strictly closed unital *-subalgebra of $M(A)$ separating representations of A: if φ, φ' are different elements of* $\mathrm{Rep}\,(A, H)$ *then $\varphi(q) \neq \varphi'(q)$ for some $q \in Q$. Then $Q = M(A)$.*

Proof: It is sufficient to show that $A \subset Q$. Assume on the contrary that $a \notin Q$ for some $a \in A$. Then there exists a strictly continuous linear functional ϕ on $M(A)$ such that $\phi|_Q = 0$ and $\phi(a) \neq 0$.

Taking into account the definition of the strict topology given at the begining of Section 2 we see that ϕ is of the form

$$
\phi(x) = \sum_{i=1}^{N} \phi_i(xa_i) + \sum_{i=N+1}^{M} \phi_i(a_i x),
$$

where $N \leq M$ are nonnegative integers, $a_i \in A$ and ϕ_i are norm-continuous linear functionals on A ($i = 1, 2, \ldots, M$). Using the GNS construction one can find $\varphi \in \mathrm{Rep}\,(A, H)$ and trace class operators ρ_i acting on H such that $\phi_i(c) = \mathrm{Tr}\, \rho_i \varphi(c)$ ($c \in A$, $i = 1, 2, \ldots, M$). Therefore for any $x \in M(A)$ we have:

$$
\phi(x) = \mathrm{Tr}\, \rho \varphi(x)
$$

where

$$
\rho = \sum_{i=1}^{N} \varphi(a_i)\rho_i + \sum_{i=N+1}^{M} \rho_i \varphi(a_i)
$$

is a trace class operator acting on H. Remembering that ϕ vanishes on Q and that $\phi(a) \neq 0$ we see that $\varphi(a)$ does not belong to the weak closure of $\varphi(Q)$. By virtue of the von Neumann double commutant theorem, there exists a unitary operator U such that $U^*\varphi(x)U = \varphi(x)$ for all $x \in Q$ and $U^*\varphi(a)U \neq \varphi(a)$. Let $\varphi'(y) = U^*\varphi(y)U$ for any $y \in A$. Then $\varphi' \in \mathrm{Rep}\,(A, H)$, $\varphi'|_Q = \varphi|_Q$ and $\varphi'(a) \neq \varphi(a)$. It shows that Q does not separate representations of A.

Q.E.D.

The natural topology on A^η is the one of almost uniform convergence. It is the weakest topology such that the z-transform

$$A^\eta \ni T \longmapsto z_T \in M(A)$$

(where $M(A)$ is equipped with the strict topology) is continuous. A net $(T_\lambda)_{\lambda\in\Lambda}$ of elements affiliated with a C*-algebra A almost uniformly converges to an element $T\,\eta\,A$, if and only if the corresponding z-transforms z_{T_λ} strictly converge to z_T. Since A is separable, the unit ball in $M(A)$ equipped with the strict topology is metrizable. So is A^η equipped with the topology of almost uniform convergence. If A is commutative, then T, $T_\lambda \in C(\mathrm{Sp}\,A)$ and $T_\lambda \longrightarrow T$ almost uniformly if and only if $T_\lambda \longrightarrow T$ uniformly on any compact subset of $\mathrm{Sp}\,A$.

According to Proposition 2.1, the topology of almost uniform convergence restricted to the multiplier algebra is weaker than the strict topology (the embedding $M(A) \hookrightarrow A^\eta$ is continuous). The two topologies coincide on bounded subsets of the multiplier algebra.

Let A and B be C*-algebras. We endow $\mathrm{Mor}(A,B)$ with the weakest topology such that for all $a \in A$ the mappings

$$\mathrm{Mor}(A,B) \ni \varphi \longmapsto \varphi(a) \in M(B)$$

are strictly continuous. One can easily verify that a net $(\varphi_\lambda)_{\lambda\in\Lambda}$ of elements of $\mathrm{Mor}(A,B)$ converges to a morphism $\varphi \in \mathrm{Mor}(A,B)$ if and only if for any $a \in A$ and $b \in B$, $\|\varphi_\lambda(a)b - \varphi(a)b\| \longrightarrow 0$. Due to the separability of A and B, $\mathrm{Mor}(A,B)$ is metrizable.

Let A be a C*-algebra and H be a Hilbert space. We know that $\mathrm{Rep}\,(A,H) = \mathrm{Mor}(A, CB(H))$, where $CB(H)$ is the C*-algebra of all compact operators acting on H. We endow $\mathrm{Rep}\,(A,H)$ with the topology coming from $\mathrm{Mor}(A, CB(H))$. One can easily verify that a net $(\pi_\lambda)_{\lambda\in\Lambda}$ of elements of $\mathrm{Rep}\,(A,H)$ converges to $\pi \in \mathrm{Rep}\,(A,H)$ if and only if

$$\pi_\lambda(a)\Psi \longrightarrow \pi(a)\Psi$$

for any $a \in A$ and $\Psi \in H$. It means that the topology of $\mathrm{Rep}\,(A,H)$ coincides with the one introduced by Takesaki in [13] (except the fact that in [13] $\mathrm{Rep}\,(A,H)$ includes also degenerate representations).

Let Λ be a locally compact space and A, B be C*-algebras. For each $\lambda \in \Lambda$ we denote by $\xi_\lambda \in \mathrm{Mor}(C_\infty(\Lambda), \mathbf{C})$ the evaluation functional:

$$\xi_\lambda(a) = a(\lambda)$$

for any $a \in C_\infty(\Lambda)$. Then $\xi_\lambda \otimes \mathrm{id} \in \mathrm{Mor}(C_\infty(\Lambda)\otimes A, A)$. For any $T\,\eta\,C_\infty(\Lambda)\otimes A$ and $\varphi \in \mathrm{Mor}(B, C_\infty(\Lambda)\otimes A)$ we set

$$\begin{array}{rcl} \tilde{T}(\lambda) & = & (\xi_\lambda \otimes \mathrm{id})T \\ \tilde{\varphi}(\lambda) & = & (\xi_\lambda \otimes \mathrm{id})\varphi \end{array}$$

Clearly $\tilde{T}(\lambda)\,\eta\,A$ and $\tilde{\varphi}(\lambda) \in \mathrm{Mor}(B, A)$. Since the family of morphisms $(\xi_\lambda)_{\lambda\in\Lambda}$ is faithful, the element $T\,\eta\,C_\infty(\Lambda)\otimes A$ ($\varphi \in \mathrm{Mor}(B, C_\infty(\Lambda)\otimes A)$ respectively) is uniquely determined by the mapping $\tilde{T} : \Lambda \to A^\eta$ ($\tilde{\varphi} : \Lambda \to \mathrm{Mor}(B, A)$ respectively). In what follows we shall omit 'tilde' identifying T and φ with the corresponding mappings. With this identification

$$\begin{aligned} C_\infty(\Lambda)\otimes A &= C_\infty(\Lambda, A) && (2.\ 4)\\ M(C_\infty(\Lambda)\otimes A) &= C_{\mathrm{bounded}}(\Lambda, M(A)) && (2.\ 5)\\ (C_\infty(\Lambda)\otimes A)^\eta &= C(\Lambda, A^\eta) && (2.\ 6)\\ \mathrm{Mor}(B, C_\infty(\Lambda)\otimes A) &= C(\Lambda, \mathrm{Mor}(B, A)) && (2.\ 7)\\ \mathrm{Mor}(B, C_\infty(\Lambda)\otimes CB(H)) &= C(\Lambda, \mathrm{Rep}\,(B, H)) && (2.\ 8) \end{aligned}$$

In these formulae $C_\infty(\Lambda, A)$ is the set of all norm continuous A-valued functions on Λ tending to zero at infinity; $C_{\mathrm{bounded}}(\Lambda, M(A))$ is the set of all bounded strictly continuous $M(A)$-valued functions on Λ and $C(\Lambda, X)$ (where $X = A^\eta$, $\mathrm{Mor}(B, A)$ and $\mathrm{Rep}\,(B, H)$ endowed with the natural topologies described above) is the set of all continuous mappings from Λ into X.

To prove formula (2. 4) we notice that $(f\otimes a)(\lambda) = f(\lambda)a$ for any $f \in C_\infty(\Lambda)$, $a \in A$ and $\lambda \in \Lambda$. Therefore $C_\infty(\Lambda)\otimes_{\mathrm{alg}} A \subset C_\infty(\Lambda, A)$ and remembering that $C_\infty(\Lambda, A)$ is complete we get $C_\infty(\Lambda)\otimes A \subset C_\infty(\Lambda, A)$.

To prove the converse inclusion we have to show that $C_\infty(\Lambda)\otimes_{\mathrm{alg}} A$ is dense in $C_\infty(\Lambda, A)$. Let $x \in C_\infty(\Lambda, A)$ and $\epsilon > 0$. Then $\{x(\lambda) : \lambda \in \Lambda\} \cup \{0\}$ is a compact subset of A and there exists finite sequence $a_0 = 0, a_1, \ldots, a_n \in A$ such that

$$\Lambda \subset \bigcup_{i=0}^{n} \{\lambda \in \Lambda : \|x(\lambda) - a_i\| < \epsilon\}\,. \tag{2. 9}$$

Let $\sum f_i(\lambda) = 1$ (where f_i are nonnegative continuous functions on Λ, $i = 0, 1, \ldots, n$) be the decomposition of unit subordinated to the covering (2. 9). We may assume that $f_0(\lambda) \to 1$ for $\lambda \to \infty$. Then $f_i \in C_\infty(\Lambda)$ for $i = 1, 2, \ldots, n$,

$$(x - \sum_{i=1}^{n} f_i\otimes a_i)(\lambda) = \sum_{i=0}^{n} f_i(\lambda)(x(\lambda) - a_i)$$

for any $\lambda \in \Lambda$ and

$$\begin{aligned} \|x - \sum_{i=1}^{n} f_i\otimes a_i\| &= \sup_{\lambda\in\Lambda} \|\sum_{i=0}^{n} f_i(\lambda)(x(\lambda) - a_i)\| \\ &\le \sup_{\lambda\in\Lambda} \sum_{i=0}^{n} f_i(\lambda)\|(x(\lambda) - a_i)\| \le \epsilon. \end{aligned}$$

It shows that x belongs to the closure of $C_\infty(\Lambda)\otimes_{\mathrm{alg}} A$ and formula (2. 4) follows.

Now we shall prove formula (2. 5). Let $x \in M(C_\infty(\Lambda)\otimes A)$. Then $x(\lambda) = (\xi_\lambda\otimes\mathrm{id})x \in M(A)$ and $\|x(\lambda)\| \le \|x\|$ for any $\lambda \in \Lambda$. It shows that $x(\cdot)$ is a bounded $M(A)$-valued function on Λ. Let us fix $f \in C_\infty(\Lambda)$ such that $f > 0$ on Λ. Then for any $a \in A$,

$(f \otimes a)x$ and $x(f \otimes a)$ belong to $C_\infty(\Lambda) \otimes A = C_\infty(\Lambda, A)$. It means that $f(\lambda)ax(\lambda)$ and $f(\lambda)x(\lambda)a$ are norm continuous with respect to λ. So are $ax(\lambda)$ and $x(\lambda)a$. It shows that $x(\cdot)$ is strictly continuous: $x \in C_{\text{bounded}}(\Lambda, M(A))$. This way we showed that $M(C_\infty(\Lambda) \otimes A) \subset C_{\text{bounded}}(\Lambda, M(A))$.

We shall prove the converse inclusion. Remembering that the mappings (2. 1) and (2. 2) are continuous one can easily show that $C_{\text{bounded}}(\Lambda, M(A))$ is a C*-algebra. Clearly $C_\infty(\Lambda, A) \subset C_{\text{bounded}}(\Lambda, M(A))$. By virtue of the continuity of (2. 3) $C_\infty(\Lambda, A)$ is an ideal of $C_{\text{bounded}}(\Lambda, M(A))$. Let x be an element of $C_{\text{bounded}}(\Lambda, M(A))$ such that $xa = 0$ for any $a \in C_\infty(\Lambda, A)$. Then $x = 0$. Therefore $C_\infty(\Lambda, A)$ is an essential ideal of $C_{\text{bounded}}(\Lambda, M(A))$ and $C_{\text{bounded}}(\Lambda, M(A)) \subset M(C_\infty(\Lambda, A))$ (cf [10, Proposition 3.12.8]). This ends the proof of (2. 5).

By virtue of the Schur lemma any irreducible $\pi \in \operatorname{Rep}(C_\infty(\Lambda) \otimes A, H)$ is of the form: $\pi = \xi_\lambda \otimes \pi_0$, where $\lambda \in \Lambda$ and $\pi_0 \in \operatorname{Rep}(A, H)$. Therefore any pure state ω of $C_\infty(\Lambda) \otimes A$ is of the form $\omega = \xi_\lambda \otimes \omega_0$, where $\lambda \in \Lambda$ and ω_0 is a pure state on A. Using now the characterization of the strict inequality in terms of pure states given in Section 1 we get:

$$\left(\begin{array}{c} x > 0 \ \text{ on} \\ \operatorname{Sp}(C_\infty(\Lambda) \otimes A) \end{array}\right) \Longleftrightarrow \left(\begin{array}{c} x(\lambda) > 0 \text{ on } \operatorname{Sp} A \\ \text{for all } \lambda \in \Lambda \end{array}\right) \tag{2. 10}$$

for any $x \in M(C_\infty(\Lambda) \otimes A)$.

Now we can prove relation (2. 6). Let $T \,\eta\, C_\infty(\Lambda) \otimes A$. Then $z_T \in M(C_\infty(\Lambda) \otimes A)$ and $z_T(\lambda) = (\xi_\lambda \otimes \mathrm{id})z_T = z_{(\xi_\lambda \otimes \mathrm{id})T} = z_{T(\lambda)}$. By virtue of (2. 5), $z_{T(\lambda)}$ is strictly continuous with respect to λ. Therefore $T(\lambda)$ is almost uniformly continuous: $T \in C(\Lambda, A^\eta)$. Conversely if $T \in C(\Lambda, A^\eta)$, then $z_{T(\lambda)}$ is strictly continuous with respect to λ and $z_{T(\lambda)}{}^* z_{T(\lambda)} < I$ on $\operatorname{Sp} A$ for any $\lambda \in \Lambda$. By virtue of (2. 5), there exists $z \in M(C_\infty(\Lambda) \otimes A)$ such that $z(\lambda) = z_{T(\lambda)}$ for all $\lambda \in \Lambda$. According to (2. 10), $z^*z < I$ on $\operatorname{Sp}(C_\infty(\Lambda) \otimes A)$. Therefore z is a z-transform of an element T' affiliated with $C_\infty(\Lambda) \otimes A$. Now, for any $\lambda \in \Lambda$ we have: $z_{T'(\lambda)} = z_{T'}(\lambda) = z(\lambda) = z_{T(\lambda)}$. It shows that $T = T'$ and $T \in (C_\infty(\Lambda) \otimes A)^\eta$. The formula (2. 6) is proved.

We shall prove (2. 7). Let $\tilde{\varphi} \in C(\Lambda, \operatorname{Mor}(B, A))$. Then for any $b \in B$, the mapping $\Lambda \ni \lambda \mapsto (\tilde{\varphi}(\lambda))(b) \in M(A)$ is strictly continuous and bounded. Therefore (cf (2. 5)) there exists $\varphi(b) \in M(C_\infty(\Lambda) \otimes A)$ such that $(\varphi(b))(\lambda) = (\tilde{\varphi}(\lambda))(b)$. Clearly the mapping $\varphi : B \to M(C_\infty(\Lambda) \otimes A)$ introduced in this way is linear, multiplicative and *-preserving. Moreover if $b > 0$ on $\operatorname{Sp} B$ then by virtue of (1. 6) $(\varphi(b))(\lambda) = (\tilde{\varphi}(\lambda))(b) > 0$ on $\operatorname{Sp} A$ and (cf (2. 10)) $\varphi(b) > 0$ on $\operatorname{Sp}(C_\infty(\Lambda) \otimes A)$. Therefore $\varphi(b)(C_\infty(\Lambda) \otimes A) \subset \varphi(B)(C_\infty(\Lambda) \otimes A)$ is dense in $C_\infty(\Lambda) \otimes A$ and $\varphi \in \operatorname{Mor}(B, C_\infty(\Lambda) \otimes A)$. It shows that $C(\Lambda, \operatorname{Mor}(B, A)) \subset \operatorname{Mor}(B, C_\infty(\Lambda) \otimes A)$.

Conversely if $\varphi \in \operatorname{Mor}(B, C_\infty(\Lambda) \otimes A)$, then for any $b \in B$, $\varphi(b) \in M(C_\infty(\Lambda) \otimes A)$ and by virtue of (2. 5), $(\tilde{\varphi}(\lambda))(b) = (\xi_\lambda \otimes \mathrm{id})\varphi(b) = (\varphi(b))(\lambda)$ is strictly continuous with respect to λ. Therefore $\tilde{\varphi} \in C(\Lambda, \operatorname{Mor}(B, A))$ and $\operatorname{Mor}(B, C_\infty(\Lambda) \otimes A) \subset C(\Lambda, \operatorname{Mor}(B, A))$. Formula (2. 7) is proved.

Inserting in (2. 7) $CB(H)$ instead of A we obtain (2. 8). This way the formulae (2. 4) – (2. 8) are proved.

Let Λ be a locally compact space, $\Lambda' \subset \Lambda$ be a locally compact subspace and A be a C*-algebra. According to (2. 6), an element $T\,\eta\,C_\infty(\Lambda')\otimes A$ is identified with a continuous mapping $T : \Lambda' \to A^\eta$. Its restriction $T|_\Lambda : \Lambda \to A^\eta$ is continuous and by virtue of (2. 6), $T|_\Lambda\,\eta\,C_\infty(\Lambda)\otimes A$. One can easily verify that

$$T|_\Lambda = (\xi_\Lambda \otimes \mathrm{id})T,$$

where $\xi_\Lambda \in \mathrm{Mor}(C_\infty(\Lambda'), C_\infty(\Lambda))$ is the restriction map: $\xi_\Lambda(f) = f|_\Lambda$ for any $f \in C_\infty(\Lambda')$.

Let B is a C*-algebra. According to (2. 7), an element $\varphi \in \mathrm{Mor}(B, C_\infty(\Lambda')\otimes A$ is identified with a continuous mapping $\varphi : \Lambda' \to \mathrm{Mor}(B,A)$. Its restriction $\varphi|_\Lambda : \Lambda \to \mathrm{Mor}(B,A)$ is continuous and by virtue of (2. 7), $\varphi|_\Lambda \in \mathrm{Mor}(B, C_\infty(\Lambda)\otimes A)$. One can easily verify that

$$\varphi|_\Lambda = (\xi_\Lambda \otimes \mathrm{id})\varphi.$$

Proposition 2.3 *Let A, B be C^*-algebras. Then the evaluation map*

$$A^\eta \times \mathrm{Mor}(A,B) \ni (T,\varphi) \longmapsto \varphi(T) \in B^\eta$$

(where A^η and B^η are equipped with the almost uniform topology) is continuous.

Proof: Let $(T(n))_{n=1,2,\ldots}$ be a sequence of elements affiliated with A converging almost uniformly to $T(\infty)\,\eta\,A$ and $(\varphi(n))_{n=1,2,\ldots}$ be a sequence of morphisms from A into B converging to $\varphi(\infty) \in \mathrm{Mor}(A,B)$. Then $(\varphi(n))(T(n))\,\eta\,B$. We have to show that

$$(\varphi(n))(T(n)) \longrightarrow (\varphi(\infty))(T(\infty)) \tag{2. 11}$$

almost uniformly for $n \to \infty$. Let $\Lambda = \{1, 2, \ldots, \infty\}$ be the one point compactification of the set of natural numbers. Then $T \in C(\Lambda, A^\eta)$ and $\varphi \in C(\Lambda, \mathrm{Mor}(A,B))$. By virtue of (2. 6) and (2. 7), $T\,\eta\,C(\Lambda)\otimes A$ and $\varphi \in \mathrm{Mor}(A, C(\Lambda)\otimes B)$. Therefore $\mathrm{id}\otimes\varphi \in \mathrm{Mor}(C(\Lambda)\otimes A, C(\Lambda)\otimes C(\Lambda)\otimes B)$ and $(\mathrm{id}\otimes\varphi)T\,\eta\,C(\Lambda)\otimes C(\Lambda)\otimes B$.

It is well known that $C(\Lambda)\otimes C(\Lambda)$ may be identified with $C(\Lambda^2)$ in such a way that $\xi_n \otimes \xi_m = \xi_{(n,m)}$ for any $(n,m) \in \Lambda^2$. Using the identification (2. 6) we see that

$$\begin{aligned} ((\mathrm{id}\otimes\varphi)T)(n,m) &= (\xi_{n,m}\otimes\mathrm{id})(\mathrm{id}\otimes\varphi)T = (\xi_n\otimes\xi_m\otimes\mathrm{id})(\mathrm{id}\otimes\varphi)T \\ &= ((\xi_m\otimes\mathrm{id})\varphi)((\xi_n\otimes\mathrm{id})T) = (\varphi(m))(T(n)) \end{aligned}$$

depends continuously on (n,m) and (2. 11) follows.

Q.E.D.

Proposition 2.4 *Let A, B, C be C^*-algebras. Then the composition map*

$$\mathrm{Mor}(A,B) \times \mathrm{Mor}(B,C) \ni (\varphi,\psi) \longmapsto \psi\varphi \in \mathrm{Mor}(A,C)$$

is continuous.

Proof: Let $(\varphi(n))_{n=1,2,\dots}$ ($(\psi(n))_{n=1,2,\dots}$ respectively) be a sequence of morphisms from A into B (from B into C respectively) converging to $\varphi(\infty) \in \mathrm{Mor}(A, B)$ ($\psi(\infty) \in \mathrm{Mor}(B, C)$ respectively). Then $(\psi(n))(\varphi(n)) \in \mathrm{Mor}(A, C)$. We have to show that

$$(\psi(n))(\varphi(n)) \longrightarrow (\psi(\infty))(\varphi(\infty)) \tag{2. 12}$$

for $n \to \infty$. Let Λ be as in the previous proof. Then $\varphi \in C(\Lambda, \mathrm{Mor}(A, B))$ and $\psi \in C(\Lambda, \mathrm{Mor}(B, C))$. By virtue of (2. 7), $\varphi \in \mathrm{Mor}(A, C(\Lambda) \otimes B)$ and $\psi \in \mathrm{Mor}(B, C(\Lambda) \otimes C)$. Therefore $\mathrm{id} \otimes \psi \in \mathrm{Mor}(C(\Lambda) \otimes B, C(\Lambda^2) \otimes C)$ and $(\mathrm{id} \otimes \psi)\varphi \in \mathrm{Mor}(A, C(\Lambda^2) \otimes C)$. Using the identification (2. 7) we see that

$$\begin{aligned} ((\mathrm{id} \otimes \psi)\varphi)(n, m) &= (\xi_{n,m} \otimes \mathrm{id})(\mathrm{id} \otimes \psi)\varphi = (\xi_n \otimes \xi_m \otimes \mathrm{id})(\mathrm{id} \otimes \psi)\varphi \\ &= ((\xi_m \otimes \mathrm{id})\psi)((\xi_n \otimes \mathrm{id})\varphi) = (\psi(m))(\varphi(n)) \end{aligned}$$

depends continuously on (n, m) and (2. 12) follows.

Q.E.D.

Proposition 2.5 *Let A, B be C^*-algebras. Then the tensor product map*

$$A^\eta \times B^\eta \ni (T, S) \longmapsto T \otimes S \in (A \otimes B)^\eta$$

is continuous.

Proof: Let $(S(n))_{n=1,2,\dots}$ ($(T(n))_{n=1,2,\dots}$ respectively) be a sequence of elements affiliated with A (with B respectively) converging to $S(\infty)\,\eta\, A$ ($T(\infty)\,\eta\, B$ respectively). Then $S(n) \otimes T(n)\,\eta\, A \otimes B$. We have to show that

$$S(n) \otimes T(n) \longrightarrow S(\infty) \otimes T(\infty) \tag{2. 13}$$

for $n \to \infty$. Let Λ be as in the previous proofs. Then $S \in C(\Lambda, A^\eta)$ and $T \in C(\Lambda, B^\eta)$. By virtue of (2. 6), $S\,\eta\, C(\Lambda) \otimes A$ and $T\,\eta\, C(\Lambda) \otimes B$. Therefore $S \otimes T\,\eta\, C(\Lambda) \otimes A \otimes C(\Lambda) \otimes B$. The latter algebra is canonically isomorphic to $C(\Lambda^2) \otimes A \otimes B$. Using the identification (2. 6) we see that

$$\begin{aligned} (S \otimes T)(n, m) &= (\xi_n \otimes \mathrm{id} \otimes \xi_m \otimes \mathrm{id})(S \otimes T) \\ &= (\xi_n \otimes \mathrm{id})S \otimes (\xi_m \otimes \mathrm{id})T = S(m) \otimes T(n) \end{aligned}$$

depends continuously on (n, m) and (2. 13) follows.

Q.E.D.

Let A and B be C^*-algebras and $\varphi \in \mathrm{Mor}(A, B)$. By virtue of Proposition 2.3, the extension

$$\varphi :\ A^\eta \longrightarrow B^\eta \tag{2. 14}$$

is continuous. To make this remark more interesting we notice that A is dense in A^η. Indeed if $(e_n)_{n=1,2,\dots}$ is an approximate unit of A, then for any $a \in M(A)$, $(e_n a)$ converges strongly to a. It shows that A is dense in $M(A)$. On the other hand if $T\,\eta\, A$ and $\lambda \in]0, \pi/2[$, then $T_\lambda = (\cos\lambda)T(I + (\sin^2\lambda)T^*T)^{-\frac{1}{2}} \in M(A)$, $z_{T_\lambda} = (\cos\lambda)z_T$ and

$z_{T_\lambda} \longrightarrow z_T$ for $\lambda \longrightarrow 0$. Therefore $T_\lambda \longrightarrow T$ almost uniformly and $M(A)$ is dense in A^η.

Let $B \in C^*(H)$. Then $\mathrm{Mor}(A,B) \subset \mathrm{Rep}\,(A,H)$ and the topology of $\mathrm{Mor}(A,B)$ is stronger than the one induced from $\mathrm{Rep}\,(A,H)$. Indeed, by virtue of Proposition 2.4 for any fixed $\psi \in \mathrm{Rep}\,(B,H)$ the mapping

$$\mathrm{Mor}(A,B) \ni \varphi \longmapsto \psi\varphi \in \mathrm{Rep}\,(A,H)$$

is continuous. In particular (for $\psi = i_B$) the embedding $\mathrm{Mor}(A,B) \hookrightarrow \mathrm{Rep}\,(A,H)$ is continuous.

3 C*-algebras generated by a finite set of affiliated elements

Introducing the notion of an object generated by a set of elements one usually lists a set of procedures that allow,

starting with given generators to construct step by step all elements of generated object. Unfortunately this method does not work in our case. Instead we shall use another approach:

Definition 3.1 *Let A be a C^*-algebra and $T_1, T_2, \ldots, T_N$ be elements affiliated with A. We say, that A is generated by $T_1, T_2, \ldots, T_N$ if for any Hilbert space H, any $B \in C^*(H)$ and any $\pi \in \mathrm{Rep}\,(A,H)$ we have:*

$$\left(\begin{array}{c} \pi(T_i)\,\eta\, B \text{ for any} \\ i = 1,2,\ldots,N \end{array} \right) \Longrightarrow \Big(\pi \in \mathrm{Mor}(A,B) \Big) \tag{3. 1}$$

Proposition 3.2 *Let A, B be C^*-algebras, $j \in \mathrm{Mor}(B,A)$, $S_i\,\eta\,B$ and $T_i = j(S_i)\,\eta\,A$ $(i = 1,2,\ldots,N)$. Assume that j is an injection and that $T_1, T_2, \ldots, T_N$ generate A. Then j is an isomorphism: $j(B) = A$.*

Proof: We may assume that $A \in C^*(H)$. Then j is a faithful representation of B and identifying B with its j-image we have $B \in C^*(H)$ and $j = i_B$. Relation $i_B \in \mathrm{Mor}(B,A)$ means that

$$\overline{BA} = A.$$

On the other hand $T_i = i_B(S_i) = S_i$ $(i = 1,2,\ldots,N)$ are closed operators affiliated with B and using (3. 1) with π replaced by i_A we see that $i_A \in \mathrm{Mor}(A,B)$. It means that

$$\overline{AB} = B.$$

Comparing this result with the previous one we get $B = A$.

Q.E.D.

Later (see example 1 of Section 4) we shall prove the following

Theorem 3.3 *Let A be a C^*-algebra and $T_1, T_2, \ldots, T_N$ be elements affiliated with A. The subset of $M(A)$ composed of all elements of the form $(I + T_i{}^*T_i)^{-1}$ and $(I + T_iT_i{}^*)^{-1}$ $(i = 1, 2, \ldots, N)$ will be denoted by Γ. Assume that*

1. *$T_1, T_2, \ldots, T_N$ separate representations of A: if φ_1, φ_2 are different elements of $\mathrm{Rep}\,(A, H)$ then $\varphi_1(T_i) \neq \varphi_2(T_i)$ for some $i = 1, 2, \ldots, N$.*
2. *There exist elements $r_1, r_2, \ldots, r_k \in \Gamma$ such that the product $r_1 r_2 \ldots r_k \in A$.*

Then A is generated by $T_1, T_2, \ldots, T_N$.

Examples:

1. Let A be a unital C*-algebra and $T_1, T_2, \ldots, T_N \in A$. Assume that A coincides with the norm closure of the set of all algebraic combinations of $I, T_1, T_2, \ldots, T_N$. Then one can immediately verify that the implication (3. 1) holds. In other words, A is generated by $T_1, T_2, \ldots, T_N$ in the sense of Definition 3.1.

Conversely if A is a C^*-algebra generated by $T_1, T_2, \ldots, T_N \,\eta\, A$ such that $\|T_i\| < \infty$ for all $i = 1, 2, \ldots, N$, then A is unital, $T_i \in A$ for all $i = 1, 2, \ldots, N$ and A coincides with the norm closure of the set of all algebraic combinations of $I, T_1, T_2, \ldots, T_N$.

Indeed $T_i \,\eta\, A$ and $\|T_i\| < \infty$ imply that $T_i \in M(A)$. Let B be the norm closure of the set of all algebraic combinations of $I, T_1, T_2, \ldots, T_N$. Then B is a unital subalgebra of $M(A)$ and the embedding $j : B \hookrightarrow M(A)$ belongs to $\mathrm{Mor}(B, A)$. Using Proposition 3.2 we get $B = A$.

2. Let Λ be a locally compact space and $T_1, T_2, \ldots, T_N \in C(\Lambda)$. We claim that $C_\infty(\Lambda)$ is generated by $T_1, T_2, \ldots, T_N$ if and only if the functions $T_1, T_2, \ldots, T_N$ separate points of Λ and

$$\lim_{\lambda \to \infty} \sum_{i=1}^{N} |T_i(\lambda)|^2 = +\infty \tag{3. 2}$$

Clearly elements $T_1, T_2, \ldots, T_N$ separate representations of $C_\infty(\Lambda)$ if and only if functions $T_1, T_2, \ldots, T_N$ separate points of Λ. Let $y_i = (I + T_i{}^*T_i)^{-1}$. One can easily verify that the relation (3. 2) is equivalent to

$$\lim_{\lambda \to \infty} \prod_{i=1}^{N} \left(I + |T_i(\lambda)|^2\right)^{-1} = 0.$$

The latter means that the product $y_1 y_2 \ldots y_N$ belongs to $C_\infty(\Lambda)$. Now, the 'if' part of our statement follows immediately from Theorem 3.3.

To prove the 'only if' part we shall use Proposition 3.2. For any $\lambda \in \Lambda$ we set $T(\lambda) = (T_1(\lambda), T_2(\lambda), \ldots, T_N(\lambda))$. Then $T : \Lambda \to \mathbf{C}^N$ is a continuous map. Let Λ' be the closure of its image: $\Lambda' = \overline{T(\Lambda)}$. For any $\lambda' \in \Lambda'$, $S_i(\lambda')$ will denote the i^{th} component of λ'. In particular

$$S_i(T(\lambda)) = T_i(\lambda) \tag{3. 3}$$

for all $\lambda \in \Lambda$. Clearly S_i are continuous functions on Λ': $S_i \,\eta\, C_\infty(\Lambda')$. For any $x \in C_\infty(\Lambda')$ we set $jx = x \circ T$. Then $jx \in C_{\mathrm{bounded}}(\Lambda)$ and $j \in \mathrm{Mor}(C_\infty(\Lambda'), C_\infty(\Lambda))$. We know that $T(\Lambda)$ is dense in Λ'. Therefore j is an injection. According to (3. 3), $j(S_i) = T_i$ $(i = 1, 2, \ldots, N)$.

Assume that $C_\infty(\Lambda)$ is generated by $T_1, T_2, \dots, T_N$. Using Proposition 3.2 we see that j is an isomorphism. Therefore $T : \Lambda \to \Lambda'$ is a homeomorphism. In particular functions $T_1, T_2, \dots, T_N$ separate points of Λ (otherwise T would not be injective). Moreover

$$\begin{aligned} \lim_{\lambda\to\infty} \sum |T_i(\lambda)|^2 &= \lim_{\lambda'\to\infty} \sum \left|T_i(T^{-1}(\lambda'))\right|^2 \\ &= \lim_{\lambda'\to\infty} \sum |S_i(\lambda')|^2 = +\infty \end{aligned}$$

This ends the proof of the 'only if' part of our statement.

In Section 6 we shall prove a far reaching generalization of this result (cf Theorem 6.2).

3. Let G be a connected Lie group, Γ be the Lie algebra of G and $T_1, T_2, \dots, T_N$ (where $N = \dim G$) be a basis of Γ. According to [23], $T_1, T_2, \dots, T_N$ are skewadjoint elements affiliated with $C^*(G)$. We claim that $C^*(G)$ is generated by $T_1, T_2, \dots, T_N$.

By the representation theory, any non-degenerate representation of $C^*(G)$ is determined by a unitary representation of G which in turn is determined by its infinitesimal generators. It means that $T_1, T_2, \dots, T_N$ separate representations of $C^*(G)$.

Let $\exp : \Gamma \to G$ be the exponential map. For any $s_1, s_2, \dots s_N \in \mathbf{R}$ we set

$$F(s_1, s_2, \dots s_N) = \exp(s_1 T_1)\exp(s_2 T_2)\dots\exp(s_N T_N)$$

Then

$$F : \mathbf{R}^N \longrightarrow G$$

is a real analytic mapping. The reader should notice that the Jacobi matrix of this mapping at point $s = 0$ is of maximal rank N. Due to the analycity the same is true for almost all $s \in \mathbf{R}^N$ (all except a closed subset $Z \subset \mathbf{R}^N$ of Lebesgue measure zero). Therefore in a neighbourhood of any point of $s \in \mathbf{R}^N \backslash Z$, the mapping F is a diffeomorphism. Using this fact one can easily show that for any finite measure μ on $\mathbf{R}^N$, absolutely continuous with respect to the Lebesgue measure, the image $F(\mu)$ is absolutely continuous with respect to the Haar measure on G. In particular there exists a function $R \in L^1(G)$ such that

$$\int_G f(g)R(g)dg = \frac{1}{2^N}\int_{\mathbf{R}^N} f(F(s_1, s_2, \dots s_N))\exp\left(-\sum |s_k|\right)ds_1 ds_2 \dots ds_N \qquad (3.\ 4)$$

for any $f \in C_{\text{bounded}}(G)$.

We shall use the canonical embedding $G \hookrightarrow M(C^*(G))$ (cf [23, Section 3]). Let $r = \int_G gR(g)dg$. Then, remembering that $L^1(G) \subset C^*(G)$ we obtain $r \in C^*(G)$. On the other hand using the formula (3. 4) we have:

$$\begin{aligned} r &= \frac{1}{2^N}\int_{\mathbf{R}^N} F(s_1, s_2, \dots s_N)\exp\left(-\sum |s_k|\right)ds_1 ds_2 \dots ds_N \\ &= \prod_{k=1}^N \left\{\frac{1}{2}\int_{\mathbf{R}} \exp(sT_k)e^{-|s|}ds\right\} = y_1 y_2 \dots y_N, \end{aligned}$$

where $y_k = (I + {T_k}^* T_k)^{-1}$. Using now Theorem 3.3 we see that $C^*(G)$ is generated by $T_1, T_2, \dots, T_N$.

4. Let p and q be the momentum and position operators of a quantum mechanical system of one degree of freedom. In the Schrodinger representation $H = L^2(\mathbf{R}, dx)$, q is the multiplication operator by x and $p = \frac{1}{i}\frac{d}{dx}$. Clearly $p, q \,\eta\, CB(H)$ (any closed operator is affiliated with the algebra of all compact operators).

Using the irreducibility of the Schrodinger representation one can easily show that the pair (p, q) separates representations of $CB(H)$. Moreover the operator $r = (1+q^2)^{-1}(1+p^2)^{-1}(1+q^2)^{-1}$ is an integral operator with the kernel

$$K(x, x') = \frac{1}{2}(1+x^2)^{-1}e^{-|x-x'|}(1+x'^2)^{-1}.$$

Let us notice that $r \geq 0$ and that $\operatorname{Tr} r = \int K(x,x)dx = \frac{\pi}{4} < \infty$. Therefore $r \in CB(H)$ and using Theorem 3.3 we see that $CB(H)$ is generated by p and q.

5. The algebra A_1 of all continuous functions vanishing at infinity on the quantum $E(2)$ introduced in Section 1 of [21] is generated by affiliated elements v and n (cf Theorem 1.1.3 of that reference). Indeed one can easily check that v, n separate representations of A_1 and that $(I + n^*n)^{-1} \in A_1$.

6. The algebra A_2 of all continuous functions vanishing at infinity on the Pontryagin dual $\widehat{E(2)}$ of $E(2)$ introduced in Section 3 of [21] is generated by affiliated elements N and b (cf Theorem 3.1.3 of that reference). Indeed one can easily check that N, b separate representations of A_2 and that $(I + N^2)^{-1}(I + b^*b)^{-1} \in A_2$.

7. Let G be the quantum Lorentz group introduced in [24]. Then $A = A_1 \otimes A_2$ is the algebra of all continuous functions vanishing at infinity on G introduced in Section 5 of that paper. It is generated by elements α, β, γ and δ affiliated with A (cf Theorem 5.1.3 of [24]). This fact follows easily from the quantum Gauss decomposition and the above two examples. One can also verify independently that α, β, γ, δ separate representations of A and that the element

$$r = (I + \alpha^*\alpha)^{-1}(I + \delta^*\delta)^{-1}(I + \beta^*\beta)^{-1}(I + \gamma^*\gamma)^{-1}$$

belongs to A. Indeed taking into account formulae (28) and (33) of [24] we see that

$$r = F_\mu(z)^*(\widetilde{I} + \mu^{\widetilde{N}})^{-1}F_\mu(z)(\widetilde{I} + \mu^{-\widetilde{N}})^{-1}(\widetilde{I} + \widetilde{n}^*\widetilde{n}\mu^{-\widetilde{N}})^{-1}(\widetilde{I} + \widetilde{b}^*\widetilde{b})^{-1},$$

where $\widetilde{I} = I_1 \otimes I_2$ is the unit of $M(A)$ (I_i denotes the unit of $M(A_i)$, $i = 1, 2$), $\widetilde{N} = I_1 \otimes N$, $\widetilde{b} = I_1 \otimes b$, $\widetilde{n} = n \otimes I_2$ and $z = \mu n v^* \otimes \mu^{-\frac{N}{2}} b$. Using now the Fourier decomposition of function F_μ and the commutation relation between N and $\operatorname{Phase} b$ (cf formula (22v) of [24]) we get

$$r = \sum_{k=-\infty}^{+\infty} F_\mu(z)^* f_k(z)(\widetilde{I} + \mu^{\widetilde{N}-2k})^{-1}(\widetilde{I} + \mu^{-\widetilde{N}})^{-1}(\widetilde{I} + \widetilde{n}^*\widetilde{n}\mu^{-\widetilde{N}})^{-1}(\widetilde{I} + \widetilde{b}^*\widetilde{b})^{-1}, \quad (3.\ 5)$$

where f_k are quantum Bessel functions introduced by the formula (58) of [22]. Each term in (3. 5) belongs to A. Remembering that $\widetilde{N}$ is a selfadjoint element with the integer spectrum we obtain:

$$\|(\widetilde{I} + \mu^{\widetilde{N}-2k})^{-1}(\widetilde{I} + \mu^{-\widetilde{N}})^{-1}\| < \frac{1}{1+\mu^{-k}}.$$

On the other hand, using the last estimate on page 650 of [22] we get

$$\|f_k(z)\| \leq C\mu^{-\frac{k}{2}},$$

where C is a numerical constant. Combining the last two relations we see that the norm of the k^{th} term in (3. 5) is estimated by $C(\mu^{\frac{k}{2}} + \mu^{-\frac{k}{2}})^{-1}$. Hence the series (3. 5) is norm convergent and $r \in A$.

8. According to Example 1, the algebra A_c of all continuous functions on the quantum $SU(2)$ introduced in [11, 18] is generated by elements $\alpha, \gamma \in A_c$.

9. Let $A_d = \sum^{\oplus} B(H^s)$ be the algebra of all continuous functions vanishing at infinity on the Pontryagin dual of quantum $SU(2)$ considered in Section 5 of [11] and $a, n \, \eta \, A_d$ be distinguished elements introduced in that reference. By Theorem 5.1.3 of [11] a, n separate representations of A_d. Moreover one can easily verify that $a^*a, (a^{-1})^* a^{-1}$ and n^*n mutually strongly commute. Therefore

$$(I + a^*a)(I + (a^{-1})^* a^{-1})(I + n^*n) \geq \mu^2 a^*a + (a^{-1})^* a^{-1} + n^*n$$

and

$$r = (I + a^*a)^{-1}(I + (a^{-1})^* a^{-1})^{-1}(I + n^*n)^{-1} \leq (\mu^2 a^*a + (a^{-1})^* a^{-1} + n^*n)^{-1}$$

On the other hand for any spin $s = 0, \frac{1}{2}, 1, 1\frac{1}{2}, \ldots$ the corresponding component of $(\mu^2 a^*a + (a^{-1})^* a^{-1} + n^*n)^{-1}$ equals to $(\mu^{-2s} + \mu^{2s+2})^{-1} I \in B(H^s)$ (cf [11, Corollary 5.2]) and tends to 0 for $s \to \infty$. It means that $(\mu^2 a^*a + (a^{-1})^* a^{-1} + n^*n)^{-1}$ belongs to A_d. Therefore $r \in A_d$ and using Theorem 3.3 we see that A_d is generated by affiliated elements a, a^{-1}, n.

10. Let G be the quantum Lorentz group introduced in [11]. Then $A = A_c \otimes A_d$ is the algebra of all continuous functions vanishing at infinity on G introduced in Section 5 of that paper. It is generated by matrix elements α, β, γ, $\delta \, \eta \, A$ of the fundamental two-dimensional representation of G. This fact follows easily from the quantum Iwasawa decomposition and the above two examples.

4 C*-algebras generated by a quantum family of affiliated elements

As it was pointed out in Section 1, a sequence $T = (T_1, T_2, \ldots, T_N)$ of elements affiliated with A may be considered as an element affiliated with $\mathbf{C}^N \otimes A$: $T \, \eta \, \mathbf{C}^N \otimes A$. This remark leads to the following generalization of Definition 3.1:

Definition 4.1 *Let A and C be C^*-algebras and T be an element affiliated with $C \otimes A$. We say, that A is generated by T if for any Hilbert space H, any $B \in C^*(H)$ and any $\pi \in \mathrm{Rep}\,(A, H)$ we have:*

$$\left(\; ((\mathrm{id} \otimes \pi) T) \; \eta \, (C \otimes B) \; \right) \Longrightarrow \left(\; \pi \in \mathrm{Mor}(A, B) \; \right) \tag{4. 1}$$

If $C = C_\infty(\Lambda)$, where Λ is a locally compact space, then T is a continuous family of elements affiliated with A labeled by an index running over Λ (cf (2. 6)). In the general case, using the language introduced in [16] one may say that T is a quantum family (pseudo-family) of unbounded elements affiliated with A labeled by an index running over the quantum space (pseudo-space) $\mathrm{Sp}\, C$.

Let H be a Hilbert space, C be a C*-algebra and $T \,\eta\, C \otimes CB(H)$. Then there exists at most one $A \in C^*(H)$ such that $T \,\eta\, C \otimes A$ and A is generated by T. Indeed assume that $T \,\eta\, C \otimes A_1$ and $T \,\eta\, C \otimes A_2$, where A_1 and A_2 are generated by T. Inserting in (4. 1) $A = A_1$, $\pi = i_{A_1}$ and $B = A_2$ we see that $i_{A_1} \in \mathrm{Mor}(A_1, A_2)$. It means that $\overline{A_1 A_2} = A_2$. In the same way one shows that $\overline{A_2 A_1} = A_1$ and the equality $A_1 = A_2$ follows.

Remark: The existence of $A \in C^*(H)$ generated by an element $T \,\eta\, C \otimes CB(H)$ is not guaranteed.

Conditions (3. 1) and (4. 1) are very powerful. As a result it is not obvious how to verify them in concrete cases. The theorem presented below is in many situations helpful for this purpose.

Theorem 4.2 *Let A and C be C^*-algebras and $T \,\eta\, C \otimes A$. Assume that*

I. T separates representations of A: if φ_1, φ_2 are different elements of $\mathrm{Rep}\,(A, H)$ then $(\mathrm{id} \otimes \varphi_1)(T) \neq (\mathrm{id} \otimes \varphi_2)(T)$.

II. There exist a C^*-algebra F and an element $r \in M(F \otimes A)$ satisfying the following two condtions:

1. For any Hilbert space H, any $B \in C^*(H)$ and any $\pi \in \mathrm{Rep}\,(A, H)$ we have:

$$\left((\mathrm{id} \otimes \pi) T \,\eta\, C \otimes B \right) \Longrightarrow \left(\begin{array}{c} (\mathrm{id} \otimes \pi) r \in M(F \otimes B) \\ \textit{and } [(\mathrm{id} \otimes \pi) r](F \otimes B) \\ \textit{is dense in } F \otimes B \end{array} \right) \tag{4. 2}$$

2. There exists a nonzero continuous linear functional ω on F such that

$$(\omega \otimes \mathrm{id})\{r(f \otimes I)\} \in A. \tag{4. 3}$$

for any $f \in F$.

Then A is generated by T.

Proof: Let H be a Hilbert space, $\pi \in \mathrm{Rep}\,(A, H)$, $B \in C^*(H)$ and $(\mathrm{id} \otimes \pi) T \,\eta\, C \otimes B$. We have to show that

$$\pi \in \mathrm{Mor}(A, B) \tag{4. 4}$$

By virtue of (4. 2), $[(\mathrm{id} \otimes \pi) r](F \otimes B)$ is dense in $F \otimes B$. Let f be an element of F such that $\omega(f) = 1$ and $b \in B$. Then for any $\epsilon > 0$ there exist $f_1, f_2, \ldots, f_n \in F$ and $b_1, b_2, \ldots, b_n \in B$ such that

$$\| f \otimes b - \sum_{m=1}^{n} (\mathrm{id} \otimes \pi)(r)(f_m \otimes b_m) \| < \epsilon \tag{4. 5}$$

Let F' be the set of all continuous linear functionals defined on F. Any $\omega \in F'$ is a linear combination of four states. On the other hand if ω is a state, then $\omega \otimes \mathrm{id}$ is a normalised completely positive map acting from $F \otimes B$ into B and $\|\omega \otimes \mathrm{id}\| = 1$. Combining the two informations we see that $\|\omega \otimes \mathrm{id}\| < \infty$ for any $\omega \in F'$.

Taking into account (4. 5) we obtain:

$$\|b - (\omega \otimes \mathrm{id}) \left\{ \sum_{m=1}^{n} (\mathrm{id} \otimes \pi)(r)(f_m \otimes b_m) \right\} \| < \epsilon \|\omega \otimes \mathrm{id}\|$$

and

$$\|b - \sum_{m=1}^{n} \pi(a_m) b_m\| < \epsilon \|\omega \otimes \mathrm{id}\|, \tag{4. 6}$$

where

$$a_m = (\omega \otimes \mathrm{id}) \left\{ r(f_m \otimes I) \right\}$$

for $m = 1, 2, \ldots, n$.

According to (4. 3), $a_m \in A$ and (4. 6) shows that b belongs to the norm closure of $\pi(A)B$. This way we proved that

$$B \subset \overline{\pi(A)B}. \tag{4. 7}$$

We have to prove the converse inclusion. Let

$$Q = \{x \in M(A) : \pi(x) \in M(B)\}$$

Clearly Q is a unital *-subalgebra of $M(A)$. We shall prove that Q is closed with respect to the strict topology. Let $(x_n)_{n=1,2,\ldots}$ be a strictly converging sequence of elements of Q: strict-lim $x_n = x \in M(A)$. Then for any $a \in A$ and $b \in B$ the sequence $\pi(x_n)\pi(a)b = \pi(x_n a)b$ converges in norm to $\pi(xa)b = \pi(x)\pi(a)b$. Taking into account (4. 7) we see that norm-lim $\pi(x_n)b = \pi(x)b$ for any $b \in B$. Remembering that $x_n \in Q$ we obtain $\pi(x_n)b \in B$ and $\pi(x)b \in B$. Passing to the hermitian conjugate operators we obtain $b\pi(x) \in B$. It shows that $\pi(x) \in M(B)$ and $x \in Q$. It shows that Q is a strictly closed subset of $M(A)$.

We assumed (cf the first line of this proof) that $(\mathrm{id} \otimes \pi)T \,\eta\, C \otimes B$. Therefore $(\mathrm{id} \otimes \pi)z_T \in M(C \otimes B)$, $\pi(\chi \otimes \mathrm{id})z_T \in M(B)$ and $(\chi \otimes \mathrm{id})z_T \in Q$ for any continuous linear functional χ defined on C. It shows that for any $\varphi \in \mathrm{Rep}\,(A, H)$, $(\mathrm{id} \otimes \varphi)z_T$ is determined by $\varphi|_Q$. So is $(\mathrm{id} \otimes \varphi)T$. Taking into account Assumption I we see that Q separates representations of A. By virtue of Proposition 2.2 we have $A \subset Q$. Therefore $\pi(A)B \subset B$. Combining this result with (4. 7) we get $\overline{\pi(A)B} = B$ and (4. 4) holds.

Q.E.D.

In concrete applications of Theorem 4.2 we have to show that an element $r \in M(F \otimes A)$ satisfies the assumption II.1. To this end we shall use the following obvious

Remark 4.3 For any C*-algebra F, the set of all elements $r \in M(F \otimes A)$ satisfying the condition 1 of the assumption II of Theorem 4.2 will be denoted by $\mathcal{R}_F$. Then

1. $(I + T^*T)^{-1/2}$, $(I + TT^*)^{-1/2} \in \mathcal{R}_C$.

2. If r is an invertible element of the smallest unital C*-subalgebra of $M(C \otimes A)$ containing the z-transform of T, then $r \in \mathcal{R}_C$.

3. If $r \in \mathcal{R}_F$ and $\Phi \in \mathrm{Mor}(F, F_1)$ (where F_1 is another C*-algebra) then $(\Phi \otimes \mathrm{id})r \in \mathcal{R}_{F_1}$.

4. If $r \in \mathcal{R}_F$ and I_{F_1} is the unit of $M(F_1)$ then $I_{F_1} \otimes r \in \mathcal{R}_{F_1 \otimes F}$.

5. If $r \in \mathcal{R}_{M_N \otimes F}$ and $r \geq 0$, then all diagonal matrix elements of r belong to $\mathcal{R}_F$.

6. If $\begin{pmatrix} p & , q \\ q^* & , s \end{pmatrix} \in \mathcal{R}_{M_2 \otimes F}$ and $\begin{pmatrix} p & , q \\ q^* & , s \end{pmatrix} \geq 0$ then $p - qs^{-1}q^* \in \mathcal{R}_F$. This point follows easily from Proposition 1.2

7. $\mathcal{R}_F$ is closed under multiplication: $r_1 r_2 \in \mathcal{R}_F$ for any $r_1, r_2 \in \mathcal{R}_F$.

8. If r is an invertible element of the smallest unital C*-subalgebra of $M(F \otimes A)$ containing $F \otimes I_A$ and $\mathcal{R}_F$, then $r \in \mathcal{R}_F$.

Remark 4.4 In the simplest case $F = \mathbf{C}$ the condition 2 of assumption II of Theorem 4.2 means that $r \in A$. Therefore the assumption II is satisfied if the intersection $\mathcal{R}_{\mathbf{C}} \cap A$ is not empty.

Examples:

1. Let A be a C*-algebra, $T \,\eta\, M_N \otimes A$ and Γ be the set of diagonal matrix elements of $(I + T^*T)^{-1}$ and $(I + TT^*)^{-1}$. Assume that T separates representations of A and that there exist elements $r_1, r_2, \ldots, r_k \in \Gamma$ such that the product $r = r_1 r_2 \ldots r_k \in A$. Then according to Remark 4.3 points 1,5 and 7, $r \in \mathcal{R}_{\mathbf{C}}$, the intersection $\mathcal{R}_{\mathbf{C}} \cap A$ is not empty and Theorem 4.2 shows that A is generated by T. The reader should notice that this result contains Theorem 3.3.

2. Let A be a unital C*-algebra and $T \in M_N \otimes A$. Assume that A coincides with the norm closure of the set of all algebraic combinations of I and matrix elements of T. Then one can immediately verify that the implication (4. 1) holds. In other words, A is generated by T in the sense of Definition 4.1.

Conversely if A is a C^*-algebra generated by $T \,\eta\, M_N \otimes A$ such that $\|T\| < \infty$ then A is unital, $T \in M_N \otimes A$ and A coincides with the norm closure of the set of all algebraic combinations of I and matrix elements of T. The proof uses Proposition 4.5 formulated at the end of this Section. Details are left to the reader.

The above statements remain valid if M_N is replaced by any finite-dimensional C*-algebra C: any such algebra is a subalgebra of M_N. The finite-dimensionality of C is essential, more generally C must be of 'discret type' (cf Example 8).

3. Let Λ be a locally compact space, C be a C*-algebra and $T : \Lambda \to C^\eta$ (where C^η is endowed with the topology of almost uniform convergence) be a continuous mapping. Then (cf (2. 6)) $T \,\eta\, C \otimes C_\infty(\Lambda)$. We claim that $C_\infty(\Lambda)$ is generated by T if and only if $T(\Lambda)$ is a closed subset of C^η and $T : \Lambda \to T(\Lambda)$ is a homeomorphism (cf Theorem 6.2).

4. Let A_1 be the algebra of all continuous functions vanishing at infinity on the quantum $E(2)$-group, v, n be distinguished elements affiliated with A_1 (cf Example 5 in the previous Section) and $C = M_2$. Then

$$T = \begin{pmatrix} v \, , \, n \\ 0 \, , \, v^* \end{pmatrix}$$

is the fundamental representation of $E_\mu(2)$. Using [23, Theorem 6.1] and [20, Example 1, page 412] one can easily show that $T = \begin{pmatrix} 0 \, , \, 1 \\ 0 \, , \, 0 \end{pmatrix} \otimes n + \begin{pmatrix} v \, , \, 0 \\ 0 \, , \, v^* \end{pmatrix}$ is affiliated with $C \otimes A_1$. Clearly T separates representations of A_1. Moreover, by elementary computations

$$(I + T^*T)^{-1} = \begin{pmatrix} \dfrac{2I + \mu^{-2}n^*n}{4I + \mu^{-2}n^*n} \, , & -v^* \dfrac{n}{4I + n^*n} \\ -\dfrac{n^*}{4I + n^*n} v \, , & \dfrac{2I}{4I + n^*n} \end{pmatrix}.$$

The right lower corner (diagonal) element of this matrix belongs to A_1 (cf the definition of $A_1 = A$ given in [21, Section 1]). Taking into account Example 1 we conclude that A_1 is generated by T.

5. Let A_2 be the algebra of all continuous functions vanishing at infinity on the Pontryagin dual $\widehat{E_\mu(2)}$ of $E_\mu(2)$, N, b be distinguished elements affiliated with A_2 (cf Example 6 in the previous Section) and $C = M_2$. Then

$$T = \begin{pmatrix} \mu^{N/2} \, , & 0 \\ b \, , & \mu^{-N/2} \end{pmatrix}$$

is the fundamental representation of $\widehat{E_\mu(2)}$. We shall prove that

$$T \, \eta \, C \otimes A_2 \tag{4. 8}$$

and that

$$A_2 \text{ is generated by } T. \tag{4. 9}$$

For any real numbers $x > 0$ and $y \geq 0$, we denote by $z_{11}(x, y)$, $z_{12}(x, y)$, $z_{21}(x, y)$ and $z_{22}(x, y)$ the matrix elements of the z-transform of the matrix $\begin{pmatrix} x \, , & 0 \\ y \, , & (\mu x)^{-1} \end{pmatrix}$. Clearly $z_{ij}(x, y)$ $(i, j = 1, 2)$ are bounded continuous functions of (x, y). The z-transform of a diagonal matrix is diagonal. Therefore

$$z_{12}(x, 0) = z_{21}(x, 0) = 0 \tag{4. 10}$$

for any $y \in]0, \infty[$. With the notation introduced above

$$z_T = \begin{pmatrix} z_{11}(\mu^{\frac{N}{2}}, |b|) & , \; (\mathrm{Phase}\, b)^* z_{12}(\mu^{\frac{N}{2}-1}, |b|) \\ (\mathrm{Phase}\, b) z_{21}(\mu^{\frac{N}{2}}, |b|) \; , & z_{22}(\mu^{\frac{N}{2}-1}, |b|) \end{pmatrix}. \tag{4. 11}$$

The reader should notice that the diagonal matrix elements of z_T belong to $M(A_2)$. Due to (4. 10) the same holds for off diagonal elements (cf the definition of $A_2 = B$ given in [21, Section 3]). Therefore $z_T \in M(C \otimes A_2)$. Let $y = (I + T^*T)^{-1} = I - z_T^* z_T$. By simple (but rather boring) computations

$$y = \begin{pmatrix} \dfrac{I + \mu^{-N-2I}}{I + \mu^{-2I} + \mu^{N} + \mu^{-N-2I} + b^*b} \;, & b^* \dfrac{-\mu^{-\frac{N}{2}}}{I + \mu^{-2I} + \mu^{N-2I} + \mu^{-N} + b^*b} \\ \\ b \dfrac{-\mu^{-\frac{N}{2}-I}}{I + \mu^{-2I} + \mu^{N} + \mu^{-N-2I} + b^*b} \;, & \dfrac{I + \mu^{N-2I} + b^*b}{I + \mu^{-2I} + \mu^{N-2I} + \mu^{-N} + b^*b} \end{pmatrix} \tag{4. 12}$$

and

$$y \begin{pmatrix} \mu^{\frac{N}{2}} & , \; 0 \\ \\ b \dfrac{\mu^{-1}}{I + \mu^{N} + b^*b} & , \; I \end{pmatrix} = m, \tag{4. 13}$$

where m is an element of $M(C \otimes A_2)$ given by

$$m = \begin{pmatrix} \dfrac{\mu^{\frac{N}{2}}}{I + \mu^{N} + b^*b} \;, & -b^* \dfrac{\mu^{-\frac{N}{2}}}{I + \mu^{-2I} + \mu^{N-2I} + \mu^{-N} + b^*b} \\ \\ 0 \;, & \dfrac{I + \mu^{N-2I} + b^*b}{I + \mu^{-2I} + \mu^{N-2I} + \mu^{-N} + b^*b} \end{pmatrix}.$$

The matrix m is triangular and its diagonal elements are strictly positive on $\mathrm{Sp}\, A_2$. Therefore $m(C \otimes A_2)$ is dense in $C \otimes A_2$ and by virtue of (4. 13), $y(C \otimes A_2)$ is dense in $C \otimes A_2$. It shows that $y > 0$, $z_T^* z_T < I$ on $\mathrm{Sp}\,(C \otimes A_2)$ and (4. 8) follows.

To prove (4. 9) it is sufficient to notice that the product of diagonal elements of (4. 12) belongs to A_2 (cf Example 1).

6. Let A_d be the algebra of all continuous functions vanishing at infinity on the Pontryagin dual $\widehat{S_\mu U(2)}$ of $S_\mu U(2)$, a, n be distinguished elements affiliated with A_d (cf Example 9 in the previous Section) and $C = M_2$. Then

$$T = \begin{pmatrix} a \;, & n \\ 0 \;, & a^{-1} \end{pmatrix}$$

is the fundamental representation of $\widehat{S_\mu U(2)}$. Remembering that A_d is a direct sum of unital C* algebras one can easily show that $T \,\eta\, C \otimes A_d$. Clearly T separates the representations of A_d. By simple (but rather boring) computations we get

$$(I + T^*T)^{-1} = \begin{pmatrix} I - \dfrac{\mu^2 I + a^2}{M} \;, & -\dfrac{an}{M} \\ -\dfrac{n^*a}{M} \;, & \dfrac{I + \mu^2 a^2}{M} \end{pmatrix}, \tag{4. 14}$$

where $M = (1 + \mu^2)I + \mu^2 a^2 + a^{-2} + n^* n$ is a central element affiliated with A_d (one can easily verify that M commutes with all elements of A_d). Denoting by p, q, q^* and s the matrix elements of (4. 14) we obtain

$$p - qs^{-1}q^* = (I + a^2)^{-1}.$$

Therefore $\frac{I+\mu^2 a^2}{M}$, $(I + a^2)^{-1} \in \mathcal{R}_{\mathbf{C}}$ and $\frac{I+\mu^2 a^2}{M(I+a^2)} \in \mathcal{R}_{\mathbf{C}}$ (cf Remark 4.3 points 1,5,6 and 7). On the other hand $\frac{I+\mu^2 a^2}{M(I+a^2)} \leq M^{-1}$ and $M^{-1} \in A_d$. Therefore $\frac{I+\mu^2 a^2}{M(I+a^2)} \in A_d$ and $\mathcal{R}_{\mathbf{C}} \cap A_d$ is not empty. Combining Theorem 4.2 and Remark 4.4 we see that A_d is generated by T.

7. Let A be the algebra of all continuous functions vanishing at infinity on the quantum Lorentz group G introduced in [11] and

$$T = \begin{pmatrix} \alpha \ , \ \beta \\ \gamma \ , \ \delta \end{pmatrix}$$

be the fundamental two-dimensional (spinor) representation of G. According to [11, Theorem 5.4.4] T separates representations of A. We know (cf Example 10 in the previous Section) that $A = A_c \otimes A_d$. Since A_c is unital, A_d is in a natural way a subalgebra of A. By virtue of the quantum Iwasawa decomposition, formula (4. 14) holds in the present case and repeating the arguments presented in the previous Example we see that A is generated by T.

8. Let G be a commutative, locally compact group, $\widehat{G}$ be the Pontryagin dual of G and $T(\hat{g}, g)$ be the value of the character $\hat{g} \in \widehat{G}$ at the point $g \in G$. Then T is a unitary element affiliated with $C \otimes A$, where $C = C_\infty(\widehat{G})$ and $A = C_\infty(G)$. We call it the universal bicharacter associated with the pair $(\widehat{G}, G)$.

We shall use Theorem 4.2. It is known that characters separate points of G. Therefore T separates representations of A and assumption I holds. To verify assumption II we set $F = C$, $r = T$ and $\omega = ch$, where h is the Haar measure on $\widehat{G}$ and c is a non-negative element of C such that $h(c) < \infty$. Inserting these data to condition 1 we get an obvious tautology (due to the unitarity of $r = T$, $[(\mathrm{id} \otimes \pi)r](C \otimes B) = C \otimes B$ for any B such that $(\mathrm{id} \otimes \pi)r \in M(C \otimes B)$; the latter is equivalent to $(\mathrm{id} \otimes \pi)T \,\eta\, C \otimes B$). The left hand side of (4. 3) is the Fourier transform of $fc \in L^1(\widehat{G})$. By the famous Lebesgue-Riemann lemma, the Fourier transform of any L^1-function is continuous and vanishes at infinity. Therefore the relation (4. 3) is fulfilled. This way we showed that A is generated by T.

This example shows that a non-unital C*-algebra A may be generated by a bounded element $T \,\eta\, C \otimes A$. In such a case $\dim C = \infty$ (more precisely C must be of 'continuous type').

The group G may be replaced by any topological (locally compact) quantum group for which the Lebesgue-Riemann lemma holds. The class of these groups includes all locally compact topological groups, the compact quantum groups (in this case A is unital and (4. 3) is obviously fulfilled), duals of compact quantum groups (cf the theory of Fourier transform presented in [11, Section 2]), quantum $E(2)$ group and its Pontryagin dual (cf next point).

9. With the notation used in Examples 4 and 5 (and Examples 5, 6 and 7 of Section 3), the universal bicharacter associated with the pair $(\widehat{E(2)}, E(2))$ is given by the formula (cf [21, formula (25)]):

$$T = F_\mu(\mu^{N/2} b \otimes vn)(I \otimes v)^{N\otimes I}. \tag{4. 15}$$

Let ω_1 and ω_2 be states on A_1 and A_2 respectively, introduced by the formulae:

$$\omega_1(v^k f(n)) = \begin{cases} 0 & \text{for} \quad k \neq 0, \\ \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(e^{i\theta}) d\theta & \text{for} \quad k = 0, \end{cases}$$

$$\omega_2((\mathrm{Phase}\, b)^k g(N, |b|)) = \begin{cases} 0 & \text{for} \quad k \neq 0, \\ g(0,1) & \text{for} \quad k = 0. \end{cases}$$

Then for any $a_1 \in A_1$ and $a_2 \in A_2$:

$$\begin{aligned} (\omega_2 \otimes \mathrm{id})\{T(a_2 \otimes I)\} &\in A_1 \\ (\mathrm{id} \otimes \omega_1)\{T(I \otimes a_1)\} &\in A_2 \end{aligned}$$

In other words the Riemann-Lebesgue lemma holds for $E(2)$ and $\widehat{E(2)}$.

10. Let C and A be C*-algebras, T be an invertible element of $M(C \otimes A)$, H be a Hilbert space and $\omega_\phi(x) = (\phi|x\phi)$ for any $\phi \in H$ and $x \in B(H)$. Then ω_ϕ is a positive linear functional on any C*-algebra belonging to $C^*(H)$. Assume that $C \in C^*(H)$, $(\omega_\phi \otimes \mathrm{id})T \in A$ for any $\phi \in H$ and that A is the smallest C*-algebra containing $(\omega_\phi \otimes \mathrm{id})T$ for all $\phi \in H$. Inserting in Theorem 4.2 $F = C$, $r = T$ and $\omega = \omega_\phi$ (where ϕ is a fixed non-zero vector of H) one can easily verify the assumptions I and II. Therefore A is generated by T.

In particular using the terminology introduced in [4], if V is a regular multiplicative unitary operator, then (cf [4, Proposition 3.6]) $V \in M(\hat{S} \otimes S)$ (where S and $\hat{S}$ are the reduced algebra and dual reduced algebra of V) and S is generated by V.

There exists the close relation with the examples considered in Point 8. Universal bicharacters become multiplicative unitary operators, if the function algebras of G and $\hat{G}$ are represented on the same Hilbert space in the suitable way (cf [12, Section 1b]). In most cases we obtain regular multiplicative unitary operators. It is known however that (4. 15) leads to a non regular multiplicative unitary operator [1, 2].

We end this Section with the following (free of Hilbert space) version of Definition 4.1:

Proposition 4.5 *Let A, C, D, D' be C^*-algebras, $T \,\eta\, C \otimes A$, $S' \,\eta\, C \otimes D'$, $\pi \in \mathrm{Mor}(A, D)$ and $j \in \mathrm{Mor}(D', D)$. Assume that A is generated by T, j is an injection and*

$$(\mathrm{id} \otimes \pi)T = (\mathrm{id} \otimes j)S'. \tag{4. 16}$$

Then there exists $\pi' \in \mathrm{Mor}(A, D')$ such that

$$\pi = j\pi'. \tag{4. 17}$$

Proof: We may assume that $D \in C^*(H)$. Then $j(D') \in C^*(H)$, $\pi \in \mathrm{Rep}\,(A, H)$ and

$$(\mathrm{id} \otimes \pi)T = (\mathrm{id} \otimes j)S' \,\eta\, C \otimes j(D').$$

By virtue of Definition 4.1, $\pi \in \mathrm{Mor}(A, j(D'))$. We assumed that j is an injection. Let $j^{-1} \in \mathrm{Mor}(j(D'), D')$ be the inverse map. Then $\pi' = j^{-1}\pi \in \mathrm{Mor}(A, D')$ and (4. 17) follows.

Q.E.D.

We shall use the above Proposition in the following context:

Let $D = C_\infty(\mathbf{N}) \otimes CB(H)$ and $D' = C(\Lambda) \otimes CB(H)$, where $\mathbf{N}$ is the set of natural numbers (with discrete topology), $\Lambda = \mathbf{N} \cup \{\infty\}$ is the one point compactification of $\mathbf{N}$ and H is a separable, infinite dimensional Hilbert space. Denote by $j \in \mathrm{Mor}(D', D)$ the restriction map. Clearly j is injective ($\mathbf{N}$ is dense in Λ). Let $(\pi(n))_{n=1.2,\dots}$ be a sequence of elements of $\mathrm{Rep}\,(A, H)$. According to (2. 8) the sequence $(\pi(n))_{n=1.2,\dots}$ is determined by a morphism $\pi \in \mathrm{Mor}(A, D)$. If the sequence $((\mathrm{id} \otimes \pi(n))T)_{n=1.2,\dots}$ is converging, then setting

$$S'(n) = \begin{cases} (\mathrm{id} \otimes \pi(n))T & \text{for} \quad n = 1, 2, \dots \\ \lim(\mathrm{id} \otimes \pi(n))T & \text{for} \quad n = \infty \end{cases}$$

and using (2. 6) we introduce $S' \,\eta\, C \otimes D'$ satisfying formula (4. 16). On the other hand formula (4. 17) means that the sequence $(\pi(n))_{n=1.2,\dots}$ is converging. This way we get

Corollary 4.6 *Let A, C be C^*-algebras, $T \,\eta\, C \otimes A$, H be a Hilbert space and $\pi(n) \in \mathrm{Rep}\,(A, H)$ for $n = 1, 2, \dots$. Assume that A is generated by T. Then*

$$\left(\begin{array}{c}\text{Sequence } ((\mathrm{id} \otimes \pi(n))T)_{n=1,2,\dots} \\ \text{is converging in } (C \otimes CB(H))^\eta\end{array}\right) \Longrightarrow \left(\begin{array}{c}\text{Sequence } (\pi(n))_{n=1,2,\dots} \text{ is} \\ \text{converging in } \mathrm{Rep}\,(A, H)\end{array}\right).$$

5 Topological W*-categories

This Section is devoted to the duality in the C*-algebra theory in the spirit of [13, 9, 17]. An object dual to a C*-algebra A is by definition the W*-category $\mathrm{Rep}\,(A, H)$. Then elements affiliated with A may be identified with continuous operator functions on the dual object.

The notion of a topological W*-category is introduced in [17]. We refer to this paper for the more complete outlook of the theory. In this Section we present a simplified approach. Firstly, we fix an infinite-dimensional separable Hilbert space H; all considered objects will be related to H. In particular working with subobjects (direct sums of objects respectively) we have to use unitary operators acting from H onto a subspace of H (a direct sum of a number of copies of H respectively). Secondly, isometries will be the only morphisms that we shall consider. Thirdly we shall deal exclusively with topological W*-categories of the form $\mathrm{Rep}\,(A, H)$ and $(C \otimes CB(H))^\eta$ where A and C

are C*-algebras. Therefore we need not to give general definitions; all the notions introduced in this Section will be directly related to $\mathrm{Rep}\,(A,H)$ or $(C\otimes CB(H))^\eta$. On the other hand this Section is selfconsistent, even more, it contains the proofs of some theorems anounced in [17] ([17] contains only definitions and statements, no proof is included).

We start with a short discussion of the adjoint action of isometries. Let V be an isometry acting on H: $V\in B(H)$ and $V^*V=I$. Then VV^* is an orthogonal projection and

$$D_V=\{x\in CB(H): xVV^*=VV^*x\}$$

is a C*-algebra: $D_V\in C^*(H)$. For any $x\in D_V$ we set

$$\mathrm{ad}_V(x)=V^*xV$$

Then $\mathrm{ad}_V\in\mathrm{Rep}\,(D_V,H)$.

Let C be a C*-algebra and $S\,\eta\,C\otimes CB(H)$. We say that S is compatible with V if $S\,\eta\,C\otimes D_V$. In such a case

$$(\mathrm{id}\otimes\mathrm{ad}_V)S\,\eta\,C\otimes CB(H).$$

To simplify the notation we shall write $\mathrm{ad}_V S$ instead of $(\mathrm{id}\otimes\mathrm{ad}_V)S$ (the trivial action on C is understood by itself). The reader should notice that $\mathrm{ad}_V S$ makes sense only if S is compatible with V. However if V is unitary, then $D_V=CB(H)$ and any S is compatible with V.

Let V be an isometry acting on H and $U_V=I-2VV^*$. Then U_V is unitary and for any $S\,\eta\,C\otimes CB(H)$ we have

$$\left(\begin{array}{c}S\text{ is compa-}\\ \text{tible with }V\end{array}\right)\Longleftrightarrow(\mathrm{ad}_{U_V}S=S)\,. \tag{5. 1}$$

Let Λ be a denumerable set,

$$V:\sum_{\lambda\in\Lambda}{}^{\oplus}H\longrightarrow H \tag{5. 2}$$

be a unitary operator (V acts from the direct sum of a number of copies of H onto H) and V_λ be the restriction of V to the λ^{th}-component of the direct sum. Then $(V_\lambda)_{\lambda\in\Lambda}$ is a family of isometries acting on H such that

$$\sum_{\lambda\in\Lambda}V_\lambda {V_\lambda}^*=I.$$

One can easily verify that for any family $(S_\lambda)_{\lambda\in\Lambda}$ of elements affiliated with $C\otimes CB(H)$ there exists unique $S\,\eta\,C\otimes CB(H)$ compatible with all V_λ such that $\mathrm{ad}_{V_\lambda}S=S_\lambda$ for all $\lambda\in\Lambda$. We say that S is a direct sum of the family $(S_\lambda)_{\lambda\in\Lambda}$:

$$S=\sum_{\lambda\in\Lambda}{}^{\oplus}S_\lambda.$$

Clearly the above decomposition implicitely refers to the unitary (5. 2).

Let R be a subset of $(C \otimes CB(H))^\eta$. We say that R is complete if it is closed under the adjoint action of isometries and the direct sums. More precisely, R is complete if

$$\left(\begin{array}{c} S_\lambda \in R \text{ for} \\ \text{all } \lambda \in \Lambda \end{array}\right) \Longrightarrow \left(\sum_{\lambda\in\Lambda}^{\oplus} S_\lambda \in R\right)$$

and

$$\left(\begin{array}{c} S \text{ is an element of } R \\ \text{compatible with an isometry} \\ V \text{ acting on } H \end{array}\right) \Longrightarrow \left(\ \mathrm{ad}_V S \in R\ \right)$$

In particular R is invariant under unitary transformations acting on H.

Let C and C' be C*-algebras, R and R' be complete subsets of $(C \otimes CB(H))^\eta$ and $(C' \otimes CB(H))^\eta$ respectively and $F : R \to R'$ be a mapping. We say that F commutes with the adjoint action of isometries if for any isometry $V \in B(H)$ and any $S \in R$ compatible with V, $F(S)$ is compatible with V and

$$F(\mathrm{ad}_V S) = \mathrm{ad}_V F(S).$$

Then F maps direct sums into direct sums.

The case $C' = \mathbf{C}$ is particularly interesting. For any complete $R \subset (C \otimes CB(H))^\eta$, the set of all mappings $R \to CB(H)^\eta$ commuting with the adjoint action of isometries will be denoted by $\mathcal{F}(R)$. Elements of $\mathcal{F}(R)$ are called operator functions. This is very important concept. It appeared in [13] under the name *admissible operator fields* and in [8] under the name *decomposable functions.* The name *operator functions* was used in [15, 9, 17]

An operator function $F \in \mathcal{F}(R)$ is called bounded if $F(S) \in B(H)$ for any $S \in R$. If F is bounded, then (cf [9])

$$\|F\| = \sup_{S\in R} \|F(S)\| < \infty. \tag{5. 3}$$

The set of all bounded operator functions on R will be denoted by $\mathcal{F}_{\text{bounded}}(R)$. $\mathcal{F}_{\text{bounded}}(R)$ carries a natural *-algebra structure: for any F, $F' \in \mathcal{F}_{\text{bounded}}(R)$, $t \in \mathbf{C}$ and $S \in R$ we set:

$$\begin{array}{rl} (tF)(S) & = tF(S) \\ (F+F')(S) & = F(S) + F'(S) \\ (FF')(S) & = F(S)F'(S) \\ F^*(S) & = F(S)^* \end{array}$$

Using the von Neumann double commutant theorem one can easily show that $\mathcal{F}_{\text{bounded}}(R)$ endowed with the algebraic operations introduced above and with the norm (5. 3) is a W*-algebra.

So much about $(C \otimes CB(H))^\eta$. We shall relate the same concepts with $\mathrm{Rep}\,(A,H)$. Let A be a C*-algebra, $\pi \in \mathrm{Rep}\,(A,H)$ and V be an isometry acting on H. We say that π is compatible with V if $\pi \in \mathrm{Mor}(A, D_V)$. In such a case

$$\mathrm{ad}_V \pi \in \mathrm{Rep}\,(A,H).$$

The reader should notice that $\mathrm{ad}_V\pi$ makes sense only if π is compatible with V. However if V is unitary, then $D_V = CB(H)$ and any π is compatible with V. We also have

$$\left(\begin{array}{c}\pi \text{ is compa-}\\ \text{tible with } V\end{array}\right) \Longleftrightarrow (\mathrm{ad}_{U_V}\pi = \pi)\,. \tag{5. 4}$$

Now repeating the definitions given above one can easily introduce the notion of direct sum of elements of $\mathrm{Rep}\,(A,H)$, the notion of complete subset of $\mathrm{Rep}\,(A,H)$, the notion of mapping commuting with the adjoint action of isometries (clearly one may also consider mappings acting between $\mathrm{Rep}\,(A,H)$ and complete subsets of $(C \otimes CB(H))^\eta$) and the notion of (bounded) operator function. In particular $\mathcal{F}(\mathrm{Rep}\,(A,H))$ will denote the set of all operator functions on $\mathrm{Rep}\,(A,H)$ and $\mathcal{F}_{\mathrm{bounded}}(\mathrm{Rep}\,(A,H))$ is the subset of bounded functions.

Let A, B and C be C*-algebras, $T\,\eta\, C \otimes A$ and $\varphi \in \mathrm{Mor}(A,B)$. One can easily verify that the mappings

$$\begin{array}{l}\mathrm{Rep}\,(A,H) \ni \pi \longmapsto (\mathrm{id}\otimes\pi)T \in (C \otimes CB(H))^\eta\\ \mathrm{Rep}\,(B,H) \ni \pi \longmapsto \pi\varphi \in \mathrm{Rep}\,(A,H)\end{array}$$

commute with the adjoint action of isometries.

We shall use the following very nontrivial result:

Theorem 5.1 *Let A be a separable C^*-algebra and R be a closed complete subset of* $\mathrm{Rep}\,(A,H)$*. Assume that R contains a faithful representation. Then $R = \mathrm{Rep}\,(A,H)$.*

Proof: Let $\rho \in R$ be a faithful representation and ρ_∞ be a direct sum of countable number of copies of ρ. Clearly $\rho_\infty \in R$.

For any $\pi \in \mathrm{Rep}\,(A,H)$, both representations ρ_∞ and $\rho_\infty \oplus \pi$ are infinite in the following sense: zero is the only element of A that is represented by a compact operator. Using the Voiculescu theorem ([14, Theorem 1.5]) we see[3] that there exists a sequence of unitaries $(U_n)_{n=1,2,\ldots}$ such that

$$\rho_\infty \oplus \pi = \lim_{n\to\infty} \mathrm{ad}_{U_n}(\rho_\infty).$$

Therefore $\rho_\infty \oplus \pi \in R$ and consequently $\pi \in R$.

Q.E.D.

The reader easily obtain

Corollary 5.2 *If A is a separable C^*-algebra and R is a closed complete subset of* $\mathrm{Rep}\,(A,H)$*, then $R = \mathrm{Rep}\,(A/J,H)$, where $J = \bigcap_{\pi\in R} \ker\pi$.*

[3] if A is not unital then we apply the Voiculescu theorem to the one-dimensional unital extension of A

Let A be a C*-algebra, A'' be the von Neumann enveloping algebra of A and $x \in A''$. For any $\pi \in \mathrm{Rep}\,(A, H)$ we set

$$F_x(\pi) = \pi''(x), \tag{5. 5}$$

where π'' is the normal extension of π. Then F_x is a bounded operator function defined on $\mathrm{Rep}\,(A, H)$: $F_x \in \mathcal{F}_{\text{bounded}}(\mathrm{Rep}\,(A, H))$. Using the von Neumann double commutant theorem one can easily show that any bounded operator function is of this form. In short

$$A'' = \mathcal{F}_{\text{bounded}}(\mathrm{Rep}\,(A, H)). \tag{5. 6}$$

The reader should notice that with this identification the W*-algebra structures of A'' and $\mathcal{F}_{\text{bounded}}(\mathrm{Rep}\,(A, H))$ coincide.

For any C*-algebra A we set

$$QM(A) = \{x \in A'' : a_1 x a_2 \in A \text{ for any } a_1,\ a_2 \in A\}$$

$$LM(A) = \{x \in A'' : xa \in A \text{ for any } a \in A\}$$

$$RM(A) = \{x \in A'' : ax \in A \text{ for any } a \in A\}$$

Elements of $QM(A)$ ($LM(A)$ and $RM(A)$ respectively) are called quasimultipliers (left and right multipliers respectively). Clearly $QM(A)$ is a *-invariant vector subspace of A'', $LM(A)$ and $RM(A)$ are subalgebras, $LM(A)^* = RM(A)$, $LM(A)$ and $RM(A)$ are subsets of $QM(A)$ and $LM(A) \cap RM(A) = M(A)$. We shall use the following

Proposition 5.3 *Let A be a C^*-algebra. Then*

$$LM(A) = \{x \in QM(A) : x^* x \in QM(A)\}$$

Proof:

We have to show that

$$LM(A) \supset \{x \in QM(A) : x^* x \in QM(A)\}, \tag{5. 7}$$

the converse inclusion is trivial.

Let $c \in A$, $0 < c \leq I$ on $\mathrm{Sp}\, A$ and $e_n = c^{\frac{1}{n}}$ $(n = 1, 2, \ldots)$. Then $(e_n)_{n=1,2,\ldots}$ and $({e_n}^2)_{n=1,2,\ldots}$ are approximate units for A and both sequences are increasing. We denote by S the set of positive functionals on A of norm ≤ 1. Clearly S is a compact set (with respect to the weak topology).

Let x be an element of $QM(A)$ such that $x^* x \in QM(A)$. Then for any $a \in A$ the products $a^* x^* x a$, $e_n x a$, $a^* x^* e_n x a$ and $a^* x^* {e_n}^2 x a$ belong to A. We shall prove that

$$\operatorname*{norm\text{-}lim}_{n \longrightarrow \infty} a^* x^* e_n x a = a^* x^* x a. \tag{5. 8}$$

By virtue of the Dini theorem ([5, Theorem 7.2.2]) it is sufficient to show that for any $\omega \in S$ we have

$$\lim_{n \to \infty} \omega(a^* x^* e_n x a) = \omega(a^* x^* x a). \tag{5. 9}$$

By the GNS construction one can find $\pi \in \mathrm{Rep}\,(A, H)$ and $\Omega \in H$ such that $\omega(a') = (\Omega|\pi(a')\Omega)$ for any $a' \in A$. One can easily show that $\pi(e_n)$ converges weakly to I. Therefore

$$\begin{aligned}\omega(a^*x^*e_nxa) &= (\Omega|\pi(a^*x^*e_nxa)\Omega)\\ &= (\pi''(xa)\Omega|\pi(e_n)\pi''(xa)\Omega) \longrightarrow (\pi''(xa)\Omega|\pi''(xa)\Omega)\\ &\qquad = (\Omega|\pi(a^*x^*xa)\Omega) = \omega(a^*x^*xa)\end{aligned}$$

and (5. 9) follows. The proof of (5. 8) is complete. Inserting in (5. 8) ${e_n}^2$ instead of e_n we get

$$\text{norm - lim}\, a^*x^*{e_n}^2xa = a^*x^*xa. \tag{5. 10}$$

Now, using (5. 8) and (5. 10) one can easily verify that

$$\text{norm - lim}\,(e_nxa - xa)^*(e_nxa - xa) = 0.$$

Therefore

$$\text{norm - lim}\, e_nxa = xa.$$

and $xa \in A$. This relation holds for any $a \in A$. Therefore $x \in LM(A)$ and (5. 7) follows.

Q.E.D.

Operator functions related (via formula (5. 5)) to elements of different classes of multipliers have different continuity property (cf [17, Theorem 7]):

Theorem 5.4 *Let A be a C^*-algebra and $R = \mathrm{Rep}\,(A, H)$. Then using the identification (5. 6) we have:*

$$QM(A) = \left\{ F \in \mathcal{F}_{\text{bounded}}(R) : \begin{array}{c}\text{For any } \phi,\ \phi' \in H, \text{ the map}\\ R \ni \pi \longmapsto (\phi|F(\pi)\phi') \in \mathbf{C}\\ \text{is continuous}\end{array} \right\}, \tag{5. 11}$$

$$LM(A) = \left\{ F \in \mathcal{F}_{\text{bounded}}(R) : \begin{array}{c}\text{For any } \phi \in H, \text{ the map}\\ R \ni \pi \longmapsto F(\pi)\phi \in H\\ \text{is continuous}\end{array} \right\}, \tag{5. 12}$$

$$RM(A) = \left\{ F \in \mathcal{F}_{\text{bounded}}(R) : \begin{array}{c}\text{For any } \phi \in H, \text{ the map}\\ R \ni \pi \longmapsto F(\pi)^*\phi \in H\\ \text{is continuous}\end{array} \right\}, \tag{5. 13}$$

$$M(A) = C_{\text{bounded}}(R) = \left\{ F \in \mathcal{F}_{\text{bounded}}(R) : \begin{array}{c}F : R \longrightarrow B(H)\\ \text{is a continuous map}\end{array} \right\}, \tag{5. 14}$$

$$A^\eta = C(R) = \left\{ F \in \mathcal{F}(R) : \begin{array}{c}F : R \longrightarrow CB(H)^\eta\\ \text{is a continuous map}\end{array} \right\}, \tag{5. 15}$$

where $B(H) = M(CB(H))$ in (5. 14) is endowed with the strict topology (the one of *-strong convergence in $B(H)$) and $CB(H)^\eta$ in (5. 15) is endowed with the topology of almost uniform convergence.

Proof: One can easily show that for any $B(H)$-valued function F on any topological space R we have the equivalence:

$$\left(\begin{array}{c} F \text{ is strongly} \\ \text{continuous} \end{array}\right) \Longleftrightarrow \left(\begin{array}{c} F \text{ and } F^*F \text{ are} \\ \text{weakly continuous} \end{array}\right)$$

Using this simple observation and Proposition 5.3 we see that (5. 11) implies (5. 12). We know that $RM(A) = LM(A)^*$. Therefore (5. 13) is equivalent to (5. 12). (5. 14) follows immediately from (5. 12) and (5. 13). Finally using the obvious formula $z_{F(\pi)} = (z_F)(\pi)$ we see that (5. 14) implies (5. 15). Summing up, it is sufficient to prove (5. 11).

Let $x \in QM(A)$. Then for any a belonging to A, $a^*xa \in A$. Recalling the definition of the topology of $\mathrm{Rep}\,(A,H)$ we see that for any ϕ, $\phi' \in H$, $(\phi|\pi(a^*xa)\phi') = (\pi(a)\phi|\pi''(x)\pi(a)\phi')$ depends continuously on $\pi \in \mathrm{Rep}\,(A,H)$. We know that the uniform limit of a sequence of continuous functions is a continuous function. Setting $a = e_n$ (where $(e_n)_{n=1,2,\ldots}$ is an approximate unit) and letting $n \to \infty$ we see that $(\phi|F_x\phi') = (\phi|\pi''(x)\phi')$ is continuous with respect to π. It shows that the left hand side of (5. 11) is contained in the right one. To prove the converse inclusion we shall use the following

Proposition 5.5 *Let A be a C^*-algebra, $(e_n)_{n=1,2,\ldots}$ be an approximate unit for a C^*-algebra A and $x \in A''$. Assume that the operator function F_x (related to x via formula (5. 5)) is weakly continuous and that*

$$\left.\begin{array}{c} \|e_nx - x\| \longrightarrow 0 \\ \|xe_n - x\| \longrightarrow 0 \end{array}\right\} \tag{5. 16}$$

for $n \to \infty$. Then $x \in A$.

Let $x \in A''$. Assume that F_x is weakly continuous. Then for any a_1, $a_2 \in A$ the product $x' = a_1xa_2$ satisfies the assumption of Proposition 5.5. Therefore $a_1xa_2 \in A$ and $x \in QM(A)$. This way we showed that the left hand side of (5. 11) contains the right one.

Q.E.D.

Proof of Proposition 5.5: We may assume that $A \in C^*(H)$. Let B be the smallest C*-algebra containing A and x. Due to (5. 16), $(e_n)_{n=1,2,\ldots}$ is an approximate unit for B. Therefore the embedding $A \hookrightarrow B$ belongs to $\mathrm{Mor}(A,B)$ and for any $\pi \in \mathrm{Rep}\,(B,H)$ the restriction $\pi|_A \in \mathrm{Rep}\,(A,H)$. We claim that

$$\pi(x) = F_x(\pi|_A). \tag{5. 17}$$

Indeed one can easily verify that the set of all $\pi \in \mathrm{Rep}\,(B,H)$ satisfying the above relation is complete and (we assumed that F_x is weakly continuous) closed. Moreover this set contains i_B (the right hand side of (5. 17) equals to $(\pi|_A)''(x)$). Using Theorem 5.1 we see that this set coincides with $\mathrm{Rep}\,(B,H)$. In other words (5. 17) holds for all $\pi \in \mathrm{Rep}\,(B,H)$.

Now we repeat the argument used in the proof of Proposition 2.2. Assume for the moment that $x \notin A$. Then there exists a continuous linear functional ω on B such that

$\omega|_A = 0$ and $\omega(x) \neq 0$. By the GNS construction one can find $\pi \in \mathrm{Rep}\,(B, H)$ and a trace-class operator $\rho \in B(H)$ such that

$$\omega(b) = \mathrm{Tr}\,\rho\pi(b)$$

for all $b \in B$. In particular $\mathrm{Tr}\,\rho\pi(a) = 0$ for all $a \in A$ and $\mathrm{Tr}\,\rho\pi(x) \neq 0$. It shows that $\pi(x)$ does not belong to the weak closure of $\pi(A)$. Using the von Neumann double commutant theorem one can find unitary $U \in \pi(A)'$ such that $[U, \pi(x)] \neq 0$. Let $\pi' = \mathrm{ad}_U \pi$. Then $\pi'|_A = \pi|_A$ and $\pi'(x) \neq \pi(x)$ and we get a contradiction with (5. 17). It shows that $x \in A$.

Q.E.D.

Theorem 5.6 *Let A and B be C^*-algebras and*

$$f : \mathrm{Rep}\,(B, H) \longrightarrow \mathrm{Rep}\,(A, H) \tag{5. 18}$$

be a continuous mapping commuting with the adjoint action of isometries. Then there exists unique $\varphi \in \mathrm{Mor}(A, B)$ such that

$$f(\pi) = \pi\varphi$$

for any $\pi \in \mathrm{Rep}\,(B, H)$.

Proof: Let $a \in A$. Then the mapping

$$\mathrm{Rep}\,(B, H) \ni \pi \longmapsto (f(\pi))(a) \in B(H)$$

is a bounded continuous operator function defined on $\mathrm{Rep}\,(B, H)$. By virtue of (5. 14), there exists unique element $\varphi(a) \in M(B)$ such that

$$(f(\pi))(a) = \pi(\varphi(a))$$

for all $\pi \in \mathrm{Rep}\,(B, H)$. Clearly the mapping $\varphi : A \to M(B)$, introduced in this way is a *-algebra homomorphism. We have to show that $\varphi \in \mathrm{Mor}(A, B)$. Assume that this is not the case. Then the closure of $\varphi(A)B$ is a proper right ideal in B and (cf [6, Theorem 2.9.5]) there exists a pure state ω on B such that

$$\omega(\varphi(a)) = 0$$

for all $a \in A$. By the GNS-construction there exists $\pi \in \mathrm{Rep}\,(B, H)$ and $\Omega \in H$ such that

$$\omega(b) = (\Omega|\pi(b)\Omega)$$

for all $b \in B$. Combining the last three formulae we see that $(\Omega|(f(\pi))(a)\Omega) = 0$ for all $a \in A$. On the other hand, since $f(\pi)$ is a nondegenerate representation of A, $(f(\pi))(e_n)$ (where $(e_n)_{n=1,2,\ldots}$ is an approximate unit of A) tends strongly to I and $(\Omega|(f(\pi))(e_n)\Omega) > 0$ for sufficiently large n. The contradiction that we have got shows that $\varphi \in \mathrm{Mor}(A, B)$.

Q.E.D.

Theorem 5.6 shows that any C*-algebra A is determined by $\mathrm{Rep}\,(A,H)$ equipped with the topology and the adjoint action of isometries. The same conclusion follows from (5. 14) combined with Proposition 1.1. Another way of reconstructing A from $\mathrm{Rep}\,(A,H)$ is proposed in [17].

Let R be a complete closed subset of $\mathrm{Rep}\,(A,H)$ or $(C\otimes CB(H))^\eta$ and $C_{\mathrm{bounded}}(R)$ be the set of all continuous bounded operator functions defined on R. Then $C_{\mathrm{bounded}}(R)$ is a (non-separable) unital C*-algebra. For any $S\in R$ and any $F\in C_{\mathrm{bounded}}(R)$ we set

$$\pi_S(F)=F(S).$$

Clearly π_S is a representation of $C_{\mathrm{bounded}}(R)$.

Let ρ be a representation of $C_{\mathrm{bounded}}(R)$ acting on a (not necessarily separable) Hilbert space K. We say that ρ is singular if it is disjoint with π_S for all $S\in R$. Let

$$C_\infty(R)=\left\{F\in C_{\mathrm{bounded}}(R):\ \begin{array}{c}\rho(F)=0 \text{ for any sin-}\\ \text{gular representation } \rho\end{array}\right\}.$$

Clearly $C_\infty(R)$ is a closed ideal in $C_{\mathrm{bounded}}(R)$. In general $C_\infty(R)$ may be very small. However if R is *'locally compact'* [17] then representations π_S restricted to $C_\infty(R)$ are non-degenerate for all $S\in R$ and the mapping

$$R\ni S\longrightarrow \pi_S|_{C_\infty(R)}\in \mathrm{Rep}\,(C_\infty(R))$$

is a homeomorphism.

With this notation, for any separable C*-algebra A, $\mathrm{Rep}\,(A,H)$ is *'locally compact'* and

$$A=C_\infty(\mathrm{Rep}\,(A,H)). \tag{5. 19}$$

Indeed one can easily show that a representation ρ of $C_{\mathrm{bounded}}(\mathrm{Rep}\,(A,H))=M(A)$ is singular if and only if $\ker\rho\supset A$. On the other hand there exists a representation ρ of $M(A)$ such that $\ker\rho=A$ (it comes from a faithful representation of the Calkin algebra ${M(A)}/{A}$) and (5. 19) follows.

6 Topological characterization

This Section contains the main result of our paper. We shall use the following simple lemma:

Lemma 6.1 *Let X and Y be metric spaces and $f:X\longrightarrow Y$ be a continuous map. Assume that for any sequence $(x_n)_{n=1,2,\dots}$ of elements of X, we have:*

$$\left(\begin{array}{c}\text{Sequence } (f(x_n))_{n=1,2,\dots}\\ \text{is converging in } Y\end{array}\right)\Longrightarrow\left(\begin{array}{c}\text{Sequence } (x_n)_{n=1,2,\dots}\\ \text{is converging in } X\end{array}\right). \tag{6. 1}$$

Then $f(X)$ is a closed subset of Y and f is a homeomorphism of X onto $f(X)$.

Proof:

At first we notice that f is an injection. Indeed if $f(x) = f(x')$ for some $x, x' \in X$, then the sequence $(f(x), f(x'), f(x), f(x'), f(x), \ldots)$ is converging in Y. By virtue of (6. 1), the sequence $(x, x', x, x', x, \ldots)$ is converging in X and $x = x'$. The closedness of $f(X)$ and the continuity of the inverse follows immediately from (6. 1).

Q.E.D.

Theorem 6.2 *Let A, C be C^*-algebras, $T\,\eta\,C \otimes A$, H be a infinite-dimensional separable Hilbert space and*

$$R = \{(\mathrm{id} \otimes \pi)T : \pi \in \mathrm{Rep}\,(A, H)\}\,. \tag{6. 2}$$

Then the following two statements are equivalent:

I. A is generated by T.

II. R is a closed subset of $(C \otimes CB(H))^\eta$ and the mapping

$$\mathrm{Rep}\,(A, H) \ni \pi \longmapsto (\mathrm{id} \otimes \pi)T \in R \tag{6. 3}$$

is a homeomorphism (bijective, continuous with the continuous inverse).

Proof:

Assume that the statement II holds. The mapping (6. 3) commutes with the adjoint action of isometries. In particular it maps direct sums into direct sums and consequently any direct sum of elements of R belongs to R. If $S = (\mathrm{id} \otimes \pi)T$ is an element of R compatible with an isometry V, then (cf (5. 1)) $\mathrm{ad}_{U_V} S = S$ and $(\mathrm{id} \otimes \mathrm{ad}_{U_V}\pi)T = (\mathrm{id} \otimes \pi)T$. Using the injectivity of (6. 3) we see that $\mathrm{ad}_{U_V}\pi = \pi$. By virtue of (5. 4), π is compatible with V, $\mathrm{ad}_V\pi \in \mathrm{Rep}\,(A, H)$ and $\mathrm{ad}_V S = (\mathrm{id} \otimes \mathrm{ad}_V\pi)T \in R$. This way we showed that R is a complete subset of $(C \otimes CB(H))^\eta$.

Now, let $\pi \in \mathrm{Rep}\,(A, H)$ and $B \in C^*(H)$ such that $S = (\mathrm{id} \otimes \pi)T\,\eta\,C \otimes B$. We claim that the image of the mapping

$$\mathrm{Rep}\,(B, H) \ni \rho \longmapsto (\mathrm{id} \otimes \rho)S\,\eta\,C \otimes CB(H) \tag{6. 4}$$

is contained in R. Indeed, remembering that R is complete and closed one can easily verify that the set $\{\rho \in \mathrm{Rep}\,(B, H) : (\mathrm{id} \otimes \rho)S \in R\}$ is also complete and closed. Moreover this set contains i_B. Using Theorem 5.1 we see that this set coincides with $\mathrm{Rep}\,(B, H)$.

Composing (6. 4) with the inverse of (6. 3) we obtain a mapping $f : \mathrm{Rep}\,(B, H) \to \mathrm{Rep}\,(A, H)$ satisfying all the assumptions of Theorem 5.6. Therefore there exists unique $\varphi \in \mathrm{Mor}(A, B)$ such that

$$f(\rho) = \rho\varphi$$

for any $\rho \in \mathrm{Rep}\,(B, H)$. Clearly (6. 3) - image of π and (6. 4) - image of i_B coincide (both are equal to S). Therefore $f(i_B) = \pi$ and using the last formula we get $\pi = \varphi$ and finally $\pi \in \mathrm{Mor}(A, B)$. Taking into account Definition 4.1 we see that A is generated by T. This way we showed that Statement I follows from Statement II.

We shall prove the converse implication. Assume that A is generated by T. We have to show that R is closed and that (6. 3) is a homeomorphism. The continuity of (6. 3) follows immediately from Proposition 2.3. Combining Lemma 6.1 with Corollary 4.6 we get the desired statement

Q.E.D.

7 Presentations of C*-algebras

Let C be a C*-algebra and R be a complete closed subset of $(C \otimes CB(H))^\eta$. We say that R is manageable if there exists a separable C*-algebra A and $T \,\eta\, C \otimes A$ such that A is generated by T and R coincides with the set (6. 2). If this is the case, then R is called a presentation of the pair (A, T).

Let us notice that the pair (A, T) is determined uniquely (up to an isomorphism) by R. Indeed if A and A' are C*-algebras generated by elements $T \,\eta\, C \otimes A$ and $T' \,\eta\, C \otimes A'$ respectively and

$$\{(\mathrm{id} \otimes \pi)T : \pi \in \mathrm{Rep}\,(A, H)\} = \{(\mathrm{id} \otimes \pi')T' : \pi' \in \mathrm{Rep}\,(A', H)\},$$

then according to Theorem 6.2, the relation

$$(\mathrm{id} \otimes \pi)T = (\mathrm{id} \otimes \pi')T' \tag{7. 1}$$

defines $1-1$ correspondence between elements $\pi \in \mathrm{Rep}\,(A, H)$ and $\pi' \in \mathrm{Rep}\,(A', H)$. This correspondence is a homeomorphism and commutes with the adjoint action of isometries. By virtue of Theorem 5.6 there exists an isomorphism $\varphi \in \mathrm{Mor}(A, A')$ such that (7. 1) is equivalent to $\pi = \pi'\varphi$. Therefore $(\mathrm{id} \otimes \pi'\varphi)T = (\mathrm{id} \otimes \pi')T'$ and choosing π' to be faithful we see that $(\mathrm{id} \otimes \varphi)T = T'$.

Let us also notice that there exists a canonical choice of A: $A = C_\infty(R)$ (cf Theorem 6.2 and formula (5. 19)).

The examples presented below should make clear the relations between the concept of presentation known from the abstract algebra theory and the one used in this Section. In the abstract algebra theory a presentation is a list of generators and relations imposed on the generators. In our approach we use a C*-algebra C to organize the generators into a single object $T \,\eta\, C \otimes A$. The relations describe the desired properties of T. At this moment the algebra A is not fixed; the relations should be significant for any C*-algebra A. In particular one may consider the set R of all $T \,\eta\, C \otimes CB(H)$ satisfying the relations. The set R is a presentation in the sense of this Section.

Examples:

1. Let

$$R = \left\{ \begin{pmatrix} \alpha\ ,\ \beta \\ \gamma\ ,\ \delta \end{pmatrix} : \begin{array}{c} \alpha, \beta, \gamma, \delta \text{ are strongly commuting} \\ \text{normal operators acting on } H \text{ such that} \\ \alpha\delta - \beta\gamma = I \end{array} \right\}.$$

Then R is a manageable subset of $(M_2 \otimes CB(H))^\eta$ and $C_\infty(R)$ coincides with the algebra of all continuous functions vanishing at infinity on the classical $SL(2, \mathbf{C})$ (cf Example 3 of Section 4).

2. Let G be a connected, simply connected Lie group, c^i_{kl} $(i, k, l = 1.2. \ldots, N$:

$N = \dim G$) be the structure constants of the Lie algebra of G and

$$R = \left\{ (T_1, T_2, \ldots, T_N) : \begin{array}{c} T_1, T_2, \ldots, T_N \text{ are skew-adjoint} \\ \text{operators acting on } H \text{ such that} \\ \text{on a dense invariant domain} \\ [T_k, T_l] = \sum_i c^i_{kl} T_i \\ \text{and } \sum_i T_i^2 \text{ is essentially} \\ \text{selfadjoint on the same domain} \end{array} \right\}.$$

Then R is a manageable subset of $(\mathbf{C}^N \otimes CB(H))^\eta$ and $C_\infty(R)$ coincides with $C^*(G)$ (cf Example 3 of Section 3).

3. Let μ be a real number in the interval $]0,1[$ and

$$R = \left\{ u : \begin{array}{c} u \text{ is a unitary matrix of the form} \\ \begin{pmatrix} \alpha \ , & -\mu\gamma^* \\ \gamma \ , & \alpha^* \end{pmatrix}, \text{ where } \alpha, \gamma \in B(H) \end{array} \right\}.$$

Then R is a manageable subset of $(M_2 \otimes CB(H))^\eta$ and $C_\infty(R)$ coincides with the algebra A_c of all continuous functions on the quantum $SU(2)$ (cf Example 8 of Section 3 and Example 2 of Section 4).

4. Let

$$R = \left\{ (v, n) : \begin{array}{c} v, n \text{ are operators acting on } H, \\ n \text{ is normal, } \mathrm{Sp}\,(|n|) \subset \mu^{\mathbf{Z}} \cup \{0\}, \\ v \text{ is unitary and } vnv^* = \mu n \end{array} \right\}.$$

Then R is a manageable subset of $(\mathbf{C}^2 \otimes CB(H))^\eta$ and $C_\infty(R)$ coincides with the algebra A_1 of all continuous functions vanishing at infinity on the quantum $E(2)$ (cf Example 5 of Section 3).

5. Let

$$R = \left\{ (N, b) : \begin{array}{c} N, b \text{ are normal operators on } H, \\ N \text{ and } |b| \text{ strongly commute,} \\ \text{the joint spectrum of } (N, |b|) \text{ is} \\ \text{contained in the closure of the set} \\ \{(s, \mu^r) : s, r - \frac{s}{2} \in \mathbf{Z}\}, \\ N(\mathrm{Phase}\, b) = (\mathrm{Phase}\, b)(N + 2I) \end{array} \right\}.$$

Then R is a manageable subset of $(\mathbf{C}^2 \otimes CB(H))^\eta$ and $C_\infty(R)$ coincides with the algebra A_2 of all continuous functions vanishing at infinity on the Pontryagin dual $\widehat{E(2)}$ of quantum $E(2)$ (cf Example 6 of Section 3).

6. At this point we present a quantum version of Example 1. Let

$$R_0 = \left\{ \begin{pmatrix} \alpha \ , & \beta \\ \gamma \ , & \delta \end{pmatrix} : \begin{array}{c} \alpha, \beta, \gamma, \delta \text{ are bounded operators acting on } H \\ \text{satisfying the 17 relations (1.9) – (1.25) of [11]} \end{array} \right\}.$$

and R be the smallest complete subset of $(M_2 \otimes CB(H))^\eta$ containing R_0. Elements of R are direct sums of a (finite or denumerable) number of elements of R_0. Then R is manageable and $C_\infty(R)$ coincides with the algebra of all continuous functions vanishing at infinity on the quantum Lorentz group introduced in [11] (cf Example 7 of Section 4).

7. In this point G is a commutative, locally compact group, $C = C_\infty(G)$ and $\Delta \in \mathrm{Mor}(C, C \otimes C)$ is the corresponding comultiplication: $\Delta(c)(g, g') = c(gg')$ for any $c \in C$ and $g, g' \in G$. We shall use the leg numbering notation [11]. Let

$$R = \left\{ u \in M(C \otimes CB(H)) : \begin{array}{c} u \text{ is unitary and} \\ (\Delta \otimes \mathrm{id})u = u_{13}u_{23} \end{array} \right\}.$$

Then R is a manageable subset of $(C \otimes CB(H))^\eta$ and $C_\infty(R)$ coincides with the algebra of all continuous functions vanishing at infinity on the Pontryagin dual of G. (cf Example 8 of Section 4).

The group G may be replaced by a number of topological (locally compact) quantum groups. The class of these groups includes all locally compact topological groups (in this case $C_\infty(R)$ coincides with $C^*(G)$), the compact quantum groups, duals of compact quantum groups, quantum $E(2)$ group and its Pontryagin dual.

8. It is an open question, whether every complete closed subset R of $(C \otimes CB(H))^\eta$ is manageable. In particular, is the set $CB(H)^\eta$ of all closed operators acting on H manageable? We belive that the answer is negative: There is no C*-algebra A generated by a single element $T \,\eta\, A$ such that any closed operator on H is of the form $\pi(T)$, where $\pi \in \mathrm{Rep}\,(A, H)$. It seems that $C_\infty(CB(H)^\eta) = \{0\}$.

The reader should notice that the set $R = CB(H)^\eta$ considered in this example corresponds to the presentation (in the sense of abstract algebra theory) with the list of generators containing one item and the empty list of relations: one generator, no relations.

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