REPRESENTATIONS OF QUANTUM LORENTZ GROUP ON GELFAND SPACES *

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Abstract

A large class of representations of the quantum Lorentz group *QLG* (the one admitting Iwasawa decomposition) is found and described in detail. In a sense the class contains all irreducible unitary representations of *QLG*.

Parabolic subgroup P of the group *QLG* is introduced. It is a smooth deformation of the subgroup of $SL(2,\mathbf{C})$ consisting of the upper-triangular matrices. A description of the set of all 1-dimensional representations (the characters) of P is given. It turns out that the topological structure of this set is not the same as for the parabolic subgroup of the classical Lorentz group.

The class of (in general non-unitary) representations of *QLG* induced by characters of its parabolic subgroup P is investigated. Representations act on spaces of smooth sections of (quantum) line boundles over the homogeneous space $P \setminus QLG$ (Gelfand spaces) as in the classical case. For any pair of Gelfand spaces the set of all non-zero invariant bilinear forms is described. This set is not empty only for certain pairs. We give a complete list of such pairs. Using this list we solve the problems of equivalence and irreducibility of the representations. We distinguish a class of Gelfand spaces carrying unitary representations of *QLG*.

*Partially supported by Polish KBN grant No P301 020 07 and 2 PO3A 030 14

0 Introduction

The paper contains a systematic study of the representation theory of the quantum Lorentz group. The main results were announced in [12]. The classical $SL(2, \mathbf{C})$ group of unimodular matrices is a very interesting object of the theory of complex semisimple Lie groups. It is the simplest example of a noncommutative and noncompact group of this kind. On the other hand it appears as symmetry group of many important spaces. Among them we have the Riemann sphere (compactified complex plane), 3-dimensional Lobaczewski space and 4-dimensional vector Minkowski space. The last example is important for relativistic physics. For this reason $SL(2, \mathbf{C})$ is often called the Lorentz group. Due to these facts the study of the representation theory of $SL(2, \mathbf{C})$ is of great importance.

In the group representation theory one may distinguish two approaches, the local and the global one. In the local approach at first one studies the representations of infinitesimal operators (the Lie algebra representation theory). Next one investigates the problem of integrability using for example the Nelson theorem. In particular this method is very fruitful in the study of unitary representations. To illustrate the power of this approach in the context of the classical Lorentz group we mention the beautiful book of Naimark [8].

The global approach is based on the theory of induced representations. It uses homogeneous spaces, sections of vector boundles etc. The method is very effective also in the case when one deals with not necessarily unitary representations. In an excellent book of Gelfand, Graev and Vilenkin (cf.[3]) such a method was developed to describe and investigate a large class of representations of $SL(2, \mathbf{C})$.

The quantum Lorentz group considered in the paper is the quantum deformation of the Lorentz group described in [10]. It is obtained as the result of double group construction applied to quantum $SU(2)$ group corresponding to a fixed value of the deformation parameter $\mu = q \in]0, 1[$. The group will be denoted by *QLG*. For the convenience of the reader the basic facts concernig the $S_qU(2)$ group, its representation theory and Pontryagin dual $\widehat{S_qU(2)}$ group are collected in Appendix B. Appendix C contains the basic information concerning *QLG*.

As in the classical case one may study the representation theory of *QLG*. In [11] the local method corresponding to the one of Naimark was developed to describe unitary representations of *QLG*. As the result of this approach the complete classification of irreducible unitary representations of *QLG* was obtained. In some sense the results of [11] were suprising. The structure of the set of irreducible unitary representations of *QLG* turned out to be essentially different from that of the classical Lorentz group. In both cases we have principal and supplementary series of representations. Representations of principal series are labeled by discrete (minimal spin) and coninuous parameter. In the classical case the continuous parameter runs over $\mathbf{R}$, whereas for *QLG* it belongs to $S^1 = {}^{\mathbf{R}}/_{\frac{2\pi}{\log q}\mathbf{Z}}$. *QLG* has two (instead of one) supplementary series of representations. Moreover for *QLG* we have a new one-dimensional (nontrivial) unitary representation which does not exist in classical case (cf. Theorem 6.1 in Section 6).

The aim of the present paper is to explain this sudden change of the representation theory. Inspired by the classsical results presented in [3] we tried to use the global approach.

In this framework we investigate a large family of (not necessarily unitary) representations of *QLG* induced by 1-dimensional representations of the quantum parabolic subgroup P of *QLG*. The representations act on spaces of smooth sections of (quantum) line boundles over the homogeneous space ${}_P\backslash^{QLG}$. Such spaces denoted by D_χ (where χ runs over the set of 1-dimensional representations of P) play the fundamental role in [3]. We call them Gelfand spaces.

A deeper investigation (with the technique of invariant bilinear forms) of D_χ shows that in principle all results concerning the classical Lorentz group contained in Chapter 3 of [3] remains are valid in the quantum case. In particular the conditions distinguishing unitary representations are of the same form and lead in a natural way to principal and supplementary series.

The difference between the classical and quantum case consists in a slightly different topological structure of the set of 1-dimensional representations (characters) of the group P. In the classical case those representations are labeled by pairs (n_1, n_2) where $n_1, n_2 \in \mathbf{C}$ with $n_1 - n_2 \in \mathbf{Z}$ and the different pairs correspond to the different representations.

In the quantum case the correspondence between the pairs (n_1, n_2) and the representations of P is no longer one-to-one: the pairs (n_1, n_2) and $(n_1 + \frac{2\pi i}{\log q}, n_2 + \frac{2\pi i}{\log q})$ give rise to the same representations of P.

The above difference between the classical and the quantum case explains in a simple way all the suprising features of the theory of unitary representations of the quantum Lorentz group such as the new topological structure of the principal series, the existence of two (instead of one) supplementary series and the existence of non-trivial 1-dimensional representation.

We shall use the following notation.

For a C^*-algebra A, $M(A)$ denotes its multiplier algebra i.e. the largest C^*-algebra containing A as an essential ideal. This assignment is functorial and $M(A) = A$ if and only if A is unital. The multiplier functor M is an algebraic counterpart of the Čech-Stone compactification of a locally compact space.

We shall also consider elements affiliated with A. They should be regarded as unbounded multipliers acting on A. For a precise definition of the affiliation relation we refer to [17]. This relation is denoted by η and A^η is the set of all affiliated elements: $a\,\eta\,A \Leftrightarrow a \in A^\eta$. In general A^η is not even a vector space but for the C^*-algebra A related to *QLG*, A^η is a *-algebra. In any case $A^\eta \supset M(A)$. Moreover $A^\eta = A$ if and only if A is unital.

For C^*-algebras A and B a morphism Φ from A to B is a non-degenerate *-algebra homomorphism $\Phi : A \longrightarrow M(B)$ (non-degeneracy means that $\Phi(A)B$ is dense in B). This notion of morphism corresponds to that of a continuous map between locally compact spaces in the category of commutative C^*-algebras. The set of morphism is denoted by $\mathrm{Mor}(A, B)$. In particular if $B = CB(H)$ is the algebra of compact operators acting on the Hilbert space H then $\mathrm{Mor}(A, CB(H)) = \mathrm{Rep}(A, H)$ where $\mathrm{Rep}(A, H)$ is the set of all non-degenerate representations of A in H. If $G = (A, \Delta)$ is a quantum group then the comultiplication $\Delta \in \mathrm{Mor}(A, A \otimes A)$. In this definition A is the (non-commutaive) C^*-algebra of *"continuous functions vanishing at infinity"* on G and the comultiplication encodes the group structure on it.

It is known that any $\Phi \in \mathrm{Mor}(A, B)$ has the canonical extension to A^η. It maps A^η into B^η and $M(A)$ into $M(B)$. Φ restricted to $M(A)$ ia a *-algebra homomorphism.

The basic notion used in the paper is that of representation (an action of a quantum group on a vector space). At first we recall (e.g.[14], [19], [11]) that a unitary representation of a quantum group G acting on the Hilbert space H is by definition a unitary element $u \in M(CB(H) \otimes A)$ satisfying the equality

$$(\mathrm{id} \otimes \Delta)u = u_{12}u_{13}. \tag{0.1}$$

For our purposes this definition is too restrictive since we would like to deal also with non unitary representations of the quantum Lorentz group acting on vector spaces with no scalar product structure. We shall consider representations acting on *smooth vector spaces*. By definition a smooth vector space is a countable Cartesian product of at most countable dimensional vector spaces endowed with natural topology. The basic facts concerning this class of vector spaces are collected in Appendix A.

To define a larger class of representations of *QLG* we introduce a $*$-subalgebra $\mathcal{A} \subset A^{\eta}$. It is a smooth vector space and $\mathcal{A}$ is called the algebra of *"smooth continuous functions"* on *QLG* (cf. Appendix C). We show that $\mathcal{A}$ is invariant under the left action of *QLG* on A^{η} (cf. (C.7) of Appendix C):

$$\Delta : \mathcal{A} \longrightarrow \mathcal{A}\hat{\otimes}\mathcal{A}, \tag{0.2}$$

where $\hat{\otimes}$ is a projective tensor product of topological locally convex vector spaces. The counit e related to *QLG* belongs to $\mathrm{Mor}(A, \mathbf{C})$. Its natural extension is a multiplicative linear functional on $\mathcal{A}$. Clearly $(\mathrm{id}_{\mathcal{A}} \otimes e)\Delta(a) = (e \otimes \mathrm{id}_{\mathcal{A}})\Delta(a) = a$ for any $a \in \mathcal{A}$. We call (0.2) the smooth regular action of *QLG*.

More generally, let D be a smooth vector space and

$$v \ : D \ \longrightarrow \ D\hat{\otimes}\mathcal{A}$$

be a continuous linear map. We say that v is *a smooth representation of QLG acting on* D whenever $(\mathrm{id}_D \otimes e)v \ = \ \mathrm{id}_D$ and the diagram

$$\begin{array}{ccc} D & \xrightarrow{\quad v \quad} & D\hat{\otimes}\mathcal{A} \\ v \Big\downarrow & & \Big\downarrow v \otimes \mathrm{id}_{\mathcal{A}} \\ D\hat{\otimes}\mathcal{A} & \xrightarrow{\ \mathrm{id}_D \otimes \Delta\ } & D\hat{\otimes}\mathcal{A}\hat{\otimes}\mathcal{A} \end{array} \tag{0.3}$$

is commutative. This condition replaces (0.1). Let us note that if the classical group G acts on the vector space D by linear operators v_g $(g \in G)$ then the commutativity of the diagram (0.3) means that $v_{g_1}(v_{g_2}x) = v_{g_1 g_2}x$ for any $x \in D$ and $g_1, g_2 \in G$.

A very interesting example of such situation arises when D is a closed *invariant subspace* of $\mathcal{A}$ i.e.

$$\Delta(D) \ \subset \ D\hat{\otimes}\mathcal{A}.$$

Then $v = \Delta\,|_D$ makes the diagram (0.3) commutative by the co-associativity of Δ. In particular all Gelfand spaces are countable dimensional invariant subspaces of $\mathcal{A}$.

Let $\mathcal{A}'$ be the space of all continuous linear functionals on $\mathcal{A}$. For any $\psi \in \mathcal{A}'$ we set

$$v_{\psi} \ = \ (\mathrm{id}_D \otimes \psi)v.$$

Then v_ψ is a linear continuous operator on D and linearly depends on ψ. Using the convolution product of functionals (cf.e.g.[16] p.626)

$$\psi_1 * \psi_2 := (\psi_1 \otimes \psi_2)\Delta$$

for any $\psi_1, \psi_2 \in \mathcal{A}'$ one can immediately check that $\psi_1 * \psi_2 \in \mathcal{A}'$ and by commutativity of the diagram (0.3) that

$$v_{\psi_1} v_{\psi_2} = v_{\psi_1 * \psi_2}. \tag{0.4}$$

Therefore the map $\psi \to v_\psi$ is a homomorpsim of the convolution algebra $\mathcal{A}'$ into the algebra of continuous operators on D. We say that v_ψ $(\psi \in \mathcal{A}')$ are *operators of the representation* v.

If $\Psi \in \mathcal{A}'$ is in the convolution center of $\mathcal{A}'$ then v_Ψ commutes with all operators of v. In this case we say that v_Ψ is a Casimir operator.

We shall briefly describe the content of the paper.

Section 1 is devoted to the parabolic subgroup P of the Lorentz group. At first we describe quantum version of P (on Hopf *-algebra level and C^*-algebra level). Next by natural embeding we identify P with a subgroup of *QLG*. An important result of the Section is the description of the set of all characters (i.e. 1-dimensional representations) of P.

In Section 2 we investigate representations of *QLG* induced by characters of P. For any character χ of P the corresponding representation v_χ of *QLG* acts on the Gelfand space D_χ. This space is realized as a countable dimensional subspace of the algebra $\mathcal{A}$ of *"smooth functions"* on *QLG* and $v_\chi = \Delta|_{D_\chi}$. The invariants of v_χ such as spin spectrum and Casimir operators are computed (cf.Theorem 2.4). It turns out that the spin spectrum of v_χ is simple and Casimir operators are multiples of the identity.

The basic computational tools used in the paper are developed in Section 3 and 4. For any pair (χ, χ') of characters of the parabolic group P we investigate the set of all Lorentz invariant bilinear functionals on $D_\chi \times D_{\chi'}$. If the set contains non-zero functionals then the pair (χ, χ') is called admissible. In Section 3 we analyze the general form of such functionals. The Lorentz invariance is equivalent to $S_qU(2)$-invariance and $\widehat{S_qU}(2)$-invariance. It turns out that any Lorentz invariant bilinear functional f on the pair $D_\chi \times D'_\chi$ of Gelfand spaces may be expressed in terms of the Haar measure h on $S_qU(2)$ and some special functional ψ on the algebra of smooth functions on $S_qU(2)$. The functional ψ satisfies certain equality (involving χ and χ'). It is called a (χ, χ')-spherical functional (cf.Definition 3.4). Therefore the admissibility is equivalent to the existence of non-zero (χ, χ')-spherical functional (cf.Theorem 3.5 and Proposition 3.2). In Section 4 the (χ, χ')-spherical functionals are studied in more detail. As a result the complete list of admissible pairs is obtained. It is shown that for any admissible pair (χ, χ') the space of (χ, χ')-spherical functionals is one dimensional (cf.Theorem 4.9). Moreover we give explicite formulae for these functionals.

These results are used in Section 5 to consider the equivalence and irreducibility of representations of *QLG* on Gelfand spaces. We proceed in the same way as in the classical approach of [3]. For any pair of characters (χ, χ') we consider a space $\mathrm{Mor}(\chi, \chi')$ of all linear operators $T: D_\chi \longrightarrow D_{\chi'}$ intertwining the representations v_χ and $v_{\chi'}$. The main result of this Section is formulated in Theorem 5.7. It shows that $\dim \mathrm{Mor}(\chi, \chi') \leq 1$. In particular any v_χ is irreducible i.e. v_χ does not split into the direct sum of two nontrivial subrepresentations. Moreover Theorem 5.7 reveals also the role of positive integer points (cf.Definition 5.3). If neither χ nor $-\chi$ is a positive integer point then v_χ is equivalent to $v_{\chi'}$ (i.e. $\mathrm{Mor}(\chi,\chi')$ contains a bijection) iff $\chi'=\chi$ or $\chi'=-\chi$.

In Section 6 we select all characters χ such that D_χ admits v_χ-invariant scalar product. Such scalar product is unique up to a positive factor. Applying completion procedure based on the theory of Hilbert C^*-modules we show that v_χ gives rise to the unitary representation of *QLG* acting on the Hilbert space H_χ obtained by the standard completion of D_χ. Comparing this result with Theorem 6.1 (proved in [11]) we see that all infinite-dimensional representations of *QLG* may be obtained in this way.

We belive that the methods developed in this paper will be also useful in the representation theory of other quantum deformations of the Lorentz group. In particular in a forthcoming paper we shall investigate representations of the quantum Lorentz group having Gauss decomposition property.

1 Parabolic subgroup and its characters

The parabolic subgroup P of the classical Lorentz group $G=SL(2,\mathbf{C})$ consists of all upper-triangular matrices

$$P \;=\; \left\{ \begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix} \in SL(2,\mathbf{C}) \;:\; \gamma=0 \right\}.$$

The algebra $\mathrm{Poly}\,(P)$ of polynomial functions on P coincides with $\mathrm{Poly}\,(SL(2,\mathbf{C}))/I_\gamma$ where I_γ is the ideal generated by the relation $\gamma=0$. We follow this idea in the quantum case.

The Hopf *-algebra $\mathcal{A}_P$ of polynomials on the parabolic subgroup of the quantum Lorentz group is generated by three elements $\dot\alpha,\dot\beta,\dot\delta$ subject to relations

$$\begin{array}{ll}
\dot\alpha^*\dot\alpha = \dot\alpha\dot\alpha^*, & \dot\delta\dot\delta^* = \dot\delta^*\dot\delta, \\
\dot\alpha\dot\beta = q\dot\beta\dot\alpha, & \dot\beta\dot\alpha^* = q^{-1}\dot\alpha^*\dot\beta, \\
\dot\beta\dot\delta = q\dot\delta\dot\beta, & \dot\delta\dot\alpha^* = \dot\alpha^*\dot\delta, \\
\dot\alpha\dot\delta = I = \dot\delta\dot\alpha, & \dot\delta\dot\beta^* = q\dot\beta^*\dot\delta, \\
\dot\beta\dot\beta^* = \dot\beta^*\dot\beta + (1-q^2)(\dot\delta^*\dot\delta - \dot\alpha^*\dot\alpha). &
\end{array} \tag{1.6}$$

These are the Podleś relations (cf.[10] eq.(1.9)-(1.25)) supplemented by the relation $\dot\gamma=0$.

The group structure of P is imposed by the requirement that

$$u_P \;=\; \begin{pmatrix} \dot\alpha, & \dot\beta \\ 0, & \dot\delta \end{pmatrix} \tag{1.7}$$

should be the fundamental representation of P. The comultiplication $\Delta_P : \mathcal{A}_P \to \mathcal{A}_P\otimes\mathcal{A}_P$ is uniquely defined by its values on the generators:

$$\begin{array}{ll}
\Delta_P(\dot\alpha) = \dot\alpha\otimes\dot\alpha, & \Delta_P(\dot\beta) = \dot\alpha\otimes\dot\beta + \dot\beta\otimes\dot\delta, \\
 & \Delta_P(\dot\delta) = \dot\delta\otimes\dot\delta.
\end{array} \tag{1.8}$$

On the C^*-level the generators $\dot\alpha, \dot\beta$ and $\dot\delta$ are unbounded elements affiliated to a C^*-algebra A_P of "continuous functions vanishing at infinity on P." To construct this algebra we use the method which is a quantum version of semi-direct product construction known in group theory. Such a method was used in [10] to introduce the quantum double group construction.

Let w be a 2×2 - matrix with entries being bounded operators acting on the Hilbert space H : $w \in M_2(B(H))$. We say that w is a $P-$matrix if w is of the form (1.7) and its matrix elements satisfy the $P-$relations (1.6). If in addition $\dot\alpha$ is positive then matrix elements of w satisfy the relations corresponding to $S_q\widehat{U(2)}-$group. We refer to this particular case by saying that w is a $S_q\widehat{U(2)}-$matrix. A $P-$matrix w is unitary if and only if $\dot\alpha$ is unitary, $\dot\beta = 0$ and $\dot\delta = \dot\alpha^*$. We shall refer to such situation by saying that w is a S^1-matrix.

Matrix elements of any $P-$matrix w satisfy also the relations for the quantum Lorentz group. Therefore the Iwasawa decomposition for quantum Lorentz group-matrices ([10] Theorem 1.3) holds for $P-$matrices:

Proposition 1.1
Let $w \in M_2(B(H))$ be a $P-$matrix. Then there exist the unique matrices w_d, $w_{S^1} \in M_2(B(H))$ such that

$$w \;=\; w_d w_{S^1},$$

where w_d is a $S_q\widehat{U(2)}-$matrix and w_{S^1} is S^1-matrix. Matrix elements of w_d commute with matrix elements of w_{S^1}. Moreover the C^*- subalgebra of $B(H)$ generated by matrix elements of w contains the matrix elements of w_d and w_{S^1}.

This shows that the quantum space P is homeomorphic to the Cartesian product $S_q\widehat{U(2)}\times S^1$. In other words the C^*-algebra of functions on P is the tensor product of the corresponding C^*-algebras:

$$A_P \;=\; A_d \otimes C(S^1).$$

The group structure of P is related to that of $S_q\widehat{U(2)}$ and S^1 in a nontrivial manner. Nevertheless it is possible to give a description of the group structure of P using the group structures of $S_q\widehat{U(2)}$ and S^1. As in the case of the quantum double construction desription involves a canonical bicharacter $\dot u$ defined on $S_q\widehat{U(2)}\times S^1$.

Let

$$\dot u \;:=\; (I_d \otimes z)^{2J_3 \otimes I_{S^1}} \tag{1.9}$$

Then $\dot u$ is a unitary element of $M(A_d \otimes C(S^1))$ and

$$\begin{aligned} (\Delta_d \otimes \mathrm{id}_{S^1})\dot u \;&=\; \dot u_{23}\,\dot u_{13}, \qquad & (\mathrm{id}_d \otimes \Delta_{S^1})\dot u \;&=\; \dot u_{12}\,\dot u_{13} \\ (e_d \otimes \mathrm{id}_{S^1})\dot u \;&=\; I_{S^1}, & (\mathrm{id}_d \otimes e_{S^1})\dot u \;&=\; I_d. \end{aligned} \tag{1.10}$$

It means that $\dot u$ is a bicharacter. Now for any $f \in C(S^1)$ and $x \in A_d$ we set

$$\dot\sigma(f \otimes x) \;:=\; \dot u(x \otimes f)\,\dot u^*. \tag{1.11}$$

Then $\dot\sigma \in \mathrm{Mor}(C(S^1) \otimes A_d,\, A_d \otimes C(S^1))$. Moreover

$$(e_d \otimes \mathrm{id}_{S^1}) \circ \dot\sigma \;=\; \mathrm{id}_{S^1} \otimes e_d, \qquad (\mathrm{id}_d \otimes e_{S^1}) \circ \dot\sigma \;=\; e_{S^1} \otimes \mathrm{id}_d. \tag{1.12}$$

By definition the comultiplication

$$\Delta_P \ := \ (\mathrm{id}_d \otimes \dot{\sigma}^{-1} \otimes \mathrm{id}_{S^1})\,(\Delta_d \otimes \Delta_{S^1}). \tag{1.13}$$

Clearly $\Delta_P \in \mathrm{Mor}(A_P,\, A_P \otimes A_P)$ and one can show (cf.the proof of Theorem 4.1 of [10]) that Δ_P is coassociative:

$$(\mathrm{id}_P \otimes \Delta_P)\,\Delta_P \ = \ (\Delta_P \otimes \mathrm{id}_P)\,\Delta_P.$$

Therefore Δ_P defines the group structure on P. The counit is

$$e_P \ := \ e_d \otimes e_{S^1}.$$

The formula for the coinverse can be also presented as in [10] but is omitted since it will not be used.

$S_q\widehat{U(2)}$ and S^1 are subgroups of P. The embedings $S_q\widehat{U(2)} \hookrightarrow P$ and $S^1 \hookrightarrow P$ are related to the morphisms

$$\dot{p}_d = \mathrm{id}_d \otimes e_{S^1} \in \mathrm{Mor}(A_P, A_d),$$
$$\dot{p}_{S^1} = e_d \otimes \mathrm{id}_{S^1} \in \mathrm{Mor}(A_P, C(S^1)).$$

One can easily verify that

$$\begin{aligned} \Delta_d \dot{p}_d &= (\dot{p}_d \otimes \dot{p}_d)\Delta_P, & \Delta_{S^1}\dot{p}_{S^1} &= (\dot{p}_{S^1} \otimes \dot{p}_{S^1})\Delta_P \\ e_d \dot{p}_d &= e_P, & e_{S^1}\dot{p}_{S^1} &= e_P. \end{aligned}$$

This means that group structures on $S_q\widehat{U(2)}$ and S^1 are restrictions of that on P.

Now remembering that the set of affiliated elements $(A_d \otimes C(S^1))^\eta$ is a *-algebra we are able to connect our construction with relations (1.6) - (1.8) (cf.Theorem 5.4 of [10]).

Theorem 1.2
Let $\dot{\alpha}, \dot{\beta}$ and $\dot{\delta}$ be elements affiliated with $A_P = A_d \otimes C(S^1)$ introduced by

$$\dot{\alpha} := q^{J_3} \otimes \bar{z}, \qquad \begin{aligned} \dot{\beta} &:= (1-q^2)q^{-1/2}J_+ \otimes z \\ \dot{\delta} &:= q^{-J_3} \otimes z \end{aligned} \tag{1.14}$$

and

$$u_P \ = \ \begin{pmatrix} \dot{\alpha}, & \dot{\beta} \\ 0, & \dot{\delta} \end{pmatrix} \ := \ \begin{pmatrix} q^{J_3}, & (1-q^2)q^{-1/2}J_+ \\ 0, & q^{-J_3} \end{pmatrix} ⓣ \begin{pmatrix} \bar{z}, & 0 \\ 0, & z \end{pmatrix}. \tag{1.15}$$

Then
1^o $\dot{\alpha},\ \dot{\beta},\ \dot{\delta}$ *satisfy relations* (1.6)
2^o u_P *is a representation of P i.e.*

$$\Delta_P(\dot{\alpha}) = \dot{\alpha} \otimes \dot{\alpha}, \qquad \begin{aligned} \Delta_P(\dot{\beta}) &= \dot{\alpha} \otimes \dot{\beta} + \dot{\beta} \otimes \dot{\delta}, \\ \Delta_P(\dot{\delta}) &= \dot{\delta} \otimes \dot{\delta}. \end{aligned} \tag{1.16}$$

3^o $(\dot{\alpha},\ \dot{\beta},\ \dot{\delta})$ *generate C^*-algebra A_P in the sense of* Definition 3.1 *of* [18] (cf.[18] Examples 8-10 p.500).

Let us note that $\dot{\delta}$ is an invertible element affiliated with A_P, $\dot{\delta}^*\dot{\delta} = q^{-2J_3} \otimes I_{S^1}$. Therefore

$$\mathrm{Sp}\, \dot{\delta}^*\dot{\delta} \;=\; q^{\mathbf{Z}} \cup \{0\} \tag{1.17}$$

and this was not apparent from the commutation relations (1.6).

The quantum group P may be realized as a subgroup of the quantum Lorentz group *QLG*. At first let us recall that the commutation relations for $\dot{\alpha}$, $\dot{\beta}$ and $\dot{\delta}$ were obtained by adding the relation "$\gamma = 0$" to Podleś relations for the quantum Lorentz group. Therefore it seems very natural to define A_P as

$$A_P \;=\; A/I_\gamma,$$

where I_γ is the closed two sided ideal of A "generated by γ." Unfortunately γ is the unbounded operator which does not belong to A and the rigorous meaning of the phrase inserted in quotation marks is not clear. On the other hand according to the Iwasawa decomposition $\gamma = q^{J_3} \otimes \gamma_c$ (cf.(5.15) in [10]). Since q^{J_3} is ivertible the relation $\gamma = 0$ is equivalent to $\gamma_c = 0$. Therefore the ideal I_γ may be replaced by the ideal generated by $I_d \otimes \gamma_c$. We shall follow this idea.

Let I_{γ_c} be the ideal in A_c generated (in the usual sense) by γ_c and $\dot{\pi}_c$ be the canonical epimorphism $\dot{\pi}_c : A_c \to A_c/I_{\gamma_c}$. Then clearly the C^*-algebra A_c/I_{γ_c} is isomorphic to $C(S^1)$ since $\dot{\pi}_c(\gamma_c) = 0$ and $\dot{\pi}_c(\alpha_c^*)$ may be identified with the unitary generator z of $C(S^1)$: $\dot{\pi}_c({\alpha_c}^*) = z$ where $z \in C(S^1)$, $z(\zeta) = \zeta$ for any $\zeta \in S^1$. Clearly $\dot{\pi}_c \in \mathrm{Mor}(A_c, C(S^1))$ and for a matrix element u^s_{kl} $(k, l = -s, -s+1, ..., s)$ of the unitary representation u^s of $S_qU(2)$ with spin s (cf. (B.19)) we obtain

$$\dot{\pi}_c(u^s_{kl}) = \delta_{kl} z^{2k}. \tag{1.18}$$

Moreover

$$(\dot{\pi}_c \otimes \dot{\pi}_c)\Delta_c = \Delta_{S^1} \circ \dot{\pi}_c, \qquad e_{S^1}\dot{\pi}_c = e_c.$$

It shows that $\dot{\pi}_c$ describes an embedding $S^1 \hookrightarrow S_qU(2)$ preserving the group structures.

Let

$$\dot{\pi} \;=\; \mathrm{id}_d \otimes\; \dot{\pi}_c.$$

Then $\dot{\pi} \in \mathrm{Mor}(A, A_P)$ and

$$\dot{\pi}(u^{\frac{1}{2}}) \;=\; (\mathrm{id}_d \otimes\; \dot{\pi}_c) \begin{pmatrix} \alpha_c, & -q\gamma_c^* \\ \gamma_c, & \alpha_c^* \end{pmatrix} \;=\; \begin{pmatrix} \bar{z}, & 0 \\ 0, & z \end{pmatrix}.$$

Taking this into account and comparing the Iwasawa decomposition for P and that for *QLG* we get

$$\dot{\pi}(\alpha) = \dot{\alpha}, \qquad \dot{\pi}(\beta) = \dot{\beta}, \qquad \dot{\pi}(\gamma) = 0, \qquad \dot{\pi}(\delta) = \dot{\delta}.$$

Due to (1.18) $\dot{\pi}(u^s) = z^{2J^s_3}$. Therefore for the canonical bicharacter $u = \sum^{\oplus}_{s \in S} u^s \in M(A) = M(A_d \otimes A_c)$ we obtain

$$\dot{\pi}(u) \;=\; (\mathrm{id}_d \otimes \dot{\pi}_c)u \;=\; (I_d \otimes z)^{2J_3 \otimes I_{S^1}} \;=\; \dot{u}, \tag{1.19}$$

where $\dot{u}$ is the bicharacter (1.9).
Now one can check that the diagram

$$\begin{array}{ccc} A_d\otimes A_c & \xrightarrow{\sigma^{-1}} & A_c\otimes A_d \\ \Big\downarrow{\scriptstyle \mathrm{id}_d\otimes\dot{\pi}_c} & & \Big\downarrow{\scriptstyle \dot{\pi}_c\otimes\mathrm{id}_d} \\ A_d\otimes C(S^1) & \xrightarrow{\dot{\sigma}^{-1}} & C(S^1)\otimes A_d \end{array} \tag{1.20}$$

is commutative.

Using this we have

$$\begin{aligned} \Delta_P\dot{\pi} &= (\mathrm{id}_d\otimes\dot{\sigma}^{-1}\otimes\mathrm{id}_{S^1})(\Delta_d\otimes\Delta_{S^1})\dot{\pi} \\ &= (\mathrm{id}_d\otimes\dot{\sigma}^{-1}\otimes\mathrm{id}_{S^1})(\Delta_d\otimes\Delta_{S^1}\circ\dot{\pi}_c) \\ &= (\mathrm{id}_d\otimes\dot{\sigma}^{-1}\otimes\mathrm{id}_{S^1})(\mathrm{id}_d\otimes\mathrm{id}_d\otimes\dot{\pi}_c\otimes\dot{\pi}_c)(\Delta_d\otimes\Delta_c) \\ &= (\mathrm{id}_d\otimes\dot{\pi}_c\otimes\mathrm{id}_d\otimes\dot{\pi}_c)(\mathrm{id}_d\otimes\sigma^{-1}\otimes\mathrm{id}_c)(\Delta_d\otimes\Delta_c) = \ (\dot{\pi}\otimes\dot{\pi})\Delta \end{aligned}$$

and similarly $e = e_P\circ\dot{\pi}$. It shows that $\dot{\pi}$ describes an embedding $P\hookrightarrow QLG$ preserving the group structures.

Now we shall describe characters of P i.e. 1-dimensional representations of P. We shall consider this in full generality not assuming unitarity or even boundedness. By definition a character of P is an invertible element χ affiliated with A_P such that

$$\Delta_P\chi = \chi\otimes\chi.$$

According to (1.8) $\dot{\delta}$ is a character. So is $\dot{\delta}^*$ or more generally $\dot{\delta}^{n_1-1}(\dot{\delta}^*)^{n_2-1}$ where n_1, n_2 are integers. (We inserted -1 in the exponents to have better correspondence with the Gelfand notation [3]). Using (1.14) we obtain

$$\dot{\delta}^{n_1-1}(\dot{\delta}^*)^{n_2-1} \ = \ q^{-(n_1+n_2-2)J_3}\otimes z^{n_1-n_2}.$$

The reader should notice that the right hand side is well defined for any $n_1, n_2\in\mathbf{C}$ provided n_1-n_2 is an integer.

Let

$$\chi \ = \ q^{-(n_1+n_2-2)J_3}\otimes z^{n_1-n_2} \ = \ t^{2J_3}\otimes z^n, \tag{1.21}$$

where $t = q^{-\frac{1}{2}(n_1+n_2-2)}$ and $n = n_1-n_2$. We shall prove (cf.Theorem 1.4) that (1.21) is a character of P and that any character of P is of that form.
To abbreviate the notation we shall write $\chi = (n_1, n_2)$. Let us note that due to the spectral condition (1.17) two pairs (n_1, n_2), (n_1', n_2') give rise to the same character if and only if

$$n_1-n_1' \ = \ n_2-n_2' \ = \ \frac{2k\pi i}{\log q} \quad \text{for some } k\in\mathbf{Z}.$$

In such a case we write $(n_1, n_2) \equiv (n'_1, n'_2)$.

Clearly any character of the (quantum) group restricted to its subgroup is a character of this subgroup. Our result says that any character of the parabolic group P is a product of the characters of its subgroups: $S_q\widehat{U(2)}$ and S^1 and vice versa.

At first we describe non trivial characters of the quantum $S_q\widehat{U(2)}$-group.

Proposition 1.3
Let χ_d be a non-zero element of A_d^η. Then the following conditins are equivalent:
i) $\Delta_d\chi_d \;=\; \chi_d \otimes \chi_d$
ii) $\chi_d \;=\; (\mathrm{id}_d \otimes \phi)u$ *for some nontrivial linear multiplicative functional* $\phi : \mathcal{A}_c \longrightarrow \mathbf{C}$
iii) $\chi_d \;=\; t^{2J_3}$ *for some non-zero complex number* t

Proof. i)⇒ ii). Let $\chi_d \in A_d^\eta$. Then $\chi_d = (\chi_d^s)_{s=0,1/2,1...}$ where $\chi_d^s \in B(H^s)$. Since the matrix elements of $u = (u^s)_{s=0,1/2,1...}$ form a linear basis of $\mathcal{A}_c$ there exists a linear functional ϕ on $\mathcal{A}_c$ such that for any $s : (\mathrm{id} \otimes \phi)u^s \;=\; \chi_d^s$. This means that any $\chi_d \in A_d^\eta$ is of the form

$$\chi_d \;=\; (\mathrm{id}_d \otimes \phi)u.$$

For χ_d satisfying the character equation we get

$$\begin{aligned}(\mathrm{id}_d \otimes \mathrm{id}_d \otimes \phi)u_{23}u_{13} \;&=\; (\mathrm{id}_d \otimes \mathrm{id}_d \otimes \phi)(\Delta_d \otimes \mathrm{id}_c)u \\ &=\; \Delta_d(\mathrm{id}_d \otimes \phi)u \;=\; \Delta_d\chi_d \;=\; \chi_d \otimes \chi_d \\ &=\; (\mathrm{id}_d \otimes \phi)u \otimes (\mathrm{id}_d \otimes \phi)u.\end{aligned}$$

Rewriting this equation in terms of matrix elements we have

$$\phi(u_{kl}^{s_1}u_{mn}^{s_2}) \;=\; \phi(u_{mn}^{s_2})\phi(u_{kl}^{s_1}).$$

It shows that ϕ is multiplicative.
ii)⇒ iii). We have to find all nontrivial linear multiplicative functionals on $\mathcal{A}_c$.
Obviously $\phi(I_c) = 1$. Applying ϕ to both sides of the relation $[\alpha_c^*, \alpha_c] = (1 - q^2)\gamma_c^*\gamma_c$ we get $\phi(\gamma_c)\phi(\gamma_c^*) = 0$. Therefore $\phi(\alpha_c)\phi(\alpha_c^*) = 1$. Let $t := \phi(\alpha_c^*)$. Then t is a non-zero complex number. Now using the relations $\alpha_c\gamma_c = q\gamma_c\alpha_c$, $\alpha_c\gamma_c^* = q\gamma_c^*\alpha_c$ we get $\phi(\gamma_c) \;=\; 0 \;=\; \phi(\gamma_c^*)$. Then by (B.19) $\phi(u_{kl}^s) \;=\; \delta_{kl}t^{2k}$ and

$$\chi_d \;=\; (\mathrm{id}_d \otimes \phi)u = \; t^{2J_3}.$$

iii)⇒ i) This implication follows immediately from the formula $\Delta_d J_3 = J_3 \otimes I_d + I_d \otimes J_3$. □

Now we can prove the main result of this section.

Theorem 1.4
Let χ be a non-zero element of A_P^η. Then

$$\left(\Delta_P(\chi) \;=\; \chi\otimes\chi \right) \qquad \Longleftrightarrow \qquad \left(\begin{array}{c} \chi \;=\; t^{2J_3}\otimes z^n \\ \text{where} \\ t\in \mathbf{C},\; t\neq 0 \text{ and } n\in\mathbf{Z} \end{array} \right).$$

Proof. $\Leftarrow$ Let $\chi \;=\; t^{2J_3}\otimes z^n$ then

$$\begin{aligned} \Delta_P\chi &= (\mathrm{id}_d\otimes\dot\sigma^{-1}\otimes\mathrm{id}_{S^1})(\Delta_d\otimes\Delta_{S^1})(t^{2J_3}\otimes z^n) \;=\; t^{2J_3}\otimes\dot\sigma^{-1}(t^{2J_3}\otimes z)\otimes z \\ &= t^{2J_3}\otimes[(z\otimes I_d)^{-I_{S^1}\otimes 2J_3}(z\otimes t^{2J_3})(z\otimes I_d)^{I_{S^1}\otimes 2J_3}]\otimes z \;=\; t^{2J_3}\otimes z\otimes t^{2J_3}\otimes z \\ &= \chi\otimes\chi \end{aligned}$$

since operators $(z\otimes I_d)^{I_{S^1}\otimes 2J_3}$ and $z\otimes t^{2J_3}$ commute.
$\Rightarrow$ We shall use the leg numbering notation. Applying $\mathrm{id_d}\otimes\dot\sigma\otimes\mathrm{id_{S^1}}$ to both sides of the character equation $\chi\otimes\chi=\Delta_P(\chi)$ we get:

$$\dot u_{23}\,\chi_{13}\,\chi_{24}\,\dot u_{23}^* \;=\; (\Delta_d\otimes\Delta_{S^1})\chi. \tag{1.22}$$

Let

$$\chi_d := (\mathrm{id}_d\otimes e_{S^1})\chi \quad\text{and}\quad \chi_{S^1} := (e_d\otimes\mathrm{id}_{S^1})\chi$$

be the restrictions of χ to the subgroups $S_q\widehat{U}(2)$ and S^1. Applying $\mathrm{id}_d\otimes\mathrm{id}_d\otimes e_{S^1}\otimes e_{S^1}$ and $e_d\otimes e_d\otimes\mathrm{id}_{S^1}\otimes\mathrm{id}_{S^1}$ to both sides of (1.22) we get

$$\Delta_d\chi_d \;=\; \chi_d\otimes\chi_d \quad\text{and}\quad \Delta_{S^1}\chi_{S^1} \;=\; \chi_{S^1}\otimes\chi_{S^1}.$$

This means that χ_d and χ_{S^1} are characters of the corresponding subgroups. On the other hand applying $\mathrm{id}_d\otimes e_d\otimes e_{S^1}\otimes\mathrm{id}_{S^1}$ to (1.22) we get that

$$\chi \;=\; \chi_d\otimes\chi_{S^1}.$$

Any character χ_{S^1} of S^1 is of the form $\chi \;=\; z^n$. By Proposition 1.3 $\chi_d \;=\; t^{2J_3}$ and the Statement follows.

□

2 Gelfand spaces

In this section we consider the representations of *QLG* induced by 1-dimensional representations (characters) of its parabolic subgroup P described in the previous section. They act on spaces of "smooth functions" on *QLG*. For the convenience of the reader these spaces are discussed in more detail in Appendix A.
An element a, affiliated with $A=A_d\otimes A_c$ is said to be smooth if for any $s=0,1/2,1,\ldots$:

$$(\pi^s\otimes\mathrm{id}_c)a \;\in\; B(H^s)\otimes\mathcal{A}_c.$$

The set of smooth elements will be denoted by $\mathcal{A}$. It is clear that $\mathcal{A}$ is a *-subalgebra of A^η.

For any character χ of P the representation of QLG induced by χ acts by right shifts on the space D_χ of smooth elements which transform under the left action of P according to the representation χ :

$$D_\chi \;:=\; \{a \in \mathcal{A}: \; (\dot{\pi} \otimes \mathrm{id})\Delta a = \chi \otimes a \;\}. \tag{2.1}$$

Let us note that the equation

$$(\dot{\pi} \otimes \mathrm{id})\Delta a = \chi \otimes a \tag{2.2}$$

coincides in the classical case with $a(pg) = \chi(p)a(g)$ for all $p \in P$ and $g \in G$ (cf.[1] p.473, formula (1)).
Solving this equation we shall give very explicit description of the spaces D_χ (cf.Theorem 2.2). As we know by Theorem 1.4 any character of P is of the form $\chi = t^{2J_3} \otimes z^n$ where $t \in \mathbf{C} \setminus \{0\}$ and $n \in$ Z. It turns out that the elements of $D_{t^{2J_3}\otimes z^n}$ are of the form $t^{2J_3} \otimes a_c$ where a_c are elements of $\mathcal{A}_c$ satisfying the equation

$$(\dot{\pi}_c \otimes \mathrm{id}_c)\Delta_c a_c = z^n \otimes a_c \;. \tag{2.3}$$

The space of solutions of this equation is the carrier space of the representation of $S_qU(2)$ induced by z^n (from the subgroup $S^1 \subset S_qU(2)$). For reasons that will be clear later we shall consider (2.3) in a more general setting.

Proposition 2.1
Let B be a smooth vector space. Then

$$\left\{b_c \in \mathcal{A}_c \hat{\otimes} B: \; (\dot{\pi}_c \otimes \mathrm{id}_c \otimes \mathrm{id}_B)(\Delta_c \otimes \mathrm{id}_B) b_c = z^n \otimes b_c \;\right\} = \mathcal{Z}_n \hat{\otimes} B$$

where

$$\mathcal{Z}_n \;=\; \left\{ \; u^s_{\frac{n}{2},k}: \; \begin{array}{l} s = \frac{|n|}{2}, \frac{|n|}{2}+1, \frac{|n|}{2}+2, \ldots \\ k = -s, -s+1, \ldots s \end{array} \right\}^{\text{linear span}} . \tag{2.4}$$

In particular the set of solutions of (2.3) coincides with (2.4).

Proof. We have

$$(\dot{\pi}_c \otimes \mathrm{id}_c)\Delta_c u^s_{p,k} \;=\; (\dot{\pi}_c \otimes \mathrm{id}_c) \sum_{l=-s}^{s} u^s_{p,l} \otimes u^s_{l,k} = z^{2p} \otimes u^s_{p,k} \tag{2.5}$$

by (1.18). This implies that for any $b \in B$, $b_c := u^s_{\frac{n}{2},k} \otimes b$ is a solution of

$$(\dot{\pi}_c \otimes \mathrm{id}_c \otimes \mathrm{id}_B)(\Delta_c \otimes \mathrm{id}_B) b_c = z^n \otimes b_c. \tag{2.6}$$

On the other hand remembering that the set of all matrix elements $u^s_{p,k}$ ($s \in S$, $p,k = -s, -s+1, ...s$) is a linear basis in $\mathcal{A}_c$ we see (cf.(A.1)) that any element $b_c \in \mathcal{A}_c \hat{\otimes} B$ has the unique decomposition of the form $b_c = \sum_{s,p,k} u^s_{p,k} \otimes b^s_{p,k}$ where the series is convergent in the topology of $\mathcal{A}_c \hat{\otimes} B$ and the coefficients $b^s_{p,k} \in B$ are uniqely defined by b_c. If b_c satisfies (2.6) then (2.5) shows that only the elements $u^s_{p,k}$ with $p = \frac{n}{2}$ may enter the decomposition with non-zero coefficients. Therefore $b_c \in \mathcal{Z}_n \hat{\otimes} B$.

□

For a representation v_c of $S_qU(2)$ *the spin spectrum of* v_c is the set $\mathrm{Sp} v_c \subset S$ of all $s \in S$ such that u^s is contained in v_c. We say that the spin spectrum is simple (multiplicity free) if each u^s appears in v_c at most once.

Let n be an integer and

$$S_n \;=\; \left\{ \frac{\mid n \mid}{2}, \frac{\mid n \mid}{2} + 1, \frac{\mid n \mid}{2} + 2, ... \right\}.$$

For any $s \in S_n$ we set

$$\mathcal{Z}_n^s \;=\; \left\{ \; u^s_{\frac{n}{2},k} : \; k = -s, -s+1, \ldots s \; \right\}^{\text{linear span}}.$$

Then $\dim \mathcal{Z}_n^s = 2s+1$. Since

$$\Delta_c u^s_{\frac{n}{2},k} \;=\; \sum_{l=-s}^{s} u^s_{\frac{n}{2},l} \otimes u^s_{l,k}$$

we see that

$$\Delta_c(\mathcal{Z}_n^s) \subset \mathcal{Z}_n^s \otimes \mathcal{A}_c$$

and

$$\Delta_c(\mathcal{Z}_n) \subset \mathcal{Z}_n \otimes \mathcal{A}_c \;=\; \mathcal{Z}_n \hat{\otimes} \mathcal{A}_c$$

i.e. $\mathcal{Z}_n$, $\mathcal{Z}_n^s$ $(s \in S_n)$ are $S_qU(2)$-invariant subspaces of $\mathcal{A}_c$ and $\mathcal{Z}_n^s$ is the carrier space for the irreducible representation of $S_qU(2)$ corresponding to the spin s. Clearly $\mathcal{Z}_n = \sum^{\oplus}_{s \in S_n} \mathcal{Z}_n^s$. This decomposition corresponds to the decomposition of the representation $v_c := \Delta_c|_{\mathcal{Z}_n}$ of $S_qU(2)$ on $\mathcal{Z}_n$ into irreducible components. Therefore the spin spectrum of v_c is simple and

$$\mathrm{Sp}\, v_c = S_n. \tag{2.7}$$

Now we shall describe the Gelfand spaces.

Theorem 2.2
Let $\chi = t^{2J_3} \otimes z^n$ *(*t *is a complex non-zero number and* $n \in \mathbf{Z}$*) be a character of* P *and* B *be a smooth vector space. Then*

$$\{ b \in \mathcal{A} \hat{\otimes} B : \; (\dot{\pi} \otimes \mathrm{id}_{\mathcal{A}} \otimes \mathrm{id}_B)(\Delta \otimes \mathrm{id}_B) b = \chi \otimes b \; \} = t^{2J_3} \otimes \mathcal{Z}_n \hat{\otimes} B.$$

In particular

$$D_\chi \;=\; t^{2J_3} \otimes \mathcal{Z}_n.$$

Proof. At first we notice that the equation

$$(\dot{\pi} \otimes \mathrm{id}_{\mathcal{A}} \otimes \mathrm{id}_B)(\Delta \otimes \mathrm{id}_B) b = \chi \otimes b \tag{2.8}$$

is equivalent to

$$[\Delta_d \otimes (\dot{\pi}_c \otimes \mathrm{id}_c)\Delta_c \otimes \mathrm{id}_B]\, b \;=\; t^{2J_3} \otimes [(\dot{\sigma} \otimes \mathrm{id}_c \otimes \mathrm{id}_B)(z^n \otimes b)]\,. \tag{2.9}$$

Indeed using the commutativity of (1.20) we obtain

$$
\begin{aligned}
&(\dot{\pi} \otimes \mathrm{id}_{\mathcal{A}} \otimes \mathrm{id}_B)(\Delta \otimes \mathrm{id}_B)b \\
&\quad = (\mathrm{id}_d \otimes \dot{\pi}_c \otimes \mathrm{id}_d \otimes \mathrm{id}_c \otimes \mathrm{id}_B)(\mathrm{id}_d \otimes \sigma^{-1} \otimes \mathrm{id}_c \otimes \mathrm{id}_B)(\Delta_d \otimes \Delta_c \otimes \mathrm{id}_B)b \\
&\quad = (\mathrm{id}_d \otimes \dot{\sigma}^{-1} \otimes \mathrm{id}_c \otimes \mathrm{id}_B)(\mathrm{id}_d \otimes \mathrm{id}_d \otimes \dot{\pi}_c \otimes \mathrm{id}_c \otimes \mathrm{id}_B)(\Delta_d \otimes \Delta_c \otimes \mathrm{id}_B)b \\
&\quad = (\mathrm{id}_d \otimes \dot{\sigma}^{-1} \otimes \mathrm{id}_c \otimes \mathrm{id}_B)\,[\Delta_d \otimes (\dot{\pi}_c \otimes \mathrm{id}_c)\Delta_c \otimes \mathrm{id}_B]\, b.
\end{aligned}
\tag{2.10}
$$

Inserting this result into (2.8) and applying to the both sides $\mathrm{id}_d \otimes \dot{\sigma} \otimes \mathrm{id}_c \otimes \mathrm{id}_B$ we get (2.9). Equivalence of (2.8) and (2.9) follows from invertibility of $\dot{\sigma}$.

Let $b \in \mathcal{A}\hat{\otimes}B$ satisfy (2.8). Applying $\mathrm{id}_d \otimes e_d \otimes \mathrm{id}_{S^1} \otimes \mathrm{id}_c \otimes \mathrm{id}_B$ to both sides of (2.9) and using (1.12) we obtain

$$[\mathrm{id}_d \otimes (\dot{\pi}_c \otimes \mathrm{id}_c)\Delta_c \otimes \mathrm{id}_B]\, b \;=\; t^{2J_3} \otimes z^n \otimes (e_d \otimes \mathrm{id}_c \otimes \mathrm{id}_B)b.$$

Remembering that $(\dot{\pi}_c \otimes \mathrm{id}_c)\Delta_c$ has a trivial kernel we conclude that b is of the form $b = t^{2J_3} \otimes b_c$, where $b_c \in \mathcal{A}_c\hat{\otimes}B$ and $(\dot{\pi}_c \otimes \mathrm{id}_c \otimes \mathrm{id}_B)(\Delta_c \otimes \mathrm{id}_B)b_c = z^n \otimes (e_d \otimes \mathrm{id}_c \otimes \mathrm{id}_B)b = z^n \otimes b_c$. Proposition 2.1 shows now that that $b_c \in \mathcal{Z}_n\hat{\otimes}B$. Conversely if $b = t^{2J_3} \otimes b_c$, where $b_c \in \mathcal{Z}_n\hat{\otimes}B$ then $(\dot{\pi}_c \otimes \mathrm{id}_c \otimes \mathrm{id}_B)(\Delta_c \otimes \mathrm{id}_B)b_c = z^n \otimes b_c$ by Proposition 2.1. The reader should notice that $\dot{u} = (I_d \otimes z)^{2J_3 \otimes I_{S^1}}$ commute with $t^{2J_3} \otimes z^n$. Therefore

$$t^{2J_3} \otimes z^n \;=\; \dot{\sigma}(z^n \otimes t^{2J_3})$$

and remembering that t^{2J_3} is a character of $\widehat{S_qU(2)}$ one can easily verify that (2.9) holds. □

The Gelfand spaces are a right invariant subspaces of $\mathcal{A}$.

Theorem 2.3
Let χ be a character of P. Then

$$\Delta(D_\chi) \;\subset\; D_\chi\hat{\otimes}\mathcal{A}.$$

Proof. Let $a \in D_\chi$. Then a satisfies (2.2). Using coassociativity of Δ one immediately checks that $b := \Delta a \in \mathcal{A}\hat{\otimes}\mathcal{A}$ satisfies (2.8) with B replaced by $\mathcal{A}$ and by Theorem 2.2 $\Delta a \in D_\chi\hat{\otimes}\mathcal{A}$.

□

Let $v_\chi := \Delta|_{D_\chi}$. Then

$$v_\chi : D_\chi \;\longrightarrow\; D_\chi\hat{\otimes}\mathcal{A}$$

is a smooth representation of *QLG*. This is the representation induced by χ. The space D_χ (denoted by $D_{n_1 n_2}$) in the classical setting appeared for the first time in the monograph [3] by Gelfand and collaborators. To commemorate this fact we call the spaces D_χ the Gelfand spaces.

Remark. Let us note that the set of all characters of the parabolic group P is an abelian group with involution. Indeed, if χ is a character of P then $\chi = t^{2J_3} \otimes z^n$ and $\chi^* := \bar{t}^{2J_3} \otimes z^{-n}$ is a conjugate character. If $\chi' = (t')^{2J_3} \otimes z^{n'}$ is another character then $\chi\chi' = (tt')^{2J_3} \otimes z^{n+n'}$ is again a character and clearly $\chi'\chi = \chi\chi'$. One can easily show that for $x \in D_\chi$, $x' \in D_{\chi'}$ we have $x^* \in D_{\chi^*}$, $xx' \in D_{\chi\cdot\chi'}$ and this structure on the set of Gelfand spaces reflects the structure of the set of all characters. Moreover remembering that the comultiplcation Δ is a $*$-homomorphism one can check that the conjugation $* : D_\chi \longrightarrow D_{\chi^*}$ and the multiplication $m : D_\chi \otimes D_{\chi'} \longrightarrow D_{\chi\chi'}$ are intertwining maps.

At the end of this section we compute the invariants of v_χ such as spin spectrum and the values of the Casimir operators. We shall use Casimir operators of the form (C.16). The corresponding Casimir operators for the representation v_χ are obtained by restriction to the Gelfand space D_χ and will be denoted by

$$C(v_\chi) = (\mathrm{id} \otimes \Psi)(\Delta|_{D_\chi}), \qquad C'(v_\chi) = (\mathrm{id} \otimes \Psi')(\Delta|_{D_\chi}). \tag{2.11}$$

Any representation v of *QLG* may be restricted to $S_qU(2)$. We shall use the shortened notation:

$$\mathrm{Sp}\, v \;:=\; \mathrm{Sp}\,(v|_{S_qU(2)}) \tag{2.12}$$

for the spin spectrum of the restricted representation.

Theorem 2.4

Let $\chi = q^{-(n_1+n_2-2)J_3} \otimes z^n$, $(n_1, n_2 \in \mathbf{C},\ n = n_1 - n_2 \in \mathbf{Z})$ *be the character of the parabolic subgroup* P *and* v_χ *be the induced representation of QLG acting on the Gelfand space* D_χ. *Then*

1. *The spin spectrum of* v_χ *is simple and coincides with* S_n.
2. *The Casimir operators are multiples of the identity*

$$C(v_\chi) \;=\; c_\chi\, \mathrm{id}_{D_\chi}, \qquad C'(v_\chi) \;=\; c'_\chi\, \mathrm{id}_{D_\chi} \tag{2.13}$$

and $c_\chi = -(q^{n_1} + q^{-n_1})$, $c'_\chi = -(q^{n_2} + q^{-n_2})$.

Proof. The representation v_χ restricted to $S_qU(2)$ is equivalent to $v_c = \Delta_c|_{\mathcal{Z}_n}$ and the Statement 1 follows immediately from (2.7).
To compute the Casimir operator $C(v_\chi)$ (cf.(2.11) and (C.15)) we calculate $C(v_\chi)a := (\mathrm{id} \otimes \Psi)v_\chi(a)$ for $a \in D_\chi \subset \mathcal{A}$. Remembering that the functional Ψ is in the convolution center of the algebra $\mathcal{A}'$ and using Proposition C.1 we get $C(v_\chi)a = (\Psi \otimes \mathrm{id})v_\chi(a)$. Since $v_\chi(a) \in D_\chi \otimes \mathcal{A}$ we have to find $\Psi|_{D_\chi}$.

Let $a = q^{-\lambda J_3} \otimes a_c$, where $a_c \in \mathcal{Z}_n$ and $\lambda = n_1 + n_2 - 2$. Using (B.22) we get

$$\psi_o|_{\mathcal{Z}_n} = q^{\frac{n}{2}} e_c \quad \text{and} \quad \bar{\psi}_o|_{\mathcal{Z}_n} = q^{-\frac{n}{2}} e_c.$$

By (B.27)

$$\psi_\alpha(q^{-\lambda J_3}) = q^{-\frac{\lambda}{2}}, \quad \psi_{\alpha^*}(q^{-\lambda J_3}) = q^{\frac{\lambda}{2}}, \quad \psi_\gamma(q^{-\lambda J_3}) = 0 = \psi_{\gamma^*}(q^{-\lambda J_3})$$

and it means that these functionals restricted to the one-dimensional subspace of A_d^η spanned by $q^{-\lambda J_3}$ are multiples of the functional e_d : $\psi_\alpha = q^{-\frac{\lambda}{2}} e_d,\ \psi_{\alpha^*} = q^{\frac{\lambda}{2}} e_d,\ \psi_\gamma = \psi_{\gamma^*} = 0$. Therefore

$$\begin{aligned}\Psi|_{D_\chi} &= \left[(1-q^2)\psi_\gamma\otimes\psi_+ - q\psi_{\alpha^*}\otimes\psi_o - q^{-1}\psi_\alpha\otimes\bar{\psi}_o\right]_{D_\chi} \\ &= (-q^{\frac{\lambda+n}{2}+1} - q^{-\frac{\lambda+n}{2}-1})\, e_d\otimes e_c = c_\chi e\end{aligned}$$

and

$$C(v_\chi)a = c_\chi(e\otimes \mathrm{id})\Delta a = c_\chi\, a.$$

In the same manner one shows that

$$\Psi'|_{D_\chi} = \left[(1-q^2)\psi_{\gamma^*}\otimes\psi_- - q\psi_{\alpha^*}\otimes\bar{\psi}_o - q^{-1}\psi_\alpha\otimes\psi_o\right]_{D_\chi} = c'_\chi e_d\otimes e_c = c'_\chi e$$

and $C'(v_\chi)a = c'_\chi(e\otimes\mathrm{id})\Delta a = c'_\chi\, a$. The Statement 2 is proven.

□

Remark. Let us remind that * - operation is an involutive intertwining map on the set of Gelfand spaces corresponding to the conjugation of the characters $(D_\chi)^* = D_{\chi^*}$. One can easily check that for $\chi = (n_1, n_2)$ it is given by

$$(n_1, n_2)^* \ = \ (\bar{n}_2, \bar{n}_1). \tag{2.14}$$

Therefore we have $c_{\chi^*} = \overline{c'_\chi}$ and $c'_{\chi^*} = \bar{c}_\chi$.

3 Invariant bilinear functionals on Gelfand spaces

We shall consider bilinear functionals on pairs of Gelfand spaces. Since the spaces carry the representations of the quantum Lorentz group, the subset of invariant bilinear functionals is distinguished. It turns out that for a given pair of spaces there exists at most one (up to a scalar multiple) invariant functional. Moreover a non-zero invariant bilinear functional exists only for special pairs of Gelfand spaces (cf. Theorem 4.9 and Definition 3.4).

We shall use the following notation. For a given character χ, (n_1, n_2) will denote the pair of complex numbers related to χ via (1.21):

$$\chi \ = \ q^{-(n_1+n_2-2)J_3}\otimes z^{n_1-n_2} \ = \ q^{-\lambda J_3}\otimes z^n,$$

where

$$\lambda \ := \ n_1+n_2-2 \quad \text{and} \quad n \ := \ n_1-n_2.$$

The difference n must be real integer. Therefore the imaginary parts of n_1 and n_2 are the same. They are defined mod$(\frac{2\pi}{\log q})$ and we fix (n_1, n_2) assuming that

$$\Im n_1 = \Im n_2 \in \left[0,\ \frac{-2\pi}{\log q}\right[. \tag{3.1}$$

The reader should notice that $q < 1$.
The corresponding Gelfand space is $D_\chi = q^{-\lambda J_3} \otimes \mathcal{Z}_n$ (cf. Theorem 2.2) and the induced action $v_\chi = \Delta|_{D_\chi}$ of the quantum Lorentz group is

$$v_\chi(q^{-\lambda J_3} \otimes a_c) := (\mathrm{id}_d \otimes \sigma^{-1} \otimes \mathrm{id}_c)(\Delta_d \otimes \Delta_c)(q^{-\lambda J_3} \otimes a_c) = q^{-\lambda J_3} \otimes \Big[(\sigma^{-1} \otimes \mathrm{id}_c)(q^{-\lambda J_3} \otimes \Delta_c a_c)\Big].$$

To abbreviate the notation the standard first factor $q^{-\lambda J_3}$ will often be omitted: $\mathcal{Z}_n$ carries the quantum Lorentz group representation

$$v^\lambda(a_c) := (\sigma^{-1} \otimes \mathrm{id}_c)(q^{-\lambda J_3} \otimes \Delta_c a_c) \tag{3.2}$$

and the isomorphism

$$\mathcal{Z}_n \ni a_c \longrightarrow q^{-\lambda J_3} \otimes a_c \in D_\chi$$

intertwines v^λ with v_χ. To pay attention to this omition we shall speak about *truncated notation*. In this sense v^λ action on $\mathcal{Z}_n$ is the truncated version of v_χ action on D_χ.

Let χ and χ' be characters of P. We shall use "prime" to denote the parameters related to χ' : $\lambda' := n_1' + n_2' - 2$, $n' := n_1' - n_2'$, $\chi' = q^{-\lambda' J_3} \otimes z^{n'}$.

Gelfand spaces are countable dimensional. Therefore the projective tensor product coincides with the algebraic one: $D_\chi \hat{\otimes} D_{\chi'} = D_\chi \otimes D_{\chi'}$. $D_\chi \otimes D_{\chi'}$ is subject to tensor product action $v_{\chi\chi'}$:

$$v_{\chi\chi'} = v_\chi \bigcirc\!\!\!\!\top\, v_{\chi'} = \hat{m}(v_\chi \otimes v_{\chi'}),$$

where $\hat{m}$ is the multiplication map

$$\hat{m} : D_\chi \hat{\otimes} \mathcal{A} \hat{\otimes} D_{\chi'} \hat{\otimes} \mathcal{A} \longrightarrow (D_\chi \otimes D_{\chi'}) \hat{\otimes} \mathcal{A}.$$

This is the continuous linear mapping such that

$$\hat{m}(x \otimes a \otimes y \otimes b) = x \otimes y \otimes ab$$

for any $x \in D_\chi$, $y \in D_{\chi'}$, $a, b \in \mathcal{A}$. With leg numbering notation $v_{\chi\chi'} = (v_\chi)_{13}\,(v_{\chi'})_{23}$.

Passing to level of $\mathcal{Z}_n$ spaces we obtain the truncated version of the action $v^{\lambda\lambda'} : \mathcal{Z}_n \otimes \mathcal{Z}_{n'} \longrightarrow (\mathcal{Z}_n \times \mathcal{Z}_{n'}) \hat{\otimes} \mathcal{A}$, where

$$v^{\lambda\lambda'} = v^\lambda \bigcirc\!\!\!\!\top\, v^{\lambda'} = (v^\lambda)_{13}\,(v^{\lambda'})_{23}. \tag{3.3}$$

We may identify bilinear functionals f on $D_\chi \times D_{\chi'}$ with linear functionals on $D_\chi \otimes D_{\chi'}$ and (using the truncated notation) with linear functionals on $\mathcal{Z}_n \otimes \mathcal{Z}_{n'}$

$$f(q^{-\lambda J_3} \otimes a_c,\, q^{-\lambda' J_3} \otimes a_c') = f(q^{-\lambda J_3} \otimes a_c \otimes q^{-\lambda' J_3} \otimes a_c') = f(a_c \otimes a_c')$$

for any $a_c \in \mathcal{Z}_n, a_c' \in \mathcal{Z}_{n'}$.

Let f be a linear functional on $\mathcal{Z}_n \otimes \mathcal{Z}_{n'}$. We say that f *is a $v^{\lambda\lambda'}$-invariant functional if*

$$(f \otimes \mathrm{id}_\mathcal{A}) v^{\lambda\lambda'}(x \otimes y) = f(x \otimes y) I_\mathcal{A} \tag{3.4}$$

for any $x \in \mathcal{Z}_n,\ y \in Z_{n'}$. Equivalently one can say that f intertwines the quantum Lorentz group action $v^{\lambda\lambda'}$ on $\mathcal{Z}_n \otimes \mathcal{Z}_{n'}$ and the trivial action on $\mathbf{C}$.

Let $v_c^{\lambda\lambda'},\ v_d^{\lambda\lambda'}$ be the restrictions of $v^{\lambda\lambda'}$ to the subgroups $S_qU(2)$ and $\widehat{S_qU(2)}$:

$$v_c^{\lambda\lambda'} := (\mathrm{id} \otimes (e_d \otimes \mathrm{id_c}))v^{\lambda\lambda'}, \qquad v_d^{\lambda\lambda'} := (\mathrm{id} \otimes (\mathrm{id_d} \otimes e_c))v^{\lambda\lambda'}. \tag{3.5}$$

Clearly f is an *QLG*-invariant functional if and only if it is $v_c^{\lambda\lambda'}$ and $v_d^{\lambda\lambda'}$-invariant. Due to this fact the problem of finding of all *QLG*-invariant functionals can be solved in two steps.

At the first step we consider $S_qU(2)$-invariance.

Applying $\mathrm{id} \otimes (e_d \otimes \mathrm{id_c})$ to both sides of (3.2) we see that for any $a \in \mathcal{Z}_n$

$$v_c^{\lambda}(a) \ = \ \Delta_c a$$

i.e. $v_c^{\lambda} = \Delta_c \mid_{\mathcal{Z}_n}$. Therefore $v_c^{\lambda\lambda'}$ is the natural action of $S_qU(2)$ on a tensor product $\mathcal{Z}_n \otimes \mathcal{Z}_{n'}$. Using the leg numbering notation we have (cf.(3.3))

$$v_c^{\lambda\lambda'}(x \otimes y) \ = \ \Delta_c(x)_{13}\Delta_c(y)_{23}$$

for any $x \in \mathcal{Z}_n,\ y \in \mathcal{Z}_{n'}$. A linear functional f on $\mathcal{Z}_n \otimes \mathcal{Z}_{n'}$ is $S_qU(2)$-invariant if

$$(f \otimes \mathrm{id_c})(\Delta_c(x)_{13}\Delta_c(y)_{23}) \ = \ f(x \otimes y)I_c. \tag{3.6}$$

To describe $S_qU(2)$-invariant functionals on $\mathcal{Z}_n \otimes \mathcal{Z}_{n'}$ we recall the basic concepts concerning $S_qU(2)$ group.

For any $a \in \mathcal{A}_c,\ \phi,\ \phi' \in \mathcal{A}_c'$ we set (cf. e.g. [15] p.129-130; [16] p.626)

$$\begin{aligned} \phi' * a \ &:= \ (\mathrm{id_c} \otimes \phi')\Delta_c(a), \\ a * \phi \ &:= \ (\phi \otimes \mathrm{id_c})\Delta_c(a), \\ \phi * \phi' \ &:= \ (\phi \otimes \phi')\Delta_c \end{aligned} \tag{3.7}$$

Then $\phi' * a,\, a * \phi \in \mathcal{A}_c,\ \phi * \phi' \in \mathcal{A}_c'$ and

$$(\phi' * \phi)(a) \ = \ \phi(a * \phi') \ = \ \phi'(\phi * a). \tag{3.8}$$

If h is the Haar measure on $S_qU(2)$ then

$$h * a \ = \ a * h \ = \ h(a)I_c \tag{3.9}$$

for any $a \in \mathcal{A}_c$.

We briefly describe the graded structure of the algebra $\mathcal{A}_c$.

Let $k, l \in \frac{\mathbf{1}}{\mathbf{2}}\mathbf{Z}$ be any half-integers and $(f_z)_{z\in\mathbf{C}}$ be the family of multiplicative functionals in $\mathcal{A}_c'$ defined in [16], Theorem 5.6. We say that $a \in \mathcal{A}_c$ is *a left-homogeneous element of degree* k if

$$a * f_z \ = \ q^{2kz}a \tag{3.10}$$

and $a \in \mathcal{A}_c$ is a *right-homogeneous element of degree l* if

$$f_z * a \;=\; q^{2lz} a. \tag{3.11}$$

Clearly the set of left(right)-homogeneous elements of a given degree is a vector space. Moreover by Theorem 5.6,4^o of [16] one can easily check that if a is a left(right)-homogeneous element of the degree k (l respectively) then a^* is a left(right)-homogeneous element of the degree (-k) ((-l) respectively).

We say that $a \in \mathcal{A}_c$ is a *homogeneous element of the degree (k,l)* if it is a left-homogeneous element of the degree k and a right-homogeneous element of the degree l. We denote by $J_{k,l}$ the vector space of homegeneous elements of degree (k,l). Then ${J_{k,l}}^* = J_{-k,-l}$ and from multiplicativity of f_z it follows that $J_{k,l} \cdot J_{k',l'} \subset J_{k+k',l+l'}$.

It is known (cf.Appendix B) that any matrix element u^s_{kl} is a homogeneous element of degree (k,l). These elements form a linear basis of $\mathcal{A}_c$. Therefore

$$(\, J_{k,l} \neq \{0\}\,) \iff (\, k-l \in \mathbf{Z}\,).$$

Let $k,l \in \frac{\mathbf{1}}{\mathbf{2}}\mathbf{Z}$, $k-l \in \mathbf{Z}$ and

$$s(k,l) := \max\{\mid k \mid, \mid l \mid\}. \tag{3.12}$$

Then

$$J_{k,l} \;= \{u^s_{kl} : s = s(k,l),\, s(k,l)+1,\, s(k,l)+2, ...\}^{\text{linear span}}. \tag{3.13}$$

Moreover $\mathcal{Z}_n = \sum_l^{\oplus} J_{\frac{n}{2},l}$ is the space of all left-homegeneous elements of the degree $\frac{n}{2}$ (cf.Proposition 2.1) and $\mathcal{A}_c = \sum_{k,l}^{\oplus} J_{k,l}$.

By Theorem 1.2 of [15] the elements of the form

$$a_{pnm} = (\alpha_c)_p (\gamma_c)_n (\gamma_c\gamma_c^*)^m,$$

where $p,n \in \mathbf{Z}$, $m = 0,1,2,...$ ($(\alpha_c)_p$ denotes α_c^p for $p = 0,1,2,3,...$ and $(\alpha_c^*)^{-p}$ for $p = -1,-2,....$ The same rule applies to $(\gamma_c)_n$.) also form a linear basis of $\mathcal{A}_c$ consisting of homogeneous elements since they are products of the matrix elements of the fundamental representation. It is clear that a_{pmn} is of degree $(\frac{1}{2}(-p+m),\, \frac{1}{2}(-p-m))$.

For $k,l \in \frac{\mathbf{1}}{\mathbf{2}}\mathbf{Z}$, $k-l \in \mathbf{Z}$ and $m = 0,1,2...$ we set

$$x^{(m)}_{2k,2l} := (\alpha_c)_{-(k+l)} (\gamma_c)_{k-l} (\gamma_c\gamma_c^*)^m. \tag{3.14}$$

Then

$$J_{k,l} \;= \{x^{(m)}_{2k,2l} :\; m = 0,1,2,...\}^{\text{linear span}}. \tag{3.15}$$

Let $\psi \in \mathcal{A}_c'$. We say that ψ *is supported by* $J_{k,l}$ if ψ vanishes on $J_{m,n}$: $J_{m,n} \subset \ker\psi$ for all $(m,n) \neq (k,l)$. For example the Haar functional h is supported by $J_{0,0}$ because

$$h\left((\alpha_c)_p(\gamma_c)_n(\gamma_c\gamma_c^*)^m\right) = \delta_{p0}\,\delta_{n0}\frac{1-q^2}{1-q^{2(m+1)}}, \tag{3.16}$$

(cf.[16, page 660]).

We shall use the bimodule structure of $\mathcal{A}_c'$. For any $a, b \in \mathcal{A}_c$ and $\phi \in \mathcal{A}_c'$, $b\,\phi\,a$ is the linear functional such that

$$(b\,\phi\,a)(x) \;:=\; \phi(axb)$$

for any $x \in \mathcal{A}_c$.

One can easily prove

Lemma 3.1

$$\{\ (bh) * a :\ a \in \mathcal{Z}_n,\ b \in \mathcal{Z}_{n'}\ \}^{\text{linear span}} \;=\; \begin{cases} J_{\frac{n}{2},-\frac{n'}{2}} & \text{for } n = n'(\text{mod}2) \\ \{0\} & \text{for } n \neq n'(\text{mod}2) \end{cases} . \tag{3.17}$$

Proof. Indeed setting $a = u^s_{\frac{n}{2}j}$ $(s \in S_n,\ j = -s, ...s)$ and $b = u^{s'}_{\frac{n}{2}'l}$ $(s' \in S_{n'},\ l = -s', ...s')$ we get

$$(bh) * a \;=\; (\text{id}_c \otimes h)\Delta_c(a)(I_c \otimes b) \;=\; \sum_{p=-s}^{s} u^s_{\frac{n}{2}p} h(u^s_{pj} u^{s'}_{\frac{n}{2}'l}).$$

Using the orthogonality relations [16, Theorem 5.7] and (B.21) we have

$$h(u^s_{ij} u^{s'}_{kl}) \;=\; \delta^{ss'}\delta_{i,-k}\delta_{j,-l}(-1)^{k-l} q^{-(k+l)}\, q^{2s} \frac{1-q^2}{1-q^{2(2s+1)}}.$$

Therefore

$$(bh) * a = \delta^{ss'}\delta_{j,-l}(-1)^{\frac{n}{2}'-l} q^{-(\frac{n}{2}'+l)} q^{2s} \frac{1-q^2}{1-q^{2(2s+1)}}\, u^s_{\frac{n}{2},-\frac{n}{2}'}$$

and the result follows by (2.4) and (3.13).

□

We shall formulate the main result concerning the $S_qU(2)$-invariant functionals.

Proposition 3.2
Let $n, n' \in \mathbf{Z}$ and f be a $S_qU(2)$ invariant functional on $\mathcal{Z}_n \otimes \mathcal{Z}_{n'}$.
Then there exists the unique linear functional $\psi \in \mathcal{A}_c'$ supported by $J_{\frac{n}{2},-\frac{n'}{2}}$ such that

$$f(a \otimes b) \;=\; \psi((bh) * a) \tag{3.18}$$

for any $a \in \mathcal{Z}_n,\ b \in \mathcal{Z}_{n'}$.
Conversely for any linear functional $\psi \in \mathcal{A}_c'$, formula (3.18) defines a $S_qU(2)$-invariant functional on $\mathcal{Z}_n \otimes \mathcal{Z}_{n'}$.

Remark 1. Equivalently the right hand side of (3.18) may be written as $h((a * \psi)b)$.

Remark 2. Any $S_qU(2)$-invariant linear functional on $\mathcal{A}_c \otimes \mathcal{A}_c$ defines by restriction an invariant functional on $\mathcal{Z}_n \otimes \mathcal{Z}_{n'}$. Conversely the right-hand side of (3.18) is meaningfull for any $a, b \in \mathcal{A}_c$ and gives the extension of f to the $(\Delta_c \textcircled{\top} \Delta_c)$-invariant functional on the whole $\mathcal{A}_c \otimes \mathcal{A}_c$. If ψ is supported by $J_{\frac{n}{2},-\frac{n'}{2}}$ then the extension restricted to $\mathcal{Z}_m \otimes \mathcal{Z}_{m'}$ vanishes for $(m, m') \neq (n, n')$.
On the other hand for any m, m' the space $\mathcal{Z}_m \otimes \mathcal{Z}_{m'}$ is $S_qU(2)$-invariant and $\mathcal{A}_c \otimes \mathcal{A}_c =$

$\sum_{m,m'}^{\oplus} \mathcal{Z}_m \otimes \mathcal{Z}_{m'}$. Putting $f(a\otimes b) = 0$ whenever $a \in \mathcal{Z}_m$, $b \in \mathcal{Z}_{m'}$ and $(m, m') \neq (n, n')$ we extend the left-hand side of (3.18) to the $S_qU(2)$-invariant functional on the whole $\mathcal{A}_c \otimes \mathcal{A}_c$. We call this extension the natural one and denote it by the same letter. Therefore the support property of ψ ensures that both extensions coincide and the formula (3.18) holds for any $a, b \in \mathcal{A}_c$.

Proof: The uniqueness of the functional ψ follows immediately from (3.17).

For any $x, y \in \mathcal{A}_c$ we set

$$W(x \otimes y) \ := \ \Delta_c(x)(I_c \otimes y) \tag{3.19}$$

By [16, Theorem 4.9] $W : \mathcal{A}_c \otimes \mathcal{A}_c \longrightarrow \mathcal{A}_c \otimes \mathcal{A}_c$ is a bijective linear map (denoted by s' in [16]).

Let i_c denote the trivial action of $S_qU(2)$ on $\mathcal{A}_c$: $i_c(x) = x \otimes I_c$ for any $x \in \mathcal{A}_c$. Using the coassociativity of Δ_c one can easily check that

$$(W \otimes \mathrm{id_c})(\Delta_c ⓣ \Delta_c) \ = \ (i_c ⓣ \Delta_c) W \tag{3.20}$$

i.e. W is an intertwining map for the $\Delta_c ⓣ \Delta_c$ and $i_c ⓣ \Delta_c$ actions of $S_qU(2)$ on $\mathcal{A}_c \otimes \mathcal{A}_c$.

Assume that f is a $S_qU(2)$ invariant functional on $\mathcal{Z}_n \otimes \mathcal{Z}_{n'}$. Denoting by the same letter its natural extension to a $(\Delta_c ⓣ \Delta_c)$-invariant functional on $\mathcal{A}_c \otimes \mathcal{A}_c$ we see that $f \circ W^{-1}$ is the $(i_c ⓣ \Delta_c)$-invariant functional on $\mathcal{A}_c \otimes \mathcal{A}_c$ and for any $a \in \mathcal{A}_c$ the linear functional

$$\mathcal{A}_c \ni b \ \longrightarrow f \circ W^{-1}(a \otimes b) \in \mathbf{C} \tag{3.21}$$

is Δ_c-invariant. Therefore it is a multiple of the Haar functional h :

$$f \circ W^{-1}(a \otimes b) \ = \ \psi(a) h(b) \tag{3.22}$$

where the coefficient $\psi(a)$ depends linearly on $a \in \mathcal{A}_c$: $\psi \in \mathcal{A}_c'$. We have $f \circ W^{-1} = \psi \otimes h$ and

$$f = (\psi \otimes h) \circ W. \tag{3.23}$$

Now (3.18) follows easily: for any $a \in \mathcal{Z}_n$ and $b \in \mathcal{Z}_{n'}$ we have

$$f(a \otimes b) \ = \ (\psi \otimes h) W(a \otimes b) \ = \ (\psi \otimes h)(\Delta_c(a)[I_c \otimes b]) \ = \ h((a * \psi) b).$$

We have to find the support of the functional ψ. Inserting in (3.22) $b = I_c$ we get

$$\psi(a) \ = \ f \circ W^{-1}(a \otimes I_c).$$

Taking into account the formula (4.36) of [16] we get

$$\psi(a) \ = \ f \circ (\mathrm{id_c} \otimes \kappa_c) \circ \Delta_c(a),$$

where κ_c is the co-inverse related to $S_qU(2)$.
If $a = u^s_{\frac{m}{2}, -\frac{m'}{2}}$, then using (5.5) of [16] we obtain

$$(\mathrm{id_c} \otimes \kappa_c) \Delta_c(a) \ = \ \sum_{p=-s}^{s} u^s_{\frac{m}{2}, p} \otimes (u^s_{-\frac{m'}{2}, p})^*.$$

It shows that $(\mathrm{id_c}\otimes\kappa_c)\Delta_c(a)\in\mathcal{Z}_m\otimes\mathcal{Z}_{m'}$. If $(m,m')\neq(n,n')$ then $f|_{\mathcal{Z}_m\otimes\mathcal{Z}_{m'}}=0$ and $\psi(a)=0$. It means that ψ is supported by $J_{\frac{n}{2},-\frac{n'}{2}}$.

Now let $\psi\in\mathcal{A}_c'$. Then $\psi\otimes h$ intertwines $(i_c \textcircled{\top} \Delta_c)$-action on $\mathcal{A}_c\otimes\mathcal{A}_c$ with the trivial action of $S_qU(2)$ on $\mathbf{C}$. Consequently $(\psi\otimes h)\circ W$ intertwines $(\Delta_c \textcircled{\top} \Delta_c)$-action on $\mathcal{A}_c\otimes\mathcal{A}_c$ with the trivial action on $\mathbf{C}$. In other words $(\psi\otimes h)\circ W$ is a $S_qU(2)$- invariant functional on $\mathcal{A}_c\otimes\mathcal{A}_c$. A simple computation shows that $(\psi\otimes h)\circ W$ restricted to $\mathcal{Z}_n\otimes\mathcal{Z}_{n'}$ coincides with (3.18):

$$(\psi\otimes h)W(a\otimes b)\;=\;(\psi\otimes h)\left(\Delta_c(a)(I_c\otimes b)\right)\;=\;(\psi\otimes b\cdot h)\Delta_c a\;=h((a*\psi)b)$$

for any $a\in\mathcal{Z}_n$ and $b\in\mathcal{Z}_{n'}$.

□

Now we investigate the $\widehat{S_qU(2)}$-invariance.
We know that $v_d^{\lambda\lambda'}$ is a linear map

$$v_d^{\lambda\lambda'}\;:\;\mathcal{Z}_n\otimes\mathcal{Z}_{n'}\;\longrightarrow(\mathcal{Z}_n\otimes\mathcal{Z}_{n'})\hat{\otimes}A_d^\eta\;=\;\sum_{s\in S}^{\oplus}\mathcal{Z}_n\otimes\mathcal{Z}_{n'}\otimes B(H^s).$$

Therefore $v_d^{\lambda\lambda'}$ is a collection of maps $\{(v_d^{\lambda\lambda'})^s\;:\;s\in S\}$ where $(v_d^{\lambda\lambda'})^s\;:\mathcal{Z}_n\otimes\mathcal{Z}_{n'}\longrightarrow\mathcal{Z}_n\otimes\mathcal{Z}_{n'}\otimes B(H^s)$ is the s-component of $v_d^{\lambda\lambda'}$. Using the canonical basis of $B(H^s)$ one may identify elements of $B(H^s)$ with $(2s+1)\times(2s+1)$ matrices with complex entries. Therefore

$$\left(v_d^{\lambda\lambda'}\right)^s\;=\;\left(\left(v_d^{\lambda\lambda'}\right)^s_{ab}\right)_{a,b=-s,-s+1,\dots s},$$

where $\left(v_d^{\lambda\lambda'}\right)^s_{ab}$ are linear maps acting on $\mathcal{Z}_n\otimes\mathcal{Z}_{n'}$.
Clearly

$$\left(v_d^{\lambda\lambda'}\right)^s_{ab}\;=\;(\mathrm{id}\otimes\xi^s_{ab})v_d^{\lambda\lambda'},\tag{3.24}$$

where ξ^s_{ab} are linear functionals on A_d^η considered in Appendix B.

A linear functional $f:\,\mathcal{Z}_n\otimes\mathcal{Z}_{n'}\longrightarrow\mathbf{C}$ is $v_d^{\lambda\lambda'}$-invariant if and only if

$$(f\otimes\mathrm{id_d})v_d^{\lambda\lambda'}(x\otimes y)\;=\;f(x\otimes y)I_d\tag{3.25}$$

It means that

$$(f\otimes\xi)v_d^{\lambda\lambda'}(x\otimes y)\;=\;f(x\otimes y)\xi(I_d)\tag{3.26}$$

for all $\xi\in(A_d^\eta)'$. Remembering that $v_d^{\lambda\lambda'}$ is a representation one can easily show that the set of functionals ξ satisfying (3.26) is closed under convolution product. Therefore it is enough to verify it for functionals $\xi=\xi^{\frac12}_{ab}$, $(a,b=\pm\frac12)$ (cf. Appendix B). It shows that (3.25) is equivalent to

$$(f\otimes\xi^{\frac12}_{ab})v_d^{\lambda\lambda'}(x\otimes y)\;=\;f(x\otimes y)\xi^{\frac12}_{ab}(I_d).\tag{3.27}$$

Clearly $\xi^{\frac12}_{ab}(I_d)=\delta_{ab}$. Formula (3.24) shows that

$$(f\otimes\xi^{\frac12}_{ab})v_d^{\lambda\lambda'}\;=\;f\circ(v_d^{\lambda\lambda'})^{\frac12}_{ab}.$$

Combining this result with (3.27) we see that f is $S_q\widehat{U(2)}$ invariant if and only if

$$f \circ (v_d^{\lambda\lambda'})^{\frac{1}{2}}_{ab} \;=\; \delta_{ab}\, f \tag{3.28}$$

for any $a, b = \pm\frac{1}{2}$.

Now we look at $v_d^{\lambda\lambda'}$ in more detail.
Applying $\mathrm{id} \otimes (\mathrm{id_d} \otimes e_c)$ to both sides of (3.2) we get

$$v_d^{\lambda}(x) \;=\; \sigma^{-1}(q^{-\lambda J_3} \otimes x) \;=\; \tau[\,u^*(q^{-\lambda J_3} \otimes x)u\,]$$

for any $x \in \mathcal{Z}_n$. A similar formula holds for $v_d^{\lambda'}$. Therefore using (3.3)

$$v_d^{\lambda\lambda'}(x \otimes y) \;=\; \{\tau[\,u^*(q^{-\lambda J_3} \otimes x)u\,]\}_{13}\{\tau[\,u^*(q^{-\lambda' J_3} \otimes y)u\,]\}_{23}$$

for any $x \in \mathcal{Z}_n,\ y \in \mathcal{Z}_{n'}$ and

$$(v_d^{\lambda\lambda'})^s(x \otimes y) \;=\; \{\tau[\,(u^s)^*(q^{-\lambda J_3^s} \otimes x)u^s\,]\}_{13}\{\tau[\,(u^s)^*(q^{-\lambda' J_3^s} \otimes y)u^s\,]\}_{23}.$$

Inserting $s = \frac{1}{2}$ and rewriting the formula in terms of matrix elements we get

$$(v_d^{\lambda\lambda'})^{\frac{1}{2}}_{ab}(x \otimes y) \;=\; \sum_{mkl} q^{-k\lambda - l\lambda'}(u^{\frac{1}{2}}_{ka})^* x u^{\frac{1}{2}}_{km} \otimes (u^{\frac{1}{2}}_{lm})^* y u^{\frac{1}{2}}_{lb}. \tag{3.29}$$

Remembering that $(v_d^{\lambda\lambda'})^{\frac{1}{2}}_{ab}$ is a linear mapping acting on $\mathcal{Z}_n \otimes \mathcal{Z}_{n'}$ we conclude that $\sum_{mkl} q^{-k\lambda - l\lambda'}(u^{\frac{1}{2}}_{ka})^* x u^{\frac{1}{2}}_{km} \otimes (u^{\frac{1}{2}}_{lm})^* y u^{\frac{1}{2}}_{lb} \in \mathcal{Z}_n \otimes \mathcal{Z}_{n'}$ for any $x \in \mathcal{Z}_n$ and $y \in \mathcal{Z}_{n'}$. Comparing (3.28) with (3.29) we obtain the following result

Proposition 3.3
Let f be a functional on $\mathcal{Z}_n \otimes \mathcal{Z}_{n'}$. f is $v_d^{\lambda\lambda'}$-invariant if and only if

$$\sum_{mkl} q^{-k\lambda - l\lambda'} f((u^{\frac{1}{2}}_{ka})^* x u^{\frac{1}{2}}_{km} \otimes (u^{\frac{1}{2}}_{lm})^* y u^{\frac{1}{2}}_{lb}) \;=\; f(x \otimes y)\delta_{ab} \tag{3.30}$$

for any $x \in \mathcal{Z}_n$, $y \in \mathcal{Z}_{n'}$ and $a, b = \pm\frac{1}{2}$. The summation is over all $m, k, l \in \{\frac{1}{2}, -\frac{1}{2}\}$.

Let f be $v^{\lambda\lambda'}$-invariant functional on $\mathcal{Z}_n \otimes \mathcal{Z}_{n'}$. Then f satisfies (3.30) and is of the form (3.18) where ψ is a linear functional supported by $J_{\frac{n}{2}, -\frac{n'}{2}}$. Then for any $x, y \in \mathcal{A}_c$

$$\psi\left(\sum_{klm} q^{-k\lambda - l\lambda'}\left((u^{\frac{1}{2}}_{lm})^* y u^{\frac{1}{2}}_{lb} \cdot h\right) * \left((u^{\frac{1}{2}}_{ka})^* x u^{\frac{1}{2}}_{km}\right)\right) \;=\; \psi((yh) * x)\delta_{ab}. \tag{3.31}$$

Indeed if $x \in \mathcal{Z}_p$ and $y \in \mathcal{Z}_{p'}$, then $(u^{\frac{1}{2}}_{ka})^* x u^{\frac{1}{2}}_{km} \in \mathcal{Z}_p$, $(u^{\frac{1}{2}}_{lm})^* y u^{\frac{1}{2}}_{lb} \in \mathcal{Z}_{p'}$ and by (3.17) the arguments of ψ in (3.31) belong to $J_{\frac{p}{2}, -\frac{p'}{2}}$. If $(p, p') \neq (n, n')$ then both sides of (3.31) vanish due to the support property of ψ. If $(p,p')=(n,n')$ then (3.31) follows immediately from (3.18). Conversely if ψ is a linear functional on $\mathcal{A}_c$ supported by $J_{\frac{p}{2},-\frac{p'}{2}}$ and satisfies (3.31) then the functional f on $\mathcal{Z}_n\otimes\mathcal{Z}_{n'}$ inntroduced by (3.18) is $v_d^{\lambda\lambda'}$-invariant.

We shall use Sweedler notation $\Delta_c x = x_{(1)}\otimes x_{(2)},\ yh*x = x_{(1)}\,h(x_{(2)}y)$. We know that $u^{\frac{1}{2}}$ is unitary: $\sum_m u^{\frac{1}{2}}_{pm}(u^{\frac{1}{2}}_{lm})^* = \delta_{pl}\,I_c$. Using this relation we compute

$$\begin{aligned}\sum_m \left((u^{\frac{1}{2}}_{lm})^* y u^{\frac{1}{2}}_{lb}\cdot h\right) * \left((u^{\frac{1}{2}}_{ka})^* x u^{\frac{1}{2}}_{km}\right) &= \sum_{mrp}(u^{\frac{1}{2}}_{kr})^* x_{(1)} u^{\frac{1}{2}}_{kp}\, h\left((u^{\frac{1}{2}}_{ra})^* x_{(2)} u^{\frac{1}{2}}_{pm}(u^{\frac{1}{2}}_{lm})^* y u^{\frac{1}{2}}_{lb}\right)\\ &= \sum_r (u^{\frac{1}{2}}_{kr})^* x_{(1)} u^{\frac{1}{2}}_{kl}\, h\left((u^{\frac{1}{2}}_{ra})^* x_{(2)} y u^{\frac{1}{2}}_{lb}\right)\end{aligned}$$

and

$$\sum_{klm} q^{-k\lambda-l\lambda'}\left((u^{\frac{1}{2}}_{lm})^* y u^{\frac{1}{2}}_{lb}\cdot h\right) * \left((u^{\frac{1}{2}}_{ka})^* x u^{\frac{1}{2}}_{km}\right) \;=\; \Omega_{ab}(x_{(1)}\otimes x_{(2)}y), \tag{3.32}$$

where $\Omega_{ab}:\mathcal{A}_c\otimes\mathcal{A}_c\to\mathcal{A}_c$ is the linear map such that

$$\Omega_{ab}(x\otimes y) := \sum_{klr} q^{-k\lambda-l\lambda'}(u^{\frac{1}{2}}_{kr})^* x u^{\frac{1}{2}}_{kl}\, h\left((u^{\frac{1}{2}}_{ra})^* y u^{\frac{1}{2}}_{lb}\right). \tag{3.33}$$

Therefore (3.31) is equivalent to

$$\psi\left(\Omega_{ab}(x_{(1)}\otimes x_{(2)}y)\right) = \psi\left((\mathrm{id}\otimes h)(x_{(1)}\otimes x_{(2)}y)\right)\delta_{ab}. \tag{3.34}$$

According to (3.19), $x_{(1)}\otimes x_{(2)}y = W(x\otimes y)$. Remembering that W maps $\mathcal{A}_c\otimes\mathcal{A}_c$ onto $\mathcal{A}_c\otimes\mathcal{A}_c$ we see that (3.34) is equivalent to

$$\psi\left(\Omega_{ab}(x\otimes y)\right) = (\psi\otimes h)(x\otimes y)\delta_{ab}.$$

This equation combined with (3.33) leads to the following definition

Definition 3.4
Any functional $\psi\in\mathcal{A}'_c$ supported by $J_{\frac{n}{2},-\frac{n'}{2}}$ and such that

$$\sum_{klr} q^{-k\lambda-l\lambda'}\psi\left((u^{\frac{1}{2}}_{kr})^* x u^{\frac{1}{2}}_{kl}\right)\, h\left((u^{\frac{1}{2}}_{ra})^* y u^{\frac{1}{2}}_{lb}\right) \;=\; \psi(x)\cdot h(y)\,\delta_{ab} \tag{3.35}$$

for any $x,y\in\mathcal{A}_c$ and any $a,b=\pm\frac{1}{2}$ is called a (χ,χ')-spherical functional.
We say that a pair (χ,χ') of characters is admissible whenever there exists a non-zero (χ,χ')-spherical functional.

Therefore we have proved (in the complete notation)

Theorem 3.5
The linear functional f on $D_\chi \otimes D_{\chi'}$ is $v_{\chi\chi'}$-invariant if and only if there exists a (χ,χ')-spherical functional $\psi \in \mathcal{A}_c'$ such that

$$\begin{aligned} f(q^{-\lambda J_3} \otimes a \otimes q^{-\lambda' J_3} \otimes b) &= \psi((bh) * a) \\ &= h((a * \psi) b) \end{aligned}$$

for any $a \in \mathcal{Z}_n$ and $b \in \mathcal{Z}_{n'}$. In particular $D_\chi \times D_{\chi'}$ carries a non-zero invariant bilinear form if and only if the pair (χ,χ') is admissible.

The equation (3.35) characterizing (χ,χ')-spherical functionals is still very complicated. It involves two variables a, b running over $\mathcal{A}_c$. The following Lemma introduces some simplification.

Lemma 3.6
Let $\psi \in \mathcal{A}_c'$. Then the following conditions are equivalent
i) ψ satisfies the equations (3.35),
ii)

$$\boxed{\begin{aligned} \psi(\alpha_c x \gamma_c) &= q^{1+\lambda} \psi(\gamma_c x \alpha_c) \\ \psi(\gamma_c^* x \alpha_c^*) &= q^{1+\lambda} \psi(\alpha_c^* x \gamma_c^*) \\ \psi(q^\lambda \alpha_c^* x \alpha_c + \gamma_c^* x \gamma_c) &= q^{\frac{1}{2}(\lambda-\lambda')-2} \psi(x) \\ \psi(q^{-\lambda} \alpha_c x \alpha_c^* + q^2 \gamma_c x \gamma_c^*) &= q^{\frac{1}{2}(\lambda'-\lambda)+2} \psi(x) \end{aligned}} \tag{3.36}$$

for any $x \in \mathcal{A}_c$.

Proof: To make the notation shorter we replace the indices $-\frac{1}{2}$, $\frac{1}{2}$ by $-$, $+$ respectively. For any a, b, l, $r \in \{-, +\}$ and $x, y \in \mathcal{A}_c$ we set

$$L_{rl}(x) = q^{-l\lambda'} \psi \left(\sum_{k=\pm\frac{1}{2}} q^{-k\lambda} (u^{\frac{1}{2}}_{kr})^* x u^{\frac{1}{2}}_{kl} \right), \tag{3.37}$$

$$h_{lbra}(y) = h\left((u^{\frac{1}{2}}_{ra})^* y u^{\frac{1}{2}}_{lb} \right).$$

Then L_{rl} and $h_{lbra} = u^{\frac{1}{2}}_{lb} \cdot h \cdot (u^{\frac{1}{2}}_{ra})^*$ are linear functionals on $\mathcal{A}_c$. Equations (3.35) take the form

$$\sum_{rl} L_{rl}(x)\, h_{lbra}(y) = \delta_{ab}\, \psi(x)\, h(y). \tag{3.38}$$

All summands are presented in Table 1.

Table 1

$l\ b\ r\ a$	$u_{lb}^{\frac{1}{2}}$	$u_{ra}^{\frac{1}{2}}$	h_{lbra}	L_{rl}	$l\ b\ r\ a$	$u_{lb}^{\frac{1}{2}}$	$u_{ra}^{\frac{1}{2}}$	h_{lbra}	L_{rl}
$----$	α_c	α_c	$\alpha_c \cdot h \cdot \alpha_c^*$	L_{--}	$-+-+$	$-q\gamma_c^*$	$-q\gamma_c^*$	$q^2\gamma_c^* \cdot h \cdot \gamma_c$	L_{--}
$--+-$	α_c	γ_c	$\alpha_c \cdot h \cdot \gamma_c^*$	L_{+-}	$-+++$	$-q\gamma_c^*$	α_c^*	$-q\gamma_c^* \cdot h \cdot \alpha_c$	L_{+-}
$+---$	γ_c	α_c	$\gamma_c \cdot h \cdot \alpha_c^*$	L_{-+}	$++-+$	α_c^*	$-q\gamma_c^*$	$-q\alpha_c^* \cdot h \cdot \gamma_c$	L_{-+}
$+-+-$	γ_c	γ_c	$\gamma_c \cdot h \cdot \gamma_c^*$	L_{++}	$++++$	α_c^*	α_c^*	$\alpha_c^* \cdot h \cdot \alpha_c$	L_{++}
$-+--$	$-q\gamma_c^*$	α_c	$-q\gamma_c^* \cdot h \cdot \alpha_c^*$	L_{--}	$---+$	α_c	$-q\gamma_c^*$	$-q\alpha_c \cdot h \cdot \gamma_c$	L_{--}
$-++-$	$-q\gamma_c^*$	γ_c	$-q\gamma_c^* \cdot h \cdot \gamma_c^*$	L_{+-}	$--++$	α_c	α_c^*	$\alpha_c \cdot h \cdot \alpha_c$	L_{+-}
$++--$	α_c^*	α_c	$\alpha_c^* \cdot h \cdot \alpha_c^*$	L_{-+}	$+--+$	γ_c	$-q\gamma_c^*$	$-q\gamma_c \cdot h \cdot \gamma_c$	L_{-+}
$+++-$	α_c^*	γ_c	$\alpha_c^* \cdot h \cdot \gamma_c^*$	L_{++}	$+-++$	γ_c	α_c^*	$\gamma_c \cdot h \cdot \alpha_c$	L_{++}

At first we shall show that equations (3.38) imply that

$$L_{-+} \;=\; 0 \;=\; L_{+-}. \tag{3.39}$$

Indeed taking $(a, b) = (+, -)$ in (3.38) we get

$$L_{--}(x)\, h_{---+}(y) \;+\; L_{++}(x)\, h_{+-++}(y) \;+\; L_{-+}(x)\, h_{+--+}(y) \;+\; L_{+-}(x)\, h_{--++}(y) \;=\; 0.$$

Putting $y = (\gamma_c^*)^2$ and using (3.16) we get

$$h_{---+}((\gamma_c^*)^2) \;=\; h_{+-++}((\gamma_c^*)^2) \;=\; h_{--++}((\gamma_c^*)^2) \;=\; 0,$$
$$h_{+--+}((\gamma_c^*)^2) \;=\; -q(1-q^2)(1-q^6)^{-1} \;\neq\; 0.$$

Therefore $L_{-+} = 0$. Putting $y = (\alpha_c^*)^2$ we get in the same manner that the only nonzero coefficient is

$$h_{--++}((\alpha_c^*)^2) \;=\; h(\alpha_c\alpha_c^*\alpha_c^*\alpha_c) \;=\; h((I_c - q^2\gamma_c\gamma_c^*)(I_c - \gamma_c^*\gamma_c)) \;=\; q^2(1-q^2)(1-q^6)^{-1}$$

and $L_{+-} = 0$.
Due to (3.39), the sum on the left hand side of (3.38) reduces to two terms. This way we showed that the equations (3.38) are equivalent to

$$\begin{aligned} L_{+-} \;&=\; 0 \;=\; L_{-+}, \\ L_{--} \otimes h_{-+--} \;+\; L_{++} \otimes h_{+++-} \;&=\; 0, \\ L_{--} \otimes h_{---+} \;+\; L_{++} \otimes h_{+-++} \;&=\; 0, \\ L_{--} \otimes h_{----} \;+\; L_{++} \otimes h_{+-+-} \;-\; \psi \otimes h \;&=\; 0, \\ L_{--} \otimes h_{-+-+} \;+\; L_{++} \otimes h_{++++} \;-\; \psi \otimes h \;&=\; 0. \end{aligned} \tag{3.40}$$

Now using the fact that $h(ab) = h(b(f_1 * a * f_1))$ (cf. e.g. [16], (5.20) or [19], (2.21)) and $f_1 * u^s_{kl} * f_1 = q^{2(k+l)} u^s_{kl}$ (u^s_{kl} is a homogeneous element of degree (k, l)) we obtain

$$h \cdot \alpha_c = q^{-2} \alpha_c \cdot h, \quad h \cdot \alpha_c^* = q^2 \alpha_c^* \cdot h, \quad h \cdot \gamma_c = \gamma_c \cdot h, \quad h \cdot \gamma_c^* = \gamma_c^* \cdot h.$$

Then

$$h_{-+--} = -q\gamma_c^* \cdot h \cdot \alpha_c^* = -q^3 \gamma_c^* \alpha_c^* \cdot h = -q^4 \alpha_c^* \gamma_c^* \cdot h = -q^4 (\alpha_c^* \cdot h \cdot \gamma_c^*) = -q^4 h_{+++-}.$$

In the same manner we get

$$h_{---+} = -q\alpha_c \cdot h \cdot \gamma_c = -q\alpha_c \gamma_c \cdot h = -q^2 \gamma_c \alpha_c \cdot h = -q^4 (\gamma_c \cdot h \cdot \alpha_c) = -q^4 h_{+-++}$$

and

$$h_{----} = \alpha_c \cdot h \cdot \alpha_c^* = q^2 \alpha_c \alpha_c^* \cdot h = q^2 (I_c - q^2 \gamma_c \gamma_c^*) \cdot h = q^2 h - q^4 h_{+-+-},$$

$$h_{++++} = \alpha_c^* \cdot h \cdot \alpha_c = q^{-2} \alpha_c^* \alpha_c \cdot h = q^{-2} (I_c - \gamma_c \gamma_c^*) \cdot h = q^{-2} h - q^{-4} h_{-+-+}.$$

Therefore the last four equations in (3.40) take the form

$$(L_{++} - q^4 L_{--}) \otimes h_{+++-} = 0,$$

$$(L_{++} - q^4 L_{--}) \otimes h_{+-++} = 0,$$

$$(L_{++} - q^4 L_{--}) \otimes h_{+-+-} + (q^2 L_{--} - \psi) \otimes h = 0,$$

$$(L_{--} - q^{-4} L_{++}) \otimes h_{-+-+} + (q^{-2} L_{++} - \psi) \otimes h = 0.$$

Having in mind that h_{+++-}, h_{+-++} are non-zero functionals we see that equations (3.35) are equivalent to

$$\begin{aligned} L_{+-} &= 0 = L_{-+}, \\ L_{--} = q^{-2}\psi, &\qquad L_{++} = q^2 \psi. \end{aligned} \tag{3.41}$$

In particular $L_{++} = q^4 L_{--}$.
The values of L_{rl} (cf.(3.37)) corresponding to all possible choices $r, l = \pm\frac{1}{2}$ are presented in Table 2.

Table 2

$r\ l$	$u^{\frac{1}{2}}_{+,r}$	$u^{\frac{1}{2}}_{+,l}$	$u^{\frac{1}{2}}_{-,r}$	$u^{\frac{1}{2}}_{-,l}$	$L_{rl}(x)$
$+-$	α_c^*	γ_c	$-q\gamma_c^*$	α_c	$q^{\frac{1}{2}\lambda'} \psi(q^{-\frac{1}{2}\lambda} \alpha_c x \gamma_c - q^{1+\frac{1}{2}\lambda} \gamma_c x \alpha_c)$
$-+$	γ_c	α_c^*	α_c	$-q\gamma_c^*$	$q^{-\frac{1}{2}\lambda'} \psi(q^{-\frac{1}{2}\lambda} \gamma_c^* x \alpha_c^* - q^{1+\frac{1}{2}\lambda} \alpha_c^* x \gamma_c^*)$
$--$	γ_c	γ_c	α_c	α_c	$q^{\frac{1}{2}\lambda'} \psi(q^{-\frac{1}{2}\lambda} \gamma_c^* x \gamma_c + q^{\frac{1}{2}\lambda} \alpha_c^* x \alpha_c)$
$++$	α_c^*	α_c^*	$-q\gamma_c^*$	$-q\gamma_c^*$	$q^{-\frac{1}{2}\lambda'} \psi(q^{-\frac{1}{2}\lambda} \alpha_c x \alpha_c^* + q^{2+\frac{1}{2}\lambda} \gamma_c x \gamma_c^*)$

(3.42)

Now we see that the equations (3.41) coincide with (3.36). Therefore (3.35) and (3.36) are equivalent and this ends the proof.

□

Now we are ready to formulate the result which allows to select the pairs of Gelfand spaces with non-zero Lorentz invariant bilinear functionals.

Lemma 3.7
Let $\psi \in \mathcal{A}_c'$. Then the following conditions are equivalent
i) ψ is a (χ, χ')- spherical functional;
ii) ψ is supported by $J_{\frac{n}{2},-\frac{n'}{2}}$ and for any $x \in \mathcal{A}_c$:

$$\boxed{\begin{array}{rcl} \psi(\alpha_c x) &=& q^{\frac{1}{2}(\lambda+\lambda')+2}\psi(x\alpha_c) \\ \psi(\alpha_c^* x) &=& q^{-\frac{1}{2}(\lambda+\lambda')-2}\psi(x\alpha_c^*) \\ \psi(\gamma_c x) &=& q^{-\frac{1}{2}(\lambda-\lambda')}\psi(x\gamma_c) \\ \psi(\gamma_c^* x) &=& q^{\frac{1}{2}(\lambda-\lambda')}\psi(x\gamma_c^*) \end{array}} \tag{3.43}$$

In particular

$$\psi(\gamma_c\gamma_c^* x) \;=\; \psi(x\gamma_c\gamma_c^*). \tag{3.44}$$

Proof: We have to show that equations (3.43) are equivalent to (3.36). By straightforward calculation we get that (3.43) implies (3.36). Conversely assume (3.36). By the second equation of (3.36) we have

$$\psi(\gamma_c^* x\alpha_c^*\gamma_c) \;=\; q^{-1}\psi(\gamma_c^* x\gamma_c\alpha_c^*) \;=\; q^{\lambda}\psi(\alpha_c^* x\gamma_c\gamma_c^*).$$

Inserting $x \to x\alpha_c^*$ in the third equation of (3.36) and using above equality we obtain on the left hand side

$$\begin{aligned} \psi(q^{\lambda}\alpha_c^* x\alpha_c^*\alpha_c + \gamma_c^* x\alpha_c^*\gamma_c) &= \psi(q^{\lambda}\alpha_c^* x[I_c - \gamma_c^*\gamma_c] + q^{-1}\gamma_c^* x\gamma_c\alpha_c^*) \\ &= q^{\lambda}\psi(\alpha_c^* x[I_c - \gamma_c^*\gamma_c] + \alpha_c^* x\gamma_c\gamma_c^*) \;=\; q^{\lambda}\psi(\alpha_c^* x). \end{aligned}$$

Therefore

$$q^{\lambda}\psi(\alpha_c^* x) \;= q^{\frac{1}{2}(\lambda-\lambda')-2}\psi(x\alpha_c^*).$$

Clearly this is equivalent to the second equation of (3.43). In the same manner inserting $x \to \alpha_c x$ in the third equation of (3.36) and using the first one we get

$$\begin{aligned} q^{\frac{1}{2}(\lambda-\lambda')-2}\psi(\alpha_c x) &= \psi(q^{\lambda}\alpha_c^*\alpha_c x\alpha_c + \gamma_c^*\alpha_c x\gamma_c) \;=\; \psi(q^{\lambda}\alpha_c^*\alpha_c x\alpha_c + q^{-1}\alpha_c\gamma_c^* x\gamma_c) \\ &= q^{\lambda}\psi(\alpha_c^*\alpha_c x\alpha_c + \gamma_c^*\gamma_c x\alpha_c) \;=\; q^{\lambda}\psi(x\alpha_c) \end{aligned}$$

and this is the first equation of (3.43). Combining this two equations we have $\psi(\alpha_c^*\alpha_c x) = \psi(x\alpha_c^*\alpha_c)$ and since $\gamma_c^*\gamma_c = I_c - \alpha_c^*\alpha_c$ this proves (3.44).
Now inserting $x \to \alpha_c^* x$ in the first equation of (3.36) we get on left hand side

$$\psi(\alpha_c\alpha_c^* x\gamma_c) \;=\; \psi([I_c - q^2\gamma_c\gamma_c^*]x\gamma_c) \;=\; \psi(x\gamma_c) - q^2\psi(\gamma_c^*\gamma_c x\gamma_c)$$

and on the right hand side using the third equation of (3.36)

$$\begin{aligned} q^{1+\lambda}\psi(\gamma_c\alpha_c^* x\alpha_c) \;&=\; q^2 q^{\lambda}\psi(\alpha_c^*\gamma_c x\alpha_c) \;=\; q^2[q^{\frac{1}{2}(\lambda-\lambda')-2}\psi(\gamma_c x) - \psi(\gamma_c^*\gamma_c x\gamma_c)] \\ &=\; q^{\frac{1}{2}(\lambda-\lambda')}\psi(\gamma_c x) - q^2\psi(\gamma_c^*\gamma_c x\gamma_c). \end{aligned}$$

Comparing both sides we get the third equation of (3.43). Inserting $x \to x\alpha_c$ in the second equation of (3.36) and using again the third equation of (3.36) we prove the fourth equality in (3.43).

□

Until this point we made our computations for arbitrary χ and χ'. In the further investigations our methods will depend on specific relations between χ and χ'. We shall have to consider separately different classes of pairs (χ, χ'). It will be done in the next Section.

4 Admissible pairs of Gelfand spaces

We continue the investigation of bilinear invariant functionals on Gelfand spaces.
Since the coefficients on the right hand side of (3.43) are non-zero for any λ, λ' therefore for any $a \in \{\alpha_c, \alpha_c^*, \gamma_c, \gamma_c^*\}$ and $x \in \mathcal{A}_c$ we have that $\psi(ax) = 0$ if and only if $\psi(xa) = 0$. Let us also note that it is enough to verify the relations (3.43) only for homogeneous $x \in \mathcal{A}_c$. In such a case ax and xa are homogeneous elements of the same degree. Due to the support property of ψ, (3.43) gives non-trivial conditions only if $ax, xa \in J_{\frac{n}{2},-\frac{n'}{2}}$. Let us remind (cf.(3.15), (3.14)) that

$$J_{\frac{n}{2},-\frac{n'}{2}} \;=\; \left\{(\alpha_c)_{\frac{1}{2}(n'-n)}(\gamma_c)_{\frac{1}{2}(n+n')}(\gamma_c\gamma_c^*)^m \;:\; m = 0,1,2,...\right\}^{\text{linear span}}. \tag{4.1}$$

Lemma 4.1
Let $\psi \in \mathcal{A}_c'$ be a (χ, χ')-spherical functional.
Then

$$1^o \quad n_1 \neq n_1' \;\Longrightarrow\; \psi(x\gamma_c) = \psi(\gamma_c x) = 0 \textit{ for any } x \in \mathcal{A}_c,$$

$$2^o \quad n_2 \neq n_2' \;\Longrightarrow\; \psi(x\gamma_c^*) = \psi(\gamma_c^* x) = 0 \textit{ for any } x \in \mathcal{A}_c,$$

$$3^o \quad \chi \neq \chi' \;\Longrightarrow\; \psi(x\gamma_c\gamma_c^*) = \psi(\gamma_c\gamma_c^* x) = 0 \textit{ for any } x \in \mathcal{A}_c.$$

Proof: Ad 1^o. We may assume that $x \in \mathcal{A}_c$ is a homogeneous element such that $x\gamma_c \in J_{\frac{n}{2},-\frac{n'}{2}}$. Then in view of (4.1) we have that

$$\gamma_c x \;=\; q^{\frac{1}{2}(n-n')} x\gamma_c$$

and by the third equation of (3.43) we have

$$\psi(x\gamma_c) \;=\; q^{\frac{1}{2}(\lambda-\lambda')}\psi(\gamma_c x) \;=\; q^{\frac{1}{2}(\lambda-\lambda')} q^{\frac{1}{2}(n-n')}\,\psi(x\gamma_c).$$

Since $\frac{1}{2}(\lambda+n) \;=\; \frac{1}{2}(n_1+n_2-2+n_1-n_2) \;=\; n_1-1$ and $\frac{1}{2}(\lambda'+n') \;=\; n_1'-1$ we get

$$(1-q^{n_1-n_1'})\,\psi(x\gamma_c) \;=\; 0.$$

Note that $q^{n_1-n_1'}=1$ if and only if $\Re n_1 = \Re n_1'$ and $\Im n_1 = \Im n_1'$ $(\mathrm{mod}(\frac{2\pi}{\log q}))$. Since $\Im n_1,\, \Im n_1' \in [0, \frac{-2\pi}{\log q}[$ the last condition is equivalent to $\Im n_1 = \Im n_1'$. Therefore $q^{n_1-n_1'} = 1$ if and only if $n_1 = n_1'$ and this proves 1^o.

Ad 2^o. We may assume that $x \in \mathcal{A}_c$ is a homogeneous element such that $x\gamma_c^* \in J_{\frac{n}{2},-\frac{n'}{2}}$. Then using again (4.1) we have

$$\gamma_c^* x \;=\; q^{\frac{1}{2}(n-n')} x\gamma_c^*$$

and using the fourth equation of (3.43) we obtain

$$\psi(x\gamma_c^*) \;=\; q^{-\frac{1}{2}(\lambda-\lambda')}\psi(\gamma_c^* x) \;=\; q^{\frac{1}{2}(\lambda'-\lambda)} q^{\frac{1}{2}(n-n')}\,\psi(x\gamma_c^*).$$

Now $\frac{1}{2}(\lambda-n) \;=\; \frac{1}{2}(n_1+n_2-2-n_1+n_2) \;=\; n_2-1$ and $\frac{1}{2}(\lambda'-n') \;=\; n_2'-1$. Therefore

$$(1-q^{n_2-n_2'})\,\psi(x\gamma_c^*) \;=\; 0$$

and 2^o follows. Combining 1^o and 2^o we get 3^o.

□

In the further analysis some pairs of characters play a distinguished role.

Definition 4.2
Let $\chi = (n_1, n_2)$, $\chi' = (n_1', n_2')$ be characters of the parabolic group $n := n_1 - n_2$ and $n' := n_1' - n_2'$.
We say that (χ, χ') is a singular pair whenever

$$1^o. \quad \tfrac{1}{2}\Re(n_1+n_1') \textit{ and } \tfrac{1}{2}\Re(n_2+n_2') \textit{ are non}-\textit{negative integers},$$

$$2^o. \qquad \tfrac{1}{2}\Im(n_1+n_1') \;=\; \tfrac{1}{2}\Im(n_2+n_2') \in \{0, \tfrac{-\pi}{\log q}\}.$$

If (χ, χ') is a singular pair then we set

$$m_{\chi\chi'} := \min\left\{\frac{1}{2}\Re(n_1+n_1'),\ \frac{1}{2}\Re(n_2+n_2')\right\}. \tag{4.2}$$

Remark. $m_{\chi\chi'}$ is a real number for any pair (χ,χ'). Since $\frac{1}{2}\Re(n_1+n_1')-\frac{1}{2}\Re(n_2+n_2')=\frac{1}{2}(n+n')$ therefore if (χ,χ') is singular then $n=n'(\mathrm{mod}(2))$. Conversely if $n=n'(\mathrm{mod}(2))$ then $m_{\chi\chi'}$ is a non-negative integer if and only if $\frac{1}{2}\Re(n_1+n_1')$ and $\frac{1}{2}\Re(n_1+n_1')$ are non-negative integers.
It is also obvious that $\frac{1}{2}\Im(n_1+n_1')=\frac{1}{2}\Im(n_2+n_2')$ for any (χ,χ'). Due to our choice of the representatives n_1, n_2, n_1', n_2', their imaginary parts belong to $[0,\frac{-2\pi}{\log q}[$. Therefore $\frac{1}{2}\Im(n_1+n_1')=0$ if and only if $\Im n_1=\Im n_1'=0=\Im n_2=\Im n_2'$. Also $\frac{1}{2}\Im(n_1+n_1')=\frac{1}{2}\Im(n_2+n_2')=\frac{-\pi}{\log q}$ if and only if $\Im n_1'=-\Im n_1\,(\mathrm{mod}(\frac{2\pi}{\log q}))$ and $\Im n_2'=-\Im n_2\,(\mathrm{mod}(\frac{2\pi}{\log q}))$.

The following result reveals the meaning of *the "singular pair"*.

Lemma 4.3
Let $\chi=(n_1,n_2)$, $\chi'=(n_1',n_2')$ be characters of the parabolic group such that $n=n'(\mathrm{mod}(2))$. Then

$$\left(\begin{array}{c}(\chi,\chi')\\ \textit{is a singular pair}\end{array}\right)\iff\left(\begin{array}{c}q^{\frac{1}{2}(\lambda+\lambda'+4-|n+n'|)-2m}=1\\ \textit{for some } m\in\{0,1,2,...\}\end{array}\right).$$

The nonnegative integer m appearing on the right hand side of the above equivalence coincides with $m_{\chi\chi'}$.

Proof: Since

$$\begin{aligned}\tfrac{1}{4}(\lambda+\lambda'+4-|n+n'|) &= \tfrac{1}{4}(n_1+n_1'+n_2+n_2'-|n_1+n_1'-n_2-n_2'|)\\ &= \begin{cases}\tfrac{1}{2}(n_1+n_1') & \text{for } n+n'\le 0\\ \tfrac{1}{2}(n_2+n_2') & \text{for } n+n'\ge 0\end{cases}\end{aligned}\tag{4.3}$$

we have

$$\begin{aligned}\tfrac{1}{4}\Re(\lambda+\lambda'+4-|n+n'|) &= m_{\chi\chi'},\\ \tfrac{1}{4}\Im(\lambda+\lambda'+4-|n+n'|) &= \tfrac{1}{2}\Im(n_1+n_1').\end{aligned}\tag{4.4}$$

Now

$$q^{\frac{1}{2}(\lambda+\lambda'+4-|n+n'|)-2m} = q^{2(m_{\chi\chi'}-m)+i\Im(n_1+n_1')}$$

and

$$q^{2(m_{\chi\chi'}-m)+i\Im(n_1+n_1')}=1$$

is possible if and only if $m=m_{\chi\chi'}$ and $\Im(n_1+n_1')\in\{0,\frac{-2\pi}{\log q}\}$ and this proves the lemma. □

Let us remind that for $m\in\{0,1,2,...\}$ by $x^{(m)}_{n,-n'}$ we denoted the element of the linear basis of $J_{\frac{n}{2},-\frac{n'}{2}}$:

$$x^{(m)}_{n,-n'} := (\alpha_c)_{\frac{1}{2}(n'-n)}(\gamma_c)_{\frac{1}{2}(n+n')}(\gamma_c\gamma_c^*)^m\tag{4.5}$$

cf.(4.1) and (3.14).

Lemma 4.4
Let (χ, χ') be an admissible pair of characters and ψ be a non-zero (χ, χ')-spherical functional. Assume that $n \neq n'$. Then (χ, χ') is singular and

$$\psi\left(x^{(m)}_{n,-n'}\right) \neq 0 \iff m = m_{\chi\chi'}. \tag{4.6}$$

Proof: We know that $n = n'(\mathrm{mod}(2))$. It is sufficient to consider two cases: $n' > n$ and $n' < n$.

If $n' > n$, then $n' - n \geq 2$ and by (4.5) and the first equation of (3.43) we have

$$\begin{aligned}
\psi\left(x^{(m)}_{n,-n'}\right) &= \psi\left(\alpha_c(\alpha_c)_{\frac{1}{2}(n'-n-2)}(\gamma_c)_{\frac{1}{2}(n+n')}(\gamma_c\gamma_c^*)^m\right) \\
&= q^{\frac{1}{2}(\lambda+\lambda')+2}\,\psi\left((\alpha_c)_{\frac{1}{2}(n'-n-2)}(\gamma_c)_{\frac{1}{2}(n+n')}(\gamma_c\gamma_c^*)^m\alpha_c\right) \\
&= q^{\frac{1}{2}(\lambda+\lambda')+2}q^{-2m-\frac{1}{2}|n+n'|}\,\psi\left(x^{(m)}_{n,-n'}\right).
\end{aligned}$$

Therefore

$$\left(1 - q^{\frac{1}{2}(\lambda+\lambda'+4-|n+n'|)-2m}\right)\psi\left(x^{(m)}_{n,-n'}\right) = 0. \tag{4.7}$$

We shall prove that the same relation holds for $n' < n$. In this case $n' - n \leq -2$. By (4.5) and the second equation of (3.43) we have

$$\begin{aligned}
\psi\left(x^{(m)}_{n,-n'}\right) &= \psi\left(\alpha_c^*(\alpha_c)_{\frac{1}{2}(n'-n+2)}(\gamma_c)_{\frac{1}{2}(n+n')}(\gamma_c\gamma_c^*)^m\right) \\
&= q^{-\frac{1}{2}(\lambda+\lambda')-2}\,\psi\left((\alpha_c)_{\frac{1}{2}(n'-n+2)}(\gamma_c)_{\frac{1}{2}(n+n')}(\gamma_c\gamma_c^*)^m\alpha_c^*\right) \\
&= q^{-\frac{1}{2}(\lambda+\lambda')-2}q^{2m+\frac{1}{2}|n+n'|}\,\psi\left(x^{(m)}_{n,-n'}\right)
\end{aligned}$$

and (4.7) follows. Remembering that ψ is a non-zero functional supported by $J_{\frac{n}{2},-\frac{n'}{2}}$ and that elements (4.5) form a linear basis of $J_{\frac{n}{2},-\frac{n'}{2}}$ we see that the equation

$$q^{\frac{1}{2}(\lambda+\lambda'+4-|n+n'|)-2m} = 1 \tag{4.8}$$

is satisfied by a non-negative integer m. Now Lemma 4.3 shows that (χ, χ') is singular. Remembering that $m = m_{\chi\chi'}$ is the only solution of (4.8) and using (4.7) we get (4.6).

□

Remark. Using (4.3) one can immediately check that

$$\tfrac{1}{2}(n+n') + m_{\chi\chi'} = \begin{cases} \tfrac{1}{2}\Re(n_1 + n_1') & \text{for } n + n' \geq 0 \\ \tfrac{1}{2}\Re(n_2 + n_2') & \text{for } n + n' \leq 0. \end{cases}$$

Therefore

$$x^{(m_{\chi\chi'})}_{n,-n'} = (\alpha_c)_{\frac{1}{2}(n'-n)}(\gamma_c)^{\frac{1}{2}\Re(n_1+n_1')}(\gamma_c^*)^{\frac{1}{2}\Re(n_2+n_2')}. \tag{4.9}$$

To describe all admissible pairs (χ, χ') of characters we shall consider four cases:

$$1^o \quad n_1 = n_1' \text{ and } n_2 = n_2',$$

$$2^o \quad n_1 = n_1' \text{ and } n_2 \neq n_2',$$

$$3^o \quad n_1 \neq n_1' \text{ and } n_2 = n_2',$$

$$4^o \quad n_1 \neq n_1' \text{ and } n_2 \neq n_2'.$$

In the first case we have the following result

Proposition 4.5
Let $\chi = \chi' = (n_1, n_2)$ be a character of the parabolic group. Then

1. *If (χ, χ') is not a singular pair then $q^{-(n_1+n_2)+|n_1-n_2|+2m} \neq 1$ for $m = 0, 1, 2, ...$ and any (χ, χ')-spherical functional is proportional to the one given by the formula*

$$\psi\left(x_{rs}^{(m)}\right) = \begin{cases} \dfrac{1 - q^{-(n_1+n_2)+|n_1-n_2|}}{1 - q^{-(n_1+n_2)+|n_1-n_2|+2m}} & \text{for } (r,s) = (n, -n) \\ \\ 0 & \text{otherwise.} \end{cases} \tag{4.10}$$

2. *If (χ, χ') is a singular pair then $q^{-(n_1+n_2)+|n_1-n_2|+2m} = 1$ for $m = m_{\chi\chi}$ (cf.4.2) and any (χ, χ')-spherical functional is proportional to the one given by the formula*

$$\psi\left(x_{rs}^{(m)}\right) = \begin{cases} 1 & \text{for } (r,s) = (n, -n) \text{ and } m = m_{\chi\chi} \\ \\ 0 & \text{otherwise.} \end{cases} \tag{4.11}$$

In particular the pair (χ, χ) is always admissible.

Proof. Let ψ be a (χ, χ)-sperical functional. Then ψ is supported by $J_{\frac{n}{2}, -\frac{n}{2}}$ and

$$\psi\left(x_{rs}^{(m)}\right) = 0 \tag{4.12}$$

for $(r, s) \neq (n, -n)$. For $\chi = \chi'$ the equations (3.43) reduce to

$$\begin{aligned} \psi(\alpha_c x) &= q^{n_1+n_2}\psi(x\alpha_c), \\ \psi(\alpha_c^* x) &= q^{-(n_1+n_2)}\psi(x\alpha_c^*), \\ \psi(\gamma_c x) &= \psi(x\gamma_c), \\ \psi(\gamma_c^* x) &= \psi(x\gamma_c^*) \end{aligned} \tag{4.13}$$

for any $x \in \mathcal{A}_c$.

It is sufficient to verify these relations for $x = x_{rs}^{(m)}$. Elements $\gamma_c x$ and $x\gamma_c$ belong to the support of ψ only if $(r-1, s+1) = (n, -n)$. In this case $r+s = 0, r-s = 2n,\ x = (\gamma_c)_n(\gamma_c\gamma_c)^*$ and $\gamma_c x = x\gamma_c$. It shows that (4.13.III) (i.e. the third equation of (4.13)) is automatically satisfied. In the similar way one verifies (4.13.IV).

Now we insert $x = x_{rs}^{(m)}$ into (4.13.I). Elements $\alpha_c x$ and $x\alpha_c$ belong to the support of ψ only if $(r-1, s-1) = (n, -n)$. Then $r+s = 2$, $r-s = 2n$ and $x = \alpha_c^* x_{n,-n}^{(m)}$. We compute then

$$\begin{aligned} x\alpha_c &= \alpha_c^* x_{n,-n}^{(m)}\alpha_c = q^{-(|n|+2m)}\alpha_c^*\alpha_c x_{n,-n}^{(m)} = q^{-(|n|+2m)}(I_c - \gamma_c\gamma_c^*)x_{n,-n}^{(m)} \\ &= q^{-(|n|+2m)}(x_{n,-n}^{(m)} - x_{n,-n}^{(m+1)}) \end{aligned}$$

and

$$\alpha_c x = \alpha_c\alpha_c^* x_{n,-n}^{(m)} = (I_c - q^2\gamma_c\gamma_c^*)x_{n,-n}^{(m)} = x_{n,-n}^{(m)} - q^2 x_{n,-n}^{(m+1)}.$$

Now (4.13.I) takes the form

$$q^{-(n_1+n_2)}\psi\left(x_{n,-n}^{(m)}\right) - q^{-(n_1+n_2)+2}\psi\left(x_{n,-n}^{(m+1)}\right) = q^{-(|n|+2m)}\left(\psi\left(x_{n,-n}^{(m)}\right) - \psi\left(x_{n,-n}^{(m+1)}\right)\right).$$

This is equivalent to

$$\left(1 - q^{-(n_1+n_2)+|n|+2m}\right)\psi\left(x_{n,-n}^{(m)}\right) - \left(1 - q^{-(n_1+n_2)+|n|+2(m+1)}\right)\psi\left(x_{n,-n}^{(m+1)}\right) = 0. \quad (4.14)$$

for $m = 0, 1, 2, ...$

In a similar way we analyze equation (4.13.II). Let $x = x_{rs}^{(m)}$. Elements $\alpha_c^* x$ and $x\alpha_c^*$ belong to the support of ψ only if $(r+1, s+1) = (n, -n)$. Then $r+s = -2$, $r-s = 2n$ and $x = \alpha_c x_{n,-n}^{(m)}$. Therefore

$$\begin{aligned} x\alpha_c^* &= \alpha_c x_{n,-n}^{(m)}\alpha_c^* = q^{|n|+2m}\alpha_c\alpha_c^* x_{n,-n}^{(m)} = q^{|n|+2m}\left(I_c - q^2\gamma_c\gamma_c^*\right)x_{n,-n}^{(m)} \\ &= q^{|n|+2m}x_{n,-n}^{(m)} - q^{|n|+2(m+1)}x_{n,-n}^{(m+1)} \end{aligned}$$

and

$$\alpha_c^* x = \alpha_c^*\alpha_c x_{n,-n}^{(m)} = \left(I_c - \gamma_c\gamma_c^*\right)x_{n,-n}^{(m)} = x_{n,-n}^{(m)} - x_{n,-n}^{(m+1)}.$$

Now (4.13.II) means that

$$\psi\left(x_{n,-n}^{(m)}\right) - \psi\left(x_{n,-n}^{(m+1)}\right) = q^{-(n_1+n_2)+|n|+2m}\psi\left(x_{n,-n}^{(m)}\right) - q^{-(n_1+n_2)+|n|+2(m+1)}\psi\left(x_{n,-n}^{(m+1)}\right),$$

which again is equivalent to (4.14). Clearly the equation (4.14) is satisfied if and only if

$$\left(1 - q^{-(n_1+n_2)+|n|+2m}\right)\psi\left(x_{n,-n}^{(m)}\right) = \mathrm{C}, \quad (4.15)$$

where C is a complex constant independent of m. This way we showed that ψ is a (χ,χ)-spherical functional if and only if it satisfies (4.12) and (4.15).

Inserting $\chi' = \chi$ in Lemma 4.3 we see that (χ, χ) is not singular if and only if $q^{-(n_1+n_2)+|n|+2m} \neq 1$ for all $m = 0, 1, 2, \ldots$ In this case

$$\psi(x^{(m)}_{n,-n'}) \; = \; \frac{\mathrm{C}}{1 - q^{-(n_1+n_2)+|n_1-n_2|+2m}}$$

and Statement 1 follows.

If (χ, χ) is singular then the first factor in (4.15) vanishes for $m = m_{\chi\chi}$. Therefore $\mathrm{C} = 0$, $\psi(x^{(m)}_{n,-n'}) \neq 0$ only for $m = m_{\chi\chi}$ and Statement 2 follows.

□

In the second case

Proposition 4.6
Let $\chi = (n_1, n_2)$, $\chi' = (n'_1, n'_2)$ be characters of the parabolic group. Assume that $n_1 = n'_1$, $n_2 \neq n'_2$. Then (χ, χ') is admissible if and only if

$$\Re n_1 \in \{0, 1, 2, \ldots\}, \qquad \Im n_1 \in \{0, \tfrac{-\pi}{\log q}\}, \qquad n'_2 \; = \; -n_2 \, (\mathrm{mod}(\tfrac{2\pi i}{\log q})).$$

In this case any (χ, χ')-spherical functional is proportional to the one given by the formula

$$\psi\left(x^{(m)}_{rs}\right) \; = \; \begin{cases} 1 & \text{for } (r,s) = (n, -n') \text{ and } m = 0 \\ \\ 0 & \text{otherwise.} \end{cases} \tag{4.16}$$

Remark. Let us note that $\Re n_1 \in \{0, 1, 2, \ldots\}$ implies that $\Re n_2$ is an integer (for $n = n_1 - n_2$ must be an integer).

Proof. In the present case $n \neq n'$. Assume that (χ, χ') is admissible. Then by Lemma 4.4 the pair (χ, χ') is singular. In particular (cf. Definition 4.2) $\Re n_1 \in \{0, 1, 2, \ldots\}$ and $\Im n_1 \in \{0, \frac{-\pi}{\log q}\}$.

We assumed that $n_2 \neq n'_2$. Let ψ be a non-zero (χ, χ')-spherical functional. By Statement 2 of Lemma 4.1 and formula (4.5)

$$\psi\left(x^{(m)}_{n,-n'}\right) \; = \; \psi\left((\alpha_c)_{\frac{1}{2}(n'-n)}(\gamma_c)_{\frac{1}{2}(n+n')}(\gamma_c\gamma_c^*)^m\right) \; = 0 \tag{4.17}$$

for $m \neq 0$ and $n + n' < 0$. Therefore $n + n' \geq 0$, $\Re(n_1 + n'_1) \geq \Re(n_2 + n'_2)$ and by definition (4.2)

$$m_{\chi\chi'} \; = \; \tfrac{1}{2}\Re(n_2 + n'_2).$$

On the other hand, comparing (4.17) with (4.6) we get $m_{\chi\chi'} = 0$. Therefore $\Re n'_2 = -\Re n_2$. Moreover $\Im n'_2 = \Im n'_1 = \Im n_1 = \Im n_2 \in \{0, \frac{-\pi}{\log q}\}$ and $\Im n_2 = -\Im n_2 \, (\mathrm{mod}(\frac{2\pi}{\log q}))$. Therefore $n'_2 = -n_2 \, (\mathrm{mod}(\frac{2\pi i}{\log q}))$. Relation (4.6) shows that ψ is proportional to the functional (4.16). This proves the "only if" part of the proposition.

To prove the converse it is sufficient that to show that (4.16) is a (χ,χ')-spherical functional. Clearly this functional is supported by $J_{\frac{n}{2},-\frac{n'}{2}}$ In the present case relations (3.43) take the form

$$\begin{aligned}
\psi(\alpha_c x) &= q^{\frac{1}{2}(n+n')}\psi(x\alpha_c),\\
\psi(\alpha_c^* x) &= q^{-\frac{1}{2}(n+n')}\psi(x\alpha_c^*),\\
\psi(\gamma_c x) &= q^{\frac{1}{2}(n-n')}\psi(x\gamma_c),\\
\psi(\gamma_c^* x) &= q^{-\frac{1}{2}(n-n')}\psi(x\gamma_c^*).
\end{aligned} \tag{4.18}$$

Formula (4.16) implies that $\psi(x\gamma_c^* y) = 0$ for any $x, y \in \mathcal{A}_c$. Consequently $\psi(x\alpha_c\alpha_c^* y) = \psi(xy) = \psi(x\alpha_c^*\alpha_c y)$. Now one can easily verify that (4.18) holds.

□

In the third case we get

Proposition 4.7
Let $\chi = (n_1, n_2)$, $\chi' = (n_1', n_2')$ be characters of the parabolic group. Assume that $n_1 \neq n_1'$, $n_2 = n_2'$. Then (χ,χ') is admissible if and only if

$$\Re n_2 \in \{0,1,2,...\}, \qquad \Im n_2 \in \{0, \tfrac{-\pi}{\log q}\}, \qquad n_1' = -n_1 \,(\mathrm{mod}(\tfrac{2\pi i}{\log q})).$$

In this case then any (χ,χ')-spherical functional is proportional to the one given by the formula

$$\psi(x_{rs}^{(m)}) = \begin{cases} 1 & \text{for } (r,s) = (n,-n') \text{ and } m = 0 \\ \\ 0 & \text{otherwise.} \end{cases} \tag{4.19}$$

Proof. This case is in a sense a mirror immage of the previous one and the proof is essentialy the same. We indicate the most important changes only. Assume that (χ,χ') is admissible. As before it must be singular. In particular (cf. Definition 4.2) $\Re n_2 \in \{0,1,2,...\}$ and $\Im n_2 \in \{0, \frac{-\pi}{\log q}\}$.

In the present case $n_1 \neq n_1'$. Assume that ψ is a non-zero (χ,χ')-spherical functional. By Statement 1 of Lemma 4.1 and (4.5)

$$\psi\left(x_{n,-n'}^{(m)}\right) = \psi\left((\alpha_c)_{\frac{1}{2}(n'-n)}(\gamma_c)_{\frac{1}{2}(n+n')}(\gamma_c\gamma_c^*)^m\right) = 0 \tag{4.20}$$

for $m \neq 0$ and $n + n' > 0$. Therefore $n + n' \leq 0$, $\Re(n_1 + n_1') \leq \Re(n_2 + n_2')$ and by definition (4.2)

$$m_{\chi\chi'} = \tfrac{1}{2}\Re(n_1 + n_1').$$

Repeating the arguments used in the previous proof we see that $m_{\chi\chi'} = 0$, $n_1' = -n_1 \,(\mathrm{mod}(\frac{2\pi i}{\log q}))$ and relation (4.6) shows that ψ is proportional to the functional (4.19).

Instead of (4.18) we have relations

$$\begin{aligned}\psi(\alpha_c x) &= q^{-\frac{1}{2}(n+n')}\psi(x\alpha_c),\\ \psi(\alpha_c^* x) &= q^{\frac{1}{2}(n+n')}\psi(x\alpha_c^*),\\ \psi(\gamma_c x) &= q^{-\frac{1}{2}(n-n')}\psi(x\gamma_c),\\ \psi(\gamma_c^* x) &= q^{\frac{1}{2}(n-n')}\psi(x\gamma_c^*).\end{aligned} \tag{4.21}$$

Formula (4.20) implies that $\psi(x\gamma_c y) = 0$ and $\psi(x\alpha_c\alpha_c^* y) = \psi(xy) = \psi(x\alpha_c^*\alpha_c y)$ (notice that in this case $n + n' \leq 0$). Using this formulae one can easily verify that the functional (4.19) satisfies (4.21).

□

Now we consider the last case.

Proposition 4.8
Let $\chi = (n_1, n_2)$, $\chi' = (n_1', n_2')$ be characters of the parabolic group. Assume that $n_1 \neq n_1'$ and $n_2 \neq n_2'$. Then (χ, χ') is admissible if and only if $\chi' \equiv -\chi$.
i.e.

$$n_1' = -n_1\,(\mathrm{mod}(\tfrac{2\pi i}{\log q})), \qquad n_2' = -n_2\,(\mathrm{mod}(\tfrac{2\pi i}{\log q})).$$

In this case any (χ, χ')-spherical functional is proportional to the one given by the formula

$$\psi(x_{rs}^{(m)}) = \begin{cases} 1 & \text{for } (r,s) = (n,n) \text{ and } m = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.22}$$

Proof. Assume that (χ, χ') is admissible and ψ is a non-zero (χ, χ')-spherical functional. Since $n_1 \neq n_1', n_2 \neq n_2'$ then by Statements 1 and 2 of Lemma 4.1 and formula (4.5)

$$\psi\left(x_{n,-n'}^{(m)}\right) = \psi\left((\alpha_c)_{\frac{1}{2}(n'-n)}(\gamma_c)_{\frac{1}{2}(n+n')}(\gamma_c\gamma_c^*)^m\right) = c\,\delta_{n+n',0}\delta_{m,0} \tag{4.23}$$

for some non-zero complex number c. Remembering that $\psi \neq 0$ we get $n + n' = 0$, i.e.

$$n_1 + n_1' = n_2 + n_2' \tag{4.24}$$

and (for $m = 0$) $c = \psi((\alpha_c)_{\frac{1}{2}(n'-n)})$ is the only non-zero value.

Let us consider two cases. If $n \neq 0$, then $n \neq n'$ and by Lemma 4.4 (χ, χ') is a singular pair. Comparing (4.23) with (4.6) we get $m_{\chi\chi'} = 0$.

If $n = 0$, then $n' = 0$ and $\psi(I_c) = c \neq 0$. Remembering that $\psi(\gamma_c\gamma_c^*) = 0$ (by (4.23)) we obtain

$$\psi(\alpha_c\alpha_c^*) = \psi(I_c) = \psi(\alpha_c^*\alpha_c).$$

Using the first equation of (3.43) we get

$$\psi(\alpha_c\alpha_c^*) \;=\; q^{\frac{1}{2}(\lambda+\lambda')+2}\psi(\alpha_c^*\alpha_c) \;=\; q^{\frac{1}{2}(\lambda+\lambda')+2}\psi(\alpha_c\alpha_c^*).$$

Therefore $(1-q^{\frac{1}{2}(\lambda+\lambda')+2})\,c=0$ and $1-q^{\frac{1}{2}(\lambda+\lambda')+2}=0$. Remembering that $n+n'=0$ and using Lemma 4.3 we see that (χ,χ') is a singular pair and $m_{\chi\chi'}=0$. Therefore both cases lead to the same conclusion.

Using Definition 4.2 and taking into account (4.24) we get

$$\Re(n_1+n_1') \;=\; \Re(n_2+n_2') \;=\; 0,$$

$$\Im(n_1+n_1') \;=\; \Im(n_2+n_2') \in \{0, \tfrac{-2\pi}{\log q}\}.$$

Relation (4.23) shows that the functional ψ is proportional to the one given by (4.22). This proves the the "only if" part of the proposition.

To prove the converse let us assume that $\chi'\equiv-\chi$. Then

$$n'=-n, \quad \tfrac{1}{2}(\lambda+\lambda')+2=0\,(\mathrm{mod}(\tfrac{2\pi i}{\log q})), \quad \tfrac{1}{2}(\lambda-\lambda')=n_1+n_2\,(\mathrm{mod}(\tfrac{2\pi i}{\log q}))$$

and the equations (3.43) reduce to

$$\begin{aligned}
\psi(\alpha_c x) &= \psi(x\alpha_c),\\
\psi(\alpha_c^* x) &= \psi(x\alpha_c^*),\\
\psi(\gamma_c x) &= q^{-(n_1+n_2)}\psi(x\gamma_c),\\
\psi(\gamma_c^* x) &= q^{n_1+n_2}\psi(x\gamma_c^*).
\end{aligned} \tag{4.25}$$

Observing that the functional (4.22) coincides with counit e_c on the space $J_{\frac{n}{2},\frac{n}{2}}$ one can easily verify that (4.22) satisfies (4.25). Therefore (4.22) is a $(\chi,-\chi)$-spherical functional and $(\chi,-\chi)$ is an admissible pair.

□

Now we are able to compile the list of all admissible pairs of characters.

Theorem 4.9
Let $\chi=(n_1,n_2)$, $\chi'=(n_1',n_2')$ be characters of the parabolic group. Assume that (χ,χ') is an admissible pair. Then we have four possibilities

$1^o \quad \chi' = \chi,$

$2^o \quad \chi' \equiv -\chi \;$ (*i.e.* $n'_1 = -n_1 \,(\mathrm{mod}(\frac{2\pi i}{\log q}))$ *and* $n'_2 = -n_2 \,(\mathrm{mod}(\frac{2\pi i}{\log q}))$,

$3^o \quad \chi = \left(p_1 - \epsilon\frac{\pi i}{\log q},\, p_2 - \epsilon\frac{\pi i}{\log q}\right), \quad \chi' = \left(p_1 - \epsilon\frac{\pi i}{\log q},\, -p_2 - \epsilon\frac{\pi i}{\log q}\right),$

where $p_1 \in \{0, 1, 2, ...\}$, $p_2 \in \{\pm 1, \pm 2, ...\}$ *and* $\epsilon = 0, 1$,

$4^o \quad \chi = \left(p_1 - \epsilon\frac{\pi i}{\log q},\, p_2 - \epsilon\frac{\pi i}{\log q}\right), \quad \chi' = \left(-p_1 - \epsilon\frac{\pi i}{\log q},\, p_2 - \epsilon\frac{\pi i}{\log q}\right),$

where $p_1 \in \{\pm 1, \pm 2, ...\}$, $p_2 \in \{0, 1, 2, ...\}$ *and* $\epsilon = 0, 1$.

Conversely in any such case the pair (χ, χ') *is admissible.*
For any admissible pair (χ, χ') *the set of* (χ, χ')*-spherical functionals is one dimensional.*

Remark. Case 2^o in the above list besides the pairs considered in Proposition 4.8 contains two pairs of the for (χ, χ) where $\chi = (0, 0)$ or $\chi = (\frac{-\pi i}{\log q}, \frac{-\pi i}{\log q})$. It means that cases 1^o and 2^o of the list are not disjoint.

We would like to stress the distinguished role of the case $\chi' \equiv -\chi$. In this case the (χ, χ')-spherical functional restricted to $\mathcal{Z}_n$ coincides with the counit e_c. Indeed e_c and ψ applied to $x_{nk}^{(m)} = (\alpha_c)_{-\frac{1}{2}(n+k)}(\gamma_c)_{\frac{1}{2}(n-k)}(\gamma_c\gamma_c^*)^m$ give $\delta_{nk}\delta_{m0}$ for $m = 0, 1, 2....$ Therefore $a*\psi = a*e_c = a$ for any $a \in \mathcal{Z}_n$. Proposition 3.5 shows now that any Lorentz invariant linear form on $D_\chi \otimes D_{-\chi}$ is a complex multiple of the linear form (in truncated notation)

$$\mathcal{Z}_n \otimes \mathcal{Z}_{-n} \ni (a \otimes b) \longrightarrow h(ab) \in \mathbf{C}. \tag{4.26}$$

This form is non-degenerate: $h(ab) = 0$ for all $b \in \mathcal{Z}_{-n}$ ($a \in \mathcal{Z}_n$ respectively) implies $a = 0$ ($b = 0$ respectively). This fact follows from the faithfulness oh the Haar measure on $\mathcal{A}_c$ (cf.[16], Statement 5 of Theorem 4.2).

Let us note that for $\chi = (n_1, n_2)$ and $\chi' = (n'_1, n'_2)$ their product is given by $\chi \cdot \chi' \equiv (n_1 + n'_1 - 1, n_2 + n'_2 - 1)$. In particular $\chi \cdot (-\chi) \equiv (-1, -1)$. Remembering that multiplication map $m : D_\chi \otimes D'_\chi \longrightarrow D_{\chi\chi'}$ intertwines the actions of quantum Lorentz group and taking into account the formula (4.26) we expect that the Haar measure considered on the Gelfand space $D_{(-1,-1)}$ is a Lorentz invariant linear functional. This is the case. In fact we can prove a stronger result which seems to be important in itself.

Proposition 4.10
The functional

$$D_{(-1,-1)} \ni q^{4J_3} \otimes a \longrightarrow h(a) \in \mathbf{C} \tag{4.27}$$

is the only (up to a complex coefficient) Lorentz invariant linear functional on $D_{(-1,-1)}$.
Moreover $D_{(-1,-1)}$ *is the only Gelfand space admitting a non-zero Lorentz invariant linear functional.*

Proof: Let $n_1 = n_2 = 1$. Then $\lambda = 0$ and $n = 0$. Therefore $D_{(1,1)} = I_d \otimes \mathcal{Z}_0$ and $I = I_d \otimes I_c \in D_{(1,1)}$. Clearly I is a Lorentz invariant element. Inserting in (4.26) $n = 0$ and $a = I_c$ we get the Lorentz invariance of (4.27).

Assume now that ϕ is a non-zero Lorentz invariant functional on $D_{(n_1,n_2)}$. Remembering that the multiplication map intertwines the action of the quantum Lorentz group on Gelfand spaces we see that $((0,0),(n_1+1,n_2+1)$ is an admissible pair of characters. Indeed, $D_{(0,0)} \cdot D_{(n_1+1,n_2+1)} \subset D_{(n_1,n_2)}$ and (we use the truncated notation) the linear functional

$$f \,:\, \mathcal{Z}_0 \otimes \mathcal{Z}_n \ni a \otimes b \longrightarrow \phi(ab) \in \mathbf{C}$$

is the invariant one. It is a non-zero functional because $f(I_c \otimes b) = \phi(b)$ and $\phi \neq 0$. Inspecting the list of all admissible pairs (cf.Theorem 4.9) we see that $\chi = (0,0)$ is the only character such that the pair $((0,0),\chi)$ is admissible. Therefore $n_1 = -1$, $n_2 = -1$. Moreover in this case the invariant linear functional is given by (4.26). Therefore ϕ must be a multiple of $h|_{\mathcal{Z}_0}$. □

Remark 4.11

One can easily show that $D_{(1,1)}$ is the only Gelfand space containing a non-zero Lorentz invariant element. $D_{(1,1)} = I_d \otimes \mathcal{Z}_0$ is a *-algebra isomorphic to $\mathcal{Z}_0$. Its C^*-completion (cf. Proposition 2.1 and [9, page 200])

$$\overline{\mathcal{Z}}_0 = \{\, a \in A_c :\ (\dot{\pi}_c \otimes \mathrm{id_c})\Delta_c(a) = I_{S^1} \otimes a \,\}$$

is the algebra of "all continuous functions" on Podleś sphere $S^2_q := {}_{S^1}\backslash^{S_qU(2)}$. Therefore $v_{(1,1)}$ describes the action of *QLG* on Podleś sphere. It is the quantum deformation of the well known action of $SL(2,\mathbf{C})$ on the Riemann sphere S^2. $SL(2,\mathbf{C})$ is the two-fold covering of the group of all bi-holomorphic isomorphisms of S^2.

5 Intertwining operators.

In this section we consider the equivalence and irreducibility of the representations of the quantum Lorentz group on Gelfand spaces. To this end we shall investigate intertwining operators for the corresponding actions.

Let $\chi = (n_1,n_2)$ and $\chi' = (n'_1,n'_2)$ be the characters of the parabolic subgroup P of the quantum Lorentz group, $n = n_1 - n_2$, $n' = n'_1 - n'_2$ and $T :\, D_\chi \longrightarrow D_{\chi'}$ be a linear operator. We say that T *intertwines* v_χ and $v_{\chi'}$ if

$$v_{\chi'} \circ T = (T \otimes \mathrm{id}) \circ v_\chi. \tag{5.1}$$

The set of all intertwiners will be denoted by $\mathrm{Mor}(\chi,\chi')$. Clearly $\mathrm{Mor}(\chi,\chi')$ is a complex vector space. We shall prove that $\dim \mathrm{Mor}(\chi,\chi') \leq 1$ for any χ and χ'. In particular for $\chi = \chi'$, $\dim \mathrm{Mor}(\chi,\chi) = 1$ and $\mathrm{Mor}(\chi,\chi) = \{\, \lambda\, \mathrm{id_{D_\chi}} :\ \lambda \in \mathbf{C} \,\}$. It shows that the representation of quantum Lorentz group on any Gelfand space is irreducible. In this context the irreducibility means that the representation does not split into a direct sum of non trivial subrepresentations. In particular the space of an irreducible representation may contain a non-trivial invariant subspace but if this is the case the irreducibility exludes the existence of an invariant complementary subspace.

Clearly the operator $T = 0$ acting from D_χ into $D_{\chi'}$ is an intertwiner for any characters χ and χ'. Moreover for every χ any multiple of identity acting on D_χ is an intertwiner. All other intertwiners are called *nontrivial*. Only exceptional pairs (χ, χ') admit nontrivial intertwining operators. We shall describe all such pairs. As in the previous section we shall mainly use the truncated notation. In particular a linear operator $T : \mathcal{Z}_n \longrightarrow \mathcal{Z}_{n'}$ such that $v^{\lambda'} \circ T = (T \otimes \mathrm{id}) \circ v^\lambda$ is a truncated version of an intertwiner.

Proposition 5.1
Let D, D', D'' be Gelfand spaces and v, v', v'' be the corresponding actions of quantum Lorentz group and $f : D' \otimes D \longrightarrow \mathbf{C}$ be $v' \textcircled{\top} v$ - invariant linear functional. Assume that f is left non-degenerate i.e. $x' = 0$ is the only element of D' such that $f(x' \otimes x) = 0$ for all $x \in D$. Then for any linear map $T : D'' \longrightarrow D'$,

$$\left(f \circ (T \otimes \mathrm{id_D}) \textit{ is } v'' \textcircled{\top} v - \textit{invariant} \right) \iff \left(T \textit{ intertwines } v'' \textit{ with } v' \right).$$

Proof. The implication $\Leftarrow$ is obvious. We shall prove the converse.

For any $x \in D$ and $a \in \mathcal{A}$ we set

$$\tilde{v}(x \otimes a) := v(x)(I \otimes a).$$

One can check that above formula defines a linear continuous map acting on $D \hat{\otimes} \mathcal{A}$. This map is invertible, the inverse map is given by the formula

$$\tilde{v}^{-1} := (\mathrm{id_D} \otimes \kappa)\tilde{v},$$

where κ is the coinverse related to *QLG*. A functional $f : D \longrightarrow \mathbf{C}$ is v - invariant if and only if

$$(f \otimes \mathrm{id})\tilde{v} \;=\; f \otimes \mathrm{id}.$$

We shall apply the above concepts to the tensor product $v' \textcircled{\top} v$ acting on $D' \otimes D$, $v' \textcircled{\top} v :$ $D' \otimes D \longrightarrow D' \hat{\otimes} D \hat{\otimes} \mathcal{A}$. Then $(v' \textcircled{\top} v)\tilde{}$ is a linear continuous invertible map acting on $D' \hat{\otimes} D \hat{\otimes} \mathcal{A}$. Using the leg-numbering notation we have

$$(v' \textcircled{\top} v)\tilde{} = \tilde{v}'_{13} \tilde{v}_{23}.$$

We assumed that f is $v' \textcircled{\top} v$ - invariant. Therefore

$$f_{12} \circ \tilde{v}'_{13} \circ \tilde{v}_{23} \;=\; f_{12}.$$

In the above formula $\tilde{v}'_{13}$ and $\tilde{v}_{23}$ are linear operators acting on $D' \hat{\otimes} D \hat{\otimes} \mathcal{A}$, whereas f_{12} maps $D' \hat{\otimes} D \hat{\otimes} \mathcal{A}$ into $\mathcal{A}$. Assume that $f \circ (T \otimes \mathrm{id})$ is $v'' \textcircled{\top} v$ - invariant. Then

$$f_{12} \circ T_1 \circ \tilde{v}''_{13} \circ \tilde{v}_{23} \;=\; f_{12} \circ T_1.$$

Combining this formula with the previous one we get

$$f_{12} \circ T_1 \circ \tilde{v}''_{13} \circ \tilde{v}_{23} \;= f_{12} \circ \tilde{v}'_{13} \circ \tilde{v}_{23} \circ T_1.$$

Clearly the operators $\tilde{v}_{23}$ and T_1 commute. Remembering that $\tilde{v}_{23}$ is invertible we obtain

$$f_{12} \circ T_1 \circ \tilde{v}''_{13} \;=\; f_{12} \circ \tilde{v}'_{13} \circ T_1$$

and (f is left non-degenerate!) $T_1 \circ \tilde{v}''_{13} = \tilde{v}'_{13} \circ T_1$. This shows that T intertwines v'' and v'. □

Let $\chi = (n_1, n_2)$ and $\chi' = (n'_1, n'_2)$ be characters of the parabolic subgroup. We shall use Proposition 5.1 in the following context

$$\begin{array}{lcl} D \;=\; D_{\chi'} & , & v \;=\; v_{\chi'}, \\ D' \;=\; D_{-\chi'} & , & v' \;=\; v_{-\chi'}, \\ D'' \;=\; D_{\chi} & , & v'' \;=\; v_{\chi} \end{array}$$

and f is the only non-degenerate (in both variables) $v_{-\chi'} \textcircled{\scriptsize T} v_{\chi'}$ - invariant functional on $D_{-\chi'} \otimes D_{\chi'}$:

$$f\left(q^{(n'_1+n'_2+2)J_3} \otimes a \otimes q^{-(n'_1+n'_2-2)J_3} \otimes b\right) \;=\; h(ab) \tag{5.2}$$

for any $a \in \mathcal{Z}_{-n'}$ and $b \in \mathcal{Z}_{n'}$ (cf.(4.26)).

Let $\psi \in \mathcal{A}'_c$ be a linear functional supported by $J_{\frac{n}{2},-\frac{n'}{2}}$. For any $a \in \mathcal{Z}_n$ we set

$$T_\psi\left(q^{-(n_1+n_2-2)J_3} \otimes a\right) = q^{(n'_1+n'_2+2)J_3} \otimes (a * \psi). \tag{5.3}$$

Then T_ψ is a linear map acting from D_χ into $D_{-\chi'}$. Indeed using the truncated notation:

$$\begin{aligned} u^s_{\frac{n}{2},k} * \psi \;=\; & \;(\psi \otimes \mathrm{id_c})\Delta_c(u^s_{\frac{n}{2},k}) \;=\; \textstyle\sum_l (\psi \otimes \mathrm{id_c})(u^s_{\frac{n}{2},l} \otimes u^s_{l,k}) \\ = & \left\{ \begin{array}{ll} 0 & \text{if} \;\; s < \frac{|n'|}{2} \\ \\ \psi\left(u^s_{\frac{n}{2},-\frac{n'}{2}}\right) u^s_{-\frac{n'}{2},k} \in \mathcal{Z}_{-n'} & \text{if} \;\; s \geq \frac{|n'|}{2}. \end{array} \right. \end{aligned} \tag{5.4}$$

According to the Theorem 3.5 any $v_\chi \textcircled{\scriptsize T} v_{\chi'}$ - invariant functional on $D_\chi \otimes D_{\chi'}$ is of the form

$$\begin{aligned} f'\left(q^{-(n_1+n_2-2)J_3} \otimes a \otimes q^{-(n'_1+n'_2-2)J_3} \otimes b\right) &= h((a * \psi)b) \\ &= f\left(T_\psi(q^{-(n_1+n_2-2)J_3} \otimes a) \otimes q^{-(n'_1+n'_2-2)J_3} \otimes b\right) \end{aligned} \tag{5.5}$$

for any $a \in \mathcal{Z}_n$ and $b \in \mathcal{Z}_{n'}$. In the above formula ψ is a (χ, χ') - spherical functional on $\mathcal{A}_c$.

Now using Proposition 5.1 we obtain

Theorem 5.2
Let χ, χ' be the characters of the parabolic subgroup of the quantum Lorentz group. Then

$$\mathrm{Mor}(\chi, -\chi') \;=\; \{\, T_\psi : \psi \; is \; a \; (\chi,\chi') - spherical \;\; functional \;\; on \;\; \mathcal{A}_c \}\,.$$

In particular

$$\dim \mathrm{Mor}(\chi, -\chi') \;=\; \begin{cases} 1 & \text{if } (\chi,\chi') \text{ is an admissible pair} \\ \\ 0 & \qquad\qquad \text{otherwise.} \end{cases}$$

We know that for any χ, the pair $(\chi, -\chi)$ is admissible and (cf. previous section) any $(\chi, -\chi)$ - spherical functional ψ is a multiple of e_c restricted to $J_{\frac{n}{2},\frac{n}{2}}$. Therefore in this case T_ψ is a multiple of the identity map. We shall prove that also in a generic case an intertwiner $T_\psi : D_\chi \longrightarrow D_{-\chi'}$ is an isomorphism. The exceptions will be described.

Let ψ be a (χ,χ')-spherical functional. The set

$$\mathrm{Sp}\,\psi := \left\{ s \in \mathrm{Sp}\, v_\chi \cap \mathrm{Sp}\, v_{\chi'} : \psi \left(u^s_{\frac{n}{2}, -\frac{n'}{2}} \right) \neq 0 \right\}$$

will be called *the spin-support of* ψ. Using (5.4) one can easily verify that (still using the truncated notation)

$$\begin{aligned} \ker T_\psi &= \left\{ u^s_{\frac{n}{2},k} : \begin{array}{c} k = -s, -s+1, ..., s; \\ s \in \mathrm{Sp}\, v_\chi \setminus \mathrm{Sp}\,\psi \end{array} \right\}^{\text{linear span}}, \\ \mathrm{Im}\, T_\psi &= \left\{ u^s_{-\frac{n'}{2},l} : \begin{array}{c} l = -s, -s+1, ..., s; \\ s \in \mathrm{Sp}\,\psi \end{array} \right\}^{\text{linear span}}. \end{aligned} \tag{5.6}$$

In particular T_ψ is injective iff $\mathrm{Sp}\,\psi = \mathrm{Sp}\, v_\chi$ and surjective iff $\mathrm{Sp}\,\psi = \mathrm{Sp}\, v_{\chi'}$. T_ψ is invertible iff $\mathrm{Sp}\, v_\chi = \mathrm{Sp}\, v_{\chi'} = \mathrm{Sp}\,\psi$.

As in the classical case (cf. [3]) we introduce the concept of *positive integer point.*

Definition 5.3
We call a character χ of the parabolic group a positive integer point whenever

$$\chi = (n_1, n_2) = \left(p_1 - \epsilon \frac{\pi i}{\log q}, p_2 - \epsilon \frac{\pi i}{\log q} \right) \tag{5.7}$$

for some $p_1, p_2 \in \{1, 2, 3, ...\}$ and $\epsilon = 0, 1$.

For an positive integer point χ of the form (5.7) we shall consider closed subspaces $E_\chi \subset D_\chi$ and $F_{-\chi} \subset D_{-\chi}$. In truncated notation

$$E_\chi := \left\{ u^s_{\frac{n}{2},k} : k = -s, -s+1, ..., s;\;\; s = \frac{|p_1 - p_2|}{2}, ..., \frac{p_1 + p_2}{2} - 1 \right\}^{\text{linear span}} \tag{5.8}$$

and

$$F_{-\chi} := \left\{ u^s_{-\frac{n}{2},l} : l = -s, -s+1, ..., s;\ s = \frac{p_1+p_2}{2}, \frac{p_1+p_2}{2}+1, ... \right\}^{\text{linear span}}_{,} \tag{5.9}$$

where $n = n_1 - n_2 = p_1 - p_2$. One can check that

$$\mathrm{codim} F_{-\chi} \;=\; \dim E_\chi \;=\; p_1 p_2.$$

Let us consider admisible pairs of the form (χ, χ).

Lemma 5.4
Let χ be a character of the parabolic subgroup and ψ be a non-zero (χ,χ) -spherical functional. Then
1^o

$$\left(\begin{array}{c} T_\psi : D_\chi \longrightarrow D_{-\chi}) \\ \textit{is an isomorphism} \end{array} \right) \Longleftrightarrow \left(\ \chi \textit{ and } (-\chi) \textit{ are not positive integer points} \ \right).$$

2^o. *If χ is a positive integer point then*

$$\ker T_\psi \;=\; E_\chi, \qquad \mathrm{Im}\, T_\psi \;=\; F_{-\chi}. \tag{5.10}$$

3^o. *If $(-\chi)$ is a positive integer point then*

$$\ker T_\psi \;=\; F_\chi, \qquad \mathrm{Im}\, T_\psi \;=\; E_{-\chi}. \tag{5.11}$$

Proof. Let $\chi = (n_1, n_2)$ and $n = n_1 - n_2$. In the considered case (cf.B.19)

$$\begin{aligned} u^s_{\frac{n}{2},-\frac{n}{2}} \;&=\; \frac{q^{-|n|(s-\frac{|n|}{2})}}{[|n|]_q!} \frac{[s+\frac{|n|}{2}]_q!}{[s-\frac{|n|}{2}]_q!} \\ &\times \sum_{j=0}^{s-\frac{|n|}{2}} \frac{(q^{2(-s+\frac{|n|}{2})};q^2)_j (q^{2(s+\frac{|n|}{2}+1)};q^2)_j}{(q^{2(|n|+1};q^2)_j (q^2;q^2)_j} (q^2 \gamma_c^* \gamma_c)^j \cdot \left\{ \begin{array}{ll} \gamma_c^n & \text{if } n \geq 0 \\ (-q\gamma_c^*)^{|n|} & \text{if } n < 0. \end{array} \right. \end{aligned} \tag{5.12}$$

Suppose that (χ, χ) is a non-singular pair and that $\mathrm{Sp}\,\psi$ is a proper subset $\mathrm{Sp}\, v_\chi$ (i.e. T_ψ is not an isomorphism). Then we may assume that ψ is given by the formula (4.10). It means that

$$\psi\left((\gamma_c^* \gamma_c)^m (\gamma_c)_n \right) \;=\; \frac{1 - q^{2z_n}}{1 - q^{2(m+z_n)}}$$

where

$$z_n := \frac{1}{2}(|n_1 - n_2| - (n_1 + n_2)) \;=\; \left\{ \begin{array}{ll} -n_2 & \text{if } n \geq 0 \\ -n_1 & \text{if } n < 0 \end{array} \right. .$$

Applying ψ to both sides of (5.12) and using Lemma B.2 we obtain after simple algebraic transformations

$$\psi\left(u^s_{\frac{n}{2},-\frac{n}{2}}\right) = q^{-(n_1+n_2)(s-\frac{|n|}{2})}\frac{\left(q^{2(|n|-z_n+1)};q^2\right)_{s-\frac{|n|}{2}}}{\left(q^{2(z_n+1)};q^2\right)_{s-\frac{|n|}{2}}}\cdot\begin{cases}1 & \text{if } n\geq 0\\ (-q)^{|n|} & \text{if } n<0.\end{cases} \tag{5.13}$$

In particular

$$\psi\left(u^{\frac{|n|}{2}}_{\frac{n}{2},-\frac{n}{2}}\right) = \begin{cases}1 & \text{for } n\geq 0\\ (-q)^{|n|} & \text{for } n<0\end{cases}$$

and is different from zero. Therefore $\frac{|n|}{2}\in \operatorname{Sp}\psi$ and by virtue of (5.13) $s\in\operatorname{Sp}\psi$ if and only if

$$\left(q^{2(|n|-z_n+1)};q^2\right)_{s-\frac{|n|}{2}} \neq 0. \tag{5.14}$$

Let s_0 be the minimal element of $\operatorname{Sp} v_\chi \setminus \operatorname{Sp}\psi$. Then $s_0-\frac{|n|}{2}$ is a positive integer and

$$q^{2(\frac{|n|}{2}-z_n+s_0)} = 1. \tag{5.15}$$

Solving the equation (5.15) we get $s_0 = -\frac{1}{2}(n_1+n_2) \bmod(\frac{\pi i}{\log q})$. Noting that $n_1-n_2=n$ we obtain $n_1=-p_1-\epsilon\frac{\pi i}{\log q}$, $n_2=-p_2-\epsilon\frac{\pi i}{\log q}$ where $\epsilon\in\{0,1\}$, $p_1=s_0-\frac{n}{2}$ and $p_2=s_0+\frac{n}{2}$. Remembering that $s_0-\frac{|n|}{2}$ is a positive integer we conclude that p_1 and p_2 are positive integers. It means that $(-\chi)$ is a positive integer point.

Conversely if $(-\chi)=(p_1-\epsilon\frac{\pi i}{\log q}, p_2-\epsilon\frac{\pi i}{\log q})$ is a positive integer point then the condition 1^o of Definition 4.2 is violated for the pair (χ,χ). Therefore (χ,χ) is not singular. Moreover $s_0=\frac{1}{2}(p_1+p_2)$ satisfies the equation (5.15) and $s_0-\frac{|n|}{2}=\min(p_1,p_2)$ is a positive integer. Therefore $s_0\in\operatorname{Sp} v_\chi\setminus\operatorname{Sp}\psi$ and $\operatorname{Sp}\psi$ is a proper subset of $\operatorname{Sp} v_\chi$. Using (5.15) one can easily show that (5.14) is satisfied if and only if $s<s_0$. Therefore

$$\operatorname{Sp}\psi = \left\{\frac{|n|}{2},\frac{|n|}{2}+1,\ldots\frac{p_1+p_2}{2}-1\right\}$$

and using (5.6) we obtain relation (5.11). Moreover we have showed that

$$\left(\begin{array}{c}(\chi,\chi)\ \textit{is not singular and}\\ T_\psi\ \textit{is not an isomorphism}\end{array}\right)\Longleftrightarrow\left(\ (-\chi)\ \textit{is a positive integer point}\ \right). \tag{5.16}$$

Now suppose that (χ,χ) is a singular pair and that $\operatorname{Sp}\psi$ is a proper subset $\operatorname{Sp} v_\chi$. Then (cf.Definition 4.2)

$$\chi=(n_1,n_2)=(p_1-\epsilon\frac{\pi i}{\log q},p_2-\epsilon\frac{\pi i}{\log q}), \tag{5.17}$$

where $p_1,p_2\in\{0,1,2,\ldots\}$. In this case $m_{\chi\chi}=\min\{p_1,p_2\}$ and

$$x^{(m_{\chi\chi})}_{n,-n} = (\gamma_c)_n(\gamma_c\gamma_c^*)^{m_{\chi\chi}}. \tag{5.18}$$

We may assume that the (χ,χ)-spherical functional ψ is given by the formula (4.11). Let $s=\frac{|n|}{2}+m$, where m is a non-negative integer. $s\in \mathrm{Sp}\,\psi$ if and only if $\psi(u^{s}_{\frac{n}{2},-\frac{n}{2}})\neq 0$. This is the case if and only if the basis element (5.18) does appear on the right hand side of (5.12) i.e. iff $m\geq m_{\chi\chi}$. It shows that

$$\mathrm{Sp}\,\psi=\left\{\frac{|n|}{2}+m_{\chi\chi},\frac{|n|}{2}+m_{\chi\chi}+1,...\right\}. \tag{5.19}$$

Remembering that $\mathrm{Sp}\,\psi$ is a proper subset of $\mathrm{Sp}\,v_\chi$ we conclude that $m_{\chi\chi}>0$. Therefore $p_1>0$, $p_2>0$ and (5.17) shows that χ is a positive integer point. Conversely if χ is a positive integer point then we have (5.17) with $p_1,\,p_2=1,2,3,....$ Then (χ,χ) is singular, $m_{\chi\chi}=\min\{p_1,p_2\}>0$ and $\frac{|n|}{2}\notin \mathrm{Sp}\,\psi$. It shows that $\mathrm{Sp}\,\psi$ is a proper subset of $\mathrm{Sp}\,v_\chi$. The reader should notice that the first element of (5.19) equals $\frac{1}{2}(p_1+p_2)$. Using (5.19) and (5.6) we obtain relation (5.10). Moreover we have shown that

$$\left(\begin{array}{c}(\chi,\chi)\ \textit{is singular and}\\ T_\psi\ \textit{is not an isomorphism}\end{array}\right)\Longleftrightarrow\left(\ \chi\ \textit{is a positive integer point}\ \right). \tag{5.20}$$

Now combining (5.16) and (5.20) we get Statement 1^o.

□

Now we consider the third case (cf.Propostion 4.6) of admisible pairs. For this case we have

Lemma 5.5
Let $p_1\in\{0,1,2,...\}$, $p_2\in\{1,2,...\}$, $\epsilon=0,1$ and

$$\chi=(n_1,n_2):=\left(p_1-\epsilon\frac{\pi i}{\log q},\,p_2-\epsilon\frac{\pi i}{\log q}\right),\quad \chi'=(n_1',n_2'):=\left(p_1-\epsilon\frac{\pi i}{\log q},\,-p_2-\epsilon\frac{\pi i}{\log q}\right)$$

be characters of the parabolic subgroup. Then (χ,χ') and $(\chi'\chi)$ are admissible pairs. Let ψ be a non-zero (χ,χ') -spherical functional and ψ' be a non-zero (χ',χ) -spherical functional. Then

1^o

$$\left(\begin{array}{c}T_\psi: D_\chi\longrightarrow D_{-\chi'}\\ \textit{is an isomorphism}\end{array}\right)\Longleftrightarrow\left(\ \chi\ \textit{is not a positive integer point}\ \right).$$

If χ is a positive integer point then T_ψ is a surjective map and

$$\ker T_\psi\ =\ E_\chi.$$

2^o

$$\left(\begin{array}{c}T_{\psi'}: D_{\chi'}\longrightarrow D_{-\chi}\\ \textit{is an isomorphism}\end{array}\right)\Longleftrightarrow\left(\ \chi\ \textit{is not a positive integer point}\ \right).$$

If χ is an integer point then $T_{\psi'}$ is an injective map and

$$\mathrm{Im}\,T_{\psi'}\ =\ F_{-\chi}.$$

Remark: Clearly χ is a positive integer point if and only if $p_1 \neq 0$.
Proof. In the present case $n = n_1 - n_2 = p_1 - p_2,\ n' = n_1' - n_2' = p_1 + p_2 > 0$. Clearly $|n| \leq n'$. Therefore $\mathrm{Sp}\, v_{\chi'} \subseteq \mathrm{Sp}\, v_\chi$. The equality holds if and only if $|n| = n'$. This is equivalent to $p_1 = 0$. We may assume that ψ is given by (4.16). Using (B.19) we get

$$\psi\left(u^s_{\frac{n}{2},-\frac{n'}{2}}\right) = \frac{q^{-\frac{1}{2}(n+n')(s-\frac{n}{2})}}{[\frac{n+n'}{2}]_q!}\sqrt{\frac{[s+\frac{n}{2}]_q![s+\frac{n'}{2}]_q!}{[s-\frac{n}{2}]_q![s-\frac{n'}{2}]_q!}} \neq 0.$$

Therefore $\mathrm{Sp}\, v_{\chi'} = \mathrm{Sp}\, \psi$ and T_ψ is surjective. T_ψ is an isomorphism only if $p_1 = 0$. If $p_1 > 0$ then

$$\mathrm{Sp}\, v_\chi \setminus \mathrm{Sp}\, v_{\chi'} = \left\{\frac{|n|}{2}, \frac{|n|}{2}+1, ..., \frac{n'}{2}-1\right\}$$

and using (5.6) we see that the kernel of T_ψ coincides with (5.8). This proves Statement 1^0.

The proof of 2^0 is similar. In this case we may assume that ψ' is given by (4.16) with $(n, -n')$ replaced by $(n', -n)$. Again using (B.19) one can show that no coefficient $\psi'(u^s_{\frac{n'}{2},-\frac{n}{2}})$ vanishes. Remembering that $n' \geq |n|$ we see that $\mathrm{Sp}\, \psi' = \mathrm{Sp} v_{\chi'}$ and $T_{\psi'}$ is injective. $T_{\psi'}$ is an isomorphism only if $|n| = n'$. This is equivalent to $p_1 = 0$. If $p_1 > 0$ then by (5.6) the image of $T_{\psi'}$ coincides with (5.9). The Statement 2^0 is proved.

□

Using Proposition 4.7 by the similar argumentation we consider the fourth case of admissible pairs. We state the result only.

Lemma 5.6
Let $p_1 \in \{1, 2, ...\}$, $p_2 \in \{0, 1, 2, ...\}$, $\epsilon = 0, 1$ *and*

$$\chi = (n_1, n_2) := \left(p_1 - \epsilon\frac{\pi i}{\log q},\ p_2 - \epsilon\frac{\pi i}{\log q}\right), \quad \chi'' = (n_1'', n_2'') := \left(-p_1 - \epsilon\frac{\pi i}{\log q},\ p_2 - \epsilon\frac{\pi i}{\log q}\right)$$

be characters of the parabolic subgroup. Then (χ, χ'') *and* (χ'', χ) *are admissible pairs. Let* ψ *be a non-zero* (χ, χ'') *-spherical functional and* ψ'' *be a non-zero* (χ'', χ) *-spherical functional. Then*
1^o

$$\left(\begin{array}{c} T_\psi : D_\chi \longrightarrow D_{-\chi''} \\ \textit{is an isomorphism}\end{array}\right) \Longleftrightarrow \left(\ \chi \textit{ is not a positive integer point}\ \right).$$

If χ *is an integer point then* T_ψ *is a surjective map and*

$$\ker T_\psi \ =\ E_\chi.$$

2^o

$$\left(\begin{array}{c} T_{\psi''} : D_{\chi''} \longrightarrow D_{-\chi} \\ \textit{is an isomorphism}\end{array}\right) \Longleftrightarrow \left(\ \chi \textit{ is not a positive integer point}\ \right).$$

If χ is an integer point then $T_{\psi''}$ is an injective map and

$$\mathrm{Im}\, T_{\psi'} \; = \; F_{-\chi}.$$

Remark 1: Clearly in this case χ is a positive integer point if and only if $p_2 \neq 0$.

Remark 2: According to Lemmas 5.4, 5.5 and 5.6 for a character χ being an integer point the subspaces E_χ of D_χ and $F_{-\chi}$ of $D_{-\chi}$ are kernels or images of some intertwiners. One can see that as in the classical case the image and the kernel of an intertwiner are invariant subspaces. In fact for a linear map $T : D \longrightarrow D'$ between two Gelfand spaces one has

$$(T \otimes \mathrm{id}) D \hat{\otimes} \mathcal{A} = (T \otimes \mathrm{id}) \sum_{s \in S}^{\oplus} D \otimes B(H^s) \otimes \mathcal{A}_c = T(D) \hat{\otimes} \mathcal{A} \subset D' \hat{\otimes} \mathcal{A}$$

since Gelfand spaces are countable dimensional vector spaces. If T is an intertwiner: $v' \circ T = (T \otimes \mathrm{id}) \circ v$ for the corresponding actions $v : D \longrightarrow D \hat{\otimes} \mathcal{A}$ and $v' : D' \longrightarrow D' \hat{\otimes} \mathcal{A}$ respectively this implies that

$$v'(T(D)) \subset T(D) \hat{\otimes} \mathcal{A} \;\; \text{and} \;\; v'(\ker T) \subset (\ker T) \hat{\otimes} \mathcal{A}$$

and the invariance follows.

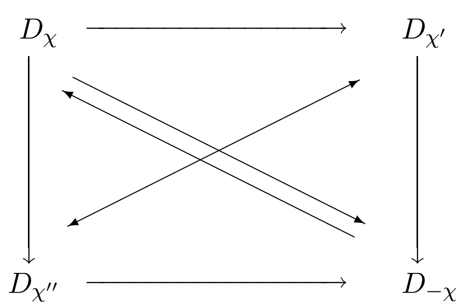
Now we have the following description of non-trivial intertwining operators acting between Gelfand spaces.

Theorem 5.7

1. Let $\chi = (n_1, n_2)$ be a positive integer point and

$$\chi' \; \equiv \; (-n_1, n_2), \qquad \chi'' \; \equiv \; (n_1, -n_2), \qquad -\chi \; \equiv \; (-n_1, -n_2).$$

Then we have the following commutative (up to a complex factor) diagram of nontrivial intertwiners:



(5.21)

The intertwiners are unique up to a complex factor.

The kernels of the intertwiners starting from D_χ and images of the intertwiners ending at D_χ coincide with the subspace $E_\chi \subset D_\chi$ introduced by (5.8).

The images of the intertwiner ending at $D_{-\chi}$ and kernels of the intertwiners starting from $D_{-\chi}$ coincide with the subspace $F_{-\chi} \subset D_{-\chi}$ introduced by (5.9).

The intertwiners starting from $D_{\chi'}$ or $D_{\chi''}$ are epimorphisms and the intertwiners ending *in $D_{\chi'}$ or $D_{\chi''}$ are monomorphisms. In particular the intertwiners between $D_{\chi'}$ and $D_{\chi''}$ are bijections.*

The subpaces E_χ and $F_{-\chi}$ are Lorentz invariant and

$$\dim E_\chi = \Re n_1 \cdot \Re n_2 = \operatorname{codim} F_{-\chi}.$$

2. Let $\chi = (n_1, n_2)$ be a character of the parabolic subgroup such that the space D_χ has appeared in no diagram of the form (5.21) *(none of the characters $(\pm n_1, \pm n_2)$ is an integer point). Then there exists unique (up to a scalar factor) bijective intertwiner*

$$D_\chi \overset{T}{\longleftrightarrow} D_{-\chi}.$$

3. The intertwiners listed in the above two points are the only non-trivial intertwiners acting between the Gelfand spaces.

Remark: For the character χ being the integer point the restricted finite-dimensional representations acting on E_χ were studied in [10] . They are non-unitary except the cases of two 1-dimensional representations appearing when $\chi = (1 - \epsilon\frac{\pi i}{\log q}, 1 - \epsilon\frac{\pi i}{\log q})$. The representations acting on $F_{-\chi}$, $D_{\chi'}$ and $D_{\chi''}$ are equivalent. Moreover we can identify the quantum Lorentz group action on $F_{-\chi}$ with the quotient action on D_χ/E_χ and the action on E_χ with the quotient action on $D_{-\chi}/F_{-\chi}$.

6 Gelfand spaces with unitary actions of QLG

In this section we find all characters χ such that corresponding Gelfand spaces D_χ admit a v_χ - invariant scalar product. If χ is such a character then the invariant scalar product is unique (up to a positive factor). Then by completion procedure D_χ is a dense subset of a Hilbert space H_χ and v_χ gives rise to a unitary representation $\tilde{v}_\chi$ of the quantum Lorentz group acting on H_χ. We shall show that in this way we obtain all infinite dimensional irreducible unitary representations of the quantum Lorentz group.

For the convenience of the reader we remind the basic results of the theory of unitary representations of the quantum Lorentz group [11].

By definition a unitary strongly continuous representation of the quantum Lorentz group acting on the Hilbert space H is a unitary element $\tilde{v} \in \mathrm{M}(CB(H) \otimes A)$ such that

$$(\mathrm{id} \otimes \Delta)\tilde{v} = \tilde{v}_{12}\tilde{v}_{13}.$$

For any unitary representation $\tilde{v}$ one introduces the spin spectrum $\mathrm{Sp}\tilde{v} := \mathrm{Sp}\,(\tilde{v}|_{S_qU(2)})$ (cf.(2.12)) and the Casimir operators $C(\tilde{v})$ and $C'(\tilde{v})$ (cf.C.16):

$$C(\tilde{v}) = (\mathrm{id} \otimes \Psi)\tilde{v}, \quad C'(\tilde{v}) = (\mathrm{id} \otimes \Psi')\tilde{v}$$

in a similar manner as for Gelfand actions.[1] We shall see that due to the special choice of central functionals Ψ and Ψ' (cf.(C.15)) the corresponding Casimir operators are mutually

[1]The Casimir operator $C(\tilde{v})$ is a multiple of the Casimir operator X used in [11], $C(\tilde{v}) = q^{-1}\sqrt{1+q^2}X$ (cf.[11], equation (3.3)).

adjoint: $C(\tilde{v})^* = C'(\tilde{v})$. If $\tilde{v}$ is an irreducible representation then $C(\tilde{v}) = cI$ where $c \in \mathbf{C}$. It is known (cf. [11], Theorem 0.1) that $\tilde{v}$ is completely characterized by its spin spectrum and the value of the Casimir operator. More precisely we have the following *classification theorem:*

Theorem 6.1
Let $\tilde{v}$ be an irreducible unitary representation of the quantum Lorentz group, $\mathrm{Sp}\,\tilde{v}$ be its spin spectrum and $c \in \mathbf{C}$ be the value of its Casimir operator $C(\tilde{v})$.
Then $\mathrm{Sp}\tilde{v}$ is simple and we have the following possibilities:

1. *Sp $\tilde{v} = \{0\}$ and $c = -(q + q^{-1})$. In this case $\tilde{v}$ is the trivial one-dimensional representation.*
2. *Sp $\tilde{v} = \{0\}$ and $c = q+q^{-1}$. In this case $\tilde{v}$ is a nontrivial one-dimensional representation.*
3. *Sp $\tilde{v} = \{p, p+1, p+2, \ldots\}$ where p is non-negative half-integer and $\mid c-2 \mid + \mid c+2 \mid = 2(q^p + q^{-p})$. In this case $\tilde{v}$ is an infinite-dimensional representation. It belongs to the principal series of representations.*
4. *Sp $\tilde{v} = \{0, 1, 2, \ldots\}$, $c \in \mathbf{R}$ and $2 < |c| < q+q^{-1}$. In this case $\tilde{v}$ is an infinite-dimensional representation. It belongs to the supplementary series of representations.*

Moreover any of these possibilities does occur and the pair $(\mathrm{Sp}\,\tilde{v}, c)$ completely determines (up to the unitary equivalence) the irreducible unitary representation $\tilde{v}$.

Let $\chi = (n_1, n_2)$ and $\chi' = (n'_1, n'_2)$ be characters of the parabolic subgroup P and $(\cdot \mid \cdot)$ be a sesquilinear form defined on $D_{\chi'} \times D_\chi$. By definition sesquilinear forms considered in this paper are linear with respect to the second variable. Scalar product is a strictly positive sesquilinear form. For any $x \in D_{\chi'}$, $y \in D_\chi$ and $a, b \in \mathcal{A}$ we set

$$(x \otimes a \mid y \otimes b)_{\mathcal{A}} \;:=\; a^* \,(x \mid y)\, b. \tag{6.1}$$

Clearly the above formula defines a continuous sesquilinear map $(D_{\chi'}\hat{\otimes}\mathcal{A}) \times (D_\chi\hat{\otimes}\mathcal{A}) \longrightarrow \mathcal{A}$. We say that $(\cdot \mid \cdot)$ is Lorentz invariant if

$$(v_{\chi'}(x) \mid v_\chi(y))_{\mathcal{A}} = (x \mid y)\, I_{\mathcal{A}} \tag{6.2}$$

for any $x \in D_{\chi'}$ and $y \in D_\chi$.

Let $(\chi')^* \equiv (\bar{n}'_2, \bar{n}'_1)$. Then

$$(D_{n'_1 n'_2})^* \;= \left\{ x^* \,:\, x \in D_{n'_1 n'_2} \right\} \;=\; D_{\bar{n}'_2 \bar{n}'_1}.$$

Remembering that $v_{\chi'}$ is the restriction of Δ to $D_{\chi'}$ we get $v_{(\chi')^*}(x^*) = v_{\chi'}(x)^*$ for any $x \in D_{\chi'}$ (Δ is a *-homomorphism). It means that $D_{\chi'} \ni x \longrightarrow x^* \in D_{(\chi')^*}$ is an (antilinear, invertible) intertwiner. Therefore linear functionals f on $D_{\bar{n}'_2 \bar{n}'_1} \hat{\otimes} D_{n_1 n_2}$ are in one-to-one correspondence with sesquilinear forms $(\cdot \mid \cdot)$ on $D_{n'_1 n'_2} \times D_{n_1 n_2}$. This correspondence is given by the formula

$$(x \mid y) := f(x^* \otimes y), \tag{6.3}$$

where $x \in D_{n'_1 n'_2}$, $y \in D_{n_1 n_2}$. Clearly f is a $v_{(\chi')^*} \bigoplus\!\!\!\!\!\!\top\; v_\chi$ - invariant functional if and only if the corresponding sesquilinear functional is Lorentz invariant (cf.(3.4)).

Using Theorem 4.9 one can easily select all pairs of Gelfand spaces admitting invariant sesquilinear form.

Theorem 6.2

Let $\chi = (n_1, n_2)$, $\chi' = (n'_1, n'_2)$ be characters of the parabolic subgroup. Assume that there exists a non-zero invariant sesquilinear form on $D_{n'_1 n'_2} \times D_{n_1 n_2}$. Then we have the following four possibilities:

1.
$$(n'_1, n'_2) \ \equiv (\bar{n}_2, \bar{n}_1); \tag{6.4}$$

2.
$$(n'_1, n'_2) \ \equiv (-\bar{n}_2, -\bar{n}_1); \tag{6.5}$$

3.
$$(n'_1, n'_2) = \left(p_1 - \epsilon\frac{\pi i}{\log q}, p_2 - \epsilon\frac{\pi i}{\log q}\right), \qquad (n_1, n_2) = \left(p_1 - \epsilon\frac{\pi i}{\log q}, -p_2 - \epsilon\frac{\pi i}{\log q}\right), \tag{6.6}$$

where $p_1 = 0, 1, 2, \ldots,\ p_2 = \pm 1, \pm 2, \ldots$ *and* $\epsilon = 0, 1$;

4.
$$(n'_1, n'_2) = \left(p_1 - \epsilon\frac{\pi i}{\log q}, p_2 - \epsilon\frac{\pi i}{\log q}\right), \qquad (n_1, n_2) = \left(-p_1 - \epsilon\frac{\pi i}{\log q}, p_2 - \epsilon\frac{\pi i}{\log q}\right), \tag{6.7}$$

where $p_1 = \pm 1, \pm 2, \ldots,\ p_2 = 0, 1, 2, \ldots$ *and* $\epsilon = 0, 1$.

In all these cases the invariant sesquilinear form is unique (up to a scalar factor).

Using this result we can select all Gelfand spaces D_χ such that $D_\chi \times D_\chi$ admits non-zero invariant sesquilinear form. Let us note that if such a form exits then it can be chosen hermitian by a suitable choice of the phase of the numerical factor. This is an easy consequence of the uniqueness. In particular we can select all D_χ admitting a v_χ - invariant scalar product (i.e. nondegenerete positive definite invariant sesquilinear form).

Proposition 6.3

Let $\chi = (n_1, n_2)$ be a character of the parabolic subgroup. Then

1. $$\left(\begin{array}{c} D_\chi \times D_\chi \textit{ admits an invariant} \\ \textit{sesquilinear form} \end{array} \right) \iff \left(\begin{array}{c} \chi = \left(\frac{n}{2} + i\omega, -\frac{n}{2} + i\omega\right) \\ \textit{where } n \in \mathbf{Z} \textit{ and } \omega \in [0, \frac{-2\pi}{\log q}[\\ \textit{or} \\ \chi = \left(\rho - \epsilon\frac{\pi i}{\log q}, \rho - \epsilon\frac{\pi i}{\log q}\right) \\ \textit{where } \rho \in \mathbf{R} \setminus \{0\} \textit{ and } \epsilon = 0, 1 \end{array} \right).$$

2. *If* $\chi = \left(\frac{n}{2} + i\omega, -\frac{n}{2} + i\omega\right)$ *where* $n \in \mathbf{Z}$ *and* $\omega \in [0, \frac{-2\pi}{\log q}[$ *then the invariant sesquilinear form can be chosen positive.*

3. *If* $\chi = \left(\rho - \epsilon\frac{\pi i}{\log q}, \rho - \epsilon\frac{\pi i}{\log q}\right)$ *where* $\rho \in \mathbf{R} \setminus \{0\}$ *and* $\epsilon = 0, 1$ *then the invariant sesquiliner form can be chosen positive if and only if* $|\rho| < 1$.

Proof: Let in Theorem 6.2 $\chi' = \chi$. $D_\chi \times D_\chi$ admits an invariant sesquilinear form iff this condition is compatible with one of the conditions (6.4)-(6.7). It is easy to see that $\chi' = \chi$ is incompatible with (6.6) and with (6.7). Indeed assuming (6.6) we get

$$\left(p_1 - \epsilon\frac{\pi i}{\log q}, p_2 - \epsilon\frac{\pi i}{\log q}\right) = \left(p_1 - \epsilon\frac{\pi i}{\log q}, -p_2 - \epsilon\frac{\pi i}{\log q}\right).$$

Therefore $p_2 = 0$ and this is a contradiction since p_2 should be a non zero integer. In the similar way one can rule out (6.7).

Consider now (6.5). Then

$$(n_1, n_2) \equiv (-\bar{n}_2, -\bar{n}_1).$$

This condition involves only real parts of n_1 and n_2. Solving it we get $(n_1, n_2) = (\frac{n}{2} + i\omega, -\frac{n}{2} + i\omega)$ where $\omega \in \mathbf{R}$ and $n = n_1 - n_2 \in \mathbf{Z}$. By (3.1) $\omega \in [\,0, \frac{-2\pi}{\log q}\,[$.

Consider now (6.4). Then

$$(n_1, n_2) \equiv (\bar{n}_2, \bar{n}_1).$$

Therefore $\Re n_1 = \Re n_2$, $\Im n_1 = -\Im n_2 = -\Im n_1 \bmod (\frac{2\pi}{\log q})$ and $(n_1, n_2) = (\rho - \epsilon\frac{\pi i}{\log q}, \rho - \epsilon\frac{\pi i}{\log q})$, where $\rho = \Re n_1 \in \mathbf{R}$ and $\epsilon \in \mathbf{Z}$. By (3.1) $\epsilon = 0, 1$. We may assume that $\rho \neq 0$, because $\chi = (-\epsilon\frac{\pi i}{\log q}, -\epsilon\frac{\pi i}{\log q})$ is included in the previous case. This proves Statement 1^o.

To prove Statements 2 and 3 we have to write the invariant sesquilinear form $(\cdot \mid \cdot)$ (cf.(6.3)) in an explicite way.

Let $n = n_1 - n_2$, $\lambda = n_1 + n_2 - 2$. By Theorem 3.5 and (6.3), for any $a, b \in \mathcal{Z}_n$ we have

$$\left(q^{-\lambda J_3} \otimes a | q^{-\lambda J_3} \otimes b\right) = h((a^* * \psi)b), \tag{6.8}$$

where ψ is a (χ^*, χ) - spherical functional on $\mathcal{A}_c$.

If $\chi = (\frac{n}{2} + i\omega, -\frac{n}{2} + i\omega)$ then $\chi^* \equiv -\chi$ and the (χ^*, χ) - spherical functional ψ coincides (up to a multiplicative constant) with e_c restricted to $J_{-\frac{n}{2}, -\frac{n}{2}}$. In this case the formula (6.8) simplifies to

$$\left(q^{-\lambda J_3} \otimes a | q^{-\lambda J_3} \otimes b\right) := h(a^* b) \tag{6.9}$$

and $(\cdot \mid \cdot)$ is positive. Statement 2^o is proved.

Let $\chi = (\rho - \epsilon\frac{\pi i}{\log q}, \rho - \epsilon\frac{\pi i}{\log q})$ where $\rho \in \mathbf{R}$, $\rho \neq 0$ and $\epsilon = 0, 1$. Then $\chi^* \equiv \chi$ and (χ^*, χ) - spherical functionals are described by Proposition 4.5 with $n = 0$. If ρ is a non-zero integer then either χ or $(-\chi)$ is an integer point and by Lemma 5.4 the linear mapping $\mathcal{Z}_0 \ni a \longrightarrow a * \psi \in \mathcal{Z}_0$ (the truncated version of the T_ψ operator) has a nontrivial kernel and $(\cdot \mid \cdot)$ is degenerate. To discuss the case when ρ is not an integer we rewrite (6.8) for elements of linear basis of $\mathcal{Z}_0$: $a = u^s_{0,k}$, $b = u^{s'}_{0,l}$. Using the identity $(u^s_{0,k})^* = (-q)^k\, u^s_{0,-k}$ (cf.(B.21)) and (5.4) we get

$$(u^s_{0,k})^* * \psi = (-q)^k\, \psi(u^s_{0,0}) u^s_{0,-k} = \psi(u^s_{0,0})\, (u^s_{0,k})^*$$

and

$$\left(q^{-\lambda J_3} \otimes u^s_{0,k} \mid q^{-\lambda J_3} \otimes u^{s'}_{0,l}\right) = \psi(u^s_{0,0})\, h\left((u^s_{0,k})^*\, u^{s'}_{0,l}\right) \tag{6.10}$$

where $s, s' = 0, 1, 2, ...; k = -s, -s+1, ..., s; l = -s', -s'+1, ..., s'$. Remembering that the pair (χ, χ) is non-singular one may assume that ψ is given by (5.13):

$$\psi(u^s_{00}) = q^{-2\rho s}\frac{(q^{2(\rho+1)}; q^2)_s}{(q^{2(-\rho+1)}; q^2)_s}. \tag{6.11}$$

Analyzing this expression we see that it has the same sign for all $s = 0, 1, 2, ...$ iff $|\rho| < 1$. The reader should notice that for $|\rho| < 1$, the sign of (6.11) is positive. The Statement 3^o is proved.

□

Remark. Let us note in the case $\chi = (\rho - \epsilon\frac{\pi i}{\log q}, \rho - \epsilon\frac{\pi i}{\log q})$, where $\rho \in \mathbf{R}$, $|\rho| \geq 1$, $\epsilon = 0, 1$, the formula (6.10) describes an invariant sesquilinear form $(\cdot|\cdot)$. If in addition $|\rho| \neq 1, 2, ...$ then the formula (6.11) applies and $\psi(u^s_{00}) \neq 0$. More precisely

$$\operatorname{sign}\psi(u^s_{00}) = \begin{cases} (-1)^s & \text{for } s < |\rho| \\ \\ (-1)^r & \text{for } s > |\rho|, \end{cases}$$

where r is an integer part of $|\rho|$. Using this result one can easily show that signature of the sesquilinear form $(-1)^r(\cdot|\cdot)$ consists of $\frac{r(r+1)}{2}$ minuses and infinitely many pluses.

If $\rho = -1, -2, -3, ...$ then formula (6.11) still applies and

$$\operatorname{sign}\psi(u^s_{00}) = \begin{cases} (-1)^s & \text{for } s < -\rho \\ \\ 0 & \text{for } s \geq -\rho. \end{cases}$$

Now $r = -\rho$ and signature of the sesquilinear form $(-1)^r(\cdot|\cdot)$ consists of $\frac{r(r+1)}{2}$ minuses, $\frac{r(r-1)}{2}$ pluses and infinitely many zeros. It means that the sesquilinear form is degenerate. The corresponding null space clearly coincides with F_χ.

If $\rho = 1, 2, ...,$ then χ is a positive integer point. In this case the functional ψ is given by the formula (4.11) (the pair (χ, χ) is singular). Taking into account the expression for u^s_{00} (cf.B.19) we obtain

$$\psi(u^s_{00}) = \begin{cases} 0 & \text{for } s < \rho \\ \\ q^{2\rho}\frac{(q^{-2s}; q^2)_\rho (q^{2(s+1)}; q^2)_\rho}{(q^2; q^2)^2_\rho} & \text{for } s \geq \rho \end{cases}$$

Therefore

$$\operatorname{sign}\psi(u^s_{00}) = \begin{cases} 0 & \text{for } s < \rho \\ \\ (-1)^r & \text{for } s \geq \rho. \end{cases}$$

In this case $r = \rho$ and signature of the sesquilinear form $(-1)^r(\cdot|\cdot)$ contains infinitely many pluses and r^2 zeros. The sesquilinear form is degenerate and the corresponding null space coincides with E_χ.

Assume that D_χ admits a v_χ - invariant scalar product $(\cdot \mid \cdot)$. We describe briefly a procedure of extending v_χ to a unitary representation $\tilde{v}_\chi$ of the quantum Lorentz group. This representation acts on the Hilbert space H_χ which is the completion of D_χ.

Let $\mathcal{A}_0 := \sum^{\oplus}_{\text{finite}} B(H^s) \otimes \mathcal{A}_c$. Then $\mathcal{A}_0$ is a dense *-subalgebra of A : $\overline{\mathcal{A}_0} = A$. Clearly $\mathcal{A}_0 \subset \mathcal{A}$. We restrict the sesquilinear form (6.1) to $(D_\chi \otimes \mathcal{A}_0) \times (D_\chi \otimes \mathcal{A}_0)$:

$$(x \otimes a \mid y \otimes b)_A := (x \mid y) a^* b$$

for any $x, y \in D_\chi$ and $a, b \in \mathcal{A}_0$. Then $(\cdot \mid \cdot)_A$ is $\mathcal{A}_0$ valued scalar product and $D_\chi \otimes \mathcal{A}_0$ is pre-Hilbert $\mathcal{A}_0$-module. Let H_χ be Hilbert space obtained by the completion of D_χ with respect to $(\cdot|\cdot)$. Then the Hilbert A-module $H_\chi \otimes A$ is a completion of $D_\chi \otimes \mathcal{A}_0$ (cf.[7]). In particular $D_\chi \otimes \mathcal{A}_0$ is a dense subset of $H_\chi \otimes A$.

A map $\Theta : H_\chi \otimes A \longrightarrow H_\chi \otimes A$ is called *adjointable* if there is a map $\Theta^* : H_\chi \otimes A \longrightarrow H_\chi \otimes A$ such that

$$(\hat{y} | \Theta\, \hat{x})_A = (\Theta^*\, \hat{y} | \hat{x})_A$$

for any $\hat{x}, \hat{y} \in H_\chi \otimes A$. Any adjointable map is bounded and A-linear. The set of all adjointable maps is a C^*-algebra denoted by $\mathcal{L}(H_\chi \otimes A)$. In what follows the adjointable maps will be also called adjointable operators.

At first we have to recall how Kasparov (cf.[4]) identifies adjointable operators acting on $H_\chi \otimes A$ with elements of $\mathrm{M}(CB(H_\chi) \otimes A)$.

For any $x, y, z \in H_\chi$ we set

$$\theta_{x,y} z \;:=\; x(y|z).$$

Then $\theta_{x,y}$ is a finite rank operator acting on H_χ : $\theta_{x,y} \in CB(H_\chi)$. Similarly for any $\hat{x}, \hat{y}, \hat{z} \in H_\chi \otimes A$ we set

$$\Theta_{\hat{x},\hat{y}} \hat{z} \;:=\; \hat{x}\,(\hat{y}|\hat{z})_A.$$

Then $\Theta_{\hat{x},\hat{y}}$ is an adjointable operator acting on $H_\chi \otimes A$. Identifying $\theta_{x,y} \otimes ab^*$ with $\Theta_{x\otimes a, y\otimes b}$ $(x, y \in H_\chi,\ a, b \in A)$ we define an embeding $CB(H_\chi) \otimes A$ into $\mathcal{L}(H_\chi \otimes A)$. By the famous Kasparov Theorem (cf.[4],[7] Theorem 2.4 and p.10) this embeding extends uniquely to the isomorphism of $\mathrm{M}(CB(H_\chi) \otimes A)$ onto $\mathcal{L}(H_\chi \otimes A)$.

The action $v_\chi : D_\chi \longrightarrow D_\chi \hat{\otimes} \mathcal{A}$ gives rise to a linear map $\tilde{v}_\chi : D_\chi \otimes \mathcal{A}_0 \longrightarrow D_\chi \otimes \mathcal{A}_0$:

$$\tilde{v}_\chi(x \otimes a) := v_\chi(x)(I \otimes a)$$

for any $x \in D_\chi$ and $a \in \mathcal{A}_0$. The map $\tilde{v}_\chi$ is invertible: the inverse mapping is given by the formula $(\tilde{v}_\chi)^{-1} := (\mathrm{id} \otimes \kappa)\tilde{v}_\chi$. In particular $\tilde{v}_\chi(D_\chi \otimes \mathcal{A}_0) = D_\chi \otimes \mathcal{A}_0$. Using (6.2) we check that

$$\left(\tilde{v}_\chi(x \otimes a) \mid \tilde{v}_\chi(y \otimes b)\right)_A = (x \otimes a \mid y \otimes b)_A \tag{6.12}$$

for any $x, y \in D_\chi$ and $a, b \in \mathcal{A}_0$. Therefore $\tilde{v}_\chi$ is an isometry acting on the dense subset $D_\chi \otimes \mathcal{A}_0$ of $H_\chi \otimes A$. It can be extended to the isometry (denoted by the same symbol $\tilde{v}_\chi$) mapping $H_\chi \otimes A$ onto itself. Replacing in (6.12) $x \otimes a$ by $\tilde{v}_\chi^{-1}(x \otimes a)$ we get

$$\left(x \otimes a \mid \tilde{v}_\chi(y \otimes b)\right)_A = \Big((\tilde{v}_\chi)^{-1}(x \otimes a) \mid y \otimes b\Big)_A . \tag{6.13}$$

It shows that $\tilde{v}_\chi$ is an adjointable map and $(\tilde{v}_\chi)^* = (\tilde{v}_\chi)^{-1}$. By the Kasparov Theorem $\tilde{v}_\chi \in \mathrm{M}(CB(H_\chi) \otimes A)$. To show that $\tilde{v}_\chi$ is a strongly continuous representation of the quantum Lorentz group it remains to verify that

$$(\mathrm{id} \otimes \Delta)\tilde{v}_\chi = (\tilde{v}_\chi)_{12}(\tilde{v}_\chi)_{13}. \tag{6.14}$$

To this end we consider linear functionals on A with finite spin support. We say that spin $s \in \{0, 1/2, 1, 3/2, ...\}$ belongs to the spin support of a functional $\xi \in A'$ whenever ξ restricted to $B(H^s) \otimes A_c$ is a non-zero functional.

Lemma 6.4
For any functional $\xi \in A'$ with finite spin support and $x \in D_\chi$:

$$[(\mathrm{id} \otimes \xi)\tilde{v}_\chi]\, x = (\mathrm{id} \otimes \xi) v_\chi(x). \tag{6.15}$$

Proof: Let $x, y \in H_\chi;\ a, b \in A$. Using the identification $\Theta_{x\otimes a, y\otimes b} = \theta_{x,y} \otimes ab^*$ we get

$$(\mathrm{id} \otimes \xi)\Theta_{x\otimes a, y\otimes b} = \xi(ab^*)\theta_{x,y} = \theta_{\xi(ab^*)x,y} = \theta_{(\mathrm{id}\otimes \mathrm{b}^*\xi)(x\otimes a),y}$$

for any $\xi \in A'$ and by continuity

$$(\mathrm{id} \otimes \xi)\Theta_{\hat{x}, y\otimes b} = \theta_{(\mathrm{id}\otimes \mathrm{b}^*\xi)\hat{x},y} \tag{6.16}$$

for any $\hat{x} \in H_\chi \otimes A$.

Let $x \in D_\chi, a \in \mathcal{A}_0$. Remembering that $\tilde{v}_\chi$ is adjointable operator we check that

$$\tilde{v}_\chi \Theta_{x\otimes a, y\otimes b} = \Theta_{\tilde{v}_\chi(x\otimes a), y\otimes b} = \Theta_{v_\chi(x)(I\otimes a), y\otimes b}$$

and by (6.16)

$$(\mathrm{id} \otimes \xi)\left[\tilde{v}_\chi \Theta_{x\otimes a, y\otimes b}\right] = \theta_{(\mathrm{id}\otimes b^*\xi)[v_\chi(x)(I\otimes a)],y} = \theta_{(\mathrm{id}\otimes ab^*\xi)v_\chi(x),y}.$$

On the other hand

$$(\mathrm{id} \otimes \xi)\left[\tilde{v}_\chi \Theta_{x\otimes a, y\otimes b}\right] = (\mathrm{id} \otimes \xi)\left[\tilde{v}_\chi(\theta_{x,y} \otimes ab^*)\right] = \left[(\mathrm{id} \otimes ab^*\xi)\tilde{v}_\chi\right]\theta_{x,y} = \theta_{[(\mathrm{id}\otimes ab^*\xi)\tilde{v}_\chi]x,y}.$$

Therefore comparing the right hand sides of the above expressions we obtain

$$[(\mathrm{id} \otimes ab^*\xi)\tilde{v}_\chi]\, x = (\mathrm{id} \otimes ab^*\xi) v_\chi(x).$$

This proves (6.15) since any continuous linear functional on A with finite spin support is of the form $ab^*\xi$ ($\xi \in A', a \in \mathcal{A}_0, b \in A$).

□

Remark: Since $v_\chi = \Delta|_{D_\chi}$ then for any $x \in D_\chi$:

$$(\mathrm{id} \otimes \xi) v_\chi(x) \;=\; (\mathrm{id} \otimes \xi)\Delta(x) \;=\; \xi * x$$

and (6.15) can be written as

$$[(\mathrm{id} \otimes \xi)\tilde{v}_\chi]\, x = \xi * x \tag{6.17}$$

Now using Lemma 6.4 we get for any $\xi, \xi' \in A'$ with finite spin supports and $x \in D_\chi$ that

$$\left[(\mathrm{id} \otimes \xi \otimes \xi')(\mathrm{id} \otimes \Delta)\tilde{v}_\chi\right] x \;=\; \left[(\mathrm{id} \otimes \xi * \xi')\tilde{v}_\chi\right] x \;=\; \xi * \xi' * x$$

(note that the functional $\xi * \xi'$ has also finite spin support) and

$$\left[(\mathrm{id} \otimes \xi \otimes \xi')(\tilde{v}_\chi)_{12}(\tilde{v}_\chi)_{13}\right] x \;=\; \left[(\mathrm{id} \otimes \xi)\tilde{v}_\chi\right]\left[(\mathrm{id} \otimes \xi')\tilde{v}_\chi\right] x \;=\; \left[(\mathrm{id} \otimes \xi)\tilde{v}_\chi\right](\xi' * x) \;=\; \xi * \xi' * x.$$

Therefore

$$[(\mathrm{id} \otimes \xi \otimes \xi')(\mathrm{id} \otimes \Delta)\tilde{v}_\chi]x \;=\; [(\mathrm{id} \otimes \xi \otimes \xi')(\tilde{v}_\chi)_{12}(\tilde{v}_\chi)_{13}]x$$

and (6.14) is proved.

One can easily show that

$$\mathrm{Sp}\, \tilde{v}_\chi = \mathrm{Sp}\, v_\chi, \qquad C(\tilde{v}_\chi) = C(v_\chi).$$

Moreover the irreducibility of v_χ implies the irreducibility of $\tilde{v}_\chi$. To see this suppose that

$$\tilde{v}_\chi\, (\hat{T} \otimes \mathrm{id}_A) = (\hat{T} \otimes \mathrm{id}_A)\, \tilde{v}_\chi$$

for some $\hat{T} \in B(H_\chi)$. In particular $\hat{T} \otimes \mathrm{id}_A$ commutes with the representation of $S_qU(2)$ group obtained from $\tilde{v}_\chi$ by restriction. Clearly the linear span of the subspaces of irreducible components of this representation coincides with D_χ. Let $T := \hat{T}|_{D_\chi}$. Then T maps D_χ into itself. Using the fact that D_χ is also v_χ - invariant we conclude that $T \in \mathrm{Mor}\,(\chi, \chi)$. Therefore T is a multiple of the identity map and the same holds for $\hat{T}$. This proves the irreducibility of $\tilde{v}_\chi$.

Due to Proposition 6.3, Theorem 2.4 and Theorem 6.1 we get

Theorem 6.5

1. *Let* $\chi = \left(\frac{n}{2} + i\omega, -\frac{n}{2} + i\omega\right)$ *where* $n \in \mathbf{Z}$, $\omega \in [\,0, \frac{-2\pi}{\log q}\,[$ *and let* $p = \frac{|n|}{2}$. *Then* $\tilde{v}_\chi$ *is an irreducible unitary representation belonging to the principal series with*

   $\mathrm{Sp}\, \tilde{v}_\chi = \{p, p+1, p+2, ...\}$ and $c = -(q^{\frac{n}{2}} + q^{-\frac{n}{2}}) \cos(\omega \log q) - i(q^{\frac{n}{2}} - q^{-\frac{n}{2}}) \sin(\omega \log q).$

2. *Let* $\chi = \left(\rho - \epsilon\frac{\pi i}{\log q}, \rho - \epsilon\frac{\pi i}{\log q}\right)$ *where* $\rho \in \mathbf{R},\ 0 < |\rho| < 1$ *and* $\epsilon = 0, 1$. *Then* $\tilde{v}_\chi$ *is an irreducible unitary representation belonging to the supplementary series with*

$$\operatorname{Sp}\tilde{v}_\chi = \{0, 1, 2, ...\} \text{ and } c = (-1)^{\epsilon+1}(q^\rho + q^{-\rho}).$$

3. *Representations described above exhaust all irreducible infinite-dimensional unitary representations of the quantum Lorentz group.*

Remark: For $\chi = (\frac{n}{2} + i\omega, -\frac{n}{2} + i\omega)$ we have $n_2 = -\bar{n}_1$ and the corresponding eigenvalue c' of the Casimir operator $C'(v_\chi)$ is $c' = \bar{c}$. For $\chi = (\rho - \epsilon\frac{\pi i}{\log q}, \rho - \epsilon\frac{\pi i}{\log q})$ we have $n_2 = n_1$ and the corresponding value $c' = c$ and is real. Therefore for any unitary representation $\tilde{v}_\chi$ we have $C'(\tilde{v}_\chi) = C(\tilde{v}_\chi)^*$.

Let us also note that v_χ is unitarizable if and only if $v_{-\chi}$ is unitarizable and clearly $\operatorname{Sp}\tilde{v}_\chi = \operatorname{Sp}\tilde{v}_{-\chi},\ C(\tilde{v}_\chi) = C(\tilde{v}_{-\chi})$. Therefore by Theorem 6.1 representations $\tilde{v}_\chi$ and $\tilde{v}_{-\chi}$ are unitarily equivalent (the same result follows also from Statement 2^o of Theorem 5.7). Conversely if $\tilde{v}_{\chi'}$ and $\tilde{v}_\chi$ are unitarily equivalent then their spin spectra and eigenvalues of Casimir operators coincide and this implies that $\chi' = \chi$ or $\chi' \equiv -\chi$.

A Appendix: Smooth vector spaces

In this section we collected auxiliary topological results needed for the description of smooth action of the quantum Lorentz group.

Beside the Banach spaces (Hilbert spaces, C^*-algebras) the topological vector spaces which appear in the paper are of the very special kind. For the basic topological notions and results we refer to [13], one can also find a short review of main results used in this presentation also in [2] (Appendices 1-3).

For topological locally convex vector spaces X and Y : $X \otimes Y$ is their algebraic tensor product and $X\hat{\otimes}Y$ will denote the complete projective tensor product. The complete projective tensor product is associative and by definition the canonical bilinear mapping

$$X \times Y \ni (x, y) \to x \otimes y \in X\hat{\otimes}Y$$

is continuous (cf.[13, Definition 43.2]). For any continuous linear maps $T_i : X_i \to Y_i,\ (i = 1, 2)$ the tensor product map $T_1 \otimes T_2 : X_1 \otimes X_2 \longrightarrow Y_1 \otimes Y_2$ has the unique extension to the continuous linear map (cf.[13, Definition 43.6 and Proposition 43.6])

$$T_1 \otimes T_2 : X_1\hat{\otimes}X_2 \longrightarrow Y_1\hat{\otimes}Y_2.$$

In particular for any continuous functional ϕ on X, $\phi \in X'$ the linear continuous map $(\phi \otimes \mathrm{id}_Y) : X\hat{\otimes}Y \longrightarrow Y$ is uniquely defind by $(\phi \otimes \mathrm{id}_Y)(x \otimes y) = \phi(x)\, y$ for any $x \in X$ and $y \in Y$.

If X is a countable dimensional vector space then providing it with the strongest locally convex topology respecting the vector structure we turn X to a topological locally convex vector space. We refer to this as a natural topology of X. Equivalently, the natural topology is the strict inductive limit topology defined by any increasing sequence of finite-dimensional vector subspaces of X. It is known that the natural topology of X is complete, nuclear and Montel. We mention other very nice properties:

- Any linear map from X into a topological vector space is continuous.
- For two countable dimensional vector spaces X_1, X_2 there is an isomorphism $X_1 \hat{\otimes} X_2 = X_1 \otimes X_2$ with natural topology. In particular any bilinear map from $X_1 \times X_2$ into a topological vector space is continuous.
- Any linear subset of X is a closed subspace and has a topological complementary subspace.

If $X_j \ (j \in J)$ are Hausdorff complete locally convex topological vector spaces, then the Cartesian product $X = \prod_{j \in J} X_j$ is a Hausdorff complete locally convex topological vector space. The topology of X is by definition the weakest topology such that all projections $X \longrightarrow X_j$ are continuous (Tichonov topology). The topological vector spaces that are at most countable Cartesian products of countable dimensional vector spaces are called (in this paper) *smooth vector spaces.*

If $X = \prod_{i \in I} X_i, \ Y = \prod_{j \in J} Y_j$ is a pair of smooth vector spaces then any continuous linear map $T : X \longrightarrow Y$ is represented by a $J \times I$ matrix (T_{ji}) consisting of linear maps $T_{ji} : X_i \longrightarrow Y_j$. A matrix $(T_{ji})_{(j,i) \in J \times I}$ represents a continuous linear map $T : X \longrightarrow Y$ if and only if any row contains only a finite number of non-zero elements. In particular a continuous linear functional on X is a row of linear functionals $\phi = (\phi_i)_{i \in I}$ such that $\phi_i \neq 0$ for finite number of $i \in I$.

The class of smooth vector spaces is closed under the complete projective tensor product operation. Indeed, for any topological vector space Y there is the natural isomorphism:

$$(\prod_{j \in J} X_j) \hat{\otimes} Y \ = \ \prod_{j \in J} (X_j \hat{\otimes} Y).$$

Therefore for any pair of smooth vector spaces we have: $X = \prod_{i \in I} X_i, \ Y = \prod_{j \in J} Y_j$:

$$X \hat{\otimes} Y = (\prod_{i \in I} X_i) \hat{\otimes} (\prod_{j \in J} Y_j) \ = \ \prod_{(i,j) \in I \times J} (X_i \otimes Y_j).$$

Using the above isomorphism one can easily see that the set of linear functionals $\{\phi \otimes \psi : \phi \in X', \ \psi \in Y'\} \ \subset (X \hat{\otimes} Y)'$ separates the points of $X \hat{\otimes} Y$.

Let X be a countable dimensional vector space with a fixed (countable) linear basis $\{e_i, i \in I\}$, Y be a smooth vector space. Then any element $b \in X \hat{\otimes} Y$ is of the form

$$b = \sum_{i \in I} e_i \otimes b^i, \tag{A.1}$$

where $b^i \in Y$ and the series is convergent in the topology of $X \hat{\otimes} Y$. Moreover vectors $b^i \ (i \in I)$ are uniquely defined.

Indeed, let for $i \in I : \epsilon^i$ be a linear functional on X such that $\epsilon^i(e_{i'}) = \delta^i_{i'},\ (i' \in I)$. Clearly ϵ^i is continuous. Applying $\epsilon^i \otimes \mathrm{id}_Y$ to both sides of (A.1) we get $(\epsilon^i \otimes \mathrm{id}_Y)(b) = b^i$ and the uniqueness follows.

Now let $b \in X \hat{\otimes} Y$ and $b^i := (\epsilon^i \otimes \mathrm{id}_Y)(b)$. Assume that $Y = \prod_{j \in J} Y_j$ where $Y_j,\ (j \in J)$ are countable dimensional vector spaces and denote by $\pi_j : Y \longrightarrow Y_j$ the canonical projections. Then $(\mathrm{id}_X \otimes \pi_j)(b) \in X \otimes Y_j$ and (A.1) means that

$$(\mathrm{id}_X \otimes \pi_j)(b) = \sum_{i \in I} e_i \otimes \pi_j(b^i) \tag{A.2}$$

for all $j \in J$. In particular the series (A.1) is convergent if and only if (A.2) is convergent for all $j \in J$. We compute:

$$(\mathrm{id}_X \otimes \pi_j)(b) = \sum_{i \in I} e_i \otimes y^i_j,$$

where $y^i_j \in Y_j$ and the sum contains only finite number of non-zero terms. Clearly

$$y^i_j = (\epsilon^i \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes \pi_j)(b) = (\mathrm{id}_X \otimes \pi_j)(\epsilon^i \otimes \mathrm{id}_Y)(b) = \pi_j(b^i).$$

Therefore $\sum_{i \in I} e_i \otimes \pi_j(b^i) = \sum_{i \in I} e_i \otimes y^i_j$ and the series (A.2) is convergent.

B Appendix: $S_qU(2)$ and $S_q\widehat{U}(2)$ quantum groups

In this section we briefly sketch the basic results concerning the quantum $S_qU(2)$ group and its dual group $S_q\widehat{U}(2)$. For more detailed description we refer the reader to [15], [10] and [16].

The algebra $\mathcal{A}_c$ of *"smooth continuous functions"* on $G_c := S_qU(2)$ is the *-algebra generated by two elements α_c, γ_c satisfying relations:

$$\begin{gathered} \alpha_c^*\alpha_c + \gamma_c^*\gamma_c = I, \quad \alpha_c\alpha_c^* + q^2\gamma_c^*\gamma_c = I, \\ \alpha_c\gamma_c = q\gamma_c\alpha_c, \quad \alpha_c\gamma_c^* = q\gamma_c^*\alpha_c, \quad \gamma_c\gamma_c^* = \gamma_c^*\gamma_c. \end{gathered} \tag{B.1}$$

The algebra of all *"continuous functions"* on G_c, denoted by A_c, is the C^*-completion of $\mathcal{A}_c$, $A_c := \overline{\mathcal{A}_c}$. Let us note that $\mathcal{A}_c$ is a countable dimensional vector space and its natural locally convex topology (cf. the preceding Appendix) is stronger than the topology induced by the C^*-norm.

The group structure on G_c is encoded by the comultiplication $\Delta_c : \mathcal{A}_c \longrightarrow \mathcal{A}_c \otimes \mathcal{A}_c$, counit $e_c : \mathcal{A}_c \longrightarrow \mathbf{C}$ and coinverse $\kappa_c : \mathcal{A}_c \longrightarrow \mathcal{A}_c$. By definition Δ_c and e_c are *-homomorphisms such that

$$\begin{aligned} &\Delta_c(\alpha_c) = \alpha_c \otimes \alpha_c - q\gamma_c^* \otimes \gamma_c, && \Delta_c(\gamma_c) = \gamma_c \otimes \alpha_c + \alpha_c^* \otimes \gamma_c, \\ &e_c(\alpha_c) = 1, && e_c(\gamma_c) = 0. \end{aligned} \tag{B.2}$$

and κ_c is a linear antimultiplicative map such that

$$\kappa_c(\alpha_c) = \alpha_c^*, \quad \kappa_c(\alpha_c^*) = \alpha_c, \quad \kappa_c(\gamma_c) = -q\gamma_c, \quad \kappa_c(\gamma_c^*) = -\frac{1}{q}\gamma_c^*. \tag{B.3}$$

Clearly (cf. the properties of countable dimensional vector spaces listed in Appendix A) the multiplication $\mathcal{A}_c \times \mathcal{A}_c \ni (a,b) \mapsto ab \in \mathcal{A}_c$, involution $\mathcal{A}_c \ni a \mapsto a^* \in \mathcal{A}_c$, comultiplication Δ_c, counit e_c and coinverse κ_c are continuous maps. In other words $\mathcal{A}_c$ is a topological Hopf *-algebra.

Δ_c and e_c admit continuous extensions to C^*-algebra morphisms from A_c into $A_c \otimes A_c$ and $\mathbf{C}$ respectively. In this place $A_c \otimes A_c$ denotes the spatial tensor product of C^*-algebras.

Any irreducible unitary representation of G_c is finite dimensional. It is uniquely (up to a unitary equivalence) determined by the dimension of the underlying Hilbert space. The irreducible unitary representations are labeled by the spin parameter $s \in S$ where S is the set of non-negative half-integers:

$$S := \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, ...\right\}.$$

The corresponding irreducible unitary representation u^s acts on $(2s+1)$-dimensional Hilbert space H^s, $u^s \in B(H^s) \otimes \mathcal{A}_c$. Let

$$A_d := \sum_{s \in S}^{\oplus} B(H^s).$$

and let π^s $(s \in S)$ be the canonical projection $\pi^s \in \mathrm{Mor}(A_d, B(H^s))$. The set of all elements affiliated with A_d will be denoted by A_d^η ([10], [18]). Any element $T \in A_d^\eta$ is uniquely determined by the sequence $(\pi^s(T))_{s\in S}$ and any sequence $(T^s)_{s \in S}$ where $T^s \in B(H^s)$ can be obtained in this way. Therefore

$$A_d^\eta = \prod_{s \in S} B(H^s). \tag{B.4}$$

Clearly in this case A_d^η carries a natural *-algebra structure. It contains A_d and the multiplier algebra $M(A_d)$ as *-subalgebras. An element $T = (T^s)_{s\in S}$ belongs to A_d $(M(A_d)$ respectively) if and only if the sequence $(||T^s||)_{s \in S}$ tends to 0 for $s \to \infty$ (is bounded respectively). The reader should notice that in this case A_d^η endowed with the product topology is a smooth vector space (cf. preceding Appendix). With this topology A_d^η is a complete topological *-algebra. On the other hand it can be equipped with a topology of almost uniform convergence (cf.[18, p.491]) as a space of affiliated elements of a C^*-algebra. To describe this topology we consider the $z-$transform.

Let B be a C^*-algebra and B^η the set of its affiliated elements. Then $z-$transform is a map (cf. [18, (1.3)])

$$B^\eta \ni T \mapsto z_T := T(I + T^*T)^{-\frac{1}{2}} \in M(B),$$

where the multiplier algebra $M(B)$ is equipped with the strict topology. The almost uniform topology on B^η is by definition the weakest topology such that $z-$transform is continuous. If B is a unital C^*-algebra then $B^\eta = M(B) = B$ and the strict topology coincides with norm topology. Moreover for any $b \in B$, $||b|| = M(1 - M^2)^{-\frac{1}{2}}$ where $M = ||z_b||$. Proposition 2.1 of [18] shows now that the $z-$transform is a homemorphic map of B onto the open unit ball of B. This leads to more general result

Proposition B.1
Let
$$B = \sum_s^{\oplus} B^s$$
be a countable direct sum of unital separable C^*-algebras. Then the set of affiliated elements
$$B^\eta = \prod_s B^s. \tag{B.5}$$
*Moreover the almost uniform topology on B^η coincides with the Tichonov product of the norm topologies on B^s $(s \in S)$. In this case B^η is a topological *-algebra with unity.*

Proof. Let π^s denote the canonical projections $\pi^s : B \to B^s$. Remembering that $M(B^s) = B^s$ and identifying $T \in B^\eta$ with $(\pi^s(T))_s \in \prod\limits_s B^s$ we obtain (B.5). Let $(T_n)_{n\in\mathbf{N}}$ be a sequence in B^η converging almost uniformly to $T_\infty \in B^\eta$ (by ([18, p.491] the topology of almost uniform convergence is metrizable). It means that $z_{T_n} \to z_{T_\infty}$ strictly. It is equivalent to the strict convergence $\pi^s(z_{T_n}) \to \pi^s(z_{T_\infty})$ for all s. Clearly $\pi^s \circ z = z \circ \pi^s$. Remembering that B^s is unital we get that strict convergence is a norm convergence $z_{\pi^s(T_n)} \to z_{\pi^s(T_\infty)}$. Therefore $\pi^s(T_n) \to \pi^s(T_\infty)$ in norm for any s i.e. $T_n \to T_\infty$ in the Tichonov product of norm topologies. □

All vector space topologies on a finite dimensional vector space are equivalent. Therefore applying the above proposition to the case $B^s = B(H^s)$ $(s \in S)$ we conclude that the smooth vector space topology on A_d^η coincides with the topology of almost uniform convergence.

Now following [10] eqs.(5.1),(5.2) (cf.also [11] pp.598-599) we introduce four (continuous) linear functionals: $\psi_o,\ \overline{\psi}_o,\ \psi_+,\ \psi_- = -q^{-1}\overline{\psi}_+$ on $\mathcal{A}_c$ where by $\overline{\psi}$ we denote the linear functional conjugate to ψ : $\overline{\psi}(a) := \overline{\psi(a^*)}$.

The functional ψ_o is multiplicative i.e.
$$\psi_o(ab) \ = \ \psi_o(a)\psi_o(b) \tag{B.6}$$
for any $a, b \in \mathcal{A}_c$. Therefore it is completely determined by its values on the generators $\alpha_c,\ \alpha_c^*,\ \gamma_c,\ \gamma_c^*$ of $\mathcal{A}_c$. By definition
$$\psi_o(I_c) = 1, \quad \psi_o(\alpha_c) = q^{-\frac{1}{2}}, \quad \psi_o(\alpha_c^*) = q^{\frac{1}{2}}, \quad \psi_o(\gamma_c) = 0, \quad \psi_o(\gamma_c^*) = 0. \tag{B.7}$$
The functional ψ_+ is a skew-derivation i.e.
$$\psi_+(ab) \ = \ \psi_+(a)\psi_o(b) \ + \ \overline{\psi}_o(a)\psi_+(b) \tag{B.8}$$
for any $a, b \in \mathcal{A}_c$ and by definition
$$\psi_+(I_c) = 0, \quad \psi_+(\alpha_c) = 0, \quad \psi_+(\alpha_c^*) = 0, \quad \psi_+(\gamma_c) = q^{-\frac{1}{2}}, \quad \psi_+(\gamma_c^*) = 0. \tag{B.9}$$
Let us note that $\overline{\psi}_o$ is also multiplicative and ψ_- is a skew-derivation. One can show that
$$\begin{array}{llll} \psi_o * \overline{\psi}_o \ = \ e_c \ = \ \overline{\psi}_o * \psi_o, & & \\ \psi_o * \psi_+ \ = \ q\psi_+ * \psi_o, & & \psi_o * \psi_- \ = \ q^{-1}\psi_- * \psi_o, \\ \overline{\psi}_o * \psi_+ \ = \ q^{-1}\psi_+ * \overline{\psi}_o, & & \overline{\psi}_o * \psi_- \ = \ q\psi_- * \overline{\psi}_o, \\ \psi_+ * \psi_- \ = \ \psi_- * \psi_+ \ + \dfrac{1}{1-q^2}(\overline{\psi}_o * \overline{\psi}_o - \psi_o * \psi_o). & & \end{array} \tag{B.10}$$

Indeed, one can check that $\psi_o * \overline{\psi}_o$ and $\overline{\psi} * \psi_o$ are multiplicative functionals which coincide with e_c on the set of generators $\{\alpha_c,\ \alpha_c^*,\ \gamma_c,\ \gamma_c^*\}$ of $\mathcal{A}_c$. This proves the first formula in (B.10). Now using (B.6) and (B.8) one checks by simple computations that the set of all elements $a \in \mathcal{A}_c$ such that

$$\psi_o * \psi_+(a) = q\psi_+ * \psi_o(a), \quad \psi_o * \psi_-(a) \ = \ q^{-1}\psi_- * \psi_o(a), \quad \overline{\psi}_o * \psi_+(a) \ = \ q^{-1}\psi_+ * \overline{\psi}_o(a),$$
$$\overline{\psi}_o * \psi_-(a) \ = \ q\psi_- * \overline{\psi}_o(a), \quad \psi_+ * \psi_-(a) \ = \ \psi_- * \psi_+(a) \ + \ \tfrac{1}{1-q^2}(\overline{\psi}_o * \overline{\psi}_o(a) - \psi_o * \psi_o(a)) \ ,$$

is an algebra containing $\alpha_c,\ \alpha_c^*,\ \gamma_c$ and γ_c^*. Therefore this set coincides with $\mathcal{A}_c$ and (B.10) is proved.

Let

$$u \ = \ \sum_{s\in S}^{\oplus} u^s. \tag{B.11}$$

Then $u \in M(A_d \otimes A_c)$ and u is a representation of the group $S_qU(2)$:

$$(\mathrm{id} \otimes \Delta_c)u \ = \ u_{12}u_{13}. \tag{B.12}$$

The map

$$\mathcal{A}_c' \ni \phi \longrightarrow (\mathrm{id}_d \otimes \phi)u \in A_d^\eta \tag{B.13}$$

is a bijection from $\mathcal{A}_c$ onto A_d^η. The convolution algebra $(\mathcal{A}_c')_o$ generated by functionals $\psi_o,\ \overline{\psi}_o,\ \psi_+,\ \psi_-$ is weakly dense in $\mathcal{A}_c'$. Therefore it separates the points of $\mathcal{A}_c$: for $a \in \mathcal{A}_c$

$$\big(\psi(a) = 0 \text{ for all } \psi \in (\mathcal{A}_c')_o\big) \iff (a = 0).$$

By (B.13) functionals $\psi_o,\ \overline{\psi}_o,\ \psi_+,\ \psi_-$ define distiguished elements of A_d^η :

$$\begin{aligned} &(\mathrm{id}_d \otimes \psi_o)u = q^{J_{d3}}, && (\mathrm{id}_d \otimes \overline{\psi}_o)u = q^{-J_{d3}}, \\ &(\mathrm{id}_d \otimes \psi_+)u = q^{-\frac{1}{2}}J_{d+}, && (\mathrm{id}_d \otimes \psi_-)u = q^{-\frac{1}{2}}J_{d-}. \end{aligned} \tag{B.14}$$

Due to (B.10) they satisfy the corresponding commutation relations (remember that u is a representation):

$$\left.\begin{aligned} q^{J_{d3}}J_{d+} \ &= qJ_{d+}q^{J_{d3}}, \qquad q^{J_{d3}}J_{d-} \ = q^{-1}J_{d-}q^{J_{d3}}, \\ [J_{d+}, J_{d-}] \ &= \ \frac{q^{-2J_{d3}} - q^{2J_{d3}}}{q^{-1} - q}, \\ (J_{d+})^* \ &= \ J_{d-}, \qquad\qquad q^{J_{d3}} \ > \ 0. \end{aligned}\right\} \tag{B.15}$$

Moreover operators $q^{J_{d3}}, q^{-J_{d3}}, J_{d+}$ and J_{d-} generate the algebra A_d in the sense [18, Definition 3.1] (cf. example 9, p.500 of [18]).

Let $q^{\pm J_3^s},\ J_+^s,\ J_-^s \in B(H^s)$ denotes the components of $q^{\pm J_{d3}}$, J_{d+} and J_{d-} i.e.

$$q^{J_{d3}} = \sum_{s\in S}^{\oplus} q^{J_3^s}, \quad q^{-J_{d3}} = \sum_{s\in S}^{\oplus} q^{-J_3^s}, \quad J_{d+} = \sum_{s\in S}^{\oplus} J_+^s, \quad J_{d-} = \sum_{s\in S}^{\oplus} J_-^s.$$

It is known (cf.[10, Corollary 5.2]) that for any $s \in S$ there exists the canonical orthonormal basis

$$\{f^s_m : m = -s, -s+1, \dots s-1, s\} \tag{B.16}$$

in the Hilbert space H^s such that

$$\begin{aligned} J^s_3 f^s_m &= m f^s_m, \\ J^s_+ f^s_m &= q^{\frac{1}{2}-s} \sqrt{[s-m]_q [s+m+1]_q}\; f^s_{m+1}, \\ J^s_- f^s_m &= q^{\frac{1}{2}-s} \sqrt{[s+m]_q [s-m+1]_q}\; f^s_{m-1}, \end{aligned} \tag{B.17}$$

where

$$[n]_q = \frac{1-q^{2n}}{1-q^2}. \tag{B.18}$$

Using the basis (B.16) we identify H^s with $\mathbf{C}^{2s+1}$. Then u^s $(s \in S)$ becomes a matrix $(u^s_{kl})_{k,l=-s,-s+1,\dots s}$ with matrix elements u^s_{kl} belonging to $\mathcal{A}_c$. We refer to them as standard matrix elements of the unitary representations of $S_qU(2)$.

To write explicite formulae for u^s_{kl} we shal use the following notation: For any complex c and non-negative integer j we set

$$(c;q^2)_j = \begin{cases} 1 & \text{for } j = 0 \\ \prod_{m=0}^{j-1} (1 - cq^{2m}) & \text{for } j \geq 1. \end{cases}$$

Then $(1;q^2)_j = \delta_{j0}$ and $(q^2;q^2)_j = (1-q^2)^j [j]_q!$ where $[0]_q! = 1$ and $[j]_q! := [1]_q [2]_q \dots [j]_q$. With this notation (cf.e.g.[6] [5])

$$\left.\begin{aligned}
u^s_{kl} &= \frac{q^{-(k-l)(s-k)}}{[k-l]_q!}\sqrt{\frac{[s+k]_q![s-l]_q!}{[s-k]_q![s+l]_q!}}(\alpha^*_c)^{k+l}\gamma_c^{k-l} \\
&\qquad\qquad \times\sum_{j=0}^{s-k}\frac{(q^{2(-s+k)};q^2)_j(q^{2(s+k+1)};q^2)_j}{(q^{2(k-l+1)};q^2)_j(q^2;q^2)_j}(q^2\gamma^*_c\gamma_c)^j, \\
u^s_{lk} &= \frac{q^{-(k-l)(s-k)}}{[k-l]_q!}\sqrt{\frac{[s+k]_q![s-l]_q!}{[s-k]_q![s+l]_q!}}(\alpha^*_c)^{k+l}(-q\gamma^*_c)^{k-l} \\
&\qquad\qquad \times\sum_{j=0}^{s-k}\frac{(q^{2(-s+k)};q^2)_j(q^{2(s+k+1)};q^2)_j}{(q^{2(k-l+1)};q^2)_j(q^2;q^2)_j}(q^2\gamma^*_c\gamma_c)^j, \\
u^s_{-k,l} &= \frac{q^{-(k+l)(s-l)}}{[k+l]_q!}\sqrt{\frac{[s+k]_q![s+l]_q!}{[s-k]_q![s-l]_q!}} \\
&\quad \times\sum_{j=0}^{s-k}\frac{(q^{2(-s+k)};q^2)_j(q^{2(s+k+1)};q^2)_j}{(q^{2(k+l+1)};q^2)_j(q^2;q^2)_j}(q^2\gamma^*_c\gamma_c)^j\ (\alpha_c)^{k-l}(-q\gamma^*_c)^{k+l}, \\
u^s_{l,-k} &= \frac{q^{-(k+l)(s-l)}}{[k+l]_q!}\sqrt{\frac{[s+k]_q![s+l]_q!}{[s-k]_q![s-l]_q!}} \\
&\quad \times\sum_{j=0}^{s-k}\frac{(q^{2(-s+k)};q^2)_j(q^{2(s+k+1)};q^2)_j}{(q^{2(k+l+1)};q^2)_j(q^2;q^2)_j}(q^2\gamma^*_c\gamma_c)^j\ (\alpha_c)^{k-l}(\gamma_c)^{k+l}.
\end{aligned}\right\} \quad \text{(B.19)}$$

In this formulae $s \in S,\ |l| \le k \le s$. In particular

$$u^{\frac{1}{2}} = \begin{pmatrix} \alpha_c & -q\gamma^*_c \\ \gamma_c & \alpha^*_c \end{pmatrix} \quad \text{(B.20)}$$

is the standard 2-dimensional (unitary) representation of $S_qU(2)$.

Using the above expressions for the matrix elements one can check that

$$(u^s_{kl})^* = (-q)^{l-k}u^s_{-k,-l} \quad \text{(B.21)}$$

for any $s \in S$ and $k, l = -s, -s+1, ...s$. One may also compute the values of our functionals on the standard matrix elements (cf. (B.14) and (B.17)):

$$\begin{aligned}
\psi_o(u^s_{kl}) &= q^k\delta_{k,l} = \overline{\psi_o}((u^s_{kl})^*), \\
\overline{\psi}_o(u^s_{kl}) &= q^{-k}\delta_{k,l} = \psi_o((u^s_{kl})^*), \\
\psi_+(u^s_{kl}) &= q^{-s}\sqrt{[s-l]_q[s+l+1]_q}\delta_{k,l+1} = -q\psi_-((u^s_{kl})^*), \\
\psi_-(u^s_{kl}) &= q^{-s}\sqrt{[s+l]_q[s-l+1]_q}\delta_{k,l-1} = -q^{-1}\psi_+((u^s_{kl})^*).
\end{aligned} \quad \text{(B.22)}$$

Now we describe the the Pontryagin dual $G_d = \widehat{S_qU(2)}$ of the quantum group $S_qU(2)$. The algebra of *"continuous functions on $\widehat{S_qU(2)}$ tending to zero at infinity"* is by definition the algebra A_d. The group structure of G_d is encoded by the comultiplication Δ_d, counit e_d and coinverse κ_d described below.

By definition the comultiplication and the counit are the only morphisms $\Delta_d \in \mathrm{Mor}(A_d, A_d \otimes A_d)$ and $e_d \in \mathrm{Mor}(A_d, \mathbf{C})$ such that

$$(\Delta_d \otimes \mathrm{id})u \;=\; u_{23}u_{13}, \tag{B.23}$$

$$(e_d \otimes \mathrm{id})u = I_c.$$

Taking into account (B.6) and (B.8) we get

$$\begin{aligned} \Delta_d(J_{d\pm}) &= q^{J_{d3}} \otimes J_{d\pm} \;+\; J_{d\pm} \otimes q^{-J_{d3}}, \qquad & e_d(q^{J_{d\pm}}) = 0, \\ \Delta_d(q^{J_{d3}}) &= q^{J_{d3}} \otimes q^{J_{d3}}, & e_d(q^{J_{d3}}) = 1. \end{aligned}$$

The coinverse κ_d is the linear antimultiplicative map acting on A_d^η such that

$$(\kappa_d \otimes \mathrm{id})u \;=\; u^*. \tag{B.24}$$

One may verify that

$$\kappa_d(q^{J_{d3}}) = q^{-J_{d3}}, \quad \kappa_d(J_{d+}) = -\tfrac{1}{q}J_{d+}, \quad \kappa_d(J_{d-}) = -qJ_{d-}, \quad \kappa_d(q^{-J_{d3}}) = q^{J_{d3}}. \tag{B.25}$$

Relations (B.12) and (B.23) show that u is a bicharacter on $G_d \times G_c$. It establishes the Pontryagin duality between G_d and G_c.

We have (cf.(B.4))

$$A_d^\eta \hat{\otimes} A_d^\eta = \prod_{(s,s')\in S\times S} B(H^s) \otimes B(H^{s'}) = (A_d \otimes A_d)^\eta.$$

Since $\Delta_d \in \mathrm{Mor}(A_d, A_d \otimes A_d)$ then by ([18, Proposition 2.3])

$$\Delta_d : \; A_d^\eta \;\longrightarrow A_d^\eta \hat{\otimes} A_d^\eta$$

is a continuous coassociative mapping. It endows A_d^η with a bialgebra structure. Moreover $e_d : A_d^\eta \to \mathbf{C}$ and $\kappa_d : A_d^\eta \to A_d^\eta$ are continuous maps.

Any element $x \in A_d^\eta$ is of the form $x = \sum_{s\in S}^{\oplus} x^s$ where $x^s \in B(H^s) = B(\mathbf{C}^{2s+1})$ is a matrix $x^s = (x^s_{ij})_{i,j=-s,-s+1,\ldots,s}$ with numerical entries. For any $s \in S$ and $k, l \in \{-s, -s+1, \ldots s\}$ let

$$\xi^s_{kl}(x) \;=\; x^s_{kl}.$$

Then ξ^s_{kl} is a continuous linear functional on A_d^η. Clearly (ξ^s_{kl}) $(s \in S, k, l = -s, -s+1, \ldots .s)$ is a linear basis in $(A_d^\eta)'$ ($(A_d^\eta)'$ is a countable dimensional vector space). Since matrix elements of u^* form a linear basis of $\mathcal{A}_c$ we have the canonical linear isomorphism

$$(A_d^\eta)' \ni \xi \longrightarrow (\xi \otimes \mathrm{id_c})u^* \in \mathcal{A}_c.$$

For any $a \in \mathcal{A}_c$, ξ_a will denote the corresponding linear functional on A_d^η. In particular (cf.(B.20) and (B.3))

$$\left(u^{\frac{1}{2}}\right)^* = \begin{pmatrix} \alpha_c^* & \gamma_c^* \\ -q\gamma_c & \alpha_c \end{pmatrix} \tag{B.26}$$

and

$$\psi_\alpha = \xi^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}}, \qquad \psi_{\alpha^*} = \xi^{\frac{1}{2}}_{-\frac{1}{2}-\frac{1}{2}}, \qquad \psi_\gamma = -q^{-1}\xi^{\frac{1}{2}}_{\frac{1}{2}-\frac{1}{2}}, \qquad \psi_{\gamma^*} = \xi^{\frac{1}{2}}_{-\frac{1}{2}\frac{1}{2}}. \tag{B.27}$$

By (B.23)

$$\begin{aligned}(\xi_1 * \xi_2 \otimes \mathrm{id_c})u^* &:= (\xi_1 \otimes \xi_2 \otimes \mathrm{id_c})(\Delta_d \otimes \mathrm{id_c})u^* = (\xi_1 \otimes \xi_2 \otimes \mathrm{id_c})(u_{23}u_{13})^* \\ &= (\xi_1 \otimes \xi_2 \otimes \mathrm{id_c})u_{13}^* u_{23}^* \ = \ [(\xi_1 \otimes \mathrm{id_c})u^*]\,[(\xi_2 \otimes \mathrm{id_c})u^*].\end{aligned}$$

This shows that

$$\xi_{ab} = \xi_a * \xi_b$$

for any $a, b \in \mathcal{A}_c$. Therfore

$$\begin{array}{ll} \psi_\alpha * \psi_{\alpha^*} + q^2\psi_{\gamma^*} * \psi_\gamma = e_d, & \psi_{\alpha^*} * \psi_\alpha + \psi_{\gamma^*} * \psi_\gamma = e_d, \\ \psi_\alpha * \psi_\gamma \ = q\psi_\gamma * \psi_\alpha, & \psi_{\alpha^*} * \psi_\gamma \ = q^{-1}\psi_\gamma * \psi_{\alpha^*}, \\ \psi_\alpha * \psi_{\gamma^*} = q\psi_{\gamma^*} * \psi_\alpha, & \psi_{\alpha^*} * \psi_{\gamma^*} = q^{-1}\psi_{\gamma^*} * \psi_{\alpha^*}, \\ \psi_\gamma * \psi_{\gamma^*} = \psi_{\gamma^*} * \psi_\gamma. & \end{array} \tag{B.28}$$

We know that that $\mathcal{A}_c$ is generated by $\alpha_c, \alpha_c^*, \gamma_c$ and γ_c^*. Therefore the set of all linear combinations of convolution products of $e_d, \psi_\alpha, \psi_{\alpha^*}, \psi_\gamma$ and ψ_{γ^*} coincides with $(A_d^\eta)'$.

We end the section by proving the identities used for calculating (χ, χ) - spherical functional in the case of a non-singular pair:

Lemma B.2
Let z be a complex number such that $\Re z \neq 0, -1, -2...$ or $\Im z \neq \frac{\pi}{\log q}p$, $(p = 0, \pm 1, \pm 2, ...)$ and k, m be non-negative integers. Then

$$\sum_{j=0}^{k} \frac{(q^{-2k}; q^2)_j (q^{2(m+k)}; q^2)_j}{(q^{2m}; q^2)_j (q^2; q^2)_j} \frac{q^{2j}}{1 - q^{2(j+z)}} \ = \ q^{2kz} \frac{(q^2; q^2)_k (q^{2(m-z)}; q^2)_k}{(q^{2z}; q^2)_{k+1} (q^{2m}; q^2)_k}. \tag{B.29}$$

Proof. At first we observe that due to the obvious identity

$$\frac{1}{1 - q^{2(z+j)}} = \frac{1}{1 - q^{2z}} \frac{(q^{2z}; q^2)_j}{(q^{2(z+1)}; q^2)_j},$$

the left hand side L of (B.29) reduces to a multiple of a value of the basic hypergeometric function:

$$L = \frac{1}{1 - q^{2z}} \ {}_3\phi_2(q^{-2k}, q^{2z}, q^{2(m+k)}; q^{2(z+1)}, q^{2m}; q^2, q^2).$$

Let us recall (e.g.[6], [5]) that for $a_1, a_2, ...a_{r+1}; b_1, b_2, ...b_r$ the corresponding basic hypergeometric function ${}_{r+1}\phi_r$ is defined as

$${}_{r+1}\phi_r(a_1, a_2, ...a_{r+1}; b_1, b_2, ...b_r; q^2, t) := \sum_{j=0}^{\infty} \frac{(a_1; q^2)_j ... (a_{r+1}; q^2)_j}{(b_1; q^2)_j ... (b_r; q^2)_j} \frac{t^j}{(q^2; q^2)_j}.$$

For $a_1 = q^{-2k}$ $(k = 0, 1, 2, ...)$ it is a polynomial of degree k since $(q^{-2k}; q_2)_j = 0$ for $j = k+1, k+2, ...$ Using succesively identities (cf.(9) of [5] and (2.2.2) of [6])

$$ {}_3\phi_2(q^{-2k}, a, b, ; c, d; q^2, q^2) = \frac{(ca^{-1}; q^2)_k}{(c; q^2)_k} a^k \; {}_3\phi_2(q^{-2k}, a, db^{-1}; ac^{-1}q^{2(-k+1)}, d; q^2, bc^{-1}q^2) $$

and

$$ {}_3\phi_2(q^{-2k}, a, b, ; d, e; q^2, de(ab)^{-1}q^2) = \frac{(ea^{-1}; q^2)_k}{(e; q^2)_k} \, {}_3\phi_2(q^{-2k}, a, db^{-1}; d, ae^{-1}q^{2(-k+1)}; q^2, q^2) $$

we obtain

$$ \begin{aligned} L &= \frac{1}{1-q^{2z}} \frac{q^{2zk}(q^2; q^2)_k}{(q^{2(z+1)}; q^2)_k} \; {}_3\phi_2(q^{-2k}, q^{2z}, q^{-2k}; q^{-2k}, q^{2m}; q^2, q^{2(m-z+k)}) \\ &= \frac{q^{2zk}(q^2; q^2)_k}{(q^{2z}; q^2)_{k+1}} \frac{(q^{2(m-z)}; q^2)_k}{(q^{2m}; q^2)_k} \; {}_3\phi_2(q^{-2k}, q^{2z}, 1; q^{-2k}, q^{2(z-m-k+1)}; q^2, q^2) \end{aligned} $$

and this proves the result since

$$ {}_3\phi_2(q^{-2k}, q^{2z}, 1; q^{-2k}, q^{2(z-m-k+1)}; q^2, q^2) = 1. $$

□

C Appendix: Quantum Lorentz group

The quantum Lorentz group *QLG* considered in this paper is defined in [10]. It appears as the result of the double group construction applied to the quantum $SU(2)$ group.

We shall use the bicharacter $u = \sum_s^{\oplus} u^s \in \mathrm{M}(A_d \otimes A_c)$ introduced in the previous section. Let $\sigma \in \mathrm{Mor}(A_c \otimes A_d, A_d \otimes A_c)$ be the isomorphism such that

$$ \sigma(a \otimes x) \; := \; u(x \otimes a)u^{-1} \tag{C.1} $$

for any $a \in A_c$ and $x \in A_d$.

In [10] the algebra A of all *"continuous functions vanishing at infinity"* on *QLG* was identified with $A_c \otimes A_d$. In the present paper it is more convenient to assume that

$$ A \; := \; A_d \otimes A_c. $$

The connection between the two approaches is given by the isomorphism σ. In particular the formulae (4.16) − (4.18) of [10] take the following form

$$ \begin{aligned} \Delta &:= (\mathrm{id_d} \otimes \sigma^{-1} \otimes \mathrm{id_c})(\Delta_d \otimes \Delta_c), \\ e &:= e_d \otimes e_c, \\ \kappa &:= \tau(\kappa_c \otimes \kappa_d)\sigma^{-1}, \end{aligned} \tag{C.2} $$

where $\tau : A_c \otimes A_d \longrightarrow A_d \otimes A_c$ is the flip isomorphism ($\tau(x \otimes a) = a \otimes x$ for any $x \in A_d$ and $a \in A_c$).

Let

$$ p_c = \mathrm{id} \otimes e_d, \qquad p_d = e_c \otimes \mathrm{id}. $$

Then $p_c \in \mathrm{Mor}(A, A_c)$ and $p_d \in \mathrm{Mor}(A, A_d)$. One can easily verify that

$$\Delta_c p_c = (p_c \otimes p_c)\Delta, \qquad \Delta_d p_d = (p_d \otimes p_d)\Delta. \tag{C.3}$$

The morphisms p_c and p_d correspond to the embedings

$$S_qU(2) \longrightarrow QLG, \qquad \widehat{S_qU(2)} \longrightarrow QLG.$$

Due to (C.3) these embedings respect the group structures.

Let $\alpha, \beta, \gamma, \delta$ denote matrix elements of

$$w = \begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix} := \begin{pmatrix} q^{J_3}, & (1-q^2)q^{-1/2}J_+ \\ 0, & q^{-J_3} \end{pmatrix} \textcircled{\scriptsize T} \begin{pmatrix} \alpha_c, & -q\gamma_c^* \\ \gamma_c, & \alpha_c^* \end{pmatrix}. \tag{C.4}$$

It means that

$$\begin{array}{ll} \alpha := q^{J_3} \otimes \alpha_c + (1-q^2)q^{-1/2}J_+ \otimes \gamma_c, & \beta := -qq^{J_3} \otimes \gamma_c^* + (1-q^2)q^{-1/2}J_+ \otimes \alpha_c^* \\ \gamma := q^{-J_3} \otimes \gamma_c, & \delta := q^{-J_3} \otimes \alpha_c^*. \end{array} \tag{C.5}$$

Using (B.1) and (B.15) we obtain Podleś commutation relations:

$$\begin{array}{ll}
\alpha\beta = q\beta\alpha, & \alpha\gamma = q\gamma\alpha, \\
\alpha\delta - q\beta\gamma = I, & \delta\alpha - \frac{1}{q}\beta\gamma = I, \\
\beta\gamma = \gamma\beta, & \\
\beta\delta = q\delta\beta, & \gamma\delta = q\delta\gamma, \\
\beta\alpha^* = \frac{1}{q}\alpha^*\beta + \frac{1-q^2}{q}\gamma^*\delta, & \\
\gamma\alpha^* = q\alpha^*\gamma, & \\
\delta\alpha^* = \alpha^*\delta, & \gamma\beta^* = \beta^*\gamma, \\
\delta\beta^* = q\beta^*\delta - q(1-q^2)\alpha^*\gamma, & \\
\delta\gamma^* = \frac{1}{q}\gamma^*\delta, & \gamma\gamma^* = \gamma^*\gamma, \\
\alpha\alpha^* = \alpha^*\alpha + (1-q^2)\gamma^*\gamma, & \delta\delta^* = \delta^*\delta - (1-q^2)\gamma^*\gamma, \\
\multicolumn{2}{l}{\beta\beta^* = \beta^*\beta + (1-q^2)\left[\delta^*\delta - \alpha^*\alpha\right] - (1-q^2)^2\gamma^*\gamma.}
\end{array} \tag{C.6}$$

One may prove that the elements $\alpha, \beta, \gamma, \delta$ are affiliated with A. They generate A in the sense of Definition 3.1 of [18] (cf. [18, Example 10, p.500]). Moreover w is a fundamental representation of *QLG*

$$\Delta(\alpha) = \alpha \otimes \alpha + \beta \otimes \gamma, \qquad \Delta(\beta) = \alpha \otimes \beta + \beta \otimes \delta,$$

$$\Delta(\gamma) = \gamma \otimes \alpha + \delta \otimes \gamma, \qquad \Delta(\delta) = \gamma \otimes \beta + \delta \otimes \delta.$$

The formula (C.4) is the quantum version of the Iwasawa decomposition.

Let

$$\mathcal{A} \;:=\; \prod_{s\in S}(B(H^s)\otimes\mathcal{A}_c) \;=A_d^\eta\hat{\otimes}\mathcal{A}_c.$$

Then $\mathcal{A}$ is a topological *-algebra. By definition $\mathcal{A}$ is a countable Cartesian product of countable dimensional vector spaces $B(H^s)\otimes\mathcal{A}_c$ $(s\in S)$. Therefore $\mathcal{A}$ is a smooth vector space. Let us note that $\mathcal{A}\subset A^\eta$. It is the fundamental algebra in our construction. Elements of $\mathcal{A}$ are called *smooth functions* on the quantum Lorentz group. Clearly the bi-character u is the unitary element of $\mathcal{A}$. Therefore the mapping (C.1) extends to the isomorphism $\sigma\in\mathrm{Mor}(\mathcal{A}_c\hat{\otimes}A_d^\eta, A_d^\eta\hat{\otimes}\mathcal{A}_c)$. This implies (cf.(C.2))

$$\Delta:\mathcal{A}\;\longrightarrow\mathcal{A}\hat{\otimes}\mathcal{A}. \tag{C.7}$$

Δ is a coassociative morphism and $\mathcal{A}$ is a bialgebra. We refer to (C.7) as *a smooth action of QLG* (on itself). The Gelfand spaces are countable dimensional subspaces of $\mathcal{A}$ invariant under the smooth action of *QLG*. The induced representations considered the paper are restrictions of (C.7) to Gelfand spaces.

We use the functionals introduced in the previous Section to define eight continuous linear functionals on $\mathcal{A}=A_d^\eta\hat{\otimes}\mathcal{A}_c$. Let

$$\begin{array}{llll}\Phi_\alpha \;:=\; \psi_\alpha\otimes e_c, & \Phi_{\alpha^*} \;:=\; \psi_{\alpha^*}\otimes e_c, & \Phi_\gamma \;:=\; \psi_\gamma\otimes e_c, & \Phi_{\gamma^*} \;:=\; \psi_{\gamma^*}\otimes e_c\\ \Phi_o \;:=\; e_d\otimes\psi_o, & \bar{\Phi}_o \;:=\; e_d\otimes\bar{\psi}_o, & \Phi_+ \;:=\; e_d\otimes\psi_+, & \Phi_- \;:=\; e_d\otimes\psi_-\end{array} \tag{C.8}$$

We shall prove that

$$\begin{array}{ll}\Phi_\alpha*\Phi_{\alpha^*}+q^2\Phi_{\gamma^*}*\Phi_\gamma=e, & \Phi_{\alpha^*}*\Phi_\alpha+\Phi_{\gamma^*}*\Phi_\gamma=e,\\ \Phi_\alpha*\Phi_\gamma\;=q\Phi_\gamma*\Phi_\alpha, & \Phi_{\alpha^*}*\Phi_\gamma\;=q^{-1}\Phi_\gamma*\Phi_{\alpha^*},\\ \Phi_\alpha*\Phi_{\gamma^*}=q\Phi_{\gamma^*}*\Phi_\alpha, & \Phi_{\alpha^*}*\Phi_{\gamma^*}=q^{-1}\Phi_{\gamma^*}*\Phi_{\alpha^*},\\ \Phi_\gamma*\Phi_{\gamma^*}=\Phi_{\gamma^*}*\Phi_\gamma, & \end{array} \tag{C.9}$$

$$\begin{array}{ll}\Phi_o*\bar{\Phi}_o\;=\;e\;=\;\bar{\Phi}_o*\Phi_o, & \\ \Phi_o*\Phi_+\;=\;q\Phi_+*\Phi_o, & \Phi_o*\Phi_-\;=\;q^{-1}\Phi_-*\Phi_o,\\ \bar{\Phi}_o*\Phi_+\;=\;q^{-1}\Phi_+*\bar{\Phi}_o, & \bar{\Phi}_o*\Phi_-\;=\;q\Phi_-*\bar{\Phi}_o,\\ \Phi_+*\Phi_-\;=\;\Phi_-*\Phi_+\;+\;(1-q^2)^{-1}(\bar{\Phi}_o*\bar{\Phi}_o-\Phi_o*\Phi_o), & \end{array} \tag{C.10}$$

$$\begin{array}{ll}\Phi_o*\Phi_\alpha\;=\Phi_\alpha*\Phi_o, & \bar{\Phi}_o*\Phi_\alpha\;=\Phi_\alpha*\bar{\Phi}_o,\\ \Phi_o*\Phi_\gamma\;=\;q^{-1}\Phi_\gamma*\Phi_o, & \bar{\Phi}_o*\Phi_\gamma\;=\;q\Phi_\gamma*\bar{\Phi}_o,\\ \Phi_o*\Phi_{\alpha^*}\;=\Phi_{\alpha^*}*\Phi_o, & \bar{\Phi}_o*\Phi^*_\alpha\;=\Phi^*_\alpha*\bar{\Phi}_o,\\ \Phi_o*\Phi_{\gamma^*}\;=\;q\Phi_{\gamma^*}*\Phi_o, & \bar{\Phi}_o*\Phi_{\gamma^*}\;=\;q^{-1}\Phi_{\gamma^*}*\bar{\Phi}_o,\\ \bar{\Phi}_+*\Phi_{\gamma^*}\;=\Phi_{\gamma^*}*\Phi_+, & \Phi_-*\Phi_\gamma\;=\;\Phi_\gamma*\Phi_-,\\ \Phi_+*\Phi_\alpha\;=\;q\Phi_\alpha*\Phi_+\;-\;q\Phi_{\gamma^*}*\Phi_o, & \\ \Phi_-*\Phi_\alpha\;=\;q\Phi_\alpha*\Phi_-\;-\;q\Phi_\gamma*\bar{\Phi}_o, & \\ \Phi_+*\Phi_{\alpha^*}\;=\;q^{-1}\Phi_{\alpha^*}*\Phi_+\;+\;q^{-1}\Phi_{\gamma^*}*\bar{\Phi}_o, & \\ \Phi_-*\Phi_{\alpha^*}\;=\;q^{-1}\Phi_{\alpha^*}*\Phi_-\;+\;q^{-1}\Phi_\gamma*\Phi_o, & \\ \Phi_+*\Phi_\gamma\;=\;\Phi_\gamma*\Phi_+\;+\;q^{-1}(\Phi_{\alpha^*}*\Phi_o-\Phi_\alpha*\bar{\Phi}_o), & \\ \Phi_-*\Phi_{\gamma^*}\;=\;\Phi_{\gamma^*}*\Phi_-\;+\;q^{-1}(\Phi_{\alpha^*}*\bar{\Phi}_o-\Phi_\alpha*\Phi_o). & \end{array} \tag{C.11}$$

For linear functionals $\psi_c \in \mathcal{A}_c'$, $\psi_d \in (\mathcal{A}_d^\eta)'$ we denote by Φ_c and Φ_d the corresponding functionals on $\mathcal{A}$, $\Phi_c := e_d \otimes \psi_c \in \mathcal{A}'$ and $\Phi_d := \psi_d \otimes e_c \in \mathcal{A}'$. Then for any $b \in \mathcal{A}$ we have

$$\Phi_c * \Phi_c'(b) \;=\; (\Phi_c \otimes \Phi_c')\Delta(b) \;=\; (e_d \otimes \psi_c \otimes e_d \otimes \psi_c')(\mathrm{id}_d \otimes \sigma^{-1} \otimes \mathrm{id}_c)(\Delta_d \otimes \Delta_c)(b)$$

Remembering that $(e_d \otimes \mathrm{id}_d)\Delta_d = \mathrm{id}_d$ and $(\mathrm{id}_c \otimes e_d)\sigma^{-1} = e_d \otimes \mathrm{id}_c$ we get

$$\Phi_c * \Phi_c'(b) \;=\; (e_d \otimes \psi_c \otimes \psi_c')(\mathrm{id}_d \otimes \Delta_c)(b) \;=\; (e_d \otimes \psi_c * \psi_c')(b).$$

Therefore (C.10) follows immediately from (B.10). In the same way one shows that

$$\Phi_d * \Phi_d'(b) \;=\; (\psi_d \otimes \psi_d' \otimes e_c)(\Delta_d \otimes \mathrm{id}_c)(b) \;=\; (\psi_d * \psi_d' \otimes e_c)(b).$$

Therefore (B.28) implies (C.9).
To prove (C.11) we note that

$$\begin{aligned}\Phi_c * \Phi_d(b) \;&=\; (e_d \otimes \psi_c \otimes \psi_d \otimes e_c)(\mathrm{id}_d \otimes \sigma^{-1} \otimes \mathrm{id}_c)(\Delta_d \otimes \Delta_c)(b)\\ &=\; (\psi_d \otimes \psi_c)(u^*bu).\end{aligned} \tag{C.12}$$

On the other hand using $(e_c \otimes e_d)\sigma^{-1} = e_d \otimes e_c$ we get

$$\begin{aligned}\Phi_d * \Phi_c(b) \;&=\; (\psi_d \otimes e_c \otimes e_d \otimes \psi_c)(\mathrm{id}_d \otimes \sigma^{-1} \otimes \mathrm{id}_c)(\Delta_d \otimes \Delta_c)(b)\\ &=\; (\psi_d \otimes \psi_c)(b)\end{aligned} \tag{C.13}$$

Now for $b = x \otimes a \in A_d^\eta \otimes \mathcal{A}_c$ we have (to abbreviate the notation we put for lower indices $(-,+)$ instead of $(-\frac{1}{2}, \frac{1}{2})$)

$$\begin{aligned}(u^*bu)^{\frac12}_{--} \;&=\; x^{\frac12}_{--}\alpha_c^* a\alpha_c + x^{\frac12}_{-+}\alpha_c^* a\gamma_c + x^{\frac12}_{+-}\gamma_c^* a\alpha_c + x^{\frac12}_{++}\gamma_c^* a\gamma_c,\\ (u^*bu)^{\frac12}_{-+} \;&=\; -qx^{\frac12}_{--}\alpha_c^* a\gamma_c^* + x^{\frac12}_{-+}\alpha_c^* a\alpha_c^* - qx^{\frac12}_{+-}\gamma_c^* a\gamma_c^* + x^{\frac12}_{++}\gamma_c^* a\alpha_c^*,\\ (u^*bu)^{\frac12}_{+-} \;&=\; -qx^{\frac12}_{--}\gamma_c a\alpha_c - qx^{\frac12}_{-+}\gamma_c a\gamma_c + x^{\frac12}_{+-}\alpha_c a\alpha_c + x^{\frac12}_{++}\alpha_c a\gamma_c,\\ (u^*bu)^{\frac12}_{++} \;&=\; q^2x^{\frac12}_{--}\gamma_c a\gamma_c^* - qx^{\frac12}_{-+}\gamma_c a\alpha_c^* - qx^{\frac12}_{+-}\alpha_c a\gamma_c^* + x^{\frac12}_{++}\alpha_c a\alpha_c^*.\end{aligned} \tag{C.14}$$

Now we shall prove the last formula in (C.11). By (C.12), (B.27) and (C.14) we obtain

$$\Phi_- * \Phi_{\gamma^*}(x \otimes a) \;=\; -qx^{\frac12}_{--}\psi_-(\alpha_c^* a\gamma_c^*) + x^{\frac12}_{-+}\psi_-(\alpha_c^* a\alpha_c^*) - qx^{\frac12}_{+-}\psi_-(\gamma_c^* a\gamma_c^*) + x^{\frac12}_{++}\psi_-(\gamma_c^* a\alpha_c^*).$$

Using skew-derivation property of ψ_- (cf.(B.8)) and (B.22) we get

$$\psi_-(\alpha_c^* a\gamma_c^*) \;=\; \bar\psi_o(\alpha_c^*)\bar\psi_o(a)\psi_-(\gamma_c^*) \;=\; -q^{-2}\bar\psi_o(a)$$

and similarly

$$\psi_-(\gamma_c^* a\gamma_c^*) \;=\; 0, \qquad \psi_-(\alpha_c^* a\alpha_c^*) \;=\; \psi_-(a), \qquad \psi_-(\gamma_c^* a\alpha_c^*) \;=\; -q^{-1}\psi_o(a).$$

Therefore

$$\Phi_- * \Phi_{\gamma^*}(x \otimes a) \;=\; q^{-1}x^{\frac12}_{--}\bar\psi_o(a) + x^{\frac12}_{-+}\psi_-(a) - q^{-1}x^{\frac12}_{++}\psi_o(a).$$

On the other hand by (C.13)

$$\Phi_{\gamma^*} * \Phi_-(x \otimes a) \ = \ x_{-+}^{\frac{1}{2}} \psi_-(a), \quad \Phi_{\alpha^*} * \ \bar{\Phi}_o(x \otimes a) \ = \ x_{--}^{\frac{1}{2}} \bar{\psi}(a), \quad \Phi_\alpha * \Phi_o(x \otimes a) \ = \ x_{++}^{\frac{1}{2}} \psi_o(a).$$

Comparison of these expressions ends the proof. The proof of other formulae in (C.11) goes the same way.

We give simple characterization of convolution center of $\mathcal{A}'$.

Proposition C.1
For $\Psi \in \mathcal{A}'$ the following conditions are equivalent
i) $\Psi * \Phi \ = \ \Phi * \Psi$ *for any $\Phi \in \mathcal{A}'$.*
ii) $\Psi * \Phi \ = \ \Phi * \Psi$ *for all eight functionals Φ from the set* (C.8)*.*
iii) $(\mathrm{id}_\mathcal{A} \otimes \Psi)\Delta = (\Psi \otimes \mathrm{id}_\mathcal{A})\Delta$.

Proof. The implications iii) $\Rightarrow$ i) $\Rightarrow$ ii) are clear. Assuming ii) we get that $\Psi * \Phi \ = \ \Phi * \Psi$ for any Φ from the convolution algebra generated by the functionals from the set (C.8). By (C.13) elements of this algebra separate points of $\mathcal{A}$. Since for any $a \in \mathcal{A}$

$$\Phi((\Psi \otimes \mathrm{id}_\mathcal{A})\Delta(a)) \ = \ \Psi * \Phi(a) \ = \ \Phi * \Psi(a) \ = \ \Phi((\mathrm{id}_\mathcal{A} \otimes \Psi)\Delta(a))$$

we get $(\Psi \otimes \mathrm{id}_\mathcal{A})\Delta(a) \ = \ (\mathrm{id}_\mathcal{A} \otimes \Psi)\Delta(a)$ and this proves iii).

□

Now using (C.9), (C.10) and (C.11) one can easily check that the fuctionals

$$\begin{aligned} \Psi \ &:= \ (1-q^2)\Phi_\gamma * \Phi_+ \ - q\Phi_{\alpha^*} * \Phi_o - q^{-1}\Phi_\alpha * \bar{\Phi}_o, \\ \Psi' \ &:= \ (1-q^2)\Phi_{\gamma^*} * \Phi_- - q\Phi_{\alpha^*} * \bar{\Phi}_o - q^{-1}\Phi_\alpha * \Phi_o \end{aligned} \tag{C.15}$$

belong to the center of the convolution algebra $\mathcal{A}$. The corresponding Casimir operators on $\mathcal{A}$ are denoted by

$$C := (\mathrm{id} \otimes \Psi)\Delta, \qquad C' := (\mathrm{id} \otimes \Psi')\Delta. \tag{C.16}$$

Restricting these operators to the Gelfand spaces $D_\chi \subset \mathcal{A}$ we obtain the Casimir operators related to the representation v_χ.

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