

Quantum deformations of the Lorentz group. The Hopf $*$ -algebra level

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Abstract

Three properties characteristic for the Lorentz group are selected and all quantum groups with the same properties are found. As a result, a number of one, two and three parameter quantum deformations of the Lorentz group are discovered. The deformations described in [1] and [2] are among them. Only the Hopf $*$ -algebra level is discussed.

0 Introduction

The existence of several different quantum deformations of the Lorentz group (cf. [1], [2]) rises the question of their classification. In this paper we give a complete answer to this question. More precisely we describe (on the Hopf $*$ -algebra level) all quantum groups of 2×2 matrices having the following properties:

1. The tensor square of the fundamental representation splits into a direct sum of two components, one of which is the one-dimensional trivial representation.
2. The tensor product of the fundamental representation by the complex conjugate one is irreducible and does not depend on the order of factors.
3. The group is not a proper subgroup of a group satisfying two above conditions.

It is not difficult to show that among the classical groups, $SL(2, \mathbf{C})$ is the only one having the above properties.

Among quantum groups, the solution is given by a few families described in Section 3. Half (roughly speaking) of those families turn out to be continuous deformations

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of $SL(2, \mathbf{C})$. Among those continuous deformations we have one three-parameter family (relations (29)–(35) and those in subsection 3.1.1). It would be interesting to compare our classification with that of possible Poisson group structures [3] on the Lorentz group (the latter classification is not known to us at the moment; general statements on the classification of simple Poisson groups, like Theorem 1 in [4] concerning the compact case, could be very useful).

The paper is organized as follows. In Section 1 a general strategy is presented. The quantum group satisfying properties characteristic for Lorentz group is shown to be determined by two basic intertwiners, one of which is the ‘twisted volume element’ $E: \mathbf{C} \rightarrow K \otimes K$ (K – the space of the fundamental representation) related also to well known R -matrix $R: K \otimes K \rightarrow K \otimes K$ for complex $SL_q(2, \mathbf{C})$ (cf. [5]). The second intertwiner, $X: K \otimes \bar{K} \rightarrow \bar{K} \otimes K$ (\bar{K} is the complex conjugate of K), tells how to commute the matrix elements of the fundamental representation with their adjoints. The intertwiners have to satisfy certain compatibility conditions. In Section 2 we classify the intertwiners (satisfying required conditions) modulo the action of the general linear group of K . In Section 3 we give a list of the corresponding commutation relations defining the algebra of polynomials on the deformed Lorentz group. Lengthy proofs are pushed to the last two sections.

Quantum groups considered in this paper are formulated on the (preliminary) Hopf $*$ -algebra level. A *Hopf $*$ -algebra* is a (complex) Hopf algebra (\mathcal{A}, Δ) with an additional star operation $*$ (making \mathcal{A} a $*$ -algebra) such that $\Delta* = (* \otimes *)\Delta$ (for Hopf algebras see [3] and references therein; more about Hopf $*$ -algebras can be found in [9, 13]). If a quantum group G is given by a Hopf $*$ -algebra (\mathcal{A}, Δ) then elements of \mathcal{A} are called *polynomials on G* . G is said to be a *quantum group of matrices* if there is a distinguished finite-dimensional representation (of G) whose matrix elements generate \mathcal{A} (the representation is called *fundamental*).

For the basic notation we refer to [10, 1]. In particular, we shall use symbols \oplus and \oplus introduced in [10].

Let us mention that the approach to study quantum groups which are close to a given classical group — by selecting some properties of representations — has been used effectively in previous papers [1, 11]. The procedure described in the present paper may be applied to study complex quantum groups of the series A_n, B_n, C_n, D_n (as given by [5]) as real groups (see the discussion of a realification procedure in [13]).

1 General framework

Let G be a quantum group satisfying the requirements 1, 2 and 3 of Section 0, \mathcal{A} be the $*$ -algebra of polynomials on G , u be the fundamental representation of G and K be the two-dimensional vector space carrying u : $u \in B(K) \otimes \mathcal{A}$. The $*$ -algebra \mathcal{A} is generated by matrix elements of u .

Let \bar{K} be the complex conjugate of K . It means that an invertible antilinear mapping $K \ni x \mapsto \bar{x} \in \bar{K}$ is given. For any $m \in B(K)$ and $x \in K$ we set $m^j \bar{x} = \overline{m x}$. Clearly, $m^j \in B(\bar{K})$ and $B(K) \ni m \mapsto m^j \in B(\bar{K})$ is an antilinear multiplicative bijective map. Let $B(\bar{K}) \ni n \mapsto n^{j^{-1}} \in B(K)$ be the inverse map. The complex conjugate (of the

fundamental) representation is introduced by the formula

$$\bar{u} = u^{j \otimes *}$$

Property 1 means that there exist linear mappings $E: \mathbf{C} \rightarrow K \otimes K$ and $E': K \otimes K \rightarrow \mathbf{C}$ such that $E'E \neq 0$ and

$$(u \oplus u)(E \otimes I) = (E \otimes I) \tag{1}$$

$$(E' \otimes I)(u \oplus u) = (E' \otimes I). \tag{2}$$

Let us notice that $(E' \otimes \text{id}_K)(\text{id}_K \otimes E) \in B(K \otimes \mathbf{C}, \mathbf{C} \otimes K) = B(K)$ intertwines the representation u with itself. It follows immediately from property 2 that u is irreducible. Therefore (Schur lemma), $(E' \otimes \text{id}_K)(\text{id}_K \otimes E) = \lambda \text{id}_K$, where $\lambda \in \mathbf{C}$. Assume for the moment that $\lambda = 0$. Then $E(1)$ is of rank 1 tensor: $E(1) = x \otimes y$, where $x, y \in K$ and using (1) one can easily show that the subspaces $\mathbf{C}x$ and $\mathbf{C}y$ are u -invariant, so we get a contradiction with the irreducibility of u . Therefore $\lambda \neq 0$. Rescaling E' we may assume that

$$(E' \otimes \text{id}_K)(\text{id}_K \otimes E) = \text{id}_K. \tag{3}$$

Due to this relation E' is determined by E .

According to Property 2, the representations $u \oplus \bar{u}$ and $\bar{u} \oplus u$ are equivalent. It means that there exists an invertible $X \in B(K \otimes \bar{K}, \bar{K} \otimes K)$ such that

$$(X \otimes I)(u \oplus \bar{u}) = (\bar{u} \oplus u)(X \otimes I). \tag{4}$$

Let us notice that $(X \otimes E')(\text{id}_K \otimes X \otimes \text{id}_K)(E \otimes \text{id}_{\bar{K}} \otimes \text{id}_K) \in B(\mathbf{C} \otimes \bar{K} \otimes K, \bar{K} \otimes K \otimes \mathbf{C}) = B(\bar{K} \otimes K)$ intertwines $\bar{u} \oplus u$ with itself. Remembering that $\bar{u} \oplus u$ is irreducible (cf. Property 2) we get

$$(X \otimes E')(\text{id}_K \otimes X \otimes \text{id}_K)(E \otimes \text{id}_{\bar{K}} \otimes \text{id}_K) \sim \text{id}_{\bar{K}} \otimes \text{id}_K,$$

where \sim denotes the equality modulo a non-zero complex numerical factor (we use this notation also in the sequel). Tensoring both sides (from the right) by id_K , composing with $\text{id}_{\bar{K}} \otimes E$ and using (3) we get

$$(X \otimes \text{id}_K)(\text{id}_K \otimes X)(E \otimes \text{id}_{\bar{K}}) \sim (\text{id}_{\bar{K}} \otimes E). \tag{5}$$

Let $\sigma: \bar{K} \otimes K \rightarrow K \otimes \bar{K}$ denote the linear bijection such that $\sigma(\bar{x} \otimes y) = y \otimes \bar{x}$ for all $x, y \in K$. One can easily verify that

$$(\bar{u} \oplus u)^{j^{-1} \otimes j \otimes *} = (\sigma \otimes I)(\bar{u} \oplus u)(\sigma^{-1} \otimes I)$$

$$[(\sigma^{-1} \otimes I)(u \oplus \bar{u})(\sigma \otimes I)]^{j^{-1} \otimes j \otimes *} = u \oplus \bar{u}.$$

Multiplying the both sides of (4) by $\sigma \otimes I$ we get

$$(X\sigma \otimes I)(\sigma^{-1} \otimes I)(u \oplus \bar{u})(\sigma \otimes I) = (\bar{u} \oplus u)(X\sigma \otimes I).$$

Applying now $j^{-1} \otimes j \otimes *$ to the both sides and using the two preceding formulae we get

$$(\sigma^{-1}(X\sigma)^{j^{-1} \otimes j} \otimes I)(u \oplus \bar{u}) = (\bar{u} \oplus u)(\sigma^{-1}(X\sigma)^{j^{-1} \otimes j} \otimes I).$$

Since $u \oplus \bar{u}$ and $\bar{u} \oplus u$ are irreducible, there exists at most one (up to numerical factor) operator intertwining them. Therefore $\sigma^{-1}(X\sigma)^{j^{-1} \otimes j} \sim X$ and

$$(X\sigma)^{j^{-1} \otimes j} \sim \sigma X. \quad (6)$$

Remark: By a choice of basis in K , relation (4) is equivalent to a system of 16 relations containing matrix elements of u and their conjugates. Due to (6) this system is selfadjoint: applying $*$ to the both sides of any relation of the system we obtain a relation belonging to the system.

Property 3 means that (1), (2) and (4) are the only algebraic relations that are imposed on matrix elements of u .

Theorem 1.1 *Let $E: \mathbf{C} \rightarrow K \otimes K$, $E': K \otimes K \rightarrow \mathbf{C}$ and $X: K \otimes \bar{K} \rightarrow \bar{K} \otimes K$ be linear maps. Assume that E and E' are of rank 2. Let \mathcal{A} be the universal $*$ -algebra generated by matrix elements of u satisfying relations (1), (2) and (4). Then there exists unique unital $*$ -homomorphism $\Delta: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ such that $(\text{id} \otimes \Delta)u = u \oplus u$. (\mathcal{A}, Δ) is a Hopf $*$ -algebra.*

Proof: We give only a sketch, since the procedure is quite standard. Since the value of Δ on the generators u_i^k (matrix elements of u in some basis of K) is fixed:

$$\Delta u_i^k = \sum_r u_r^k \otimes u_i^r,$$

the morphism Δ is unique, if exists. For the existence one has to show that Δu_i^k satisfy the same relations as u_i^k (relations (1), (2) and (4)). One can check it by a direct calculation (usually omitted in papers on the subject). However, one can easily observe that this follows from the form of the defining relations, which are given by *intertwiners* (for a general statement concerning this point see e.g. [9]). The existence of the antipode is related to the invertibility of u , which holds by the non-degeneracy of E and E' .

Q.E.D.

Remark: Theorem 1.1 is in fact very weak. It says nothing about the size of the algebra \mathcal{A} . If E , E' and X do not satisfy relations (3), (5) and (6) then \mathcal{A} may be very small. In a generic case \mathcal{A} is generated by single unitary element v such that $v^2 = I$ ($\dim \mathcal{A} = 2$) and $u = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}$. Such a Hopf algebra is related to the \mathbf{Z}_2 group. On the other hand relations (3), (5) and (6) imply that \mathcal{A} is as large as the algebra of polynomials on the classical Lorentz group $SL(2, \mathbf{C})$. Indeed, we have

Theorem 1.2 *Let \mathcal{A} be the $*$ -algebra introduced in Theorem 1.1 and \mathcal{A}^N denote the subspace in \mathcal{A} of all polynomials (in matrix elements of u and \bar{u}) of degree $\leq N$. Let E , E' and X satisfy relations (3), (5), (6) and assume that X is invertible. Then $\dim \mathcal{A}^N$ is the same as for the classical Lorentz group.*

For the proof we refer to Section 4, in which further information on the structure of \mathcal{A} is contained.

Let $G(E, X)$ denote the quantum group determined by a choice of E and X as in Theorem 1.2. Relations (1), (2), (4) can be written as conditions for \bar{u} :

$$(\bar{u} \oplus \bar{u})(\overline{\pi E} \otimes I) = (\overline{\pi E} \otimes I) \quad ((\overline{\pi E})' \otimes I)(\bar{u} \oplus \bar{u}) = ((\overline{\pi E})' \otimes I)$$

$$(X^{-1} \otimes I)(\bar{u} \oplus u) = (u \oplus \bar{u})(X^{-1} \otimes I),$$

where π is the permutation in $K \otimes K$. It follows that

$$G(E, X) = G(\overline{\pi E}, X^{-1}), \quad (7)$$

since \bar{u} can be treated as a fundamental representation of $G(E, X)$.

2 Classification theorem

In this section we present all solutions of (3), (5) and (6). As before, K denotes a 2-dimensional complex vector space.

For simplicity, we shall not distinguish between E and $E(1)$ in the sequel. Let E^{sym} denote the symmetric part of E . The normal form of E is given by the following algebraic lemma, stated in [11] (cf. also [7, 8]).

Lemma 2.1 *Let $E \in K \otimes K$ be of rank 2. There exists a basis e_1, e_2 in K such that either*

$$E = E_q = e_1 \otimes e_2 - qe_2 \otimes e_1, \quad (8)$$

where q is a non-zero complex number (the case of $\text{rank } E^{sym} \neq 1$),
or

$$E = E_{\text{special}} = e_1 \otimes e_2 - e_2 \otimes e_1 + e_1 \otimes e_1 \quad (9)$$

(the case of $\text{rank } E^{sym} = 1$).

In the sequel we replace X by $Q = \sigma X \in \text{End}(K \otimes \overline{K})$. In terms of Q , conditions (5) and (6) read as follows

$$Q_{13}Q_{23}E_{12} \sim E_{12}, \quad (10)$$

$$\sigma \bar{Q} \sigma^{-1} \sim Q, \quad (11)$$

where we have used the usual leg-numbering notation and $\bar{Q} = Q^{j \otimes j^{-1}}$.

In order to formulate our main result, we adopt the following notation. Given a basis e_1, e_2 of K , we denote by Q_e the matrix of Q in the basis $e_1 \otimes \bar{e}_1, e_1 \otimes \bar{e}_2, e_2 \otimes \bar{e}_1, e_2 \otimes \bar{e}_2$. (in another basis f_1, f_2 of K , the corresponding matrix of Q is Q_f , etc.). If $Q = \sum_{kl} e_k^l \otimes q^k_l$, where $e_k^l = e_k \otimes e^l$ (here e^1, e^2 is the dual basis) and $q^k_l \in \text{End}(\overline{K})$, then

$$Q_e = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad (12)$$

where a, b, c and d are the matrices of q^1_1, q^1_2, q^2_1 and q^2_2 , respectively.

Theorem 2.2 *Let $E \in K \otimes K$ be of rank 2 and let an invertible $Q \in \text{End}(K \otimes \overline{K})$ satisfy conditions (10) and (11). Then there exists a basis e_1, e_2 in K such that either*

I. E is given by (8) and Q_e has one of the following forms

$$Q_e \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad t \in \mathbf{R} \setminus \{0\} \quad (13)$$

$$Q_e \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & -t & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad t \in i\mathbf{R} \setminus \{0\} \quad (14)$$

$$Q_e \sim \begin{bmatrix} 1 & 0 & 0 & \pm 1 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (\text{only if } q \text{ is real}) \quad (15)$$

$$Q_e \sim \begin{bmatrix} 1 & 0 & 0 & \pm 1 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & -q^{-1} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (\text{only if } q \text{ is imaginary}) \quad (16)$$

For $q = -1$ we have additionally three following cases:

$$Q_e \sim \begin{bmatrix} 0 & 1 & s & 0 \\ 1 & 0 & 0 & -s \\ s & 0 & 0 & -1 \\ 0 & -s & -1 & 0 \end{bmatrix} \quad |s| = 1, s \neq \pm 1 \quad (17)$$

$$Q_e \sim \begin{bmatrix} 1 & 0 & 0 & p \\ 0 & -1 & p & 0 \\ 0 & p & -1 & 0 \\ p & 0 & 0 & 1 \end{bmatrix} \quad p \in \mathbf{R}, p \neq \pm 1 \quad (18)$$

$$Q_e \sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \quad (19)$$

or,

II. E is given by (9) and Q_e has one of the following forms

$$Q_e \sim \begin{bmatrix} 1 & 0 & 0 & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad r \in \mathbf{R} \quad (20)$$

$$Q_e \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (21)$$

For the proof we refer to Section 5.

Remark: It is possible to describe Q given by formulas (17)–(19) in a more convenient form. Let $\sigma_x, \sigma_y, \sigma_z$ be linear operators in K with matrices

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

respectively. Then Q in formulas (17)–(19) can be written as follows:

$$Q \sim \sigma_z \otimes \bar{\sigma}_x + s\sigma_x \otimes \bar{\sigma}_z \quad (\text{formula (17)}) \quad (22)$$

$$Q \sim \sigma_z \otimes \bar{\sigma}_z + p\sigma_x \otimes \bar{\sigma}_x \quad (\text{formula (18)}) \quad (23)$$

$$Q \sim I \otimes (I + \bar{\sigma}_x) + \sigma_x \otimes (I - \bar{\sigma}_x) \quad (\text{formula (19)}). \quad (24)$$

Matrices of σ_x and σ_z in the basis $f_1 = e_1 + ie_2, f_2 = e_1 - ie_2$ of eigenvectors of σ_y are the same as matrices of σ_y and σ_x in the basis e_1, e_2 . It follows that

$$Q_f \sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & t & 0 \\ 0 & t & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad t \in \mathbf{R} \setminus \{0\} \quad (25)$$

in the case (23) (or (18)), and

$$Q_f \sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & t & 0 \\ 0 & -t & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad t \in i\mathbf{R} \setminus \{0\} \quad (26)$$

in the case (22) (or (17)). In case (24) we pass to the basis $f_1 = e_1 + e_2, f_2 = e_1 - e_2$ of eigenvectors of σ_x and we have

$$Q_f \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad (27)$$

In the new basis, E has no longer the form (8). Instead, it has the following form:

$$E \sim f_1 \otimes f_1 - f_2 \otimes f_2 \quad (28)$$

(in all three above cases!).

The following table introduces notation for quantum groups $G(E, X)$ corresponding to pairs (E, Q) classified in Theorem 2.2.

$G(E, X)$	formula for E	formula for Q	parameters
$G_{q,t}$	(8)	(13) or (14)	$0 \neq q \in \mathbf{C}, 0 \neq t \in \mathbf{R} \cup i\mathbf{R}$
G_q^\pm	(8)	(15) or (16)	$0 \neq q \in \mathbf{R} \cup i\mathbf{R}$
G_{-1}^t	(28)	(25) or (26)	$0 \neq t \in \mathbf{R} \cup i\mathbf{R}$
G_{-1}	(28)	(27)	
G_{spec}^r	(9)	(20)	$r \in \mathbf{R}$
G_{spec}	(9)	(21)	

The range of parameters can be in fact restricted, because of the following equalities implied by (7):

$$G_{q,t} = G_{1/\bar{q},1/\bar{t}}, \quad G_q^\pm = G_{1/q}^\mp \quad (\text{real } q), \quad G_q^\pm = G_{1/\bar{q}}^\pm \quad (\text{imaginary } q),$$

$$G_{-1}^t = G_{-1}^{1/t}, \quad G_{\text{spec}}^r = G_{\text{spec}}^{-r}.$$

Additionally, due to the invariance of Q_e in (13), (14) with respect to the permutation $e_1 \rightarrow e_2, e_2 \rightarrow e_1$, we have the following equality:

$$G_{q,t} = G_{1/q,t}.$$

3 Commutation relations

In this section we write down explicitly the commutation rules (1), (2) and (4) corresponding to E and Q as classified in preceding section, thus giving a detailed list of Hopf *-algebra deformations of $SL(2, \mathbf{C})$.

Let e_1, e_2 be a basis as in Theorem 2.2. The commutation relations (1), (2) for the elements of $u = (u^k_l) = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ turn out to be the following (cf. [11])

$$\alpha\beta = q\beta\alpha \tag{29}$$

$$\alpha\gamma = q\gamma\alpha \tag{30}$$

$$\beta\delta = q\delta\beta \tag{31}$$

$$\gamma\delta = q\delta\gamma \tag{32}$$

$$\beta\gamma = \gamma\beta \tag{33}$$

$$\alpha\delta = I + q\beta\gamma \tag{34}$$

$$\delta\alpha = I + q^{-1}\beta\gamma \tag{35}$$

in the case $E = E_q, q \neq 0$, and

$$\alpha^2 - \beta\alpha + \alpha\beta = I \tag{36}$$

$$\alpha\gamma - \beta\gamma + \alpha\delta = I \tag{37}$$

$$\gamma\alpha - \delta\alpha + \gamma\beta = -I \quad (38)$$

$$\gamma^2 - \delta\gamma + \gamma\delta = 0 \quad (39)$$

$$\gamma^2 - \alpha\gamma + \gamma\alpha = 0 \quad (40)$$

$$\gamma\delta - \alpha\delta + \gamma\beta = -I \quad (41)$$

$$\delta\gamma - \beta\gamma + \delta\alpha = I \quad (42)$$

$$\delta^2 - \beta\delta + \delta\beta = I \quad (43)$$

in the case $E = E_{\text{special}}$ (see also [6, 7]).

The commutation relations between u^k_l and $(u^m_n)^*$ are given by

$$Q_e \begin{bmatrix} \alpha\alpha^* & \alpha\beta^* & \beta\alpha^* & \beta\beta^* \\ \alpha\gamma^* & \alpha\delta^* & \beta\gamma^* & \beta\delta^* \\ \gamma\alpha^* & \gamma\beta^* & \delta\alpha^* & \delta\beta^* \\ \gamma\gamma^* & \gamma\delta^* & \delta\gamma^* & \delta\delta^* \end{bmatrix} = \begin{bmatrix} \alpha^*\alpha & \beta^*\alpha & \alpha^*\beta & \beta^*\beta \\ \gamma^*\alpha & \delta^*\alpha & \gamma^*\beta & \delta^*\beta \\ \alpha^*\gamma & \beta^*\gamma & \alpha^*\delta & \beta^*\delta \\ \gamma^*\gamma & \delta^*\gamma & \gamma^*\delta & \delta^*\delta \end{bmatrix} Q_e. \quad (44)$$

Corresponding to different solutions for E and Q given by Theorem 2.2 we have several particular cases of commutation relations. We list them in three following subsections.

3.1 The case of $E = E_q$, where q is a complex non-zero parameter.

In this case we have relations (29)–(35) for $\alpha, \beta, \gamma, \delta$ and the following four possibilities for rules (44) (corresponding to (13), (14), (15), (16)).

3.1.1 $G_{q,t}$; t real, non-zero.

$$\begin{aligned} \alpha\alpha^* &= \alpha^*\alpha \\ \alpha\beta^* &= t\beta^*\alpha & \beta\beta^* &= \beta^*\beta \\ \alpha\gamma^* &= t^{-1}\gamma^*\alpha & \beta\gamma^* &= \gamma^*\beta & \gamma\gamma^* &= \gamma^*\gamma \\ \alpha\delta^* &= \delta^*\alpha & \beta\delta^* &= t^{-1}\delta^*\beta & \gamma\delta^* &= t\delta^*\gamma & \delta\delta^* &= \delta^*\delta. \end{aligned}$$

Remark: In the case $q = t$, we obtain the quantum deformation of $SL(2, \mathbf{C})$, related to the Gauss decomposition as investigated in [2].

3.1.2 $G_{q,t}$; t imaginary, non-zero.

$$\begin{aligned} \alpha\alpha^* &= \alpha^*\alpha \\ \alpha\beta^* &= t\beta^*\alpha & \beta\beta^* &= -\beta^*\beta \\ \alpha\gamma^* &= t^{-1}\gamma^*\alpha & \beta\gamma^* &= -\gamma^*\beta & \gamma\gamma^* &= -\gamma^*\gamma \\ \alpha\delta^* &= \delta^*\alpha & \beta\delta^* &= -t^{-1}\delta^*\beta & \gamma\delta^* &= -t\delta^*\gamma & \delta\delta^* &= \delta^*\delta. \end{aligned}$$

3.1.3 G_q^\pm ; q real.

$$\begin{aligned} \alpha\alpha^* \pm \gamma\gamma^* &= \alpha^*\alpha \\ \alpha\beta^* \pm \gamma\delta^* &= q^{-1}\beta^*\alpha & \beta\beta^* \pm \delta\delta^* &= \beta^*\beta \pm \alpha^*\alpha \\ \alpha\gamma^* &= q\gamma^*\alpha & \beta\gamma^* &= \gamma^*\beta & \gamma\gamma^* &= \gamma^*\gamma \\ \alpha\delta^* &= \delta^*\alpha & q^{-1}\beta\delta^* &= \delta^*\beta \pm \gamma^*\alpha & \gamma\delta^* &= q^{-1}\delta^*\gamma & \delta\delta^* &= \delta^*\delta \pm \gamma^*\gamma. \end{aligned}$$

Relations in this form do not lead to a commutative algebra for $q = 1$. However, passing to a new basis $f_1 = |s|^{-1/2}e_1$, $f_2 = e_2$ we obtain relations of the following form (cf. the prototype of (15) given in (72) below for $r = 1$, $t = q^{-1}$):

$$\begin{aligned} \alpha\alpha^* + s\gamma\gamma^* &= \alpha^*\alpha \\ \alpha\beta^* + s\gamma\delta^* &= q^{-1}\beta^*\alpha & \beta\beta^* + s\delta\delta^* &= \beta^*\beta + s\alpha^*\alpha \\ \alpha\gamma^* &= q\gamma^*\alpha & \beta\gamma^* &= \gamma^*\beta & \gamma\gamma^* &= \gamma^*\gamma \\ \alpha\delta^* &= \delta^*\alpha & q^{-1}\beta\delta^* &= \delta^*\beta + s\gamma^*\alpha & \gamma\delta^* &= q^{-1}\delta^*\gamma & \delta\delta^* &= \delta^*\delta + s\gamma^*\gamma, \end{aligned}$$

where s is an arbitrary real number ($s = 0$ is admitted by 3.1.1). The commutative case is now recovered in the limit $s \rightarrow 0$, $q \rightarrow 1$. Quantum groups corresponding to the same q and the same sign of s are isomorphic.

Remark: The above relations in the form corresponding to $s = q^2 - 1$ have been studied in [1] as a first example of a deformation of the Lorentz group. This case turns out to be both the quantum double and the *real complexification* (cf. [9]) of $SU_q(2)$. The case $s = 1 - q^2$ corresponds to real complexification (quantum double) of $SU_q(1, 1)$ (cf. [9], formulae (3.77)–(3.80)).

3.1.4 G_q^\pm ; q imaginary.

$$\begin{aligned} \alpha\alpha^* \pm \gamma\gamma^* &= \alpha^*\alpha \\ \alpha\beta^* \pm \gamma\delta^* &= q^{-1}\beta^*\alpha & \beta\beta^* \pm \delta\delta^* &= -\beta^*\beta \pm \alpha^*\alpha \\ \alpha\gamma^* &= q\gamma^*\alpha & \beta\gamma^* &= -\gamma^*\beta & \gamma\gamma^* &= -\gamma^*\gamma \\ \alpha\delta^* &= \delta^*\alpha & q^{-1}\beta\delta^* &= -\delta^*\beta \pm \gamma^*\alpha & \gamma\delta^* &= -q^{-1}\delta^*\gamma & \delta\delta^* &= \delta^*\delta \mp \gamma^*\gamma. \end{aligned}$$

3.2 Three additional cases for $q = -1$.

We use here the form of E given in (28), which yields the following commutation relations for α , β , γ and δ :

$$\begin{aligned} \alpha\beta &= \gamma\delta \\ \beta\alpha &= \delta\gamma \\ \alpha\gamma &= \beta\delta \\ \gamma\alpha &= \delta\beta \\ \alpha^2 &= \delta^2 \\ \beta^2 &= \gamma^2 \\ \alpha^2 - \beta^2 &= 1 \end{aligned}$$

We have then three following cases (corresponding to (25)–(27)).

3.2.1 G_{-1}^t ; t real, non-zero.

$$\begin{aligned} \alpha\alpha^* &= \delta^*\delta \\ \alpha\beta^* &= t\gamma^*\delta & \beta\beta^* &= \gamma^*\gamma \\ \alpha\gamma^* &= t^{-1}\beta^*\delta & \beta\gamma^* &= \beta^*\gamma & \gamma\gamma^* &= \beta^*\beta \\ \alpha\delta^* &= \alpha^*\delta & \beta\delta^* &= t^{-1}\alpha^*\gamma & \gamma\delta^* &= t\alpha^*\beta & \delta\delta^* &= \alpha^*\alpha. \end{aligned}$$

3.2.2 G_{-1}^t ; t imaginary, non-zero.

$$\begin{aligned}
\alpha\alpha^* &= \delta^*\delta \\
\alpha\beta^* &= t\gamma^*\delta & \beta\beta^* &= -\gamma^*\gamma \\
\alpha\gamma^* &= t^{-1}\beta^*\delta & \beta\gamma^* &= -\beta^*\gamma & \gamma\gamma^* &= -\beta^*\beta \\
\alpha\delta^* &= \alpha^*\delta & \beta\delta^* &= -t^{-1}\alpha^*\gamma & \gamma\delta^* &= -t\alpha^*\beta & \delta\delta^* &= \alpha^*\alpha.
\end{aligned}$$

3.2.3 G_{-1}

$$\begin{aligned}
\alpha\alpha^* &= \alpha^*\alpha \\
\alpha\beta^* &= \beta^*\alpha & \beta\beta^* &= -\beta^*\beta \\
\alpha\gamma^* &= \gamma^*\alpha & \beta\gamma^* &= \gamma^*\beta & \gamma\gamma^* &= -\gamma^*\gamma \\
\alpha\delta^* &= \delta^*\alpha & \beta\delta^* &= -\delta^*\beta & \gamma\delta^* &= -\delta^*\gamma & \delta\delta^* &= \delta^*\delta.
\end{aligned}$$

3.3 The case of $E = E_{\text{special}}$.

In this case we have relations (36)–(43), which can also be written in the following equivalent form (cf. [12]):

$$[\alpha, \beta] = I - \alpha^2 \quad (45)$$

$$[\alpha, \gamma] = \gamma^2 \quad [\beta, \gamma] = \gamma\alpha + \delta\gamma \quad (46)$$

$$[\alpha, \delta] = (\delta - \alpha)\gamma \quad [\beta, \delta] = \delta^2 - I \quad [\gamma, \delta] = -\gamma^2 \quad (47)$$

Now follow two cases of commutation rules (44) (corresponding to (20), (21)).

3.3.1 G_{spec}^r ; r real.

$$\begin{aligned}
\alpha\alpha^* + r\gamma\gamma^* &= \alpha^*\alpha \\
\alpha\beta^* + r\gamma\delta^* &= \beta^*\alpha & \beta\beta^* + r\delta\delta^* &= \beta^*\beta + r\alpha^*\alpha \\
\alpha\gamma^* &= \gamma^*\alpha & \beta\gamma^* &= \gamma^*\beta & \gamma\gamma^* &= \gamma^*\gamma \\
\alpha\delta^* &= \delta^*\alpha & \beta\delta^* &= \delta^*\beta + r\gamma^*\alpha & \gamma\delta^* &= \delta^*\gamma & \delta\delta^* &= \delta^*\delta + r\gamma^*\gamma.
\end{aligned}$$

3.3.2 G_{spec}

$$\begin{aligned}
[\alpha, \beta^*] &= \alpha^*\alpha - \alpha\delta^* - \gamma\beta^* \\
[\alpha, \gamma^*] &= \gamma^*\gamma & [\beta, \gamma^*] &= \gamma^*\alpha + \delta\gamma^* \\
[\alpha, \delta^*] &= \gamma^*\alpha + \gamma\delta^* & [\beta, \delta^*] &= \delta\delta^* - \delta^*\alpha - \gamma^*\beta & [\gamma, \delta^*] &= \gamma^*\gamma
\end{aligned}$$

and

$$\begin{aligned}
[\alpha, \alpha^*] &= -\alpha\gamma^* - \gamma\alpha^* \\
[\beta, \beta^*] &= -\alpha^*\beta - \beta^*\alpha - \beta\delta^* - \delta\beta^* \\
[\gamma, \gamma^*] &= 0 \\
[\delta, \delta^*] &= -\delta^*\gamma - \gamma^*\delta.
\end{aligned}$$

Remark: Relations in Section 3.3 do not contain an explicit deformation parameter (cf. remark in 3.1.3). We can introduce it passing to a new basis $f_1 = \tau^{-1}e_1$, $f_2 = e_2$, where τ is an arbitrary non-zero complex number. This is equivalent to replacing β by β/τ and γ by $\gamma\tau$ in the commutation relations. If we do this, equations (45)–(47) have again the same form with the commutator $[\cdot, \cdot]$ replaced by $1/\tau[\cdot, \cdot]$. In 3.3.1 the only change consists in replacing r by $r|\tau|^2$. In 3.3.2, if we use (78) with $r = -1$ instead of (21) and $\tau = -h \in i\mathbf{R}$, the corresponding quantum Lorentz group contains $SL_h(2, \mathbf{R})$ of [12] as a subgroup (and seems to coincide with its real complexification [9]).

4 Proof of Theorem 1.2

Theorem 1.2 follows from two following propositions.

Proposition 4.1 *Let \mathcal{A} be the $*$ -algebra introduced in Theorem 1.1 and \mathcal{A}_{hol} be the subalgebra of \mathcal{A} generated by matrix elements of u . Assume that E , E' and X satisfy relations (3), (5) and (6). Then*

1. \mathcal{A}_{hol} is the universal algebra generated by matrix elements of u satisfying the relations (1) and (2).
2. Any element $a \in \mathcal{A}$ is of the form

$$a = \sum a_r b_r^*, \quad (48)$$

where $a_r, b_r \in \mathcal{A}_{\text{hol}}$ (r runs over a finite set).

3. The decomposition (48) is unique in the sense that the sum

$$\sum a_r \otimes b_r^*$$

is determined by a .

Proof: Since K is finite-dimensional, it follows from (3) that

$$(\text{id}_K \otimes E')(E \otimes \text{id}_K) = \text{id}_K \quad (49)$$

(identifying elements of $V \otimes W$ with linear maps from W^* to V , formula (3) means that $E' \in K^* \otimes K^*$ is the inverse of $E \in K \otimes K$). By (5),

$$(X \otimes \text{id}_K)(\text{id}_K \otimes X)(E \otimes \text{id}_{\overline{K}}) = c(\text{id}_{\overline{K}} \otimes E) \quad (50)$$

for some number c . Tensoring both sides (from the right) by id_K and composing with $(\text{id}_{\overline{K}} \otimes \text{id}_K \otimes E')$ (from the left), we get

$$X(\text{id}_K \otimes \text{id}_{\overline{K}} \otimes E')(\text{id}_K \otimes X \otimes \text{id}_K)(E \otimes \text{id}_{\overline{K}} \otimes \text{id}_K) = c(\text{id}_{\overline{K}} \otimes \text{id}_K),$$

hence also

$$(\text{id}_K \otimes \text{id}_{\overline{K}} \otimes E')(\text{id}_K \otimes X \otimes \text{id}_K)(E \otimes \text{id}_{\overline{K}} \otimes \text{id}_K)X = c(\text{id}_{\overline{K}} \otimes \text{id}_K).$$

Tensoring by id_K (from the left) and composing with $(E' \otimes \text{id}_{\overline{K}})$ (also from the left) we obtain

$$(\text{id}_{\overline{K}} \otimes E')(X \otimes \text{id}_K)(\text{id}_K \otimes X) = c(E' \otimes \text{id}_{\overline{K}}). \quad (51)$$

Let $\tilde{\mathcal{A}}$ be the free $*$ -algebra generated by matrix elements of u , \mathcal{J} be the $*$ -ideal of $\tilde{\mathcal{A}}$ generated by the relations (1), (2) and (4). Then

$$\mathcal{A} = \tilde{\mathcal{A}}/\mathcal{J}.$$

Let $\tilde{\mathcal{A}}_{\text{hol}} \subset \tilde{\mathcal{A}}$ be the (free) subalgebra generated by matrix elements of u , \mathcal{J}_{hol} be the ideal in $\tilde{\mathcal{A}}_{\text{hol}}$ generated by relations (1) and (2) and \mathcal{J}_2 be the ideal of $\tilde{\mathcal{A}}$ generated by relations (4). It is sufficient to show that

$$\tilde{\mathcal{A}} = \tilde{\mathcal{A}}_{\text{hol}}\tilde{\mathcal{A}}_{\text{hol}}^* \oplus \mathcal{J}_2 \quad (52)$$

$$\mathcal{J} = \mathcal{J}_{\text{hol}}\tilde{\mathcal{A}}_{\text{hol}}^* + \tilde{\mathcal{A}}_{\text{hol}}\mathcal{J}_{\text{hol}}^* + \mathcal{J}_2 \quad (53)$$

$$\mathcal{J} \cap \tilde{\mathcal{A}}_{\text{hol}} = \mathcal{J}_{\text{hol}}. \quad (54)$$

Indeed, Statement 1 is equivalent to (54), Statement 2 follows from (52) and Statement 3 is implied by (53).

Let

$$u = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}.$$

Remembering that X is invertible one may rewrite relations (4) in the following form:

$$u_a^* u_b = \sum_{a'b'} Y^{b'a'} u_{ab} u_{b'} u_{a'}^*,$$

where $a, b, a', b' \in \{11, 12, 21, 22\}$ and $Y^{b'a'}$ are complex numbers depending on matrix elements of X .

By definition, elements of $\tilde{\mathcal{A}}$ are linear combinations of linearly independent words composed of characters $\alpha, \beta, \gamma, \delta, \alpha^*, \beta^*, \gamma^*$ and δ^* . Let t_s ($s = 0, 1, 2, \dots$) be the linear operator acting on $\tilde{\mathcal{A}}$ in the following way:

If a word $w \in \tilde{\mathcal{A}}$ is of the form

$$w = w' u_a^* u_b w'', \quad (55)$$

where w', w'' are words and the length of w' is s then

$$t_s w = \sum_{a'b'} Y^{b'a'} u_{ab} w' u_{b'} u_{a'}^* w''.$$

Otherwise (w is not of the form (55)) $t_s w = w$. One can easily verify that

$$t_s^2 = t_s \quad (56)$$

$$\left. \begin{aligned} t_s t_{s+1} t_s &= t_{s+1} t_s t_{s+1} \\ t_s t_{s'} &= t_{s'} t_s \quad (\text{for } |s - s'| > 1) \end{aligned} \right\} \quad (57)$$

$$x - t_s x \in \mathcal{J}_2 \quad (58)$$

for any $x \in \tilde{\mathcal{A}}$. Moreover, if $x \in \mathcal{J}_2$ then

$$x = \sum_s x_s, \quad (59)$$

where $x_s \in \ker t_s$. (Elements of \mathcal{J}_2 are linear combinations of elements of the form

$$x = y(X_{cd}^{ab} u_k^c \bar{u}_l^d - \bar{u}_c^a u_d^b X_{kl}^{cd})z,$$

where y, z are monomials in matrix elements of u, \bar{u} . Clearly, $t_s x = 0$ where s is the length of y .) Let $s < r$ be nonnegative integers. Using the braid relations (57) we get

$$\begin{aligned} (t_0 t_1 \dots t_r) t_s &= t_0 t_1 \dots t_s t_{s+1} t_s t_{s+2} \dots t_r \\ &= t_0 t_1 \dots t_{s+1} t_s t_{s+1} t_{s+2} \dots t_r \\ &= t_{s+1} (t_0 t_1 \dots t_r). \end{aligned}$$

By virtue of (56), $(t_0 t_1 \dots t_r) t_r = t_0 t_1 \dots t_r$. Therefore setting $T_r = (t_0 t_1 \dots t_r)^{r+1}$ we get

$$T_r t_s = T_r \quad (60)$$

for $s = 0, 1, 2, \dots, r$. Using (58) we obtain

$$x - T_r x \in \mathcal{J}_2 \quad (61)$$

for any $x \in \tilde{\mathcal{A}}$. If r is larger than the length of any word entering x then clearly $T_r x \in \tilde{\mathcal{A}}_{\text{hol}} \tilde{\mathcal{A}}_{\text{hol}}^*$ and (61) shows that

$$\tilde{\mathcal{A}} \subset \tilde{\mathcal{A}}_{\text{hol}} \tilde{\mathcal{A}}_{\text{hol}}^* + \mathcal{J}_2. \quad (62)$$

Assume that $x \in \mathcal{J}_2$. Then x is of the form (59) and choosing r larger than all s in (59) and using (60) we get

$$T_r x = \sum_s T_r x_s = \sum_s T_r t_s x_s = 0.$$

On the other hand if $x \in \tilde{\mathcal{A}}_{\text{hol}} \tilde{\mathcal{A}}_{\text{hol}}^*$ then $t_s x = x$ for all s and $T_r x = x$. This way we showed that $\tilde{\mathcal{A}}_{\text{hol}} \tilde{\mathcal{A}}_{\text{hol}}^* \cap \mathcal{J}_2 = \{0\}$ and (52) follows.

Let “ \equiv ” denote the equality in $\tilde{\mathcal{A}} \bmod \mathcal{J}_2$. Using (50) one can easily show that

$$\begin{aligned} &c\bar{u} \oplus [(u \oplus u - I_{B(K \otimes K) \otimes \tilde{\mathcal{A}}})(E \otimes I)] \\ &\equiv (X \otimes \text{id}_K \otimes I)(\text{id}_K \otimes X \otimes I) \{ [(u \oplus u - I_{B(K \otimes K) \otimes \tilde{\mathcal{A}}})(E \otimes I)] \oplus \bar{u} \}. \end{aligned}$$

Similarly using (51) one may verify that

$$\begin{aligned} &c\bar{u} \oplus [(E' \otimes I)(u \oplus u - I_{B(K \otimes K) \otimes \tilde{\mathcal{A}}})] \\ &\equiv \{ [(E' \otimes I)(u \oplus u - I_{B(K \otimes K) \otimes \tilde{\mathcal{A}}})] \oplus \bar{u} \} (\text{id}_K \otimes X^{-1})(X^{-1} \otimes \text{id}_K). \end{aligned}$$

By virtue of (62) these relations show that

$$\tilde{\mathcal{A}} \mathcal{J}_{\text{hol}} \subset \mathcal{J}_{\text{hol}} \tilde{\mathcal{A}}_{\text{hol}}^* + \mathcal{J}_2.$$

Therefore denoting by \mathcal{J}' the right hand side of (53) and using once more (62) we see that $\tilde{\mathcal{A}}\mathcal{J}' \subset \mathcal{J}'$. It means that \mathcal{J}' is a left ideal in $\tilde{\mathcal{A}}$. On the other hand \mathcal{J}_2 is $*$ -invariant (see remark following relation (6)), so is \mathcal{J}' . It shows that \mathcal{J}' is a $*$ -ideal in \mathcal{A} . Remembering that \mathcal{J} is the smallest $*$ -ideal in \mathcal{A} containing \mathcal{J}_{hol} and \mathcal{J}_2 we obtain (53). Relation (54) follows immediately from (52) and (53).

Q.E.D.

Clearly, the algebra \mathcal{A}_{hol} introduced in Theorem 1.2 depends only on E and can be identified as the algebra generated by four elements $\alpha, \beta, \gamma, \delta$ satisfying relations (29)–(35) or (36)–(43).

Proposition 4.2 *Let $\mathcal{A}_{\text{hol}}^N$ be the subspace in \mathcal{A} of all polynomials (of α, β, γ and δ) of degree $\leq N$. Then*

$$\dim \mathcal{A}_{\text{hol}}^N = \sum_{k=1}^{N+1} k^2. \quad (63)$$

Proof: Indeed, for the case of relations (36)–(43) see [11]. For relations (29)–(35): using the representation

$$\begin{aligned} \alpha e_{ijk} &= e_{i-1,j,k} \\ \beta e_{ijk} &= q^i e_{i,j+1,k} \\ \gamma e_{ijk} &= q^i e_{i,j,k+1} \\ \delta e_{ijk} &= e_{i+1,j,k} + q^{2i+1} e_{i+1,j+1,k-1} \end{aligned}$$

it is easy to show that $\{\alpha_k \beta^m \gamma^n\}_{k \in \mathbf{Z}, m, n \geq 0}$ (where $\alpha_k = \alpha^k$ for $k \geq 0$ and $\alpha_k = \delta^{-k}$ for $k < 0$) is a linearly independent set. It is therefore a basis and (63) follows easily (cf. the last paragraph of Section 3 in [11]).

Q.E.D.

Using the above propositions one can make the following remarks on representation theory of the quantum group introduced in Theorem 1.1 under assumptions of Theorem 1.2.

Remark 1: Let V be a finite-dimensional vector space and let $v_1 \in B(V) \otimes \mathcal{A}_{\text{hol}}$, $v_2 \in B(V) \otimes \mathcal{A}_{\text{hol}}^*$ be corepresentations of \mathcal{A}_{hol} and $\mathcal{A}_{\text{hol}}^*$ (respectively), such that the matrix elements of v_1 commute with the matrix elements of v_2 . Then $v = v_1 v_2 \in B(V) \otimes \mathcal{A}$ is a corepresentation of \mathcal{A} . Each corepresentation of \mathcal{A} is of this form (cf. Prop. 6.2 of [1]).

Remark 2: Except cases when $E = E_q = e_1 \otimes e_2 - q e_2 \otimes e_1$ with q being a root of unity, an analogue of Theorem 6.3 of [1] holds.

5 The proof of Theorem 2.2

The most important observation used in the proof is that equations

$$Q_{13} Q_{23} E_{12} = E_{12} \quad \text{and} \quad E'_{12} Q_{13} Q_{23} = E'_{12} \quad (64)$$

(i.e. (50) and (51); we can consider equalities in (64) instead of the \sim signs by the scaling argument) are equivalent to equations (29)–(35) (if E is given by (8)) or (36)–(43) (if E is given by (9)) with $\alpha, \beta, \gamma, \delta$ replaced by 2×2 -matrices a, b, c, d (see formula (12)).

Using explicitly the matrix elements of the blocks in (12): $a = \begin{bmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{bmatrix}$, etc., we can write (11) as follows:

$$\begin{bmatrix} \bar{a}^1_1 & \bar{b}^1_1 & \bar{a}^1_2 & \bar{b}^1_2 \\ \bar{c}^1_1 & \bar{d}^1_1 & \bar{c}^1_2 & \bar{d}^1_2 \\ \bar{a}^2_1 & \bar{b}^2_1 & \bar{a}^2_2 & \bar{b}^2_2 \\ \bar{c}^2_1 & \bar{d}^2_1 & \bar{c}^2_2 & \bar{d}^2_2 \end{bmatrix} = \tau \begin{bmatrix} a^1_1 & a^1_2 & b^1_1 & b^1_2 \\ a^2_1 & a^2_2 & b^2_1 & b^2_2 \\ c^1_1 & c^1_2 & d^1_1 & d^1_2 \\ c^2_1 & c^2_2 & d^2_1 & d^2_2 \end{bmatrix}, \quad (65)$$

where τ is a complex number of modulus 1 (because $Q \mapsto \sigma \bar{Q} \sigma^{-1}$ is an anti-linear involution).

Now we pass to considering specific cases. In each case we start with a basis e_1, e_2 , in which E has its canonical form (8) or (9). The matrix Q_e of Q has the form (12). We solve conditions (64), (65) for a, b, c, d . We sometimes change then the original basis e_1, e_2 to a ‘better’ basis f_1, f_2 , in which E has also a canonical form. The final basis f_1, f_2 has then to be taken as e_1, e_2 appearing in the Theorem.

5.1 $E = E_q, q = 1$.

In this case a, b, c, d generate a commutative subalgebra in $\text{End}(\mathbf{C}^2)$. Such a subalgebra is generated either by a projection, or by a nilpotent. We have two cases:

$$Q = A \otimes \bar{P} + B \otimes (I - \bar{P}), \quad \text{where } P \text{ is a projection, } P \neq 0 \quad (66)$$

$$Q = A \otimes I + B \otimes \bar{N}, \quad \text{where } N^2 = 0, N \neq 0. \quad (67)$$

In the first case, $\sigma \bar{Q} \sigma^{-1} = P \otimes \bar{A} + (I - P) \otimes \bar{B}$, hence from (11) it follows that A, B are functions of P . Using the canonical basis f_1, f_2 for P (the matrix of P being $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$), we obtain

$$Q_f \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & r \end{bmatrix}$$

and from $ad = I$ we have $s = tr$, hence

$$Q_f \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & tr & 0 \\ 0 & 0 & 0 & r \end{bmatrix}. \quad (68)$$

From (65) we obtain $0 \neq r \in \mathbf{R}$ and $|r| = 1$, hence $r = \pm 1$. This gives (13) and (14).

In the second case, A, B are functions of N (the same reason as before). Let f_1, f_2 be the canonical basis for N (the matrix of N being $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$). We obtain

$$Q_f \sim \begin{bmatrix} 1 & r & s & t \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and it follows from $ad = I$ that $r = 0$. From (65) we obtain $s = 0$, $t \in \mathbf{R}$, hence

$$Q_f \sim \begin{bmatrix} 1 & 0 & 0 & t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I + tf_1^2 \otimes \bar{f}_1^2.$$

We may assume that $t \neq 0$. Passing to a rescaled basis, $g_1 = kf_1$, $g_2 = k^{-1}f_2$, where $k^4 = |t|$, we obtain $f_1^2 = k^{-2}g_1^2$, hence $tf_1^2 \otimes \bar{f}_1^2 = tk^{-4}g_1^2 \otimes \bar{g}_1^2 = \pm g_1^2 \otimes \bar{g}_1^2$. This is (15) for $q = 1$.

5.2 $E = E_q$, $q = -1$.

We consider several cases.

5.2.1 The abelian case: a, b, c, d form a commuting set

First assume that Q is of the form (66). From (64) we have

$$Q_{13}Q_{23} = (A \otimes I \otimes \bar{P} + B \otimes I \otimes (I - \bar{P}))(I \otimes A \otimes \bar{P} + I \otimes B \otimes (I - \bar{P})) = A \otimes A \otimes \bar{P} + B \otimes B \otimes (I - \bar{P}),$$

hence

$$Q_{13}Q_{23}E_{12} = (A \otimes A)E \otimes \bar{P} + (B \otimes B)E \otimes (I - \bar{P}),$$

which, according to (64), has to be $E_{12} = E \otimes \bar{P} + E \otimes (I - \bar{P})$. It follows that $(A \otimes A)E = E = (B \otimes B)E$. There are two possibilities. In the basis e_1, e_2 ,

- (i) A, B are diagonal,
- (ii) A diagonal, B anti-diagonal

(the case when both are anti-diagonal is excluded because I is a combination of A and B , by (11)).

Considering (i) we see that P is also diagonal and

$$Q_e \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & r \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & tr & 0 \\ 0 & 0 & 0 & r \end{bmatrix}$$

as in (68). From (65) we obtain (13) and (14).

If we assume (ii), we obtain $A = I$ and the matrix of B is of the form $\begin{bmatrix} 0 & k \\ k^{-1} & 0 \end{bmatrix}$. It follows that B is an involution and $P = \frac{1}{2}(I \pm B)$. The minus sign is however excluded by (11). Changing the basis, $f_1 = \sqrt{k}e_1$, $f_2 = \frac{1}{\sqrt{k}}e_2$, we obtain $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ as the matrix of B , hence (19).

Now let us assume that Q is of the form (67). From $(A \otimes A)E = E$ it follows that A is either diagonal or anti-diagonal (in the basis e_1, e_2). Since A is non-degenerate and a combination of I and N , we have $\text{tr } A \neq 0$. This shows that A is diagonal, hence $A = I$. Now, $B = rI + sN$ gives $Q = I \otimes I + rI \otimes \bar{N} + sN \otimes \bar{N}$. This and (11) imply $r = 0$. Solving $(A \otimes B + B \otimes A)E = 0$ explicitly in components, we obtain also $s = 0$. This excludes the nilpotent case.

5.2.2 The case $a = 0$

By (65), in this case b , c and d have to be of the form

$$\begin{bmatrix} 0 & * \\ * & * \end{bmatrix},$$

where $*$ denotes entries not determined yet. It follows from $bc \sim I$ that b and c are anti-diagonal. From (65) we have

$$Q_e \sim \begin{bmatrix} 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \\ 0 & * & 0 & 0 \\ * & 0 & 0 & r \end{bmatrix},$$

with some number r . Since $bd = -db$, we obtain $r = 0$, hence $d = 0$, and we return to the abelian case, considered previously.

5.2.3 The case $d = 0$

This case leads to the abelian case, by similar reasons as above.

5.2.4 The case $c = 0$

By (65), in this case a , b and d have to be of the form

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$$

(as before, $*$ are not determined entries). The commutation rule $ab = -ba$ implies that b is of the form

$$b = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}.$$

It follows from (65) that a , d are diagonal. If $t = 0$, then we have an abelian case. If $t \neq 0$, then $ab = -ba$, $db = -bd$ imply $\text{tr } a = 0 = \text{tr } d$. One can assume that $a = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. From (65) we obtain

$$Q_e \sim \begin{bmatrix} 1 & 0 & 0 & t \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = (e_1^1 - e_2^2) \otimes (\bar{e}_1^1 - \bar{e}_2^2) + te_1^2 \otimes \bar{e}_1^2,$$

where t is a real number. Substituting $e_1 = xf_1$, $e_2 = x^{-1}f_2$, we obtain

$$Q_f \sim (f_1^1 - f_2^2) \otimes (\bar{f}_1^1 - \bar{f}_2^2) + |x|^4 t f_1^2 \otimes \bar{f}_1^2.$$

For an appropriate x we obtain (15).

5.2.5 The case $b = 0$

Similarly as in the preceding paragraph, we obtain

$$Q_e \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ t & 0 & 0 & 1 \end{bmatrix}.$$

The substitution $e_1 = f_2, e_2 = f_1$ leads exactly to the previous case.

5.2.6 Remaining cases

We assume that a, b, c, d do not form a commuting set and all are non-zero.

It follows that a^2, b^2, c^2, d^2 commute with the elements of a non-commutative subalgebra in $\text{End}(\mathbf{C}^2)$, hence they are multiples of I . There exist $u, v, u', v' \in \text{End}(\overline{\mathbf{C}^2})$ and numbers p, r, s, t such that

$$\begin{aligned} a &= pu & b &= rv \\ d &= tu' & c &= sv' \\ u^2 &= v^2 = u'^2 = v'^2 = I. \end{aligned}$$

By the anti-commutativity, none of operators u, v, u', v' is a multiple of I . From $uu' = u'u$ we obtain $u' = \pm u$. Similarly, $v' = \pm v$. We can assume that $u' = u, v' = v$, hence

$$Q_e = \begin{bmatrix} pu & rv \\ sv & tu \end{bmatrix} = \begin{bmatrix} p & 0 \\ 0 & t \end{bmatrix} \otimes u + \begin{bmatrix} 0 & r \\ s & 0 \end{bmatrix} \otimes v,$$

where

$$u^2 = I = v^2, \quad uv + vu = 0 \tag{69}$$

and $pt + rs \neq 0$ (since $ad + bc \neq 0$).

From (11) it follows that $\text{tr} \begin{bmatrix} p & 0 \\ 0 & t \end{bmatrix} = \text{tr}(\text{a combination of } \bar{u} \text{ and } \bar{v}) = 0$, hence $t = -p, p^2 \neq rs$. We can write Q_e in a form

$$Q_e = p \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes u + h \begin{bmatrix} 0 & k \\ k^{-1} & 0 \end{bmatrix} \otimes v,$$

with $p^2 \neq h^2$. Passing to the basis $f_1 = \sqrt{k}e_1, f_2 = \frac{1}{\sqrt{k}}e_2$, we obtain $Q_f \sim u_0 \otimes \tilde{u} + gv_0 \otimes \tilde{v}$, where $g \neq \pm 1, g \neq 0$,

$$u_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad v_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and \tilde{u} and \tilde{v} satisfy the same algebraic relations as u and v in (69). It follows from (11) that \tilde{u} and \tilde{v} are linear combinations of \bar{u}_0 and \bar{v}_0 :

$$\tilde{u} = x\bar{u}_0 + y\bar{v}_0, \quad \tilde{v} = z\bar{u}_0 + t\bar{v}_0.$$

It is easy to check that conditions (69) for \tilde{u} and \tilde{v} are fulfilled if and only if $x^2 + y^2 = 1 = z^2 + t^2$ and $xz + yt = 0$, or, equivalently, if $t = \pm x$, $z = \mp y$ and $x^2 + y^2 = 1$. We have therefore

$$Q_f \sim xu_0 \otimes \bar{u}_0 + yu_0 \otimes \bar{v}_0 + g(yv_0 \otimes \bar{u}_0 - xv_0 \otimes \bar{v}_0)$$

(\pm sign absorbed in g), where $x^2 + y^2 = 1$ and $g \neq \pm 1$, $g \neq 0$. Now, from (11) it follows that there exists a complex number s of modulus one, such that $\bar{x} = sx$, $\bar{y} = sgy$ and $\bar{g}\bar{x} = sgx$. We have the following implications

$$x \neq 0 \implies g \in \mathbf{R}, \quad y \neq 0 \implies |g| = 1.$$

We have then either $x = 0$ or $y = 0$.

If $x = 0$ and $y \neq 0$ then $|g| = 1$ and

$$Q_f \sim u_0 \otimes \bar{v}_0 + gv_0 \otimes \bar{u}_0,$$

hence (17).

If $y = 0$ and $x \neq 0$ then $g \in \mathbf{R}$ and

$$Q_f \sim u_0 \otimes \bar{u}_0 + g xv_0 \otimes \bar{v}_0,$$

hence (18).

5.3 $E = E_q$, $q \neq \pm 1$.

Lemma 5.1 *a and d are invertible.*

Proof: We have $ad \neq 0$, because if not, then $bc = -q^{-1}I$ and $da = (1 - q^{-2})I$ is invertible.

Similarly we have $da \neq 0$.

Assume now that $\text{rank}(ad) = 1$, then also $\text{rank}(da) = 1$. Then $(I + qbc)$, $(I + q^{-1}bc)$ are of rank 1, hence bc can be diagonalized in a basis v_1, v_2 :

$$bcv_1 = -q^{-1}v_1, \quad bcv_2 = -qv_2.$$

We have $bc(av_2) = q^{-2}a(bcv_2) = -q^{-1}av_2$, hence $av_2 = kv_1$ for some number k . We have also $bc(av_1) = q^{-2}a(bcv_1) = -q^{-3}av_1$, hence $av_1 = 0$ (because $q^{-1} \neq q^{-3} \neq q$). Since $a \neq 0$, we have $k \neq 0$ and

$$dv_1 = k^{-1}dav_2 = k^{-1}(I + q^{-1}bc)v_2 = 0.$$

On the other hand, $bc(dv_2) = q^2d(bcv_2) = -q^3dv_2$, hence $dv_2 = 0$. We have obtained $d = 0$ in contradiction with $ad \neq 0$.

Q.E.D.

Lemma 5.2 $b^2 = 0 = c^2$.

Proof: From $aba^{-1} = qb$ it follows that $\det b = 0$ and $\text{tr } b = 0$.

Q.E.D.

Lemma 5.3 *Either $b = 0$, or $c = 0$.*

Proof: Let us assume that $b \neq 0 \neq c$. In view of Lemma 5.2, matrices commuting with b are of the form $t_1I + t_2b$. It follows that $c = tb$ for some number t . Since ab and ba are also nilpotents commuting with b , we have

$$ba = sb \quad \text{and} \quad ab = qsb.$$

It follows that qs, s are eigenvalues of a , hence a is diagonalizable: there exists a one-dimensional projection $P \in \text{End}(\mathbf{C}^2)$ such that

$$a = qsP + s(I - P).$$

We have $Pb = b$ and $bP = 0$.

Now we shall use the following fact

$$(q^k_l \otimes q^l_m)F = z\delta^k_m F, \tag{70}$$

where $F = \overline{E}$, q^k_l are matrix components of Q : $Q = e_k^l \otimes q^k_l$ and z is a number. To prove formula (70) it is sufficient to insert $Q \sim \sigma \overline{Q} \sigma^{-1} = \overline{q^k_l} \otimes \overline{e_k^l}$ into $\overline{Q}_{13} \overline{Q}_{23} \overline{E}_{12} \sim \overline{E}_{12}$.

Using (70), we obtain

$$(a \otimes a + b \otimes c)F_e = zF_e = (c \otimes b + d \otimes d)F_e,$$

$$(a \otimes b + b \otimes d)F_e = 0 = (c \otimes a + d \otimes c)F_e,$$

where $F_e = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \overline{q} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbf{C}^2 \otimes \mathbf{C}^2$. In particular,

$$0 = (b \otimes b)(a \otimes b + b \otimes d)F_e = (ba \otimes b)F_e,$$

hence $(b \otimes b)F_e = 0$. We can choose a basis f_1, f_2 such that $Pf_1 = f_1, Pf_2 = 0, bf_1 = 0, bf_2 = f_1$ and then $F^{22} = 0$ ($F_e = F^{kl} f_k \otimes f_l$).

We have also

$$0 = (P \otimes P)(a \otimes b + b \otimes d)F_e = (sqP \otimes b + (sq)^{-1}b \otimes P)F_e,$$

$$0 = (P \otimes P)(c \otimes a + d \otimes c)F_e = (tsqb \otimes P + t(sq)^{-1}P \otimes b)F_e$$

and this means that

$$\begin{cases} sqF^{12} + (sq)^{-1}F^{21} = 0 \\ t(sq)^{-1}F^{12} + csqF^{21} = 0. \end{cases}$$

Since F^{12} and F^{21} cannot vanish simultaneously ($\text{rank } E^{sym} = 2$), the determinant of this system of equations is zero, hence

$$(sq)^4 = 1. \tag{71}$$

On the other hand, we have

$$\begin{aligned} qs^2[(I - P) \otimes P]F_e &= [(I - P) \otimes P](a \otimes a + b \otimes c)F_e \\ &= [(I - P) \otimes P](c \otimes b + d \otimes d)F_e = (qs^2)^{-2}[(I - P) \otimes P]F_e, \end{aligned}$$

$$\begin{aligned} qs^2[P \otimes (I - P)]F_e &= [P \otimes (I - P)](a \otimes a + b \otimes c)F_e \\ &= [P \otimes (I - P)](c \otimes b + d \otimes d)F_e = (qs^2)^{-2}[P \otimes (I - P)]F_e. \end{aligned}$$

Since F^{12} and F^{21} cannot vanish simultaneously, we have $(qs^2)^2 = 1$, and, by (71), we have $q^2 = 1$.

Q.E.D.

In view of the above Lemma, we consider now two cases.

5.3.1 $c = 0$

By (65), in this case a , b and d have to be of the form

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix},$$

and from Lemma 5.2, $b = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}$. Again by (65), it follows that a and d are diagonal. From $ad \sim I$ we obtain

$$Q_e \sim \begin{bmatrix} 1 & 0 & 0 & s \\ 0 & t & 0 & 0 \\ 0 & 0 & tr & 0 \\ 0 & 0 & 0 & r \end{bmatrix}. \quad (72)$$

From (65) we obtain $0 \neq r \in \mathbf{R}$ and $|r| = 1$, hence $r = \pm 1$, and

$$Q_e \sim \begin{bmatrix} 1 & 0 & 0 & s \\ 0 & t & 0 & 0 \\ 0 & 0 & \pm t & 0 \\ 0 & 0 & 0 & \pm 1 \end{bmatrix}. \quad (73)$$

For $s = 0$ we obtain (13) and (14).

If $s \neq 0$, then $ab = qba$ implies $t = q^{-1}$ and we obtain (15) and (16) (one can rescale s to be ± 1 by passing to a new basis of the form $f_1 = xe_1$, $f_2 = e_2$).

5.3.2 $b = 0$

Repeating the method of the preceding paragraph we obtain

$$Q_e \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & \pm t & 0 \\ s & 0 & 0 & \pm 1 \end{bmatrix}. \quad (74)$$

We can limit ourselves to the case $s \neq 0$. In this case $t = q$. Passing to the basis $f_1 = e_2$, $f_2 = e_1$, we return to the case of the preceding paragraph (with q replaced by q^{-1}).

5.4 $E = E_{\text{special}}$

Lemma 5.4 $c = 0$.

Proof: It is easy to see that c is not invertible, because otherwise from (40) we would have

$$c^{-1}a - ac^{-1} = I,$$

which is impossible for 2×2 matrices. From (40) we have $\text{tr } c^2 = 0$. Since also $\det c^2 = 0$, c^2 is a nilpotent, hence also c is a nilpotent.

Now, let us assume that $c \neq 0$. It follows from $[a, c] = 0$ and $[d, c] = 0$ that $a = \xi_1 I + \eta_1 c$ and $d = \xi_2 I + \eta_2 c$ for some numbers $\xi_1, \xi_2, \eta_1, \eta_2$. We have therefore $[a, d] = 0$ and $(\xi_1 - \xi_2)c = c(d - a) = [a, d] = 0$. It follows that $\xi_1 = \xi_2 = \xi$ for some ξ . We have $2 = \text{tr } I = \text{tr } a^2 = \xi^2 \text{tr } I$, hence $\xi = \pm 1$. From $bc = ac + ad - I$ and $cb = -ca + da - I$ we have

$$[b, c] = 2\xi c. \quad (75)$$

Since $ad = I + \xi(\eta_1 + \eta_2)c$ and $ac \sim c$, we have also

$$bc \sim c. \quad (76)$$

From $\text{tr } c = 0$ and $\det c = 0$ we have the following form of the matrix of c :

$$c = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}, \quad x^2 + yz = 0.$$

If $z = 0$, then $x = 0$, hence $c = \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}$, $y \neq 0$, therefore (76) implies $b = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$. It follows then from (65) then $c = 0$.

Assume that $z \neq 0$. The following change of basis does not change the form of E :

$$e_1 = f_1, \quad e_2 = f_2 - t f_1$$

(here t is a parameter). From

$$q = e_1^1 \otimes a + e_1^2 \otimes b + e_2^1 \otimes c + e_2^2 \otimes d = f_1^1 \otimes a' + f_1^2 \otimes b' + f_2^1 \otimes c' + f_2^2 \otimes d'$$

we obtain $c' = c$, and the matrix of c in the new basis equals

$$[c]_f = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} c \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x - tz & y + 2tx - t^2 z \\ z & -x + tz \end{bmatrix}.$$

If we set $t = x/z$, we obtain $[c]_f = \begin{bmatrix} 0 & 0 \\ z & 0 \end{bmatrix}$. Then, by (65), a' and d' are of the form

$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$, and, as functions of c' , have to be diagonal, hence $b' = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}$. It follows from (76) (for primed quantities), that $b' = 0$ and then, by (75), $c' = 0$.

Q.E.D.

Now we investigate the case $c = 0$. We have $d = a^{-1}$ and

$$[a, b] = I - a^2 \quad (77)$$

is the only relation to be satisfied.

We have $\text{tr } a^2 = 2 = \text{tr } a^{-2}$ and

$$a = \begin{bmatrix} x & * \\ 0 & y \end{bmatrix}, \quad d = \begin{bmatrix} x^{-1} & * \\ 0 & y^{-1} \end{bmatrix}$$

(using (65)).

Lemma 5.5 $x^2 + y^2 = 2 = x^{-2} + y^{-2} \iff x = \pm 1, y = \pm 1$.

Proof: The proof is elementary.

Q.E.D.

By Lemma 5.5, we have now two possible cases.

1.

$$a = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

By (65), we have $b = \begin{bmatrix} \bar{t} & r \\ 0 & -\bar{t} \end{bmatrix}$. It follows from (77) that $t = 0$ or $t = 1$. From $t = 0$ we obtain (20). If $t = 1$,

$$Q_e \sim \begin{bmatrix} 1 & 1 & 1 & r \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (78)$$

and passing to a new basis $f_1 = e_1, f_2 = e_2 - \frac{r}{4}e_1$, we obtain (21) as the matrix of Q .

2.

$$a = \begin{bmatrix} 1 & t \\ 0 & -1 \end{bmatrix}, \quad d = \begin{bmatrix} 1 & t \\ 0 & -1 \end{bmatrix},$$

but this is in contradiction with (65).

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