THEOREM C'. If the series (1.1) in Theorem C is of power series type, then the function \( F \) has at \( a_n \) an \( r \)-th Peano un symmetric derivative in \( L^p \), \( p < \infty \), equal to \( s \).

The proof parallels that of Theorem C. The generalization obviously adds nothing to our Theorem 2.

2. The conclusion of Theorem 1 holds if the hypothesis is replaced by the following one: at each point \( x \in E \) we have (1.4) with \( O \) instead of \( Q \), (the polynomial \( P(x) \) may then, of course, be of degree \( k-1 \) or less). The proof remains unchanged if we note that the conclusion of the lemma on p. 91 remains unchanged if we replace the \( c \) in (2.3) and (2.4) by \( O \), provided \( a > 1 \).

References


On a theorem of Mackey, Stone and v. Neumann

by

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The canonical commutation rules of the quantum mechanics of a system with \( N \) degrees of freedom have the following form:

\[
Q_n P_m - P_m Q_n = i \delta_{nm},
\]

\[
P_n P_m - P_m P_n = 0, \quad (m, n = 1, 2, \ldots, N)
\]

\[
Q_n Q_m - Q_m Q_n = 0,
\]

where \( Q_n, P_n \) \( (n = 1, 2, \ldots, N) \) are self-adjoint operators in the Hilbert space \( H \). For \( x \in \mathbb{R}^N, \; a \in \mathbb{R}^N \) we define

\[
U(\tilde{x}) = \exp \left( i \sum_{n=1}^N \tilde{x}_n Q_n \right), \quad V(x) = \exp \left( i \sum_{n=1}^N x_n P_n \right).
\]

The commutation rules were put by H. Weyl in the following correct form:

\[
U(\tilde{x}) \ U(\tilde{a}) = U(\tilde{x} + \tilde{a}),
\]

\[
V(x) \ V(z) = V(x + z),
\]

\[
V(x) \ U(\tilde{z}) \ V(z) = U(\tilde{z}) \ V(z) \exp \left( i \sum_{n=1}^N a_n \tilde{a}_n \right).
\]

Let us notice that \( x \rightarrow \exp \left[ i \sum_{n=1}^N a_n \tilde{a}_n \right] \) is a character of the group \( E^N \). We assume that the algebra generated by the operators \( U(\tilde{z}) \) is cyclic. This assumption means that this algebra has a simple spectrum. In the language of physics we say that the operators \( Q_n \) form a complete set of commuting observables (cf. [9], p. 122). From these assumptions it follows that there exists so called Schrödinger representation of the operators \( Q \) and \( P \). This means that there exists an isomorphism

\[
H \rightarrow L^2(E^N)
\]

such that

\[
(IQ_n \ I^{-1} \varphi)(y) = y_n \varphi(y), \quad (IP_n I^{-1} \varphi)(y) = -i \frac{\partial}{\partial y_n} \varphi(y)
\]

for \( \varphi \in C_0(E^N) = L^2(E^N) \).
We can rewrite these equations in the Weyl form:
\[
\begin{align*}
\{U(\hat{a})I^{-1}\phi\}(y) &= \phi(y) \exp\{i \sum \hat{a}_n y_n\}, \\
\{V(x)I^{-1}\phi\}(y) &= \phi(x + y)
\end{align*}
\]
for \(\phi \in L^2(\mathbb{R}^n)\).

This theorem (called Stone and von Neumann theorem) was proved in \cite{7} and \cite{3}. It was generalized by G. Mackey in \cite{3} for the case where \(E\) was replaced by any separable locally compact Abelian group.

H. Araki (in \cite{1}) and I. M. Gelfand (in \cite{2}), using Bochner theorem for nuclear spaces, determined all cyclic representations of the commutation rules for systems with infinite number of degrees of freedom (quantum theory of fields).

In the present note we show that Araki-Gelfand method allows to obtain a simple proof of a variant of the Mackey theorem.

Let \(G\) be a separable locally compact Abelian group, \(\hat{G}\) the character group of \(G\), \(\hat{x}, \hat{y}\) the value of the character \(x \in G\) at the point \(x \in G\).

**Theorem (Mackey).** Let \(H\) be a separable Hilbert space and let \((U, H)\) and \((V, H)\) be a strongly unitary representation of \(G\) and of \(\hat{G}\) in the space \(H\) respectively such that
1. \(V(x)U(\hat{a}) = (\hat{a}, x)U(\hat{a})V(x)\),
2. There exists a cyclic vector \(\hat{h}_0 \in H:\n\)
\[
\left\{ \left( U(\hat{a})\hat{h}_0, \hat{a} \in \hat{G} \right) \right\} = H, \quad (\hat{h}_0 | \hat{h}_0) = 1.
\]

Then there exists an isomorphism \(H \cong \hat{G} \to L^2(\mathbb{R}^n, \mu)\), the Hilbert spaces \(H\) and \(L^2(\mathbb{R}^n, \mu)\) such that
\[
\{U(\hat{a})\hat{h}\}(y) = (\hat{a}, y)\hat{h}(y), \quad \{V(x)\hat{h}\}(y) = \hat{h}(\hat{x}y).
\]

**Proof.** Put \(E(\hat{a}) = \left\{ U(\hat{a})\hat{h}_0, \hat{h}_0 \right\}\). One checks that \(E(\hat{a}) = 1, (\hat{a} \in \hat{G})\) is positive definite function on \(\hat{G}\). Applying the generalized Bochner theorem (see \cite{4}) we have
\[
E(\hat{a}) = \left\{ U(\hat{a})\hat{h}_0, \hat{h}_0 \right\} = \int \delta(\hat{a}, y) d\mu(y),
\]
where \(\mu\) is positive measure on \(\hat{G}\).

Let \(\hat{h} \in H\) and \(\hat{h} = \sum c_i U(\hat{a}_i)\hat{h}_0\). Let us put \(h(y) = \sum c_i (\hat{a}_i, y)\). One can easily check that
\[
\int h_i(y) \overline{h_j(y)} d\mu(y) = (h_i | h_j).
\]

From the assumption 2 and from the completeness of the set of characters it follows that the mapping \(h \to h(\cdot)\) can be extended to an isomorphism of the Hilbert spaces \(H\) and \(L^2(\mathbb{R}^n, \mu)\). From the definition it follows that
\[
\{U(\hat{a})\hat{h}\}(y) = (\hat{a}, y)\hat{h}(y) \quad \text{for } \mu\text{-almost all } y.
\]

(From \(h = \hat{h}\) it follows that \(h(y) = h_i(y)\) for \(\mu\)-almost all \(y\).)

Now let us consider the operators \(V(x)\). In view of the assumption 1 we have
\[
\{V(x)\hat{h}\}(y) = \sum c_i V(x)U(\hat{a}_i)\hat{h}_0(y) = \sum c_i (\hat{a}_i, x)U(\hat{a}_i)\hat{h}_0(y) = \sum c_i (\hat{a}_i, x\hat{a}_i)\hat{h}_0(y) = \hat{h}(\hat{x}y)\{V(x)\hat{h}\}(y).
\]

The mapping \(\hat{x} \to \{V(x)\hat{h}\}(\cdot)\) is a strongly measurable function on \(\hat{G}\). Its value belongs to \(L^2(\mathbb{R}^n, \mu)\). Then (cf. \cite{2}, p. 106) there exists a measurable function on the product
\[
\hat{G} \times G \to \{V(x)\hat{h}\}(\cdot) \in C^1
\]
such that \(\{V(x)\hat{h}\}(y) = a(x, y)\hat{h}(y)\) for \(\hat{h}(y) = a(x, y)\hat{h}(y)\) all \(x\) and \(\mu\)-almost all \(y\). Therefore
\[
\{V(x)\hat{h}\}(y) = \hat{h}(\hat{x}y)a(x, y).
\]

Setting in this equations \(h = \{V(x)\hat{h}\}\) and \(h(y) = a(x, y)\) we get
\[
a(x, y) = a(x, y)a(x, y).
\]

This equation is true for all \(x, y\) and \(\mu\)-almost all \(y\). Setting \(y = -x\) we get (because \(h_i(y) = 1\)) \(1 = a(x, x) = a(x, y)a(x, y)\) and \(a(x, y) \neq 0\) for \(\mu\)-almost all \(y\).

Let us write
\[
\begin{align*}
N &= \{(x, y) \in G^2 : a(x, y) = a(x, y)\}, \\
N' &= \{(x, y) \in G^2 : a(x, y) = 0\}.
\end{align*}
\]

The sets \(N\) and \(N'\) are measurable, since \(a(\cdot, \cdot)\) is measurable. Let \(\chi_N(x, y)\) and \(\chi_{N'}(x, y)\) be characteristic functions of the set \(N\) and \(N'\) respectively. Obviously,
\[
\int \chi_N(x, y) d\mu(y) = 0, \quad \int \chi_{N'}(x, y) d\mu(y) = 0.
\]

Using Fubini theorem we have
\[
\begin{align*}
\int d\mu(y) \int \chi_N(x, y) d\mu(x) dy &= \int d\mu(x) \int \chi_N(x, y) d\mu(y) = 0, \\
\int d\mu(y) \int \chi_{N'}(x, y) d\mu(x) dy &= \int d\mu(x) \int \chi_{N'}(x, y) d\mu(y) = 0.
\end{align*}
\]
(dz refers to Haar measure on \( G \)). Then there exists a \( y_0 \in G \) such that

\[
\int_{\mathcal{X}} xN(x, x, y_0) d\mu(x) dz = 0, \quad \int_{\mathcal{X}} xN(x, y_0) d\mu(x) = 0.
\]

These equations mean that \( a(x, y) = a(x, y_0) a(x, y_0) \) for \( \mu \)-almost all \( x \) and almost all \( y \) and \( a(x, y) = 0 \) for \( \mu \)-almost all \( x \). From these relations it follows that

\[
a(x, y_0) = \frac{a(x, y)}{a(x, y_0)} \quad \text{for \( \mu \)-almost all } x \text{ and almost all } y.
\]

Since the operators \( V\sigma \) are unitary, the measure \( \mu(x) \) and its translation \( \mu(\sigma y) \) are equivalent (cf. [3]). Setting \( y = \sigma y \) and

\[
K(y) = a(y y_0^{-1}, y_0)
\]

we get the equation

\[
a(x, y) = \frac{K(\sigma y)}{K(y)} \quad \text{for almost all } x \text{ and } \mu\text{-almost all } y.
\]

Now for each \( h \in \mathcal{H} \) we put

\[
h[y] = h(y) K(y).
\]

Then

\[
(U(x) h)[y] = K(y) (U(\sigma) h)(y) = K(y) (x, y) h(y) = (x, y) h[y],
\]

\[
(V(\sigma) h)[y] = K(y) (V(\sigma) h)(y) = K(y) h(\sigma y) a(x, y)
\]

\[
= h(\sigma y) K(\sigma y) = h[y].
\]

(The last equation was proved for almost all \( x \) and \( \mu \)-almost all \( y \). From the continuity of the mapping \( \sigma \rightarrow V(\sigma) h \) it follows that it is true for all \( x \) and \( \mu \)-almost all \( y \).)

We have

\[
(h_1, h_2) = \int h_1(y) h_2(y) d\mu(y) = \int h_1(y) h_2(y) dy, \quad \text{where } dy = \frac{d\mu(y)}{K(y)}.
\]

Since the operators \( V(\sigma) \) are unitary, the measure \( dy \) is invariant; therefore \( dy \) is the Haar measure on the group \( G \). This proves that \( h[\cdot] \cdot L^2(G) \). The proof is complete.

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