Operator theory in the C^*-algebra framework

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Abstract

Properties of operators affiliated with a C^*-algebra are studied. A functional calculus of normal elements is constructed. Representations of locally compact groups in a C^*-algebra are considered. Generalizations of Stone and Nelson theorems are investigated. It is shown that the tensor product of affiliated elements is affiliated with the tensor product of corresponding algebras.

0 Introduction

Let H be a Hilbert space and $CB(H)$ be the algebra of all compact operators acting on H. It was pointed out in [17] that the classical theory of unbounded closed operators acting in H [8, 9, 3] is in a sense related to $CB(H)$. It seems to be interesting to replace in this context $CB(H)$ by any non-unital C^*-algebra. A step in this direction is done in the present paper.

We shall deal with the following topics: the functional calculus of normal elements (Section 1), the representation theory of Lie groups including the Stone theorem (Sections 2,3 and 4) and the extensions of symmetric elements (Section 5). Section 6 contains elementary results related to tensor products. The perturbation theory (in the spirit of T.Kato) is not covered in this paper. The elementary results in this direction are contained the first author's previous paper (cf. [17, Examples 1, 2 and 3 pp. 412–413]).

To fix the notation we remind the basic definitions and results [17].

Let A be a C^*-algebra and T be a linear mapping acting on A defined on a dense linear domain $D(T)$. The adjoint mapping T^* is introduced by the following equivalence $(x, y \in A)$

$$\left(\begin{array}{c} x \in D(T^*) \\ \textit{and } y = T^*x \end{array} \right) \Longleftrightarrow \left(\begin{array}{c} \textit{For any } a \in D(T) \\ (Ta)^*x = a^*y \end{array} \right) \tag{0.1}$$

Clearly T^* is a closed linear mapping. Even if T is bounded there is no guarantee that $D(T^*)$ is dense in A. However, if T is bounded, so is T^*. Indeed, approximating y by $a \in D(T)$ we obtain $y^*y = (Ty)^*x$, $\|y\|^2 \leq \|Ty\|\,\|x\| \leq \|T\|\,\|y\|\,\|x\|$ and $\|y\| \leq \|T\|\,\|x\|$.

Let $B(A)$ denote the Banach algebra of all bounded linear mappings acting on A defined on the whole A ($D(a) = A$ for all $a \in B(A)$) and let

$$M(A) = \{a \in B(A) : a^* \in B(A)\}.$$

Then $M(A)$ is a norm–closed subalgebra of $B(A)$, $a^* \in M(A)$ for any $a \in M(A)$ and $M(A)$ endowed with the *-operation is a unital C^*-algebra. It coincides with the multiplier algebra of A (see e.g. [12]). Assuming that elements of A act on A by left multiplication we include $A \hookrightarrow M(A)$. A is an ideal in $M(A)$.

The natural topology on $M(A)$ is the topology of almost uniform convergence. We say that a net (a_α) of elements of $M(A)$ converges (almost uniformly) to 0 if $\|a_\alpha x\| \to 0$ and $\|a_\alpha^* x\| \to 0$ for any $x \in A$. For example if (e_α) is an approximate unity for A ([5]) then e_α converges to I almost uniformly. Using this fact one can easily show that A is dense in $M(A)$. The almost uniform topology remembers the position of A in $M(A)$:

$$A = \left\{ x \in M(A) : \begin{array}{c} \lim \|a_\alpha x\| = 0 \text{ for any net } (a_\alpha) \text{ of elements} \\ \text{of } M(A) \text{ converging almost uniformly to } 0 \end{array} \right\}.$$

To prove this relation it is sufficient to consider $a_\alpha = I - e_\alpha$, where (e_α) is the approximate unity for A. Assuming $\|a_\alpha x\| \to 0$ we get $e_\alpha x \to x$ in norm. On the other hand $e_\alpha \in A$ and A is an ideal in $M(A)$. Therefore $e_\alpha x \in A$ and $x \in A$ (A is complete!)

Remembering that T^* is bounded for any bounded T we get

Proposition 0.1 *Let A be a C^*-algebra and $T \in B(A)$. Assume that $D(T^*)$ is dense in A. Then $T \in M(A)$.*

Let A be a C^*-algebra and T be a linear mapping acting on A having a dense domain $D(T)$. We say that T is affiliated with A and write $T\eta A$ if and only if there exists $z \in M(A)$ such that $\|z\| \leq 1$, $D(T) = \sqrt{I - z^*z}A$ and $T\sqrt{I - z^*z}\, a = za$ for all $a \in A$.

It is known that z is determined by T. We call it z-transform of T and denote by z_T. If $T\eta A$ then T is closed. The set of all bounded elements affiliated with A coincides with $M(A)$. According to Theorem 1.4 of [17], if $T\eta A$ then $T^*\eta A$ and

$$z_{T^*} = {z_T}^*. \tag{0.2}$$

Let A and B be C^*-algebras. The set of all *-algebra homomorphisms φ from A into $M(B)$ such that $\varphi(A)B$ is dense in B will be denoted by $\mathrm{Mor}(A, B)$. If $T\eta A$ and $\varphi \in \mathrm{Mor}(A, B)$ then there exists unique $\varphi(T)\eta B$ such that $\varphi(D(T))B$ is a core of $\varphi(T)$ and

$$\varphi(T)\varphi(a)b = \varphi(Ta)b$$

for any $a \in D(T)$ and $b \in B$ (see [17, Theorem 1.2]). One can check that $\varphi(T) \in M(B)$ for any $T \in M(A)$. The resulting mapping $\varphi : M(A) \to M(B)$ is a continuous unital *-algebra homomorphism (the both multiplier algebras are considered with the topology of almost uniform convergence). In general case ($T\eta A$) we have $z_{\varphi(T)} = \varphi(z_T)$.

Remark. If D_0 is a core of T then $\varphi(D_0)B$ is a core of $\varphi(T)$. This follows immediately from the definition. One can easily show (cf. [17, Theorem 1.4]) that $\varphi(T^*) = \varphi(T)^*$.

The affiliation relation η was introduced by Baaj [4]. It was rediscovered in [17] where its properties were investigated in detail. The set of morphisms $\mathrm{Mor}(A, B)$ was introduced and investigated in [15] and [14].

In the following we shall need the following version of polar decomposition (cf. [17, Proposition 0.2]).

Proposition 0.2 *Let $c \in M(A)$, where A is a C^*-algebra. Assume that cA (c^*A resp.) is a dense subset of eA (fA resp.), where e, f are selfadjoint projections belonging to $M(A)$. Then there exists unique $u \in M(A)$ such that*

$$c = u\sqrt{c^*c}, \;\; uu^* = e \text{ and } u^*u = f.$$

Proof. Using [5, Lemma 2.9.4] one can easily show that $\overline{c^*cA} = \overline{c^*A}$ and $\overline{c^*cA} = \overline{\sqrt{c^*c}A}$. Therefore $\sqrt{c^*c}A$ is a dense subset of fA and the set of elements of the form

$$a = \sqrt{c^*c}\, x + (I - f)y \tag{0.3}$$

where $x, y \in A$ is dense in A. Let us notice that $a^*a = x^*c^*cx + y^*(I - f)y \geq (cx)^*cx$. Therefore there exists a unique bounded linear mapping $u : A \to A$ such that $ua = cx$ for any a of the form (0.3). In the similar way one can show that the set of elements of the form

$$b = \sqrt{cc^*}\, x' + (I - e)y' \tag{0.4}$$

where $x, y \in A$ is dense in A and construct a unique bounded linear mapping $u' : A \to A$ such that $u'b = cx'$ for any b of the form (0.4). Let us notice that

$$(ua)^*b = x^*c^*\sqrt{cc^*}\, x = x^*\sqrt{c^*c}\, c^*x' = a^*(u'b).$$

Therefore $u' = u^*$, hence $u \in M(A)$. One can easily check that u satisfies all the requirements.

Q.E.D.

1 Functional calculus of normal elements

An element T affiliated with a C^*-algebra A is called normal if $D(T) = D(T^*)$ and $(Ta)^*(Ta) = (T^*a)^*(T^*a)$ for any $A \in D(T)$. This is equivalent to $z_T^* z_T = z_T z_T^*$. T is called selfadjoint if $T = T^*$.

We remind the description of the universal normal element given in [17]. Let Λ be a closed subset of $\mathbf{C}$ and ζ_Λ be the element affiliated with $C_\infty(\Lambda)$ introduced by the formula $\zeta_\Lambda(\lambda) = \lambda$ for any $\lambda \in \Lambda$. Then we have (cf. [17, Theorem 1.6]).

Theorem 1.1 *Let A be a C^*-algebra, T be a normal element affiliated with A and $\Lambda =SpT$. Then there exists unique $\varphi_T \in \mathrm{Mor}(C_\infty(\Lambda), A)$ such that*

$$\varphi_T(\zeta_\Lambda) = T \ .$$

Let T be a normal element affiliated with a C^*-algebra A and f be a complex valued continuous function defined on $\Lambda = \mathrm{Sp}T$. Then $f \eta C_\infty(\Lambda)$ and using the notation introduced in Theorem 1.1 we set

$$f(T) = \varphi_T(f). \tag{1.1}$$

Clearly $f(T)\eta A$. $f(T)$ is a normal element (f is normal and due to [17, Theorem 1.2] any image of a normal element is normal). If $f(\lambda) = 1$ for all $\lambda \in \mathbf{C}$ then $f = I_{C_\infty(\mathbf{C})}$ and $f(T) = \varphi_T(I_{C_\infty(\mathbf{C})}) = I_A$. (For any C^*-algebra A, by I_A we denote the unity of $M(A)$.) Similarly, if $f(\lambda = \lambda$ for all $\lambda \in \mathbf{C}$ then $f = \zeta$ and $f(T) = \varphi_T(\zeta) = T$. In general the definition (1.1) gives the meaning to many algebraic and analytical expressions containing T. For example $(I + T^*T)\sin(2T) = \varphi_T(f)$ where f is the continuous function on $\mathbf{C}$ such that $f(\lambda) = (1 + \lambda\bar{\lambda})\sin(2\lambda)$ for all $\lambda \in$$\mathbf{C}$. Similarly $T(I + T^*T)^{-1/2} = \varphi_T(f)$ where $f(\lambda) = \lambda(1 + \bar{\lambda}\lambda)^{-1/2}$ for any $\lambda \in \mathbf{C}$. One can easily verify that in this case $f = z_\zeta$. Therefore

$$z_T = T(I + T^*T)^{-\frac{1}{2}} \tag{1.2}$$

for any normal $T\eta A$.

The reader easily examines how the functional calculus introduced by (1.1) works in particular examples. If A is a C^*-algebra of bounded operators acting in a Hilbert space (examples 3 and 4 in [17]) then

$$f(T) = \int_{\mathrm{Sp}\, T} f(\lambda)\, dE(\lambda)$$

where $dE(\cdot)$ is the spectral measure associated with the normal operator T:

$$T = \int_{\mathbf{C}} \lambda \, dE(\lambda).$$

Considering $A = C_\infty(\Lambda)$ where Λ is a locally compact topological space one can see that any $T \eta C_\infty(\Lambda)$ is normal, $\operatorname{Sp} T = \overline{T(\Lambda)}$ and

$$f(T) = f \circ T. \tag{1.3}$$

Let A, B be C^*-algebras and $\Phi \in \operatorname{Mor}(A, B)$. Then for any normal $T \eta A$, $\Phi(T)$ is a normal element affiliated with B. Moreover, $(\Phi \circ \varphi_T)(\zeta) = \Phi(\varphi_T(\zeta)) = \Phi(T) = \varphi_{\Phi(T)}(\zeta)$. Therefore $\varphi_{\Phi(T)} = \Phi \circ \varphi_T$ and for any $f \in C(\mathbf{C})$ we get $f(\Phi(T)) = \varphi_{\Phi(T)}(f) = \Phi(\varphi_T(f)) = \Phi(f(T))$ and

$$f(\Phi(T)) = \Phi(f(T)). \tag{1.4}$$

Let A be a C^*-algebra; $f_1, f_2 \in C(\mathbf{C})$ and $T \eta A$. Assume that T is normal. Replacing in (1.4) A, B, Φ, T and f by $C_\infty(\mathbf{C}), A, \varphi_T, f_2$ and f_1 resp. and remembering that $f_1(f_2) = f_1 \circ f_2$ (cf. (1.3)) we obtain

$$f_1(f_2(T)) = (f_1 \circ f_2)(T). \tag{1.5}$$

Clearly (cf. Section 0)

$$f(T)^* = \bar{f}(T) \tag{1.6}$$

where by definition $\bar{f}(\lambda) = \overline{f(\lambda)}$ for any $\lambda \in \operatorname{Sp} T$.

If $f \in C_{\text{bounded}}(\operatorname{Sp} T)$ then $f(T) \in M(A)$. Remembering that $\varphi_T \in \operatorname{Mor}(C_\infty(\operatorname{Sp} T), A)$ we see that

$$(\lambda_1 f_1 + \lambda_2 f_2)(T) = \lambda_1 f_1(T) + \lambda_2 f_2(T) \tag{1.7}$$

$$(f_1 \cdot f_2)(T) = f_1(T) \cdot f_2(T) \tag{1.8}$$

for any $f_1, f_2 \in C_{\text{bounded}}(\operatorname{Sp} T)$ and $\lambda_1, \lambda_2 \in \mathbf{C}$. Moreover, if $f \in C_{\text{bounded}}(\operatorname{Sp} T)$ and $(f_n)_{n=1,2,\ldots}$ is a sequence of bounded continuous functions with a common bound such that $f_n \to f$ uniformly on any compact subset of $\operatorname{Sp} T$ then endowing $M(A)$ with the almost uniform topology we have

$$f_n(T) \to f(T) \tag{1.9}$$

Due to relations (1.5)–(1.8) the notation introduced by (1.1) is selfconsistent. Moreover if $f(\lambda)$ is a continuous bounded function such that $\lambda f(\lambda)$ is bounded then

$$f(T)x \in D(T) \tag{1.10}$$

$$T(f(T)x) = (Tf(T))x \tag{1.11}$$

for any $x \in A$. The simple proof is left to the reader.

Let A be a C^*-algebra and let $T\eta A$. We say that T is skew–adjoint (skew–symmetric resp.) if $T^* = -T$ ($T^* \supset (-T)$ resp.). Like in the Hilbert space operator theory we have

Lemma 1.2 *Skew-adjoint elements have no proper skew-symmetric extensions.*

Proof. Let A be a C^*-algebra, $T, S\eta A$, $T^* = -T$, $S^* \supset (-S)$ and $T \subset S$. Then $S \subset (-S^*) \subset (-T^*) = T$ and $S = T$. Q.E.D.

Clearly any skew-adjoint element T is normal. Its spectrum is contained in $i\mathbf{R}$. Indeed, denoting by ζ the element affiliated with $C_\infty(\mathbf{C})$ introduced by the formula $\zeta(\lambda) = \lambda$ for all $\lambda \in \mathbf{C}$, we have: $\varphi_T(z_\zeta) = z_T = -z_T^* = -\varphi_T(\bar{z}_\zeta)$. Therefore φ_T vanishes on the closed ideal in $C_{\text{bounded}}(\mathbf{C})$ generated by the function $z_\zeta + \bar{z}_\zeta$, which consists of all bounded continuous functions vanishing on $i\mathbf{R}$. It means (cf. [17, formula (1.20)] that $\operatorname{Sp} T \subset i\mathbf{R}$.

Let T be a skew-adjoint element affiliated with a C^*-algebra A. Taking into account the localization of $\operatorname{Sp} T$ described above and using (1.5)–(1.9) we obtain

$$\left(e^{tT}\right)^* = e^{-tT} = \left(e^{tT}\right)^{-1} \tag{1.12}$$

$$e^{t_1 T} e^{t_2 T} = e^{(t_1+t_2)T} \tag{1.13}$$

$$\lim_{t\to 0} e^{tT} = I \tag{1.14}$$

$$\lim_{t\to 0} \frac{e^{tT} - I}{t} (I + T^*T)^{-\frac{1}{2}} = T(I + T^*T)^{-\frac{1}{2}}$$

for any $t, t_1, t_2 \in \mathbf{R}$. The above limits are understood in the sense of almost uniform convergence. Let us notice that $(I + T^*T)^{-1/2} = (I - z_T^* z_T)^{-1/2}$. Therefore for any $a \in D(T)$ there exists $x \in A$ such that $a = (I + T^*T)^{-1/2}x$ and using the last of the above formulae we get

$$\text{norm-}\lim_{t\to 0} \frac{e^{tT} - I}{t} a = Ta \tag{1.15}$$

for all $a \in D(T)$.

2 Infinitesimal representations of Lie groups

Let G be a locally compact group and A be a C^*-algebra. We say that a mapping $u : G \to M(A)$ is a unitary representation of G in A if

1. $u(g)$ is unitary for any $g \in G$.
2. $u(g_1 g_2) = u(g_1)u(g_2)$ for any $g_1, g_2 \in G$.
3. For any $a \in A$ the mapping $G \ni g \mapsto u(g)a \in A$ is norm continuous.

Let us remind that elements of $M(A)$ are continuous linear operators on A (they act by left multiplication). Therefore we may use all the concepts and results of the general theory of linear actions of locally compact groups on Banach spaces [1]. In particular in the Lie group case, we may consider infinitesimal representations of G.

From now till the end of this Section we assume that G is an N–dimensional Lie group. Let $\mathfrak{g}$ be the Lie algebra and $\mathcal{E}$ be the enveloping algebra of G. By definition $\mathfrak{g}$ consists of all right–invariant real vector fields on G. Consequently elements of $\mathcal{E}$ are differential operators on G commuting with the right shifts. $\mathcal{E}$ is equipped with an antilinear and antimultiplicative involution $\mathcal{E} \ni M \mapsto M^+ \in \mathcal{E}$ such that $X^+ = -X$ for all $X \in \mathfrak{g}$.

Let u be a unitary representation of G in a C^*-algebra A. The corresponding infinitesimal representation will be denoted by du. Operators $du(M)$ (where $M \in \mathcal{E}$) are defined on the invariant dense domain

$$D^\infty(u) = \left\{ a \in A : \begin{array}{c} \text{The mapping } G \ni g \mapsto u(g)a \in A \\ \text{is of } C^\infty\text{-class in the sense} \\ \text{of norm topology in } A. \end{array} \right\}$$

For any $M \in \mathcal{E}$ and $a \in D^\infty(u)$

$$du(M)a = Mu(g)a|_{g=e}$$

Clearly $D^\infty(u)$ is invariant under all $u(h)$, $h \in G$. Replacing in the above formula a by $u(h)a$ and remembering that M commutes with right translations we get $du(M)u(h)a = Mu(g)a|_{g=h}$. This formula may be rewritten in the following way:

$$du(M)u(h)a = Mu(h)a \tag{2.1}$$

Let $a, b \in D^\infty(u)$ and $X \in \mathfrak{g}$. Applying X to the constant function $(u(g)a)^* u(g)b = a^* b$ and using the Leibniz rule we get

$$(du(X)a)^* b = -a^*(du(X)b).$$

Therefore

$$(du(M)a)^* b = a^* du(M^+)b$$

for any $a, b \in D^\infty(u)$ and $M \in \mathcal{E}$. Using this formula one can easily show that for all $M \in \mathcal{E}$ operators $du(M)$ are closable. In what follows the closure of $du(M)$ will be denoted by $u(M)$.

Affiliation problem

Characterize all elements $M \in \mathcal{E}$ such that $u(M)\eta A$ for any unitary representation u of G in a C^*-algebra A.

The following theorem provides a simple necessary condition

Theorem 2.1 *Let u be a unitary representation of G in a C^*-algebra A and $M \in \mathcal{E}$. Assume that $f = 0$ is the only bounded C^∞–solution of the differential equation $M^+ M f = -f$. Then $u(M), u(M^+)\eta A$ and $u(M)^* = u(M^+)$.*

Proof. By virtue of [17, Proposition 2.2] it is sufficient to show that the set

$$\left\{ \begin{pmatrix} a - du(M^+)b \\ du(M)a + b \end{pmatrix} : a, b \in D^\infty(u) \right\}$$

is dense in $A \oplus A$. Let ω and ω' be continuous linear functionals on A such that

$$\omega(a - du(M^+)b) + \omega'(du(M)a + b) = 0 \tag{2.2}$$

for all $a, b \in D^\infty(u)$. We have to show that $\omega = \omega' = 0$. Let $c \in D^\infty(u)$. Inserting in (2.2) $u(g)c$ and $-du(M)u(g)c$ instead of a and b we get

$$f + M^+ M f = 0$$

where $f(g) = \omega(u(g)c)$ is a bounded C^∞-function on G. Therefore $\omega(u(g)c)$ $= 0$, $\omega(c) = 0$ and ($D^\infty(u)$ is dense in A) $\omega = 0$. Now, inserting in (2.2) $a = 0$ we get $\omega'(b) = 0$ for all $b \in D^\infty(u)$ and $\omega' = 0$. Q.E.D.

It is known that all elements $M \in \mathfrak{g}$ satisfy the assumption of Theorem 2.1. The same is true for the Nelson operator $\Delta = \sum X_i^+ X_i$, where $(X_i)_{i=1,2,\dots,N}$ is a basis in $\mathfrak{g}$. Elements $u(X_i)$ ($i = 1, 2, \dots, N$) are called infinitesimal generators of u. They are skew–adjoint and affiliated with A. Element $u(\Delta)\eta A$ is selfadjoint and positive.

Theorem 2.2 *Let A, B be C^*-algebras, $\varphi \in \mathrm{Mor}(B, A)$, v be a unitary representation of G in B and $u(g) = \varphi(v(g))$ for all $g \in G$. Then*

1. *u is a unitary representation of G in A.*
2. *If $M \in \mathcal{E}$ and $v(M)\eta B$ then $u(M)\eta A$ and*

$$u(M) = \varphi(v(M)).$$

Proof. Statement 1 is trivial. We prove Statement 2. By definition $\varphi(D^\infty(v))A$ is a core for $\varphi(v(M))$ and for any $b \in D^\infty(v)$ and $a \in A$ we have:

$$\begin{aligned}\varphi(v(M))\varphi(b)a &= \varphi(v(M)b)a = \varphi(Mv(g)b|_{g=e})a = (M\varphi(v(g)b)|_{g=e})a \\ &= (Mu(g)\varphi(b)|_{g=e})a = u(M)\varphi(b)a.\end{aligned}$$

Therefore it is sufficient to show that $\varphi(D^\infty(v))A$ is a core for $u(M)$.

Let $c \in D^\infty(u)$ and $\epsilon > 0$. Then $u(h)c$ and $u(M)u(h)c$ (=$Mu(h)c$ by virtue of (2.1)) depend continuously on h and there exists a neighbourhood U of e such that

$$\|u(h)c - c\| \le \frac{\epsilon}{2} \tag{2.3}$$

$$\|u(M)u(h)c - u(M)c\| \le \frac{\epsilon}{2} \tag{2.4}$$

for any $h \in U$. Let f be a C^∞-function on G with a compact support contained in U such that $f(h) \ge 0$ for all $h \in G$ and $\int_G f(h)\,dh = 1$ (dh denotes the left–invariant Haar measure on G). We set

$$\mathcal{R}_u = \int_G f(h)u(h)\,dh,$$

$$\mathcal{R}_v = \int_G f(h)v(h)\,dh.$$

Clearly $\mathcal{R}_u \in M(A)$, $\mathcal{R}_v \in M(B)$ and $\mathcal{R}_u = \varphi(\mathcal{R}_v)$. Moreover

$$\|\mathcal{R}_u a\| \le \|a\| \tag{2.5}$$

for all $a \in A$.

$\mathcal{R}_u$ and $\mathcal{R}_v$ are regularizing operators: $\mathcal{R}_u A \subset D^\infty(u)$ and $\mathcal{R}_v B \subset D^\infty(v)$ (cf. the concept of the Gårding domain [6]). Let $a \in A$. Using (2.1) and integrating by parts we get $u(M)\mathcal{R}_u a = u(M)\int f(h)u(h)a\,dh = \int f(h)(Mu(h)a)\,dh = \int \overline{(M^+ f)(h)}u(h)a\,dh$. Therefore

$$\|u(M)\mathcal{R}_u a\| \le Q\|a\| \tag{2.6}$$

where $Q = \int |(M^+ f)(h)|\,dh$.

Let $\theta = \frac{\epsilon}{2}\cdot\min(1, Q^{-1})$. We know (cf. Section 0) that $\varphi(B)A$ is dense in A. Therefore there exist $b_1, b_2, \ldots, b_N \in B$ and $a_1, a_2, \ldots, a_N \in A$ such that

$$\|\sum_{k=1}^{N} \varphi(b_k)a_k - c\| \le \theta.$$

Inserting in (2.5) and (2.6) $a = \sum \varphi(b_k)a_k - c$ we get

$$\|\sum_{k=1}^{N} \varphi(\mathcal{R}_v b_k)a_k - \mathcal{R}_u c\| \le \frac{\epsilon}{2}, \tag{2.7}$$

$$\|u(M)\sum_{k=1}^{N} \varphi(\mathcal{R}_v b_k)a_k - u(M)\mathcal{R}_u c\| \le \frac{\epsilon}{2}. \tag{2.8}$$

On the other hand due to (2.3) and (2.4) we have

$$\|\mathcal{R}_u c - c\| \le \frac{\epsilon}{2}, \tag{2.9}$$

$$\|u(M)\mathcal{R}_u c - u(M)c\| \le \frac{\epsilon}{2}. \tag{2.10}$$

Combining (2.7) with (2.9) and (2.8) with (2.10) we get

$$\|c' - c\| \le \epsilon, \tag{2.11}$$

$$\|u(M)c' - u(M)c\| \le \epsilon. \tag{2.12}$$

where $c' = \sum \varphi(\mathcal{R}_v b_k)a_k$. This way we showed that for any $c \in D^\infty(u)$ and any $\epsilon > 0$ there exists $c' \in \varphi(D^\infty(v))A$ such that the estimates (2.11) and (2.12) hold. It means that $\varphi(D^\infty(v))A$ is a core for $u(M)$. Q.E.D.

The remaining part of this Section is mainly devoted to the case $G = \mathbf{R}$.

Theorem 2.3 *Let u be a representation of $\mathbf{R}$ in a C^*-algebra A and $T\eta A$ be the infinitesimal generator of u. Then*

1.

$$Ta = \lim_{t\to 0} \frac{u(t)a - a}{t} \tag{2.13}$$

In particular $a \in D(T)$ if and only if the limit on the right hand side of (2.13) exists in the sense of the norm topology on A.

2. T is skew–adjoint ($T^* = -T$) and using the functional calculus of normal elements we have

$$u(t) = e^{tT} \tag{2.14}$$

for all $t \in \mathbf{R}$.

Proof.

Ad 1. Let T be the operator introduced by (2.13). One can easily see that T is a closed extension of $du(\frac{d}{dt})$. Therefore $T \supset u(\frac{d}{dt})$. On the other hand due to the Leibniz rule $(Ta)^*b + a^*Tb = 0$ for all $a, b \in D(T)$. It means that T is skew–symmetric and using Lemma 1.2 we obtain $T = u(\frac{d}{dt})$.

Ad 2. By virtue of (1.12) – (1.15) the mapping $\mathbf{R} \ni t \mapsto e^{tT} \in M(A)$ is a unitary representation of $\mathbf{R}$ in A and its infinitesimal generator contains T. Using Lemma 1.2 and remembering that any representation is uniquely determined by its generator we get (2.14). Q.E.D.

Corollary 2.4 *Let u be a representation of G in a C^*-algebra A and let $Y_1, Y_2, \ldots, Y_N$ be infinitesimal generators of u. Then*

1.

$$\bigcap_{i=1,2,\ldots,N} D(Y_i) = \left\{ a \in A : \begin{array}{c} \textit{The mapping } G \ni g \mapsto u(g)a \in A \\ \textit{is of } C^1\textit{-class} \end{array} \right\}.$$

2. For any $t_1, t_2, \ldots, t_N \in \mathbf{R}$

$$u(\exp(\sum t_i X_i)) = e^{\sum t_i Y_i}, \tag{2.15}$$

where exp : $\mathfrak{g} \to G$ *is the exponential mapping of the theory of Lie groups and the right hand side is understood in the sense of functional calculus of normal (skew–adjoint) elements (cf. Section 1).*

3 Elements affiliated with $C^*(G)$.

Let G be a locally compact group. It is well known that unitary representations of G are in 1–1 correspondence with the representations of the algebra $C^*(G)$. By Theorem 2.2, investigating the affiliation problem formulated in Section 2 it is sufficient to consider the universal representation of G in $C^*(G)$.

We refer to [12] for the definition and basic properties of $C^*(G)$. By definition $L^1(G) \subset C^*(G)$: $C^*(G)$ is the completion of $L^1(G)$ with respect to a C^*–norm. Moreover the space of all finite (complex valued) measures on G is contained in $M(C^*(G))$.

Identifying any element $g \in G$ with the probability measure concentrated at g we define the embedding $G \hookrightarrow M(C^*(G))$. One can easily show that this embedding $G \ni g \mapsto U(g) \in M(C^*(G))$ is a unitary representation of G in $C^*(G)$. It is called the universal representation for the following

Theorem 3.1 *Let u be a representation of G in a C^*-algebra A. Then there exists unique $\tilde{u} \in \mathrm{Mor}(C^*(G), A)$ such that $u(g) = \tilde{u}(U(g))$ for all $g \in G$.*

Conversely if $\tilde{u} \in \mathrm{Mor}(C^*(G), A)$ then (cf. Theorem 2.2.1) setting $u(g) = \tilde{u}(U(g))$ we define a representation u of G.

Using theorem 3.1 one can easily prove

Proposition 3.2 *There exists unique $\Phi \in \mathrm{Mor}(C^*(G), C^*(G) \otimes C^*(G))$ such that $\Phi(U(g)) = U(g) \otimes U(g)$.*

Proposition 3.3 *Let $\varphi : G_1 \to G_2$ be a homomorphism of locally compact groups. Then there exists unique $\tilde{\varphi} \in \mathrm{Mor}(C^*(G_1), C^*(G_2))$ such that $\tilde{\varphi}(U_1(g)) = U_2(\varphi(g))$ for all $g \in G_1$ (U_i denotes the universal representation of G_i, $i = 1, 2$).*

The algebra $C^*(G)$ may be considered as the algebra of all "continuous, vanishing at infinity functions" on a quantum space $\hat{G}$. The comultiplication Φ introduced in Proposition 3.2 defines a group structure on $\hat{G}$. $\hat{G}$ endowed with this structure is the Pontryagin dual of G (cf. [15]). It turns out that the pair $(C^*(G), \Phi)$ determines the group G uniquely: Using the Tatsuuma duality [2] one can show that G (or more precisely $U(G)$) coincides with $\{v \in M(C^*(G)) : v \neq 0 \text{ and } \Phi(v) = v \otimes v\}$. Moreover restricting to G the topology of almost uniform convergence on $M(C^*(G))$ we get the original topology of G.

Indeed the mapping $G \ni g \mapsto U(g) \in M(C^*(G))$ is continuous (U is a representation of G in $C^*(G)$). To prove the continuity of the reverse mapping it is sufficient to consider left regular representation L of G: one can easily show that

$$\left(\begin{array}{c} L_{g_\alpha} \to L_{g_\infty} \\ \textit{in the strong} \\ \textit{operator topology} \end{array}\right) \Longleftrightarrow \left(\begin{array}{c} g_\alpha \to g_\infty \end{array}\right)$$

for any sequence (g_α) of elements of G and any $g_\infty \in G$.

A deeper analysis (see [16]) shows that the Pontryagin dual of $\hat{G}$ coincides with G.

Assume now that G is a Lie group. Let $\mathfrak{g}$ be the Lie algebra of G, $(X_1, X_2, \ldots, X_{N=\dim G})$ be the basis in $\mathfrak{g}$ and c^i_{kl} be the corresponding structure constants:

$$[X_k, X_l] = \sum_i c^i_{kl} X_i.$$

The enveloping algebra of $\mathfrak{g}$ will be denoted by $\mathcal{E}$. In particular the Nelson operator $\Delta = \sum X_i^+ X_i \in \mathcal{E}$. In what follows we identify G with its U–image. Consequently each $M \in \mathcal{E}$ will be identified with the corresponding closed unbounded operator acting on $C^*(G)$. It means that we shall omit the symbol U writing g and M instead of $U(g)$ and $U(M)$ resp. We also set $D^\infty = D^\infty(U)$.

We already know (cf. Section 2, remarks after Theorem 2.1) that X_i $(i = 1, 2, \ldots, N)$ and Δ are affiliated with $C^*(G)$. X_i are skew–adjoint; Δ is selfadjoint and positive. Therefore $\mathrm{Sp}\,\Delta \subset [0, \infty[$. Let

$$b_0 = (I + \Delta)^{-1}. \tag{3.1}$$

By virtue of [11, Theorem 3.1] $b_0 \in C^*(G)$ (see also [13]). Inserting in (1.10) $T = \Delta^{1/2}$, $f(\lambda) = (1 + \lambda^2)^{-1/2}$ and $x = b_0^{1/2}$ we obtain $b_0 \in D(\Delta^{1/2})$.

Like in the Hilbert space operator theory one can show that any core of Δ is a core of $\Delta^{1/2}$. Using now the obvious estimate $\|X_i a\| \leq \|\Delta^{1/2} a\|$ for all $a \in D^\infty$ we see that $D(\Delta^{1/2}) \supset D(X_i)$. In particular $b_0 \in D(X_i)$ for all $i = 1, 2, \ldots, N$. Let $b_i = X_i b_0$. Then $b_0, b_1, \ldots, b_N \in C^*(G)$.

Clearly $b_0, b_1, \ldots, b_N$ coincide with the generators introduced in Lemma 2 of [13]. They satisfy the following Szymański relations

$$\begin{aligned} b_0^* &= b_0, \\ b_k^* b_0 &= -b_0 b_k, \\ b_0(I - b_0) &= \sum_{j=1}^{N} b_j^* b_j, \\ b_k^* b_l - b_l^* b_k &= \sum_{j=1}^{N} c_{kl}^j b_0 b_j, \\ b_k^* + b_k &= \sum_{i,j=1}^{N} c_{ki}^j (b_j^* b_i + b_i^* b_j). \end{aligned}$$

for all $k, l = 1, 2, \ldots, N$. If G is connected and simply connected then $C^*(G)$ coincides with the (non–unital) C^*-algebra generated by abstract elements $b_0, b_1, \ldots b_N$ satisfying the above relations [13]. We shall use the following C^*–version of Lemma 2 of [13].

Proposition 3.4 *Let A be a C^*-algebra and $b_0', b_1', \ldots, b_N'$ be elements of $M(A)$ satisfying the Szymański relations. Assume that $b_0'A$ is dense in A and that G is connected and simply connected. Then there exists unique $\varphi \in \mathrm{Mor}(C^*(G), A)$ such that $b_j' = \varphi(b_j)$ for all $j = 0, 1, \ldots, N$.*

Proof. We may assume that $A \subset B(H)$ where H is a Hilbert space and the embedding is non–degenerate. Then $b_0', b_1', \ldots, b_N' \in B(H)$ and using Lemma 2 of [13] we get $b_j' = \varphi(b_j)$ where φ is a representation of $C^*(G)$ in H. Remembering that $b_0, b_1, \ldots, b_N$ generate $C^*(G)$ we see that $\varphi(C^*(G)) \subset M(A)$. Moreover $\varphi(C^*(G))A \supset \varphi(b_0)A = b_0'A$. Therefore $\varphi(C^*(G))A$ is dense in A and $\varphi \in \mathrm{Mor}(C^*(G), A)$. Q.E.D.

4 The integrability theorem of E.Nelson

Let Y be a linear operator acting on a C^*-algebra A. W say that Y is skew–symmetric if $(Ya)^*b = -a^*(Yb)$ for all $a, b \in D(Y)$.

In this section we assume that G is a connected, simply connected Lie group. We shall use the notation introduced in Sections 2 and 3. The following theorem is the C^*–version of Theorem 5 and Corollary 9.1 of [10].

Theorem 4.1 *Let Y_i $(i = 1, 2, \ldots, N)$ be closed skew–symmetric operators acting on a C^*-algebra A and D be a dense linear subset of A such that for all $a \in D$ and $i, j = 1, 2, \ldots, N$ we have*

1. *$a \in D(Y_i)$ and $Y_i a \in D(Y_j)$,*
2. *$Y_i Y_j a - Y_j Y_i a = \sum_{k=1}^N c_{ij}^k Y_k a$.*

Assume that the closure of $\sum_{k=1}^N Y_k^2|_D$ is a selfadjoint element affiliated with A.

Then $Y_i \eta A$ $(i = 1, 2, \ldots, N)$ and there exists unique representation u of G such that $u(X_i) = Y_i$ $(i = 1, 2, \ldots, N)$.

Proof. The uniqueness of u follows from the general theory of Banach space representations of Lie groups. We have to show the existence.

Let $\tilde{\Delta}$ be the closure of $-\sum_{k=1}^N Y_k^2|_D$ and $C = I + \tilde{\Delta}$. Then D is a core of C, $C^* = C \eta A$ and $C \geq I$. It implies that $CD(C) = A$. The inverse of C will be denoted by $b_0{}'$. Clearly $b_0{}' \in M(A),\ 0 \leq b_0{}' \leq I$ and $b_0{}'A = D(C)$.

Let $i \in \{1, 2, \ldots, N\}$. For any $a \in D$ we have

$$(Y_i a)^*(Y_i a) \leq \sum_{k=1}^N (Y_k a)^*(Y_k a) = a^*(C - I)a \leq \frac{1}{4}(Ca)^*(Ca).$$

The last estimate follows from the equality $C - I = (C/2)^2 - (C/2 - I)^2$. Therefore $\|Y_i a\| \leq \frac{1}{2}\|Ca\|$ for all $a \in D$. Remembering that D is a core of C and that Y_i is closed we get $D(C) \subset D(Y_i)$ and $\|Y_i a\| \leq \frac{1}{2}\|Ca\|$ for all $a \in D(C)$. It shows that the operator $b_i{}' = Y_i b_0{}'$ is defined on the whole A and $\|b_i{}'\| \leq \frac{1}{2}$. Therefore $b_i{}' \in B(A)$.

Let $x \in D(Y_i)$. Then for any $z \in A$ we have $(b_i{}'z)^*x = (Y_i b_0{}'z)^*x = -(b_0{}'z)^*Y_i x = z^*(-b_0{}'Y_i x)$. It shows that $x \in D((b_i{}')^*),\ D(Y_i) \subset D((b_i{}')^*)$ and by virtue of Proposition 0.1 we obtain $b_i{}' \in M(A)$ for $i = 1, 2, \ldots, N$.

By simple computations one can check that ${b_0}', {b_1}', \ldots, {b_N}'$ satisfy the Szymański relations. Therefore (cf. Proposition 3.4) there exists unique $\varphi \in \mathrm{Mor}(C^*(G), A)$ such that $\varphi(b_k) = {b_k}'$ for $k = 1, 2, \ldots, N$.

Let $i \in \{1, 2, \ldots, N\}$. For any $a \in D(C)$ we have

$$\begin{aligned} Y_i a &= Y_i {b_0}' C a = {b_i}' C a = \varphi(b_i) C a = \varphi(X_i b_0) C a = \varphi(X_i)\varphi(b_0) C a \\ &= \varphi(X_i) {b_0}' C a = \varphi(X_i) a. \end{aligned}$$

According to (3.1), $b_0 C^*(G) = D(\Delta) \supset D^\infty$. Therefore $b_0 C^*(G)$ is a core of X_i and consequently (cf. Section 0, Remark before Prop. 0.2) $\varphi(b_0) A = {b_0}' A = D(C)$ is a core of $\varphi(X_i)$. Taking into account the above computations and remembering that Y_i is closed we conclude that $\varphi(X_i) \subset Y_i$. By virtue of Lemma 1.2, $\varphi(X_i) = Y_i$. In particular $Y_i \eta A$.

For any $g \in G \subset M(C^*(G))$ we set $u(g) = \varphi(g)$. By virtue of Theorem 2.2, u is a unitary representation of G in A and $u(X_i) = \varphi(X_i) = Y_i$ for $i = 1, 2, \ldots, N$. Q.E.D.

5 Selfadjoint extensions of symmetric elements

Let A be a C^*-algebra and $T\eta A$. We say that T is symmetric if $T \subset T^*$. It means that
$$y^*Tx = (Ty)^*x \tag{5.1}$$
for any $x, y \in D(T)$. Let z be the z–transform of $T\eta A$. Then $x = \sqrt{I - z^*z}\, a$, $Tx = za$, $y = \sqrt{I - z^*z}\, b$ and $Ty = zb$, where a, b are elements of A. Inserting these data into (5.1) we get
$$z^*\sqrt{I - z^*z} = \sqrt{I - z^*z}\, z. \tag{5.2}$$

Conversely, (5.2) immediately implies (5.1). Like in the theory on the Hilbert space level the problem of extensions of symmetric elements can be solved by the Cayley transform method (see e.g. [9]).

Let T be a symmetric element affiliated with a C^*-algebra and z be the z–transform of T. We set
$$c_T = w_- w_+^* \tag{5.3}$$
where $w_\pm = z \pm i\sqrt{I - z^*z}$. Due to (5.2) w_+ and w_- are isometries: $w_\pm^* w_\pm = I$. Therefore c_T is a partial isometry: $c_T^* c_T = w_+ w_+^*$ and $c_T c_T^* = w_- w_-^*$ are projections (note that an element c of a C^*-algebra is a partial isometry if and only if $cc^*c = c$). We call c_T the Cayley transform of T. To justify this terminology one can verify that
$$c_T(T + iI)x = (T - iI)x$$
for any $x \in D(T)$.

Let $b \in A$. Then
$$\frac{i}{2}(c_T - I)c_T^* = \frac{i}{2}(w_- w_+^* - I)w_+ w_-^* b = \sqrt{I - z^*z}\, w_-^* b$$
and $\frac{i}{2}(c_T - I)c_T^* b \in D(T)$. Let us notice that each element of $D(T)$ is of this form. Indeed, for any $a \in A$ we have $a = w_-^* b$, where $b = w_- a$. With the same notations we have
$$T(\frac{i}{2}(c_T - I)c_T^* b) = zw_-^* b = \frac{1}{2}(w_- + w_+)w_-^* b = \frac{1}{2}(c_T c_T^* + c_T^*)b = \frac{1}{2}(c_T + I)c_T^* b.$$
This way we showed that
$$\begin{pmatrix} x \in D(T) \\ y = Tx \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \textit{There exists } b \in c_T^* A \textit{ such that} \\ x = i(c_T - I)b \textit{ and } y = (c_T + I)b \end{pmatrix} \tag{5.4}$$

Therefore T is uniquely determined by its Cayley transform c_T. Moreover, $(c_T - I)c_T^* A$ is dense in A.

Proposition 5.1 *Let c be an element of $M(A)$ such that $cc^*c = c$ (i.e. c is a partial isometry). Assume that $(c - I)c^*A$ is dense in A. Then c is the Cayley transform of a symmetric $T\eta A$.*

Proof. $(I - c^*)cc^*A = (c - I)c^*A$ is, by assumption,dense in A. Therefore $(I - c^*)A$ is dense in A and $c(I - c^*)A$ is dense in $cA = cc^*A$ (because $cc^*A \subset cA = cc^*cA \subset cc^*A$). Applying Proposition 0.4 to the operator $c(I - c^*)$ we get

$$c(I - c^*) = v_-|c(I - c^*)|$$

where $v_-^*v_- = I$, $v_-v_-^* = cc^*$ and $|c(I - c^*)| = \sqrt{(I - c)c^*c(I - c^*)} = \sqrt{c^*c - c - c^* + cc^*}$.

Let $v_+ = c^*v_-$. Then $v_+^*v_+ = v_-^*cc^*v_- = v_-^*v_-v_-^*v_- = I$, $v_+v_+^* = c^*v_-v_-^*c = c^*cc^*c = c^*c$ and

$$v_-v_+^* = v_-v_-^*c = cc^*c = c.$$

We shall apply Proposition 0.4 to the operator $(v_+ - v_-)^*$. In this case $e = f = I$. Indeed, $(v_+ - v_-)A \supset (v_+ - v_-)v_-^*A = (c^* - cc^*)A = (I - c)c^*A$ is, by the assumption, dense in A and $f = I$. Furthermore, $(v_+ - v_-)^*A = v_+^*v_+(v_+ - v_-)^*A = v_+^*(c^*c - c^*)A = v_+^*c^*(I - c)A$. We know that $(I - c)c^*A$ is dense in A, so is $(I - c)A$, whence $(v_+ - v_-)^*A$ is dense in $v_+^*c^*A = v_-^*cc^*A \supset v_-^*cc^*cA = v_-^*cA = v_+^*A \supset v_+^*v_+A = A$ and $e = I$.

By virtue of Proposition 0.4 $(v_+ - v_-)^* = rs$, where $r, s \in M(A)$, r is unitary, $s \geq 0$ and $v_+ - v_- = sr^*$.

Let

$$w_+ = iv_+r,$$

$$w_- = iv_-r.$$

By computation we get

$$w_+^*w_+ = I, \quad w_-^*w_- = I, \tag{5.5}$$

$$w_+w_+^* = c^*c, \quad w_-w_-^* = cc^*,$$

$$\frac{1}{i}(w_+ - w_-) \geq 0, \tag{5.6}$$

$$c = w_-w_+^*.$$

Let
$$z = \frac{1}{2}(w_+ + w_-). \tag{5.7}$$
Then
$$z^*z + \left(\frac{1}{2i}(w_+ - w_-)\right)^* \left(\frac{1}{2i}(w_+ - w_-)\right) = I.$$
and taking into account (5.6) we get
$$\frac{1}{2i}(w_+ - w_-) = \sqrt{I - z^*z}, \tag{5.8}$$
Combining (5.7) and (5.8) we get
$$w_\pm = z \pm i\sqrt{I - z^*z}. \tag{5.9}$$
Since
$$\sqrt{I - z^*z}\, A = i(v_+ - v_-)rA = (v_+ - v_-)A$$
is dense in A, z is the z–transform of an element T affiliated with A. T is symmetric. Indeed, combining (5.5) and (5.9) we get (5.2). Clearly c is the Cayley transform of T. Q.E.D.

Now we are able to prove

Theorem 5.2 *Let T be a symmetric element affiliated with a C^*-algebra A, z be the z–transform of T and E^+, E^- be elements of $M(A)$ introduced by*
$$E^\pm = z^*z \pm i\left(z\sqrt{I - z^*z} - \sqrt{I - z^*z}\, z^*\right) - zz^*. \tag{5.10}$$

Then

1. E^+ and E^- are orthogonal projections: $E^{\pm *} = E^\pm = (E^\pm)^2$. Moreover, $E^+ \neq I \neq E^-$.

2. T is selfadjoint if and only if $E^\pm = 0$.

3. The set of all selfadjoint extensions of T is in one to one correspondence with the set of all elements $v \in M(A)$ such that $v^*v = E^+$ and $vv^* = E^-$.

4. The set of all symmetric extensions of T is in one to one correspondence with the set of all partial isometries $v \in M(A)$ such that $v^*v \leq E^+$ and $vv^* \leq E^-$.

Proof.

Ad 1. Using the relations derived in the proof of Proposition 5.1 one can easily verify that

$$E^+ = I - c_T^* c_T,$$
$$E^- = I - c_T c_T^*,$$

where c_T is the Cayley transform of T and the statements follow (c_T is a partial isometry).

Ad 2. If T is selfadjoint then $z^* = z$ and (cf. (5.10)) $E^{\pm} = 0$. Conversely, if $E^{\pm} = 0$ then $z^*z = zz^*$ and $z\sqrt{I - z^*z} = \sqrt{I - z^*z}\, z^*$. Therefore z^* commutes with z^*z and $z\sqrt{I - z^*z} = z^*\sqrt{I - z^*z}$. Remembering that $D(T) = \sqrt{I - z^*z}\, A$ is dense in A we get $z = z^*$ and the selfadjointness of T follows (cf. (0.2)).

It follows immediately from the above considerations that a symmetric element is selfadjoint if and only if its Cayley transform is unitary.

Ad 4. Let $v \in M(A)$ be a partial isometry such that $v^*v \leq E^+$ and $vv^* \leq E^-$. Then $v^*v \leq I - c_T^* c_T$ and $vv^* \leq I - c_T c_T^*$. Therefore

$$(vc_T^*)^*(vc_T^*) = c_T v^* v c_T^* \leq c_T(I - c_T^* c_T)c_T^* = 0,$$

$$(v^*c_T)^*(v^*c_T) = c_T^* v v^* c_T \leq c_T^*(I - c_T c_T^*)c_T = 0$$

and

$$vc_T^* = v^* c_T = 0. \tag{5.11}$$

Using these relations one can easily prove that $c = c_T + v$ is a partial isometry. Indeed,

$$c^*c = I - (E^+ - v^*v),$$
$$cc^* = I - (E^- - vv^*)$$

are orthogonal projections.

Let $b \in c_T^* A$. Then $b = c_T^* a$ (where $a \in A$), $vb = vc_T^* a = 0$ (cf. (5.11)) and $cb = c_T b$. Moreover, $b = c_T^* c_T c_T^* a = (c_T^* + v^*)c_T c_T^* a = c^* c_T c_T^* a$ and $b \in c^* A$. Therefore $(c - I)c^* A \supset (c_T - I)c_T^* A$ is dense in A, c is the Cayley transform of a symmetric $S \eta A$ (see Proposition 4.1) and using (5.4) we get $T \subset S$.

Conversely, assume that S is a symmetric extension of T. By virtue of (5.4) for any $b \in c_T^* A$ there exists $b' \in c_S^* A$ such that

$$i(c_T - I)b = i(c_S - I)b',$$

$$(c_T + I)b = (c_S + I)b'.$$

Solving these equations we get $b' = b$ and $c_S b' = c_T b$. It shows that

$$c_T^* A \subset c_S^* A \tag{5.12}$$

and

$$c_S c_T^* = c_T c_T^* \tag{5.13}$$

We know that $c_S^* c_S c_S^* = c_S^*$, hence $c_S^* c_S a = a$ for any $a \in c_S^* A$. By virtue of (5.12) the same relation holds for $a \in c_T^* A$. In particular $c_S^* c_S c_T^* c_T = c_T^* c_T$. Therefore $c_T^* c_T \leq c_S^* c_S$ and $c_S^* c_S - c_T^* c_T$ is an orthogonal projection.

Let

$$v = c_S(c_S^* c_S - c_T^* c_T).$$

Then $v^* v = c_S^* c_S - c_T^* c_T$. It shows that v is a partial isometry and $v^* v \leq I - c_T^* c_T = E^+$. Using (5.13) one can verify that $vv^* = c_S c_S^* - c_T c_T^* \leq E^-$. To end the proof of Statement 4 we notice that by virtue of (5.13)

$$v = c_S c_S^* c_S - c_T c_T^* c_T = c_S - c_T.$$

Ad 3. According to the remark at the end of the proof of Statement 2 the extension S is selfadjoint if and only if c_S is unitary. Clearly this is the case if and only if $v^* v = I - c_T^* c_T = E^+$ and $vv^* = I - c_T c_T^* = E^-$.

Q.E.D.

Corollary 5.3 *Let A be a C^*-algebra such that $M(A)$ contains no projections except* 0 *and I. Then every symmetric $T \eta A$ is selfadjoint.*

Remembering that $E^\pm = I - w_\pm w_\pm^*$ we get the following enhancement

Corollary 5.4 *Let A be a C^*-algebra such that each isometry belonging to $M(A)$ is unitary. Then each symmetric $T \eta A$ is selfadjoint.*

Let $T \eta A$. We say that T is positive if $x^* T x \geq 0$ for any $x \in D(T)$. Let z be the z–transform of T. One can easily check that T is positive if and only if

$$\sqrt{I - z^* z}\, z \geq 0.$$

In particular any positive element is symmetric.

Proposition 5.5 *Every positive element T affiliated with a C^*-algebra admits a selfadjoint extension.*

Proof. Let z be the z–transform of T. For any $\lambda \in \mathbf{C}$ such that $|\lambda| = 1$, $\Re\lambda \geq 0$ we set

$$w_\lambda = z + \lambda\sqrt{I - z^*z}.$$

Then $w_\lambda^* w_\lambda = I + (2\Re\lambda)\sqrt{I - z^*z}\, z \geq I$. It shows that $\mathrm{Sp}\,(w_\lambda^* w_\lambda) \subset [1, \infty[$. Therefore (cf. [7, Problem 69]) $\mathrm{Sp}\,(w_\lambda w_\lambda^*) \subset \{0\} \cup [1, \infty[$.

Let f be a continuous function on $[0, \infty[$ such that $f(0) = 1$ and $f(t) = 0$ for $t \geq 1$. We set

$$E(\lambda) = f(w_\lambda w_\lambda^*).$$

Then $E(\lambda)$ is an orthogonal projection belonging to $M(A)$; $E(\lambda)$ depends continuously on λ and $E(\pm i) = E^\pm$. The existence of a partial isometry v such that $v^*v = E^+$ and $vv^* = E^-$ follows now from the following lemma well known in the K–theory of C^*-algebras.

Lemma 5.6 *Let e, f be orthogonal projections belonging to $M(A)$ such that $\|e - f\| < 1$. Then there exists $u \in M(A)$ such that $u^*u = e$ and $uu^* = f$.*

Proof. Apply Proposition 0.4 to the element $c = ef$.

Q.E.D.

Remark. At the moment we are not able to show that any positive element admits a positive selfadjoint extension.

Let T be a positive element affiliated with a C^*-algebra A. We endow $D(T)$ with the norm

$$|\!|\!| x |\!|\!| = \sqrt{\|x^*(T + I)x\|}, \quad x \in D(T).$$

Let $\tilde{D}(T)$ be the completion of $D(T)$ with respect to this norm, $D_F = \tilde{D}(T) \cap D(T^*)$ and $T_F = T^*|_{D_F}$. T_F is called the Friedrichs extension of T.

Problem: Is T_F affiliated with A? It seems that the answer is negative in general. However, if A is balanced (i.e. any left multiplier is a multiplier: $LM(A) = M(A)$) then $T_F \eta A$ for any positive $T \eta A$. In this case T_F is a positive selfadjoint extension of T.

6 Tensor product of affiliated elements

In this section we shall prove the following

Theorem 6.1 *Let A_i be a C^*-algebra and $T_i \eta A_i$ $(i = 1, 2)$. Then there exists unique $T_1 \otimes T_2 \eta A_1 \otimes A_2$ such that $D(T_1) \otimes_{alg} D(T_2)$ is a core of $T_1 \otimes T_2$ and*

$$(T_1 \otimes T_2)(a_1 \otimes a_2) = T_1 a_1 \otimes T_2 a_2 \tag{6.1}$$

for any $a_1 \in D(T_1)$ and $a_2 \in D(T_2)$. Moreover

$$(T_1 \otimes T_2)^* = T_1^* \otimes T_2^*. \tag{6.2}$$

We shall use the following

Proposition 6.2 *Let C be a unital C^*-algebra; $J \subset C$ be a closed ideal; $b_1, b_2, \ldots, b_N$ be hermitian, mutually commuting elements of C, $\Lambda \subset \mathbf{R}^N$ be the joint spectrum of $(b_1, b_2, \ldots, b_N)$ and $f, g \in C(\Lambda)$. Assume that*

$$(f(\lambda) = 0) \Longrightarrow (g(\lambda) = 0) \tag{6.3}$$

for any $\lambda \in \Lambda$.

Then $g(b_1, b_2, \ldots, b_N)J$ is contained in the closure of $f(b_1, b_2, \ldots, b_N)J$.

Proof. Due to (6.3) the closed ideal of $C(\Lambda)$ generated by f contains g (cf. [5, Lemma 2.9.4]). It means that for any $\epsilon > 0$ there exists $h_\epsilon \in C(\Lambda)$ such that

$$|g(\lambda) - f(\lambda)h_\epsilon(\lambda)| \leq \epsilon$$

for all $\lambda \in \Lambda$.

Let $x \in J$. Then $y_\epsilon = h_\epsilon(b_1, b_2, \ldots, b_N)x \in J$, using the above inequality we get $\|g(b_1, b_2, \ldots, b_N)x - f(b_1, b_2, \ldots, b_N)y_\epsilon\| \leq \epsilon$ and the statement follows. Q.E.D.

We shall use Proposition 6.2 in the following situation: $C = M(A_1 \otimes A_2)$; $J = A_1 \otimes A_2$; $b_1 = z_1^* z_1 \otimes I$ and $b_2 = I \otimes z_2^* z_2$, where z_1 and z_2 are z–transforms of T_1 and T_2 resp.. Then $\Lambda \subset [0, 1]^2$ and the functions $f(\lambda_1, \lambda_2) = (1 - \lambda_1)(1 - \lambda_2) + \lambda_1 \lambda_2$ and $g(\lambda_1, \lambda_2) = \sqrt{(1 - \lambda_1)(1 - \lambda_2)}$ satisfy the relation (6.3). Indeed, the only points, where f vanishes, are $(1, 0)$ and $(0, 1)$. Notice that in this case $g(b_1, b_2)J = (\sqrt{I - z_1^* z_1} \otimes \sqrt{I - z_2^* z_2})(A_1 \otimes A_2) \supset D(T_1) \otimes_{alg} D(T_2)$ is dense in $A_1 \otimes A_2$. Therefore we have

Lemma 6.3 *The range of* $(I - z_1^* z_1) \otimes (I - z_2^* z_2) + z_1^* z_1 \otimes z_2^* z_2$ *is dense in* $A_1 \otimes A_2$.

Replacing T_1 and T_2 by T_1^* and T_2^* resp. we get

Lemma 6.4 *The range of* $(I - z_1 z_1^*) \otimes (I - z_2 z_2^*) + z_1 z_1^* \otimes z_2 z_2^*$ *is dense in* $A_1 \otimes A_2$.

Proof of Theorem 6.1

We shall use Theorem 2.3 of [17]. Let

$$\begin{aligned} a &= \sqrt{I - z_1 z_1^*} \otimes \sqrt{I - z_2 z_2^*}, \\ b &= c = z_1 \otimes z_2, \\ d &= \sqrt{I - z_1^* z_1} \otimes \sqrt{I - z_2^* z_2}, \\ Q &= \begin{pmatrix} d, & -c^* \\ b, & a^* \end{pmatrix}. \end{aligned}$$

Clearly $a, b, c, d \in M(A_1 \otimes A_2)$. By direct computation one may check that $ab = cd$. Next $a(A_1 \otimes A_2)$ contains $a(A_1 \otimes_{alg} A_2) = D(T_1^*) \otimes_{alg} D(T_2^*)$ and $d(A_1 \otimes A_2)$ contains $d(A_1 \otimes_{alg} A_2) = D(T_1) \otimes_{alg} D(T_2)$. Both these sets are dense in $A_1 \otimes A_2$. Finally

$$QQ^* = \begin{pmatrix} d^2 + c^* c & , & 0 \\ 0 & , & a^2 + bb^* \end{pmatrix}$$

and using Lemmas 6.3 and 6.4 we see that the range of Q is dense in $(A_1 \otimes A_2) \oplus (A_1 \otimes A_2)$.

By virtue of Theorem 2.3 of [17] there exists an element $T_1 \otimes T_2 \eta A_1 \otimes A_2$ such that $d(A_1 \otimes A_2)$ is a core of $T_1 \otimes T_2$ and

$$(T_1 \otimes T_2) dx = bx \tag{6.4}$$

for all $x \in A_1 \otimes A_2$. Remembering that $A_1 \otimes_{alg} A_2$ is dense in $A_1 \otimes A_2$ we see that $d(A_1 \otimes_{alg} A_2) = D(T_1) \otimes_{alg} D(T_2)$ is a core of $T_1 \otimes T_2$. Moreover, if $a_1 \in D(T_1)$ and $a_2 \in D(T_2)$ then $a_1 = \sqrt{I - z_1^* z_1}\, x_1,\ a_2 = \sqrt{I - z_2^* z_2}\, x_2$ and inserting in (6.4) $x = x_1 \otimes x_2$ we get (6.1).

Let $x, y \in A_1 \otimes A_2$. By virtue of (0.1) $x \in D((T_1 \otimes T_2)^*)$ and $y = (T_1 \otimes T_2)^* x$ if and only if

$$((T_1 \otimes T_2)c)^* x = c^* y \tag{6.5}$$

for all $c \in D(T_1 \otimes T_2)$. We may assume that c is of the form $c = dc'$ where $c' \in A_1 \otimes A_2$ ($d(A_1 \otimes A_2)$ is a core of $T_1 \otimes T_2$). Then $(T_1 \otimes T_2)c = bc'$ and relation (6.5) is equivalent to

$$b^* x = d^* y \tag{6.6}$$

On the other hand replacing T_1 and T_2 by T_1^* and T_2^* resp. (and consequently a by d and c by b^*) and using Statement 2 of Theorem 2.3 of [17] we see that (6.6) is equivalent to $x \in D(T_1^* \otimes T_2^*)$ and $y = (T_1^* \otimes T_2^*)x$. This way formula (6.2) is proved.

Q.E.D.

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