Extended $SU(1,1)$ quantum group. Hilbert space level.

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Abstract

According to a paper published in 1991, the quantum group SU(1,1) does not exists on the C*-algebra level. In the present paper we show, that the situation essentially improves, if SU(1,1) is replaced by its double covering. A first step in this direction was made by L. Korogodski. To incorporate his idea into the quantum group framework we developed a new theory of balanced extension of subbalanced operators. This is a generalization of the theory of selfadjoint extensions of symmetric operators.

The paper is not finished yet. It contains the complete construction of extended $SU(1,1)$ quantum group on the Hilbert space level. However the associativity of the tensor product stated in Theorem 1.5 is still unproven (so at the moment it has the status of conjecture), although we strongly believe that it holds.

0 Introduction.

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The quantum $SU(1,1)$-group on the level of Hopf *-algebra is an object with no problems. The deformation parameter q is a real number in the interval $]0,1[$. The Hopf *-algebra $\mathcal{A}$ of polynomial functions on quantum $SU(1,1)$ is generated by two elements α,γ subject to the following five relations:

$$\begin{array}{rcl} \alpha\gamma &=& q\gamma\alpha, \\ \alpha\gamma^* &=& q\gamma^*\alpha, \\ \gamma\gamma^* &=& \gamma^*\gamma. \end{array} \qquad \begin{array}{rcl} \alpha^*\alpha &=& I+\gamma^*\gamma, \\ \alpha\alpha^* &=& I+q^2\gamma^*\gamma, \end{array} \tag{0.1}$$ SqU11

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The comultiplication $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is the *-algebra homomorphism acting on generators in the following way:

$$\begin{aligned} \Delta(\alpha) &= \alpha \otimes \alpha + q\gamma^* \otimes \gamma, \\ \Delta(\gamma) &= \gamma \otimes \alpha + \alpha^* \otimes \gamma. \end{aligned} \tag{0.2}$$

Delta

One can easily verify that the object $(\mathcal{A}, \Delta)$ described above is a Hopf *-algebra and that the matrix

$$w = \begin{pmatrix} \alpha \ , \ q\gamma^* \\ \gamma \ , \ \alpha^* \end{pmatrix}$$

is a corepresentation of $(\mathcal{A}, \Delta)$. Clearly w is the fundamental representation of the quantum $SU(1,1)$.

On the Hilbert space level generators α and γ should be treated as unbounded[1] operators acting on a Hilbert space. Since for unbounded operators the algebraic operations are often ill defined, one has to give a more precise meaning to the formulae (0.1). The most natural way is to assume that the four operators: α, γ, α^* and γ^* have the same domain D. Then all terms in (0.1) may be understood as sesquilinear forms on $D \times D$: e.g: $\alpha\gamma$ is understood as the sesquilinear form: $D \times D \ni (x,y) \mapsto (\alpha^* x | \gamma y) \in \mathbf{C}$. In what follows, the relations (0.1) will be understood as equalities of sesquilinear forms (cf formulae (4.2) – (4.5) of [17]). On the Hilbert space level one may also formulate conditions of non-algebraic nature. For example one often supplements (0.1) by a spectral condition of the form:

$$\operatorname{Sp} \gamma^*\gamma \subset \Lambda, \tag{0.3}$$

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where Λ is a fixed closed subset of $\mathbf{R}$. Such additional spectral condition was considered by many authors (cf e.g: [9]). It is coherent with (0.1) provided Λ is invariant under multiplication by q^2 and q^{-2}. We say that (α, γ) is a $S_qU(1,1)$-pair acting on a Hilbert space H, if α and γ are closed operators acting on H satisfying (0.1) (in the sense explained above) and (0.3).

We shall use the terminology introduced in [18]. By the procedure described in Section 7 of [18], relations (0.1) and (0.3) give rise to a C*-algebra A. This C*-algebra is generated by two unbounded elements α, γ affiliated with it and

$$\pi \longleftrightarrow \Big(\pi(\alpha), \pi(\gamma)\Big) \tag{0.4}$$

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defines continuous one to one correspondence between the set $\operatorname{Rep}(A, H)$ of all representation of A acting on a Hilbert space H and the set of all $S_qU(1,1)$-pairs acting on H.

Assume now, that A is equipped with a comultiplication $\Delta \in \operatorname{Mor}(A, A \otimes A)$ such that (0.2) holds. Then for any $\pi_1 \in \operatorname{Rep}(A, H_1)$ and $\pi_2 \in \operatorname{Rep}(A, H_2)$ one may consider the tensor product:

$$\pi_1 \textcircled{$\top$} \pi_2 = (\pi_1 \otimes \pi_2) \circ \Delta.$$

Clearly $\pi_1 \textcircled{$\top$} \pi_2 \in \operatorname{Rep}(A, H_1 \otimes H_2)$. Using the one to one correspondence (0.4) we may define the tensor product for $S_qU(1,1)$-pairs. If (α_1, γ_1) is a $S_qU(1,1)$-pair acting on a

[1] one can easily check that relations (0.1) cannot be satisfied by bounded operators α and $\gamma \neq 0$.

Hilbert space H_1 and (α_2, γ_2) is a $S_qU(1,1)$-pair acting on a Hilbert space H_2, then by virtue of (0.2):

$$(\alpha_1, \gamma_1) \bigoplus\!\!\!\!\!\!\top\, (\alpha_2, \gamma_2) = (\widetilde{\alpha}, \widetilde{\gamma}),$$

where

$$\begin{aligned} \widetilde{\alpha} &= \alpha_1 \otimes \alpha_2 + q\gamma_1^* \otimes \gamma_2, \\ \widetilde{\gamma} &= \gamma_1 \otimes \alpha_2 + \alpha_1^* \otimes \gamma_2. \end{aligned}$$

One expects that $(\widetilde{\alpha}, \widetilde{\gamma})$ is a $S_qU(1,1)$-pair acting on $H_1 \otimes H_2$. Unfortunately (cf [17, Theorem 4.1] and [8, Theorem 6.1]) this is not the case. It turns out that the domains of $\widetilde{\alpha}$ and $\widetilde{\alpha}^*$ do not coincide. This failure cannot be repaired by extending the operators to larger domains. This negative result shows there is no comultiplication on A being in agreement with (0.2). Quantum $SU(1,1)$-group does not exist on C*-level.

The next important step in this subject was done by Leonid Korogodski in [8]. He discovered that the situation is more hopeful if one replaces $SU(1,1)$ by its two-fold covering $S\tilde{U}(1,1)$. On the classical level

$$S\tilde{U}(1,1) = \left\{ g \in SL(2, \mathbf{C}) : g^* \begin{pmatrix} 1 \,, & 0 \\ 0 \,, & -1 \end{pmatrix} g = \pm \begin{pmatrix} 1 \,, & 0 \\ 0 \,, & -1 \end{pmatrix} \right\}.$$

This group enters into Iwasawa-type decomposition of $SL(2, \mathbf{C})$: any element $g \in SL(2, \mathbf{C})$ such that $|g_{12}| \neq |g_{21}|$ admits unique decomposition of the form

$$g = g' g'',$$

where $g' \in S\tilde{U}(1,1)$ and g'' is an upper triangular matrix with positive elements on the diagonal.

Any element $g \in S\tilde{U}(1,1)$ is of the form

$$g = \begin{pmatrix} \alpha \,, & \epsilon\gamma^* \\ \gamma \,, & \epsilon\alpha^* \end{pmatrix},$$

where $\epsilon = \pm 1$ and α, γ are complex numbers such that

$$\alpha^* \alpha = \epsilon + \gamma^* \gamma.$$

A quantum deformation of $S\tilde{U}(1,1)$ on the Hopf *-algebra level was constructed in [8], where rather exotic terminology of *shadows* was used. We shall describe this deformation using the more standard language. The Hopf *-algebra $\mathcal{A}$ of polynomial functions on $S_q\tilde{U}(1,1)$ is generated by three elements α, γ, ϵ subject to the following set of relations:

$$\begin{aligned} \epsilon^2 &= I, & \epsilon\alpha\epsilon &= \alpha, \\ \epsilon^* &= \epsilon, & \epsilon\gamma\epsilon &= \gamma, \end{aligned} \tag{0.5} \quad \text{a}$$

$$\begin{aligned} \alpha\gamma &= q\gamma\alpha, \\ \alpha\gamma^* &= q\gamma^*\alpha, \\ \gamma\gamma^* &= \gamma^*\gamma. \end{aligned} \qquad \begin{aligned} \alpha^*\alpha &= \epsilon + \gamma^*\gamma, \\ \alpha\alpha^* &= \epsilon + q^2\gamma^*\gamma, \end{aligned} \tag{0.6} \quad \text{b}$$

The comultiplication $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is the *-algebra homomorphism acting on generators in the following way:

$$\begin{array}{rcl} \Delta(\alpha) & = & \alpha \otimes \alpha + q\epsilon\gamma^* \otimes \gamma, \\ \Delta(\gamma) & = & \gamma \otimes \alpha + \epsilon\alpha^* \otimes \gamma, \\ \Delta(\epsilon) & = & \epsilon \otimes \epsilon. \end{array} \tag{0.7}$$

c

One can easily verify that the object $(\mathcal{A}, \Delta)$ described above is a Hopf *-algebra and that the matrix

$$w = \left(\begin{array}{c} \alpha\ ,\ q\epsilon\gamma^* \\ \gamma\ ,\ \epsilon\alpha^* \end{array} \right)$$

is a corepresentation of $(\mathcal{A}, \Delta)$. Clearly w is the fundamental representation of the quantum $S\tilde{U}(1,1)$.

We pass to the Hilbert space level. Now α, γ, ϵ are operators acting on a Hilbert space H and relations (0.6) are understood as equalities of sesquilinear forms. Due to (0.5) the Hilbert space

$$H = H_+ \oplus H_-,$$

where $H_\pm$ are eigenspaces of ϵ corresponding to eigenvalues ± 1. This decomposition is respected by α and γ. We denote by $\gamma_\pm$ the restriction of γ to $H_\pm$. It turns out that $\operatorname{Sp} \gamma_-^* \gamma_-$ is uniquely determined by the commutation relations, so the spectral condition may be imposed on γ_+ only. Korogodski uses the following condition:

$$\operatorname{Sp} \gamma_+^* \gamma_+ \subset c\, q^{2\mathbf{Z}} \cup \{0\}, \tag{0.8}$$

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where c is a fixed positive number.

We say that $(\alpha, \gamma, \epsilon)$ is a $S_q\tilde{U}(1,1)$-triple acting on a Hilbert space H, if α, γ and ϵ are closed operators acting on H satisfying the conditions (0.5), (0.6) and (0.8).

Let $(\alpha, \gamma, \epsilon)$ be a $S_q\tilde{U}(1,1)$-triple acting on a Hilbert space H. We say that $(\alpha, \gamma, \epsilon)$ is bosonic (fermionic respectively) if $\epsilon = I$ ($\epsilon = -I$ respectively). Any $S_q\tilde{U}(1,1)$-triple is a direct sum of bosonic and fermionic triples. Irreducible triples are either bosonic or fermionic. Clearly any bosonic $S_q\tilde{U}(1,1)$-triple is of the form (α, γ, I), where (α, γ) is a $S_qU(1,1)$-pair.

Formulae (0.7) imply the following form of tensor product of $S_q\tilde{U}(1,1)$-triples:

$$(\alpha_1, \gamma_1, \epsilon_1) \textcircled{$\top$} (\alpha_2, \gamma_2, \epsilon_2) = (\widetilde{\alpha}, \widetilde{\gamma}, \widetilde{\epsilon}),$$

where

$$\begin{array}{rcl} \widetilde{\alpha} & = & \alpha_1 \otimes \alpha_2 + q\epsilon_1\gamma_1^* \otimes \gamma_2, \\ \widetilde{\gamma} & = & \gamma_1 \otimes \alpha_2 + \epsilon_1\alpha_1^* \otimes \gamma_2, \\ \widetilde{\epsilon} & = & \epsilon_1 \otimes \epsilon_2. \end{array}$$

We already know that this product is ill defined for bosonic triples. Korogodski has shown, that the same negative result holds[2] when both triples have definite evenness

[2]assuming that γ_1 and γ_2 do not vanish.

(i.e. are bosonic or fermionic). The situation essentially improves when one considers the reducible triples containing bosonic and fermionic components in equal number. We have the following result:

Let $(\alpha_1, \gamma_1, \epsilon_1)$ and $(\alpha_2, \gamma_2, \epsilon_2)$ be $S_q\tilde{U}(1,1)$-triples. Assume that each of the triples is a direct sum of two irreducible components: bosonic and fermionic. Assume furthermore that the bosonic components are infinite-dimensional. Then there exists $S_q\tilde{U}(1,1)$-triple $(\widetilde{\alpha}, \widetilde{\gamma}, \widetilde{\epsilon})$ such that

$$\begin{aligned} \alpha_1 \otimes \alpha_2 + q\epsilon_1\gamma_1^* \otimes \gamma_2 &\subset \widetilde{\alpha} \\ \gamma_1 \otimes \alpha_2 + \epsilon_1\alpha_1^* \otimes \gamma_2 &\subset \widetilde{\gamma} \\ \epsilon_1 \otimes \epsilon_2 &= \widetilde{\epsilon}. \end{aligned} \tag{0.9}$$

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This is the main result of the Korogodski paper [8]. It gives rise to the following problem:

How to modify the definition of $S_q\tilde{U}(1,1)$-triple to make bosonic and fermionic components inseparable? This modification should eliminate purely bosonic and purely fermionic triples for which the tensor product is ill defined. Furthermore at the moment the tensor product is not uniquely defined: there are many $S_q\tilde{U}(1,1)$-triples $(\widetilde{\alpha}, \widetilde{\gamma}, \widetilde{\epsilon})$ satisfying (0.9). *One can show that the present definition of $S_q\tilde{U}(1,1)$-triple gives no natural choice among them. So the modified definition should distinguish in a canonical way one among all $S_q\tilde{U}(1,1)$-triples $(\widetilde{\alpha}, \widetilde{\gamma}, \widetilde{\epsilon})$ satisfying* (0.9).

The present paper contains a solution of this problem. The definition is modified by including a new element denoted by Y, so we shall deal with $S_q\tilde{U}(1,1)$-quadruples $(\alpha, \gamma, \epsilon, Y)$. On the Hopf *-algebra level, Y is an algebraic combination of the first three generators:

$$Y = q^{\frac{1}{2}}\epsilon\gamma^* - \alpha.$$

Performing simple computations one can easily verify that

$$YY^* = qY^*Y + (1-q)\epsilon.$$

Therefore one expects that on the Hilbert space level the domains of Y and Y^* coincide. However this is not the case. Consider the following condition:

$$\left.\begin{array}{l} \textit{There exists a closed operator } Y \textit{ such that} \\ \quad \textit{1. } Y \textit{ is an extension of } q^{\frac{1}{2}}\epsilon\gamma^* - \alpha\textit{:} \\ \qquad\qquad q^{\frac{1}{2}}\epsilon\gamma^* - \alpha \subset Y, \\ \quad \textit{2. Domains } Y \textit{ and } Y^* \textit{ coincide} \\ \quad \textit{3. } Y \textit{ commutes with } \mathrm{Phase}\,\gamma. \end{array}\right\} \tag{C}$$

It turns out that this condition selects precisely these $S_q\tilde{U}(1,1)$-triples $(\alpha, \gamma, \epsilon)$ for which the tensor product is well defined. Moreover the choice of Y provides an additional structure that enables us to select one among many $S_q\tilde{U}(1,1)$-triples $(\widetilde{\alpha}, \widetilde{\gamma}, \widetilde{\epsilon})$ satisfying (0.9).

The paper uses heavily the theory of unbounded operators on Hilbert spaces (cf [1, 6, 7, 10]). We shall mainly use closed operators. The domain of an operator a acting on a Hilbert space H will be denoted by $D(a)$. We shall always assume that $D(a)$ is dense in H.

We shall use the continuous functional calculus for systems of strongly commuting selfadjoint operators. To explain the rather peculiar but very convenient notation used in the paper, let us consider the pair of strongly commuting selfadjoint operators a and b acting on a Hilbert space H. Then, by the spectral theorem

$$a = \int_{\mathbf{R}^2}^{\oplus} \lambda \, dE(\lambda, \mu), \qquad b = \int_{\mathbf{R}^2}^{\oplus} \mu \, dE(\lambda, \mu)$$

where $dE(\lambda)$ is the common spectral measure associated with a, b. For any measurable (complex valued) function f of two variables,

$$f(a, b) = \int_{\mathbf{R}^2}^{\oplus} f(\lambda, \lambda') \, dE(\lambda, \lambda').$$

Let χ be the logical evaluation of a sentence:

$$\begin{aligned} \chi(\text{false}) &= 0, \\ \chi(\text{true}) &= 1. \end{aligned}$$

If $\mathcal{R}$ is a two argument relation defined on real numbers, then $f(\lambda, \lambda') = \chi(\mathcal{R}(\lambda, \lambda'))$ is a characteristic function of the set $\Delta = \{(\lambda, \lambda') \in \mathbf{R}^2 : \mathcal{R}(\lambda, \lambda')\}$ and (assuming that Δ is measurable) $f(a, b) = E(\Delta)$. We shall write $\chi(\mathcal{R}(a, b))$ instead of $f(a, b)$:

$$\chi(\mathcal{R}(a, b)) = \int_{\mathbf{R}^2}^{\oplus} \chi(\mathcal{R}(\lambda, \lambda')) \, dE(\lambda, \lambda') = E(\Delta).$$

The range of this projection will be denoted by $H(\mathcal{R}(a, b))$. The letter 'H' in this expression refers to the Hilbert space, where operators a, b act.

This way we gave meaning to the expressions: $\chi(a > b)$, $\chi(a^2 + b^2 = 1)$, $\chi(a = 1)$, $\chi(b < 0)$, $\chi(a \neq 0)$ and many others of this form. They are orthogonal projections onto corresponding spectral subspaces. For example $H(a = 1)$ is the eigenspace of a corresponding to the eigenvalue 1 and $\chi(a = 1)$ is the orthogonal projection onto this eigenspace. More generally if Δ is a measurable subset of $\mathbf{R}$, then $H(a \in \Delta)$ is the spectral subspace of a corresponding to Δ and $\chi(a \in \Delta)$ is the corresponding spectral projection.

Formally, expressions of the form $\mathcal{R}(a, b)$ are sentences of a non-commutative logic, and χ is the logical evaluation of such sentences. The values of χ are orthogonal projections. We are not going to move any further in this direction.

To fix the notation we recall the basic properties of the polar decomposition of a closed operator. Let T be a closed operator acting on a Hilbert space H. The polar decomposition of T will be written in the form:

$$T = (\mathrm{Phase}\, T)|T|,$$

where $|T| = (T^*T)^{\frac{1}{2}}$ and $\mathrm{Phase}\, T$ is the partial isometry acting on H such that $(\mathrm{Phase}\, T)^*(\mathrm{Phase}\, T)$ and $(\mathrm{Phase}\, T)(\mathrm{Phase}\, T)^*$ are projections onto $H' = (\ker T)^\perp$ and $H'' = (\ker T^*)^\perp$ respectively. Restricting $\mathrm{Phase}\, T$ to the subspace H' we obtain unitary operator

$$\mathrm{phase}\, T : H' \longrightarrow H''.$$

This operator intertwines T^*T restricted to H' with TT^* restricted to H'':

$$(\mathrm{phase}\, T)\, T^*T|_{H'}\, (\mathrm{phase}\, T)^* = TT^*|_{H''}. \tag{0.10}$$ splatacz

Using this fact one can easily show (cf. [6, Problem 69]) that $\mathrm{Sp}\, T^*T$ and $\mathrm{Sp}\, TT^*$ essentially coincide:

$$\mathrm{Sp}\, T^*T \cup \{0\} = \mathrm{Sp}\, TT^* \cup \{0\}. \tag{0.11}$$ Halmos

A closed operator T is called balanced if $D(T^*) = D(T)$. Let T_{min} be a closed operator acting on a Hilbert space H, such that $D(T_{\mathrm{min}}) \subset D(T^*_{\mathrm{min}})$ and $T \supset T_{\mathrm{min}}$ be a closed extension of T_{min}. We say that T is a balanced extension of T_{min} if $D(T) = D(T^*)$. This notion generalizes the concept of selfadjoint extensions of symmetric operators: any balanced extension of a symmetric operator is selfadjoint. This paper is full of balanced operators and balanced extensions. If $(\alpha, \gamma, \epsilon, Y)$ is a $S_q\tilde{U}(1,1)$-quadruple, then α, γ and Y are balanced operators. The inclusions in (0.9) denote balanced extensions.

Let T be a balanced operator. Then $D(T) = D(T^*)$ is a Banach space with respect to each of the two graph norms $|||\cdot|||$ and $|||\cdot|||_*$: $|||x|||^2 = \|Tx\|^2 + \|x\|^2$ and $|||x|||_*^2 = \|T^*x\|^2 + \|x\|^2$. Therefore the norms are equivalent. In particular the dense sets are the same for the two topologies. This way we proved the following

thesamecore

Proposition 0.1 *Let T be a balanced operator and D be a core for T. Then D is a core for T^*.*

It turns out that a balanced extension $T_{\mathrm{min}} \subset T$ is determined, when T_{min} and $D(T)$ are given. We have even more:

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Proposition 0.2 *Let T and T' be balanced extensions of T_{min}. Assume that a core of T is contained in $D(T')$. Then $T = T'$.*

Proof: If T_{min} has a balanced extension, then clearly $D(T_{\mathrm{min}}) \subset D(T^*_{\mathrm{min}})$. Let T_{max} be the adjoint of the restriction of T^*_{min} to $D(T_{\mathrm{min}})$: $T^*_{\mathrm{max}} = T^*_{\mathrm{min}}|_{D(T_{\mathrm{min}})}$. If $T_{\mathrm{min}} \subset T$ is a balanced extension, then $T^* \subset T^*_{\mathrm{min}}$. Restricting the operators to $D(T_{\mathrm{min}})$ we get $T^*|_{D(T_{\mathrm{min}})} = T^*_{\mathrm{min}}|_{D(T_{\mathrm{min}})} = T^*_{\mathrm{max}}$. Therefore $T^* \supset T^*_{\mathrm{max}}$ and $T \subset T_{\mathrm{max}}$. This way we showed that any balanced extension of T_{min} is a restriction of T_{max}.

In particular T and T' are restrictions of the same closed operator. Remembering that a core of T is contained in $D(T')$ we see that $T \subset T'$. Therefore $T^* \supset (T')^*$, $D(T) = D(T^*) \supset D((T')^*) = D(T')$, $D(T) = D(T')$ and $T = T'$.

Q.E.D.

In the paper we shall use a number of simple results of the theory of unbounded operators. For the reader convenience we collect them in Appendix A.

1 Basic definitions and results.

S1

In the present paper we shall use the spectral condition (0.8) with $c = 1$. The other values of c require some technical changes. The notion of a $S_q\tilde{U}(1,1)$-quadruple appeared already in Section 0. Now we give the precise definition:

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Definition 1.1 *Let $\pi = (\alpha, \gamma, \epsilon, Y)$, where α, γ, ϵ and Y are closed operators acting on a Hilbert space H. We say that π is a $S_q\tilde{U}(1,1)$-quadruple on H, if the following six conditions are satisfied:*

1. ϵ is unitary and selfadjoint: $\epsilon = \epsilon^* = \epsilon^{-1}$.

2. ϵ commutes with α and γ: $\epsilon\alpha\epsilon = \alpha$ and $\epsilon\gamma\epsilon = \gamma$.

3. Operators α, γ, α^* and γ^* have the same domain: $D(\alpha) = D(\gamma) = D(\alpha^*) = D(\gamma^*)$ and

$$\begin{aligned}
(\alpha^* x|\gamma y) &= q\,(\gamma^* x|\alpha y)\,, && (1.1) \quad \text{cr1}\\
(\alpha^* x|\gamma^* y) &= q\,(\gamma x|\alpha y)\,, && (1.2) \quad \text{cr2}\\
(\gamma^* x|\gamma^* y) &= (\gamma x|\gamma y)\,, && (1.3) \quad \text{cr3}\\
(\alpha x|\alpha y) &= (x|\epsilon|y) + (\gamma x|\gamma y)\,, && (1.4) \quad \text{cr4}\\
(\alpha^* x|\alpha^* y) &= (x|\epsilon|y) + q^2\,(\gamma x|\gamma y) && (1.5) \quad \text{cr5}
\end{aligned}$$

for any $x, y \in D(\gamma)$.

4. The restriction γ_+ of γ to the subspace $H(\epsilon = 1)$ satisfies the following spectral condition:

$$\mathrm{Sp}\,(|\gamma_+|) \subset q^{\mathbf{Z}} \cup \{0\}. \tag{1.6}$$ SPCON

5. Y is a balanced extension of $q^{\frac{1}{2}}\epsilon\gamma^* - \alpha$:

$$\begin{aligned}
q^{\frac{1}{2}}\epsilon\gamma^* - \alpha &\subset Y,\\
D(Y) &= D(Y^*).
\end{aligned}$$

6. Y commutes with Phase γ.

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Remark 1.2 It follows easily from Condition 3 that the graph topologies associated with operators α, γ, α^* and γ^* coincide. Therefore α, γ, α^* and γ^* have the same cores.

Due to Condition 3, operator γ is normal. $S_q\tilde{U}(1,1)$-quadruples split into two extreme types. We say that a $S_q\tilde{U}(1,1)$-quadruple $\pi = (\alpha, \gamma, \epsilon, Y)$ is of unbounded type if $\ker\gamma = \{0\}$. Conversely π is of bounded type if $\gamma = 0$. Any $S_q\tilde{U}(1,1)$-quadruple $\pi = (\alpha, \gamma, \epsilon, Y)$ acting on H is a direct sum of quadruples of bounded and unbounded type acting on $H(\gamma = 0)$ and $H(\gamma \neq 0)$ respectively; it follows easily from the commutation relations that the direct sum decomposition

$$H = H(\gamma = 0) \oplus H(\gamma \neq 0)$$

is respected by operators $\alpha, \gamma, \epsilon, Y$.

If a $S_q\tilde{U}(1,1)$-quadruple $\pi = (\alpha, \gamma, \epsilon, Y)$ is of bounded type, then α is unitary, $\gamma = 0$, $\epsilon = I$ and $Y = -\alpha$. In other words, a $S_q\tilde{U}(1,1)$-quadruple of bounded type is determined by a single unitary operator. For $S_q\tilde{U}(1,1)$-quadruples of unbounded type the description is more complicated:

Let $H = L^2(\Lambda, m)$, where Λ is the denumerable subset of $\mathbf{R}$ introduced by

$$\Lambda = \{-q^{-n} : n \in \mathbf{N}\} \cup \{q^n : n \in \mathbf{Z}\} \tag{1.7}$$ m1

and m is the measure on Λ such that

$$m(\{\lambda\}) = |\lambda|^{-1} \tag{1.8}$$ m2

for any $\lambda \in \Lambda$. We consider operators $\alpha_0, \gamma_0, \epsilon_0, Y_0$ on H defined by

$$\left.\begin{array}{rcl}
(\alpha_0 x)(\lambda) & = & \lambda \sqrt{q + \dfrac{\operatorname{sign}(\lambda)}{q\lambda^2}}\; x(\lambda q), \\[2ex]
(\gamma_0 x)(\lambda) & = & |\lambda|\; x(\lambda), \\[2ex]
(\epsilon_0 x)(\lambda) & = & \operatorname{sign}(\lambda)\; x(\lambda), \\[2ex]
(Y_0 x)(\lambda) & = & \lambda \left\{ q^{\frac{1}{2}}\; x(\lambda) - \sqrt{q + \dfrac{\operatorname{sign}(\lambda)}{q\lambda^2}}\; x(q\lambda) \right\}.
\end{array}\right\} \tag{1.9}$$ m3

By definition, the domains of α_0 and γ_0 consist of all functions $x \in H$ such that the right hand sides of the first two formulae are square integrable over Λ (belong to H). Operator ϵ is bounded and its domain coincides with H. If the right hand side of the last formula is square integrable over Λ, then there exist two limits $\lim\limits_{\lambda \to \pm\infty} x(\lambda)$. By definition, the domain of Y_0 is the set of all $x \in H$ such that the right hand side of the last formula is square integrable and

$$\lim_{\lambda \to -\infty} x(\lambda) = \lim_{\lambda \to +\infty} x(\lambda) \tag{1.10}$$ m4

Then one can check that

$$\pi_0 = (\alpha_0, \gamma_0, \epsilon_0, Y_0)$$

is a $S_q\tilde{U}(1,1)$-quadruple of unbounded type acting on $L^2(\Lambda, m)$. In this case operators α_0, γ_0, Y_0 are not bounded. More generally if $r \in [0, 2\pi[$, then

$$\pi_r = (e^{-ir}\alpha_0, e^{ir}\gamma_0, \epsilon_0, e^{-ir}Y_0)$$

is also a $S_q\tilde{U}(1,1)$-quadruple of unbounded type

It is not difficult to show that the π_r is irreducible. It is characterized uniquely by the number $e^{ir} = \operatorname{Phase}\gamma \in \mathbf{S}^1$. In Section 3 we shall prove the following irred

Theorem 1.3 *Any $S_q\tilde{U}(1,1)$-quadruple of unbounded type is a direct integral of irreducible $S_q\tilde{U}(1,1)$-quadruples π_r ($r \in [0, 2\pi[$) described above.*

Beside the operator Y, there is another operator X playing in our theory a similar role. Defining the tensor product of two $S_q\tilde{U}(1,1)$-quadruples π_1 and π_2 we shall use operator Y related to π_1 and operator X related to π_2. Let $\pi = (\alpha, \gamma, \epsilon, Y)$ be a $S_q\tilde{U}(1,1)$-quadruple acting on a Hilbert space H. Operator X is a balanced extension of $\epsilon(q^{\frac{1}{2}}\gamma + \alpha)$:

$$\left.\begin{array}{rcl} \epsilon(q^{\frac{1}{2}}\gamma + \alpha) & \subset & X, \\ D(X) & = & D(X^*). \end{array}\right\} \tag{1.11}$$ X

If π is of bounded type, then clearly $X = \alpha$. In the unbounded case the operator X is introduced by the formula:

$$X = (\mathrm{Phase}\,\gamma)^2 V Y V^*, \tag{1.12}$$ XY

where

$$V = \left[-\epsilon(\mathrm{Phase}\,\gamma)^2\right]^{\log|q\gamma|}. \tag{1.13}$$ V

The symbol 'log' denotes the logarithm with base q: $\log q^k = k$. In Section 3 we shall prove that V is unitary and that operator X introduced by (1.12) satisfies the relations (1.11).

Now we are ready to formulate the main results of the paper.

Ilten

Theorem 1.4 *Let $\pi_1 = (\alpha_1, \gamma_1, \epsilon_1, Y_1)$ and $\pi_2 = (\alpha_2, \gamma_2, \epsilon_2, Y_2)$ be $S_q\tilde{U}(1,1)$-quadruples of unbounded type acting on Hilbert spaces H_1 and H_2 respectively and X_2 be the operator related to π_2 in the way described above.*

Then there exists unique $S_q\tilde{U}(1,1)$-quadruple $\widetilde{\pi} = (\widetilde{\alpha}, \widetilde{\gamma}, \widetilde{\epsilon}, \widetilde{Y})$ of unbounded type acting on $H_1 \otimes H_2$ such that:

1.

$$\begin{array}{lcl} \alpha_1 \otimes \alpha_2 + q\epsilon_1{\gamma_1}^* \otimes \gamma_2 & \subset & \widetilde{\alpha} \\ \gamma_1 \otimes \alpha_2 + \epsilon_1{\alpha_1}^* \otimes \gamma_2 & \subset & \widetilde{\gamma} \\ \epsilon_1 \otimes \epsilon_2 & = & \widetilde{\epsilon} \end{array}$$

where the operators on the left hand sides are defined on $D(\gamma_1) \otimes_{\mathrm{alg}} D(\gamma_2)$.

2. The domain $D(\widetilde{\gamma})$ contains all vectors of the form

$$\chi\left(\begin{array}{c} |\gamma_1 \otimes \gamma_2^{-1}| = q^m \\ \epsilon_1 \otimes \epsilon_2 = s \end{array}\right)(\epsilon_1^m x_1 \otimes x_2), \tag{1.14}$$ Dalpha

where $x_1 \in D(\gamma_1 Y_1)$, $x_2 \in D(\gamma_2 X_2)$, $m \in \mathbf{Z}$ and $s = \pm 1$. Moreover the linear span of $D(\gamma_1) \otimes_{\mathrm{alg}} D(\gamma_2)$ and all vectors of the form (1.14) is a core for $\widetilde{\alpha}, \widetilde{\gamma}, \widetilde{\alpha}^*$ and $\widetilde{\gamma}^*$.

3. The domain $D(\widetilde{Y})$ contains all vectors of the form[3]

$$\chi\left(|\widetilde{\gamma}|\,|\gamma_1 \otimes \gamma_2^{-1}| = q^n\right)(x \otimes \epsilon_2^{n-r} y), \tag{1.15}$$ DYtilde

[3]it turns out, that $|\widetilde{\gamma}|$ strongly commutes with $|\gamma_1 \otimes \gamma_2^{-1}|$ so the operator $|\widetilde{\gamma}|\,|\gamma_1 \otimes \gamma_2^{-1}|$ is positive selfadjoint

where $x \in H_1(|\gamma_1| = q^r)$, $y \in D(Y_2)$ and $n, r \in \mathbf{Z}$. Moreover the linear span of $D(\widetilde{\gamma})$ and all vectors of the form (1.15) *is a core for $\widetilde{Y}$ and $\widetilde{Y}^*$.*

Remark: The uniqueness of $\widetilde{\pi}$ follows immediately from Proposition 0.2.

In what follows, the $S_q\tilde{U}(1,1)$-quadruple $\widetilde{\pi}$ introduced in the above theorem will be denoted by $\pi_1 \mathbin{\textcircled{\scriptsize T}} \pi_2$. In order to have our exposition complete we should define the product $\textcircled{\scriptsize T}$ when one of the factors is of bounded type. We set:

$$\begin{aligned} (\alpha_1, 0, I, -\alpha_1) \mathbin{\textcircled{\scriptsize T}} (\alpha_2, 0, I, -\alpha_2) &= (\alpha_1 \otimes \alpha_2, 0, I \otimes I, -\alpha_1 \otimes \alpha_2) \\ (\alpha_1, 0, I, -\alpha_1) \mathbin{\textcircled{\scriptsize T}} (\alpha_2, \gamma_2, \epsilon_2, Y_2) &= (\alpha_1 \otimes \alpha_2, \alpha_1^* \otimes \gamma_2, I \otimes \epsilon_2, \alpha_1 \otimes Y_2) \\ (\alpha_1, \gamma_1, \epsilon_1, Y_1) \mathbin{\textcircled{\scriptsize T}} (\alpha_2, 0, I, -\alpha_2) &= (\alpha_1 \otimes \alpha_2, \gamma_1 \otimes \alpha_2, \epsilon_1 \otimes I, \widetilde{Y}), \end{aligned} \tag{1.16}$$ Ttproduct

where in the last relation

$$\widetilde{Y} = (I \otimes \alpha_2)^{2\log|\gamma_1 \otimes I| + 2}(Y_1 \otimes \alpha_2)(I \otimes \alpha_2)^{-2\log|\gamma_1 \otimes I|}. \tag{1.17}$$ YYY

It is not difficult to show, that the right hand sides of (1.16) are $S_q\tilde{U}(1,1)$-quadruples. To relax the reader disappointed with the complicated form of relation (1.17) we notice that this relation looks much simpler, when expressed in terms of operators X: Using the translation formula (1.12) one can easily show that (1.17) is equivalent to the equation: $\widetilde{X} = X_1 \otimes \alpha_2$.

The '$\textcircled{\scriptsize T}$'- product is associative:

asso

Theorem 1.5 *Let $\pi_1 = (\alpha_1, \gamma_1, \epsilon_1, Y_1)$, $\pi_2 = (\alpha_2, \gamma_2, \epsilon_2, Y_2)$ and $\pi_3 = (\alpha_3, \gamma_3, \epsilon_3, Y_3)$ be $S_q\tilde{U}(1,1)$-quadruples acting on Hilbert spaces H_1, H_2 and H_3 respectively. Then*

$$(\pi_1 \mathbin{\textcircled{\scriptsize T}} \pi_2) \mathbin{\textcircled{\scriptsize T}} \pi_3 = \pi_1 \mathbin{\textcircled{\scriptsize T}} (\pi_2 \mathbin{\textcircled{\scriptsize T}} \pi_3).$$

The proof of these theorems is given in the forthcoming sections.

2 Balanced extensions.

S2

In this Section we shall present the theory of balanced extensions of operators of some special form. Let μ be a real number such that $0 < \mu < 1$ and a be a selfadjoint and v be an isometric operator acting on a Hilbert space H. Throughout this Section we shall assume that

$$H = \sum_{n \in \mathbf{Z}}^{\oplus} H(|a| = \mu^n) \tag{2.1}$$ supro

$$va \supset \mu a v \tag{2.2}$$ rel

$$vv^* \geq \chi(|a| > \mu^{n_0}) \tag{2.3}$$ ess

for some $n_0 \in \mathbf{Z}$. Decomposition (2.1) means, that $|a|$ has a pure point spectrum with eigenvalues of the form μ^n ($n \in \mathbf{Z}$).

Relation (2.2) implies that v (v^* respectively) commutes with $\operatorname{sign} a$ and maps eigenvectors of $|a|$ into eigenvectors of $|a|$ dividing (multiplying respectively) the eigenvalue by μ:

$$v : H(|a| = \mu^n) \longrightarrow H(|a| = \mu^{n-1}) \tag{2.4}$$ v

$$v^* : H(|a| = \mu^{n-1}) \longrightarrow H(|a| = \mu^n) \tag{2.5}$$ vstar

for any $n \in \mathbf{Z}$. In general (2.4) are isometries, however due to (2.3) they are unitaries for any $n \leq n_0$. Clearly, for any $n \leq n_0$, (2.5) is the inverse of (2.4). In what follows we denote by P the orthogonal projection onto the eigenspace $K = H(|a| = \mu^{n_0})$. In this Section ϵ will denote the unitary involution introduced by the formula:

$$\epsilon = \operatorname{sign} a. \tag{2.6}$$ epsilon

One can easily show, that K is ϵ-invariant.

Let $x \in H$. We shall prove that

$$\Big(x \in D(a)\Big) \Longleftrightarrow \left(\sum_{k=0}^{\infty} \|\mu^{-k} P\,(v^*)^k\, x)\|^2 < \infty\right). \tag{2.7}$$ dziedzinaa

Indeed (cf (2.1)) $x = \sum x_n$, where $x_n \in H(|a| = \mu^n)$ and n runs over all integers. Then $P(v^*)^k x = \sum P(v^*)^k x_n$. By virtue of (2.5) the only non-zero term in this sum is the one with $n = n_0 - k$ and $P(v^*)^k x = (v^*)^k x_{n_0-k}$. Remembering that (2.5) are unitary for $n \leq n_0$ we get $\|P(v^*)^k x\| = \|x_{n_0-k}\|$. Now it is easy to see that the right hand side of (2.7) is equivalent to the convergence of the series $\sum \|\mu^n x_n\|^2$, which in turn is equivalent to the left hand side of (2.7).

Using (2.7) one can easily show, that $D(a)$ is v-invariant:

$$\Big(vx \in D(a)\Big) \Longleftrightarrow \Big(x \in D(a)\Big) \Longleftrightarrow \Big(v^* x \in D(a)\Big) \tag{2.8}$$ inv

for any $x \in H$.

Let $p > 0$. Replacing in the above demonstration a by $|a|^p$ (and μ by μ^p) one shows that for any $x \in H$ we have:

$$\Big(x \in D(|a|^p)\Big) \Longleftrightarrow \left(\sum_{k=0}^{\infty} \|\mu^{-pk} P\,(v^*)^k\, x)\|^2 < \infty\right), \tag{2.9}$$ dzinaaa

$$\Big(vx \in D(|a|^p)\Big) \Longleftrightarrow \Big(x \in D(|a|^p)\Big) \Longleftrightarrow \Big(v^* x \in D(|a|^p)\Big). \tag{2.10}$$ inv2

Let f and g be real valued functions defined on $\operatorname{Sp}(a)$. We shall assume that they are bounded on compact subsets of $\operatorname{Sp}(a)$ and that their behaviour at infinity is described by the formulae:

$$\begin{aligned} f(\lambda) &= \lambda + f_0 + f_1(\lambda), \\ g(\lambda) &= \lambda + g_0 + g_1(\lambda), \end{aligned} \tag{2.11}$$ e1

where $f_0, g_0 \in \mathbf{R}$, $\sup |\lambda f_1(\lambda)| < \infty$ and $\sup |\lambda g_1(\lambda)| < \infty$.

We shall consider the following operators:

$$\begin{aligned} T_{\min} &= f(a) - \mu^{\frac{1}{2}} g(a) v, \\ T_{\min}^{+} &= f(a) - \mu^{\frac{1}{2}} v^* g(a). \end{aligned} \tag{2.12}$$

operatory

Originally the operators $T_{\min}$ and $T_{\min}^{+}$ are defined on the domain $D_{\min} = D(a)$: $D(T_{\min}) = D(T_{\min}^{+}) = D_{\min}$. Due to (2.2) and (2.11) we have:

$$\begin{aligned} T_{\min} &= \left(I - \mu^{-\frac{1}{2}} v\right) a + b, \\ T_{\min}^{+} &= \left(I - \mu^{\frac{1}{2}} v^*\right) a + b^*, \end{aligned} \tag{2.13}$$

rozklad

where b is a bounded operator: $b \in B(H)$. Using the obvious estimates:

$$\begin{aligned} \| \left(I - \mu^{-\frac{1}{2}} v\right) x\| &\geq \left(\mu^{-\frac{1}{2}} - 1\right) \|x\| \\ \| \left(I - \mu^{\frac{1}{2}} v^*\right) x\| &\geq \left(1 - \mu^{\frac{1}{2}}\right) \|x\| \end{aligned} \qquad (x \in H),$$

one can easily show that the operators $T_{\min}$ and $T_{\min}^{+}$ are closed.

Clearly $T_{\min}$ and $T_{\min}^{+}$ are formally adjoint to each other:

$$(x|T_{\min} y) = \left(T_{\min}^{+} x \middle| y\right)$$

for all $x, y \in D_{\min}$. It shows that $T_{\min}^{+} \subset T_{\min}^{*}$ and symmetrically $T_{\min} \subset \left(T_{\min}^{+}\right)^*$. We set: $T_{\max} = \left(T_{\min}^{+}\right)^*$ and $T_{\max}^{+} = (T_{\min})^*$. Then we have the following diagram

$$\begin{tikzcd} T_{\min} \arrow[r, phantom, "\subset"] \arrow[d, leftrightarrow, "*"'] & T_{\max} \arrow[d, leftrightarrow, "*"] \\ T_{\max}^{+} \arrow[r, phantom, "\supset"] & T_{\min}^{+} \end{tikzcd}$$

where the vertical arrows denote the passing to the adjoint operators.

Taking into account (2.13) we obtain

$$\begin{aligned} T_{\max} &= a\left(I - \mu^{\frac{1}{2}} v\right) + b, \\ T_{\max}^{+} &= a\left(I - \mu^{-\frac{1}{2}} v^*\right) + b^*. \end{aligned} \tag{2.14}$$

rozkladmax

In particular

$$\begin{aligned} D(T_{\max}) &= \left\{x \in H : \left(I - \mu^{\frac{1}{2}} v\right) x \in D(a)\right\}, \\ D(T_{\max}^{+}) &= \left\{x \in H : \left(I - \mu^{-\frac{1}{2}} v^*\right) x \in D(a)\right\}. \end{aligned} \tag{2.15}$$

dziedzinamax

Using the obvious formula

$$I - \mu^{-\frac{1}{2}} v^* = -\mu^{-\frac{1}{2}} v^* \left(I - \mu^{\frac{1}{2}} v\right) \tag{2.16}$$ obvious

and taking into account (2.8) we see that

$$D(T_{\max}) = D(T^+_{\max}).$$

This common domain of $T_{\max}$ and $T^+_{\max}$ will be denoted by $D_{\max}$.

In what follows we shall consider convergent sequences of vectors of a Hilbert space. In these considerations the rate of convergence will be important. Roughly speaking a sequence $(x_n)_{n\in\mathbf{N}}$ is called ν-converging if it converges faster than $(\nu^{n/2})_{n\in\mathbf{N}}$. Precise definition is the following:

Let ν be a real number such that $0 < \nu < 1$ and $(x_k)_{k\in\mathbf{N}}$ be a sequence of elements of a Hilbert space H. We say that $(x_k)_{k\in\mathbf{N}}$ ν-converges to 0 if

$$\sum_{k=0}^{\infty} \nu^{-k} \|x_k\|^2 < \infty.$$

Consequently $(x_k)_{k\in\mathbf{N}}$ ν-converges to a vector $x_\infty \in H$ if $(x_k - x_\infty)_{k\in\mathbf{N}}$ ν-converges to 0.

One can easily check that a linear combination of ν-convergent sequences is ν-convergent. Let $(x_k)_{k\in\mathbf{N}}$ be a sequence of elements of a Hilbert space H and $(y_k)_{k\in\mathbf{N}}$ be a sequence of elements of a Hilbert space K. If one of the sequences is ν-converging to zero, while the second is bounded, then clearly $(x_k \otimes y_k)_{k\in\mathbf{N}}$ is ν-converging to zero. Now the formula

$$x_k \otimes y_k - x_\infty \otimes y_\infty = (x_k - x_\infty) \otimes y_k + x_\infty \otimes (y_k - y_\infty)$$

shows that the tensor product of ν-convergent sequences is ν-convergent.

The Cauchy criterion for ν-convergent sequences is extremely simple:

Cauchy

Proposition 2.1 *Let $(x_k)_{k\in\mathbf{N}}$ be a sequence of elements of a Hilbert space H. Then*

$$\left(\begin{array}{c} (x_k)_{k\in\mathbf{N}} \textit{ is} \\ \nu\textit{-convergent} \end{array}\right) \Longleftrightarrow \left(\begin{array}{c} (x_{k+1} - x_k)_{k\in\mathbf{N}} \textit{ is} \\ \nu\textit{-converging to } 0 \end{array}\right).$$

Proof: The implication '$\Longrightarrow$' is obvious. Conversely assume that $(x_{k+1} - x_k)_{k\in\mathbf{N}}$ is ν-converging to 0. It means that

$$\sum_{k=0}^{\infty} \nu^{-k} \|x_{k+1} - x_k\|^2 < \infty.$$

Therefore

$$\|x_{k+1} - x_k\| = c_k \nu^{\frac{k}{2}}, \tag{2.17}$$ cka

where $(c_k)_{k\in\mathbf{N}}$ is a square summable sequence. Since $\left(\nu^{\frac{k}{2}}\right)_{k\in\mathbf{N}}$ is also square summable, $\left(c_k\nu^{\frac{k}{2}}\right)_{k\in\mathbf{N}}$ is summable and (2.17) shows that $(x_k)_{k\in\mathbf{N}}$ is a Cauchy sequence. This way we showed the existence of the limit:

$$x_\infty = \lim_{k\to\infty} x_k.$$

To prove the ν-convergence we start with the following formula:

$$x_k - x_\infty = -\sum_{m=k}^{\infty}(x_{m+1}-x_m).$$

Therefore, by virtue of (2.17):

$$\nu^{-\frac{k}{2}}\|x_k - x_\infty\| \le \sum_{m=k}^{\infty} c_m \nu^{\frac{m-k}{2}} = \sum_{m=0}^{\infty} c_{m+k}\nu^{\frac{m}{2}}. \tag{2.18}$$ 58

For each $m=0,1,2,\dots$ the sequence $(c_{k+m})_{k\in\mathbf{N}}$ is square summable with the l^2-norm bounded by a constant independent of m. On the other hand the sequence $\left(\nu^{\frac{m}{2}}\right)_{m\in\mathbf{N}}$ is summable. Therefore the sequence

$$\left(\sum_{m=0}^{\infty} c_{m+k}\nu^{\frac{m}{2}}\right)_{k\in\mathbf{N}}$$

is square summable. By the estimate (2.18), the sequence $\left(\nu^{-\frac{k}{2}}\|x_k - x_\infty\|\right)_{k\in\mathbf{N}}$ is also square summable. It means that $(x_k)_{k\in\mathbf{N}}$ is ν-convergent.

Q.E.D.

The main result of this Section is contained in the following two theorems.

Dmax

Theorem 2.2

1. Let $x\in H$. Then $x\in D_{\max}$ if and only if the sequence $\left(\mu^{-\frac{k}{2}}P\,(v^*)^k\,x\right)_{k\in\mathbf{N}}$ is μ-convergent. If this is the case, then we set

$$\theta(x) = \lim_{k\to\infty}\mu^{-\frac{k}{2}}P\,(v^*)^k\,x. \tag{2.19}$$ theta

2.

$$D_{\min} = \{x\in D_{\max} : \theta(x)=0\}.$$

3.

$$D_{\max} = \left\{\left(I-\mu^{\frac{1}{2}}v\right)^{-1}y + t:\ y\in K,\ t\in D_{\min}\right\}. \tag{2.20}$$ Dprim

main2

Theorem 2.3 *Let T be a balanced extension of $T_{\min}$. Then $T\subset T_{\max}$ and there exists unique unitary involution $u: K\to K$ such that $u\epsilon = -\epsilon u$ and*

$$\begin{aligned} D(T) = &\ \left\{\left(I-\mu^{\frac{1}{2}}v\right)^{-1}y + t:\ y\in K(u=1),\ t\in D(a)\right\} \\ = &\ \left\{x\in H: \begin{array}{c}\textit{The sequence } \left(\mu^{-\frac{k}{2}}P\,(v^*)^k\,x\right)_{k\in\mathbf{N}} \textit{ is} \\ \mu\textit{-converging to a vector } y\in K(u=1)\end{array}\right\} \\ = &\ \left\{x\in D_{\max}:\ \theta(x)\in K(u=1)\right\}. \end{aligned} \tag{2.21}$$ dziedzina

Moreover the set of y appearing in the above formula coincides with $K(u=1)$.

Conversely if $u : K \to K$ is a unitary involution anticommuting with ϵ, then the domain (2.21) is contained in $D_{\max}$ and restricting $T_{\max}$ to (2.21) we obtain a balanced extension of $T_{\min}$.

If u is an unitary involution acting on K, anticommuting with ϵ, then $uK(\epsilon=1) = K(\epsilon=-1)$ and $\dim K(\epsilon=1) = \dim K(\epsilon=-1)$. This must be the case if $T_{\min}$ admits a balanced extension.

The unitary involution u appearing in (2.21) is called the linking operator associated with the balanced extension T.

To prove the above theorems we shall use the following Proposition:

qq

Proposition 2.4 *Let $p > \frac{1}{2}$, $x \in H$ and $y \in K$. Then*

$$\left(\begin{array}{c} \textit{Sequence } \left(\mu^{-\frac{k}{2}} P\,(v^*)^k\,x\right)_{k\in\mathbf{N}} \\ \textit{is } \mu^{2p-1}\textit{-converging to } y \end{array} \right) \Longleftrightarrow \left(\begin{array}{c} \textit{Vector } x \textit{ is of the form} \\ x = \left(I - \mu^{\frac{1}{2}} v\right)^{-1} y + t \\ \textit{where } t \in D(|a|^p) \end{array} \right). \tag{2.22}$$ q

Proof: Using the geometric series expansion

$$\left(I - \mu^{\frac{1}{2}} v\right)^{-1} y = \sum_{m=0}^{\infty} \left(\mu^{\frac{1}{2}} v\right)^m y$$

we get

$$(v^*)^k \left(I - \mu^{\frac{1}{2}} v\right)^{-1} y = \sum_{m=0}^{k-1} \mu^{\frac{m}{2}} (v^*)^{k-m} y + \sum_{m=k}^{\infty} \mu^{\frac{m}{2}} v^{m-k} y.$$

Applying to the both sides the projection P we see, that only the term with $m = k$ survives on the right hand side. Therefore

$$\mu^{-\frac{k}{2}} P\,(v^*)^k \left(I - \mu^{\frac{1}{2}} v\right)^{-1} y = y. \tag{2.23}$$ staly

Now the '$\Leftarrow$' part of (2.22) follows immediately from (2.9). Conversely assume that the left hand side of (2.22) holds. Put $t = x - \left(I - \mu^{\frac{1}{2}} v\right)^{-1} y$. Then by virtue of (2.23), the sequence $\left(\mu^{-\frac{k}{2}} P\,(v^*)^k\,t\right)_{k\in\mathbf{N}}$ is μ^{2p-1}-converging to 0 and using again (2.9) we see that $t \in D(|a|^p)$. The right hand side of (2.22) is proved.

Q.E.D.

Proof of the Theorem 2.2**:** Inserting $p = 1$ and $y = 0$ in Proposition 2.4 we see, that

$$\left(x \in D_{\min}\right) \Longleftrightarrow \left(\begin{array}{c} \text{The sequence } \left(\mu^{-\frac{k}{2}} P\,(v^*)^k\,x\right)_{k\in\mathbf{N}} \\ \text{is } \mu\text{-converging to } 0 \end{array} \right). \tag{2.24}$$ nnumer

We shall prove Statement 2. According to (2.15),

$$\left(x \in D_{\max}\right) \Longleftrightarrow \left(\left(I - \mu^{\frac{1}{2}} v\right) x \in D_{\min}\right).$$

By (2.24), the latter holds if and only if the sequence: $\left(\mu^{-\frac{k}{2}} P\left(v^{*}\right)^{k}\left(I-\mu^{\frac{1}{2}} v\right) x\right)_{k \in \mathbf{N}}$ is μ-converging to 0. A trivial computation shows that

$$\mu^{-\frac{k}{2}} P\left(v^{*}\right)^{k}\left(I-\mu^{\frac{1}{2}} v\right) x=\mu^{-\frac{k}{2}} P\left(v^{*}\right)^{k} x-\mu^{-\frac{k-1}{2}} P\left(v^{*}\right)^{k-1} x \tag{2.25}$$

trivial

and Proposition 2.1 shows that $\left(\mu^{-\frac{k}{2}} P\left(v^{*}\right)^{k}\left(I-\mu^{\frac{1}{2}} v\right) x\right)_{k \in \mathbf{N}}$ μ-converges to 0 if and only if the sequence $\left(\mu^{-\frac{k}{2}} P\left(v^{*}\right)^{k} x\right)_{k \in \mathbf{N}}$ is μ-convergent. Statement 1 is proven.

Now Statement 3 follows immediately from Proposition 2.4 and Statement 1 from (2.24).

Q.E.D.

For any $x, x' \in D_{\max}$ we set:

$$\Psi(x, x')=\left(T_{\max}^{+} x \middle| x'\right)-\left(x \middle| T_{\max} x'\right). \tag{2.26}$$

Psi

Clearly Ψ is a sesquilinear form on $D_{\max}$. It vanishes, when one of the arguments belongs to $D_{\min}$. If T is an operator such that $T_{\min} \subset T \subset T_{\max}$, then $T_{\min}^{+} \subset T^{*} \subset T_{\max}^{+}$ and for any $x \in D(T^{*})$ and $x' \in D(T)$ we have: $\Psi(x, x')=(T^{*} x | x')-(x | T x')=0$. A moment of reflection shows that

$$D(T^{*})=\left\{x \in D_{\max}: \begin{array}{c}\Psi(x, x')=0 \text { for } \\ \text { any } x' \in D(T)\end{array}\right\}. \tag{2.27}$$

DTstar

To use this statement we have to compute $\Psi(x, x')$ for arbitrary elements $x, x' \in D_{\max}$. According to Theorem 2.2

$$x=\left(I-\mu^{\frac{1}{2}} v\right)^{-1} y+t,$$

$$x'=\left(I-\mu^{\frac{1}{2}} v\right)^{-1} y'+t',$$

where $y, y' \in K$ and $t, t' \in D_{\min}$. We know, that $\Psi(x, x')=0$ when one of the arguments belongs to $D_{\min}$. Thus t and t' may be neglected in further computations. We shall use formulae (2.14). It is obvious that the operator b gives no contribution to (2.26). Therefore we may replace in this formula $T_{\max}$ and $T_{\max}^{+}$ by $a\left(I-\mu^{\frac{1}{2}} v\right)$ and $a\left(I-\mu^{-\frac{1}{2}} v^{*}\right)$ respectively. Now, taking into account (2.16) we have:

$$\Psi(x, x')=-\mu^{-\frac{1}{2}}\left(a v^{*} y \middle| \left(I-\mu^{\frac{1}{2}} v\right)^{-1} y'\right)-\left(\left(I-\mu^{\frac{1}{2}} v\right)^{-1} y \middle| a y'\right).$$

By (2.5), $\mu^{-\frac{1}{2}} a v^{*} y=\mu^{\frac{1}{2}} v^{*} a y$. To proceed with computations we use the geometric series expansion:

$$\Psi(x, x')=-\sum_{m=1}^{\infty}\left(a y \middle| \left(\mu^{\frac{1}{2}} v\right)^{m} y'\right)-\sum_{m=0}^{\infty}\left(\left(\mu^{\frac{1}{2}} v\right)^{m} y \middle| a y'\right).$$

We know, that y and y' are eigenvectors of $|a|$ with the same eigenvalue μ^{n_0}. Remembering that v divides the eigenvalue by μ we see that the only non-vanishing term is the one with $m = 0$:

$$\Psi(x, x') = -\left(y|ay'\right).$$

Remembering, that on K, $a = \mu^{n_0}\epsilon$ we obtain:

$$\Psi(x, x') = -\mu^{n_0}\left(y|\epsilon y'\right). \tag{2.28}$$ Psi1

Now we are able to prove our main theorem.

Proof of Theorem 2.3: Let T be balanced extension of $T_{\rm min}$. Then $T^* \subset T^*_{\rm min} = T^+_{\rm max}$. Remembering that $T^+_{\rm min}$ is a restriction of $T^+_{\rm max}$ and that $D(T^+_{\rm min}) = D(T_{\rm min}) \subset D(T) = D(T^*)$ we see that $T^+_{\rm min} \subset T^*$. Therefore $T \subset \left(T^+_{\rm min}\right)^* = T_{\rm max}$ and $D(T) \subset D(T_{\rm max})$. Using Statement 3 of Theorem 2.2 one can easily show that

$$D(T) = \left\{\left(I - \mu^{\frac{1}{2}}v\right)^{-1} y + t : \, y \in L, \, t \in D_{\rm min}\right\}, \tag{2.29}$$ Dziedzin

where L is a linear subset of K. Let u be the unitary involution acting on K such that $K(u = 1) = L$ and $K(u = -1) = L^\perp$ ($L^\perp$ denotes the orthogonal complement of L in K). Inserting in (2.27), $D(T^*) = D(T)$ and using (2.28) we see that

$$L = \left\{y \in K : (y|\epsilon y') = 0 \text{ for any } y' \in L\right\}. \tag{2.30}$$ L

It means that $L^\perp = \epsilon L$. Therefore u anticommutes with ϵ. Now (2.29) coincides with the first row of (2.21). The second equality in (2.21) follows immediately from Proposition 2.4 and the third one from Statement 1 of Theorem 2.2.

Conversely, let T be the restriction of $T_{\rm max}$ to the domain (2.21) determined by a unitary involution u acting on K. If u anticommutes with ϵ, then the eigenspace $L = K(u = 1)$ satisfies the relation $L^\perp = \epsilon L$. Therefore (2.30) holds and formula (2.27) shows that $D(T^*) = D(T)$.

Q.E.D.

Let $x \in D_{\rm max}$. Then (cf Statement 3 of Theorem 2.2 and Proposition 2.4)

$$x = \left(I - \mu^{\frac{1}{2}}v\right)^{-1} y + t,$$

where $y = \theta(x) \in K$ and $t \in D(a)$. By virtue of (2.14) and (2.16):

$$x = \left(I - \mu^{\frac{1}{2}}v\right)^{-1} y + t,$$

where $y = \theta(x) \in K$ and $t \in D(a)$. By virtue of (2.14) and (2.16):

$$T_{\rm max}(x) = \left\{ \begin{array}{c} ay + a\left(I - \mu^{\frac{1}{2}}v\right)t \\ + b\left(I - \mu^{\frac{1}{2}}v\right)^{-1} y + bt, \end{array} \right. \tag{2.31}$$ wynik

Clearly the right hand side depends continuously on $y \in K$ and $t \in D(a)$ (when $D(a)$ is equipped with the graph topology of operator a). Combining this observation with Theorem 2.3 we obtain core

Proposition 2.5 *Let T be a balanced extension of $T_{\min}$, u be the corresponding linking operator and D' be a linear subset of $D(T)$ containing a core of the operator a. Assume that the set $\{\theta(x) : x \in D'\}$ is dense in $K(u = 1)$. Then D' is a core for T.*

In particular we have:

core1

Proposition 2.6 *Let T be a balanced extension of $T_{\min}$ and $u : K \to K$ be the linking operator associated with T. For any $p \geq 1$ we set*

$$D_p = \left\{ x \in H : \begin{array}{c} \textit{The sequence } \left(\mu^{-\frac{k}{2}} P\,(v^*)^k\, x\right)_{k\in\mathbf{N}} \textit{ is} \\ \mu^p\textit{-converging to a vector } y \in K(u = 1) \end{array} \right\}. \tag{2.32}$$ Dp

Then D_p is a core for T. Moreover the set of y appearing in the above formula coincides with $K(u = 1)$. The reader should notice that $D_1 = D(T)$.

Proof: Let $p' = \frac{1}{2}(p + 1) \geq 1$. Then $p = 2p' + 1$ and by virtue of Proposition 2.4

$$D_p = \left\{ \left(I - \mu^{\frac{1}{2}} v\right)^{-1} y + t : \, y \in K(u = 1), \, t \in D(|a|^{p'}) \right\}. \tag{2.33}$$ Dpp

We know that $D(|a|^{p'})$ is a core for a. The above formula shows, that $D_p \supset D(|a|^{p'})$. If $x = \left(I - \mu^{\frac{1}{2}} v\right)^{-1} y + t$ then $y = \theta(x)$. Therefore $\{\theta(x) : x \in D_p\} = K(u = 1)$. Now, Proposition 2.5 shows, that D_p is a core for T.

Q.E.D.

For $p = 3$, the domain D_p has the following remarkable property:

DaT1

Proposition 2.7 *Let T be a balanced extension of $T_{\min}$ and $u : K \to K$ be the linking operator associated with T. Then the operators T^* and T map D_3 into $D(T)$. In other words: T^*T^*x, T^*Tx, TT^*x and TTx are well defined for all $x \in D_3$.*

Proof: Assume for the moment that the constants f_0 and g_0 entering the asymptotic formulae (2.11) do vanish: $f_0 = g_0 = 0$. Then the operators b and b^* appearing in (2.13) map H into $D_{\min}$. Let $x \in H$. Remembering that $bH \subset D(a)$ and using (2.14) we see that $x \in D(aT)$ if and only if $x \in D(T)$ and $a\left(I - \mu^{\frac{1}{2}} v\right) x \in D(a)$. The latter is equivalent to $\left(I - \mu^{\frac{1}{2}} v\right) x \in D(a^2)$. Similarly $x \in D(aT^*)$ if and only if $x \in D(T^*) = D(T)$ and (cf (2.14)) $a\left(I - \mu^{-\frac{1}{2}} v^*\right) x \in D(a)$. By (2.16) and (2.10), the latter is again equivalent to $\left(I - \mu^{\frac{1}{2}} v\right) x \in D(a^2)$. Let $x' = \left(I - \mu^{\frac{1}{2}} v\right) x$. According to Proposition 2.4:

$$\left(x' \in D(a^2) \right) \Longleftrightarrow \left(\begin{array}{c} \text{The sequence } \left(\mu^{-\frac{k}{2}} P\,(v^*)^k\, x'\right)_{k\in\mathbf{N}} \\ \text{is } \mu^3\text{-converging to } 0 \end{array} \right).$$

Using now Proposition 2.1 and formula (2.25) we see that the right hand side of the above equivalence is equivalent to the μ^3-convergence of the sequence $\left(\mu^{-\frac{k}{2}} P\,(v^*)^k\, x\right)_{k\in\mathbf{N}}$.

Since $x \in D(T)$, the limit must belong to $K(u=1)$ (cf (2.21)). This way we showed that

$$\Big(x \in D(aT)\Big) \Longleftrightarrow \Big(x \in D(aT^*)\Big) \Longleftrightarrow \Big(x \in D_3\Big).$$

In particular Tx and T^*x belong to $D(a) = D_{\min}$ for any $x \in D_3$.

In general case we set: $T_1 = T - f_0 I - \mu^{\frac{1}{2}} g_0 v$. Then

$$T = T_1 + f_0 I + \mu^{\frac{1}{2}} g_0 v. \tag{2.34}$$ adhoc

Let $x \in D_3$. By the first part of the proof $T_1 x$ and $T_1^* x$ belong to $D_{\min}$. One can easily show, that D_3 is v and v^* invariant. Therefore $(f_0 I + \mu^{\frac{1}{2}} g_0 v)x$ and $(f_0 I + \mu^{\frac{1}{2}} g_0 v)^* x$ belong to D_3. Formula (2.34) shows now, that Tx and T^*x belong to $D_{\min} + D_3 \subset D(T)$.

Q.E.D.

Notice that we have also proved the following:

DaT

Proposition 2.8 *Let T be a balanced extension of $T_{\min}$ and $u : K \to K$ be the linking operator associated with T. Assume that the constants f_0 and g_0 appearing in* (2.11) *vanish: $f_0 = g_0 = 0$. Then $D(aT^*) = D(aT)$ coincide with the domain D_3 introduced by* (2.32).

Let $x \in D(T_{\max})$. Then $\left(I - \mu^{\frac{1}{2}} v\right) x \in D(a)$. We claim that

$$\theta(x) = P\left[x + \mu^{-n_0+\frac{1}{2}} v^* \left(I - \mu^{\frac{1}{2}} v^*\right)^{-1} |a| \left(I - \mu^{\frac{1}{2}} v\right) x\right]. \tag{2.35}$$ theta1

Indeed denoting by RHS the Right Hand Side of the above relation and using the geometric power series expansion we get:

$$\mathrm{RHS} = Px + \sum_{l=1}^{\infty} \mu^{-n_0+\frac{l}{2}} P\left(v^*\right)^l |a| \left(I - \mu^{\frac{1}{2}} v\right) x.$$

$P\left(v^*\right)^l$ kills all the eigenspaces $H(|a| = \mu^n)$ of $|a|$ except the one with $n = n_0 - l$. Therefore $P\left(v^*\right)^l |a| = \mu^{n_0 - l} P\left(v^*\right)^l$ and

$$\begin{aligned} \mathrm{RHS} &= \sum_{l=1}^{\infty} \mu^{-\frac{l}{2}} P\left(v^*\right)^l \left(I - \mu^{\frac{1}{2}} v\right) x + Px \\ &= \sum_{l=1}^{\infty} \left[\mu^{-\frac{l}{2}} P\left(v^*\right)^l x - \mu^{-\frac{l-1}{2}} P\left(v^*\right)^{l-1} x\right] + Px \end{aligned}$$

Computing the partial sum from $l = 1$ to $l = k$ we see that all terms except $\mu^{-\frac{k}{2}} P\left(v^*\right)^k x$ cancel. Therefore

$$\mathrm{RHS} = \lim_{k \to \infty} \mu^{-\frac{k}{2}} P\left(v^*\right)^k x = \theta(x).$$

Formula (2.35) is shown. The reader should notice that the above computations give the alternative proof of the existence of the limit (2.19). Formula (2.35) shows that $\theta(x)$ depends continuously on $|a| \left(I - \mu^{\frac{1}{2}} v\right) x$. Taking into account (2.14) we get:

Proposition 2.9 *The mapping:* thetaciagle

$$D(T_{\max}) \ni x \longmapsto \theta(x) \in K$$

is continuous provided the source space is equipped with the graph topology of operator $T_{\max}$.

Let T be a balanced extension of $T_{\min}$ and $u : K \to K$ be the linking operator associated with this extension. Iterating (2.4) l - times we obtain a unitary operator $v^l : K = H(|a| = \mu^{n_0}) \to H\left(|a| = \mu^{n_0-l}\right)$. Therefore $v^l u (v^*)^l$ is a unitary involution acting on $H\left(|a| = \mu^{n_0-l}\right)$. Summing up we obtain a unitary involution

$$R = \sum_{l\geq 0}^{\oplus} v^l u (v^*)^l \tag{2.36}$$ R

acting on $H(|a| \geq \mu^{n_0}) = \sum_{l\geq 0}^{\oplus} H\left(|a| = \mu^{n_0-l}\right)$. We extend this operator to the whole space H putting $Rx = 0$ for all $x \in H(|a| < \mu^{n_0})$. Then

$$\left.\begin{aligned} R &= R^*, \\ R^2 &= \chi(|a| \geq \mu^{n_0}), \\ R\epsilon &= -\epsilon R, \\ R|a| &\subset |a| R, \\ vH(R = \pm 1) &\subset H(R = \pm 1). \end{aligned}\right\} \tag{2.37}$$ refl

To prove the third relation it is sufficient to use anticommutativity of u and ϵ, the forth relation means that R respects the direct sum decomposition (2.1) and the last relation follows from the formula: $H(R = \pm 1) = \sum_{l\geq 0}^{\oplus} v^l K(u = \pm 1)$, which in turn follows immediately from the definition (2.36).

The operator R is called the reflection operator associated with the balanced extension $T_{\min} \subset T$. One can easily show, that any operator R satisfying (2.37) is the reflection operator of a balanced extension of $T_{\min}$. The corresponding linking operator u is the restriction of R to $H(|a| = \mu^{n_0})$.

Let $x \in H$, $n \leq n_0$ and $l = n_0 - n$. Then $\chi(|a| = \mu^n) = v^l P (v^*)^l$. Using this equality one can easily verify that

$$\left\{\begin{array}{c} \text{Sequence } \left(\mu^{-\frac{k}{2}} P (v^*)^k x\right)_{k\in\mathbf{N}} \\ \text{is converging to a vector} \\ y \in K(u = 1) \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} \text{Sequence } \left(\mu^{-\frac{k}{2}} \chi(|a| = \mu^n) (v^*)^k x\right)_{k\in\mathbf{N}} \\ \text{is converging to a vector} \\ y' \in H(|a| = \mu^n, R = 1) \end{array}\right\}$$

The rate of convergence of the both sequences is the same. Relation between y and y' is given by the formula: $y' = \mu^{\frac{l}{2}} v^l y$.

The reader should notice that (2.3) is all the more satisfied when we replace n_0 by a smaller integer. This leads to the following:

uwaga

Remark 2.10 In the above considerations, n_0 may by replaced by any integer $n \leq n_0$. Then P, K and u should be replaced by $\chi\,(|a| = \mu^n)$, $H\,(|a| = \mu^n)$ and (as the above equivalence shows) the reflection operator R restricted to $H(|a| = \mu^n)$.

In many cases it is important to know, what are the kernels of balanced extensions considered in this Section. We shall prove:

jadro

Theorem 2.11 *Let T be a balanced extension of the operator $T_{\min}$ introduced by* (2.12) *and $x \in \ker T$. Then for any $n \in \mathbf{Z}$ and $s = \pm 1$ we have:*

$$|f(s\mu^n)|\,\|\chi(a = s\mu^n)x\| = \mu^{\frac{1}{2}}\,|g(s\mu^n)|\,\|\chi(a = s\mu^{n+1})x\|. \tag{2.38}$$ jadro1

Moreover

$$\lim_{n\to-\infty} \mu^{\frac{n}{2}}\|\chi(a = -\mu^n)x\| = \lim_{n\to-\infty} \mu^{\frac{n}{2}}\|\chi(a = \mu^n)x\| \tag{2.39}$$ jadro2

Proof: Let $x \in D(T_{\max})$. Using (2.14) and (2.4) one can easily show that

$$\begin{aligned} \chi(a = s\mu^n)T_{\max}x &= f(s\mu^n)\chi(a = s\mu^n)x - \mu^{\frac{1}{2}}g(s\mu^n)\chi(a = s\mu^n)vx \\ &= f(s\mu^n)\chi(a = s\mu^n)x - \mu^{\frac{1}{2}}g(s\mu^n)v\chi(a = s\mu^{n+1})x \end{aligned}$$

We know that $T_{\max}$ is an extension of T. Therefore $x \in \ker T$ imply $x \in D(T_{\max})$ and $T_{\max}x = 0$. Now using the above formula and remembering that v is an isometry we obtain (2.38).

3 $S_q\tilde{U}(1,1)$-quadruples.

S3

In this Section we shall examine in more detail the commutation relation entering the Definition 1.1. One of the aims is to prove Theorem 1.3. We start with the following

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Proposition 3.1 *Let α, γ and ϵ be operators acting on a Hilbert space H satisfying the first three conditions of Definition 1.1. Assume moreover that* $\ker\gamma = \{0\}$. *Then operators α and γ are not bounded and*

1.

$$\alpha\gamma = q\gamma\alpha \tag{3.1}$$ cr11

$$\alpha\gamma^* = q\gamma^*\alpha \tag{3.2}$$ cr12

$$\gamma\gamma^* = \gamma^*\gamma \tag{3.3}$$ cr13

$$\alpha^*\alpha = \epsilon + \gamma^*\gamma \tag{3.4}$$ cr14

$$\alpha\alpha^* = \epsilon + q^2\gamma^*\gamma \tag{3.5}$$ cr15

2. Phase γ *is unitary and commutes with* Phase α, *γ and ϵ.*

3. If $H(\epsilon = -1) \neq \{0\}$, then the restriction γ_- of γ to this subspace satisfies the following spectral condition:

$$\mathrm{Sp}\,(|\gamma_-|) = q^{-\mathbf{N}}.$$

4. If $H(\epsilon = 1) \neq \{0\}$*, then the restriction* γ_+ *of* γ *to this subspace have the following property:*

$$q\,\mathrm{Sp}\,(|\gamma_+|) = \mathrm{Sp}\,(|\gamma_+|).$$

5. $\mathrm{Phase}\,\alpha$ *is an isometry:*

$$\begin{aligned}(\mathrm{Phase}\,\alpha)^*\mathrm{Phase}\,\alpha &= I\\ \mathrm{Phase}\,\alpha(\mathrm{Phase}\,\alpha)^* &= \chi\left(\epsilon|\gamma| \neq -q^{-1}\right)\end{aligned} \tag{3.6}$$ izometria

6. For any measurable subset $\Delta \subset \mathbf{R}_+$*, the isometry* $\mathrm{Phase}\,\alpha$ *maps* $H(|\gamma| \in \Delta)$ *into* $H(|q\gamma| \in \Delta)$*. If* $1 \notin \Delta$ *then the mapping*

$$\mathrm{Phase}\,\alpha : H(|\gamma| \in \Delta) \longrightarrow H(|q\gamma| \in \Delta)$$

is unitary.

7. We have:

$$\begin{aligned}\alpha &= \quad \mathrm{Phase}\,\alpha \quad \sqrt{\epsilon + |\gamma|^2}\\ \alpha^* &= (\mathrm{Phase}\,\alpha)^* \sqrt{\epsilon + q^2|\gamma|^2}\end{aligned}$$

Proof: We start with Statement 1. Assume that $x \in D(\gamma^*\gamma)$. Then using (1.5) we see that $(\alpha^*y|\alpha^*x) = (y|\epsilon + q^2\gamma^*\gamma|x)$ for any $y \in D(\alpha^*)$. Therefore $\alpha^*x \in D(\alpha)$ and $\alpha\alpha^*x = (\epsilon + q^2\gamma^*\gamma)x$. It shows that $\alpha\alpha^* \supset \epsilon + q^2\gamma^*\gamma$.

Conversely if $x \in D(\alpha\alpha^*)$, then by virtue of (1.5) $(y|\alpha\alpha^* - \epsilon|x) = q^2\,(\gamma y|\gamma y)$ for any $y \in D(\gamma)$. Therefore $\gamma x \in D(\gamma^*)$ and $\alpha\alpha^* - \epsilon \subset q^2\gamma^*\gamma$ and (3.5) follows. In the similar way, using relations (1.1) – (1.4) we obtain (3.1) – (3.4). Statement 1 is proved.

Now Statement 7 is obvious: The formulae coincide with the polar decomposition of α and α^*.

Due to Condition 2 of Definition 1.1, operators α and γ respect the decomposition $H = H(\epsilon = 1) \oplus H(\epsilon = -1)$. Therefore in the proof it is sufficient to consider two cases: $\epsilon = I$ and $\epsilon = -I$. We shall use this possibility proving Statements 3, 4, 5 and 6.

Assume at first that $\epsilon = I$. Then we shall write α_+ and γ_+ instead of α and γ. Using (1.4) and (1.5) we see that $\alpha_+^*\alpha_+ \geq I$ and $\alpha_+\alpha_+^* \geq I$. Therefore $\ker\alpha_+ = \ker\alpha_+^* = \{0\}$ and $\mathrm{Phase}\,\alpha_+ = \mathrm{phase}\,\alpha_+$ is unitary. Statement 5 is proved. (notice that in the considered case $\epsilon|\gamma_+| = |\gamma_+| \geq 0$, so the right hand side of (3.6) equals to I).

According to (0.10)

$$(\mathrm{Phase}\,\alpha_+)\alpha_+^*\alpha_+(\mathrm{Phase}\,\alpha_+)^* = \alpha_+\alpha_+^*.$$

Remembering that $\epsilon = I$ and using (3.4) and (3.5) we have:

$$(\mathrm{Phase}\,\alpha_+)\gamma_+^*\gamma_+(\mathrm{Phase}\,\alpha_+)^* = q^2\gamma_+^*\gamma_+.$$

Computing the square root of the both sides we get

$$(\mathrm{Phase}\,\alpha_+)|\gamma_+|(\mathrm{Phase}\,\alpha_+)^* = q|\gamma_+|. \tag{3.7}$$ splatacz12

It shows that $\mathrm{Sp}\,|\gamma_+| = q\,\mathrm{Sp}\,|\gamma_+|$ and Statement 4 follows. Remembering that the functional calculus is covariant with respect to unitary transformations we get:

$$(\mathrm{Phase}\,\alpha_+)\chi(|\gamma_+| \in \Delta)(\mathrm{Phase}\,\alpha_+)^* = \chi(|q\gamma_+| \in \Delta).$$

It shows that $x \in H(|\gamma_+| \in \Delta)$ if and only if $(\mathrm{Phase}\,\alpha_+)x \in H(|q\gamma_+| \in \Delta)$.

Assume now, that $\epsilon = -I$. Then we shall write α_- and γ_- instead of α and γ. According to (3.4) and (3.5),

$$\begin{aligned} I + \alpha_-^*\alpha_- &= \gamma_-^*\gamma_-, \\ I + \alpha_-\alpha_-^* &= q^2\gamma_-^*\gamma_-. \end{aligned} \tag{3.8}$$ dod

Therefore $I + \alpha_-\alpha_-^* = q^2(I + \alpha_-^*\alpha_-)$ and $q^2\alpha_-^*\alpha_- = \alpha_-\alpha_-^* + (1-q^2)I \geq (1-q^2)I$. It shows that $\ker\alpha_- = \{0\}$ and the first formula of (3.6) follows. The second formula of (3.8) shows that $\ker\alpha_-^* = H\left(q^2\gamma_-^*\gamma_- = 1\right) = H\left(\epsilon|\gamma| = -q^{-1}\right)$. The proof of Statement 5 is complete.

The second relation (3.8) shows that

$$q\,\mathrm{Sp}\,(|\gamma_-|) \subset [1, \infty]. \tag{3.9}$$ dod1

Let $H'' = H(|\gamma_-| = q^{-1})^\perp$. Then by virtue of (0.10)

$$(\mathrm{phase}\,\alpha_-)\alpha_-^*\alpha_-(\mathrm{phase}\,\alpha_-)^* = \alpha_-\alpha_-^*\big|_{H''}$$

and using (3.8) we obtain:

$$(\mathrm{phase}\,\alpha_-)\gamma_-^*\gamma_-(\mathrm{phase}\,\alpha_-)^* = q^2\gamma_-^*\gamma_-\big|_{H''}.$$

Computing the square root of the both sides we get

$$(\mathrm{phase}\,\alpha_-)|\gamma_-|(\mathrm{phase}\,\alpha_-)^* = q\,|\gamma_-''|, \tag{3.10}$$ splatacz1

where γ_-'' is the operator γ_- restricted to H''. It shows that $\mathrm{Sp}\,|\gamma_-| = q\,\mathrm{Sp}\,|\gamma_-''|$. Assume for the moment that $q^{-1} \notin \mathrm{Sp}\,|\gamma_-|$. Then $H'' = H$, $\gamma_-'' = \gamma_-$ and $\mathrm{Sp}\,|\gamma_-|$ would be invariant under multiplication by q in contradiction with (3.9). Therefore $q^{-1} \in \mathrm{Sp}\,|\gamma_-|$.

Remembering that the orthogonal complement of H'' is the eigenspace of $|\gamma_-|$ corresponding to the eigenvalue q^{-1} we get:

$$q\,\mathrm{Sp}\,|\gamma_-| = \mathrm{Sp}\,|\gamma_-| \cup \{1\}.$$

Iterating this formula n-times we obtain:

$$q^n\,\mathrm{Sp}\,|\gamma_-| = \mathrm{Sp}\,|\gamma_-| \cup \{1, q, q^2, \ldots, q^{n-1}\}.$$

Now using (3.9) we see that

$$\left(q^n\,\mathrm{Sp}\,|\gamma_-|\right) \cap [0,1] = \{1, q, q^2, \ldots, q^{n-1}\}$$

and Statement 3 follows.

To prove Statement 6 we rewrite (3.10) in the following form:

$$q^{-1}(\mathrm{phase}\,\alpha_-)|\gamma_-| = |\gamma''_-|(\mathrm{phase}\,\alpha_-).$$

The only difference between phase α_- and Phase α_- lies in their target spaces: H'' for phase α_- and H for Phase α_-. Therefore

$$q^{-1}(\mathrm{Phase}\,\alpha_-)|\gamma_-| = |\gamma_-|(\mathrm{Phase}\,\alpha_-). \tag{3.11}$$ splatacz11

It shows that $x \in H(|\gamma_-| = q^n)$ if and only if $(\mathrm{Phase}\,\alpha_-)x \in H(|\gamma_-| = q^{n-1})$. If $n \neq 0$, then the second formula (3.6) shows that $H(|\gamma_-| = q^{n-1})$ is contained in the range of Phase α_- and Statement 6 follows.

We still have to prove Statement 2. (3.3) says that γ is a normal operator. Remembering that $\ker\gamma = \{0\}$ we see that Phase γ is unitary and commutes with $|\gamma|$. By virtue of Condition 2 of Definition 1.1, Phase γ commutes with ϵ.

Replacing in (3.1), operators α and γ by their polar decomposition we obtain:

$$(\mathrm{Phase}\,\alpha)\left(\epsilon + |\gamma|^2\right)^{\frac{1}{2}}(\mathrm{Phase}\,\gamma)|\gamma| = q(\mathrm{Phase}\,\gamma)|\gamma|(\mathrm{Phase}\,\alpha)\left(\epsilon + |\gamma|^2\right)^{\frac{1}{2}}.$$

We know that Phase γ commutes with $|\gamma|$. Moreover, due to (3.7) and (3.11), the product $q|\gamma|(\mathrm{Phase}\,\alpha)$ may be replaced by $(\mathrm{Phase}\,\alpha)|\gamma|$. Therefore

$$(\mathrm{Phase}\,\alpha)(\mathrm{Phase}\,\gamma)\left(\epsilon + |\gamma|^2\right)^{\frac{1}{2}}|\gamma| = (\mathrm{Phase}\,\gamma)(\mathrm{Phase}\,\alpha)\left(\epsilon + |\gamma|^2\right)^{\frac{1}{2}}|\gamma|$$

and

$$(\mathrm{Phase}\,\alpha)(\mathrm{Phase}\,\gamma)x = (\mathrm{Phase}\,\gamma)(\mathrm{Phase}\,\alpha)x \tag{3.12}$$ komu

for any $x \in \mathrm{Range}\,(\epsilon + |\gamma|^2)^{\frac{1}{2}}|\gamma|$. One can easily show that the closure of the range of $(\epsilon + |\gamma|^2)^{\frac{1}{2}}|\gamma|$ coincides with $H(\epsilon|\gamma| \neq -1)$. By Statement 3, -1 is not an eigenvalue of $\epsilon|\gamma|$. Therefore $H(\epsilon|\gamma| \neq -1) = H$ and (3.12) holds for any $x \in H$. Statement 2 is proved.

Q.E.D.

ograniczone

Proposition 3.2 *Let α, γ and ϵ be closed operators satisfying the three first conditions of Definition 1.1. Then the operators $|\alpha|-|\gamma|$, $|\alpha^*|-q|\gamma|$, $(|\alpha|-|\gamma|)|\gamma|$ and $(|\alpha^*|-q|\gamma|)|\gamma|$ are bounded. The norms of the first three operators are ≤ 1, whereas the norm of the last one is $\leq q^{-1}$.*

Proof: According to Conditions 1 and 2 of Definition 1.1, ϵ is a unitary and selfadjoint operator commuting with $|\gamma|$. Therefore the operator

$$a = \epsilon|\gamma| \tag{3.13}$$ aS3

is selfadjoint, $|a| = |\gamma|$ and $\mathrm{sign}\,a = \epsilon$. By Statement 3 Proposition 3.1 $\mathrm{Sp}\,a \subset -q^{\mathbf{N}} \cup \mathbf{R}_+$. Using Statement 7 of Proposition 3.1 one can easily verify, that $\epsilon|\alpha|$ and $\epsilon|\alpha^*|$ are functions of a:

$$\epsilon|\alpha| = h(a), \qquad\qquad q^{-1}\epsilon|\alpha^*| = g(a),$$

where

$$h(\lambda) = \lambda\left(1+\frac{1}{\lambda|\lambda|}\right)^{\frac{1}{2}}, \quad g(\lambda) = \lambda\left(1+\frac{1}{q^2\lambda|\lambda|}\right)^{\frac{1}{2}} \tag{3.14}$$ deffun

for any $\lambda \in \mathrm{Sp}\,(a)$. Elementary analysis shows that:

$$|h(\lambda)-\lambda| \le 1, \qquad |g(\lambda)-\lambda| \le q^{-1},$$

$$|\lambda|\,|h(\lambda)-\lambda| \le 1, \qquad |\lambda|\,|g(\lambda)-\lambda| \le q^{-2}$$

for any $\lambda \in -q^{\mathbf{N}} \cup \mathbf{R}_+$. Remembering that, for any function f on $\mathrm{Sp}\,a$, $\|f(a)\| \le \sup\left\{|f(\lambda)| : \lambda \in \mathrm{Sp}\,a\right\}$ we obtain the desired estimates.

Q.E.D.

Let $\pi = (\alpha, \gamma, \epsilon, Y)$ be a $S_q\tilde{U}(1,1)$-quadruple of unbounded type acting on a Hilbert space H. By Statement 2 of Proposition 3.1 and Condition 6 of Definition 1.1, the operator $\mathrm{Phase}\,\gamma$ is central: it commutes with α, γ, ϵ and Y. Combining Condition 4 of Definition 1.1 with Statement 3 of Proposition 3.1 we see that $\mathrm{Sp}\,|\gamma| \subset q^{\mathbf{Z}}$. Let a be the selfadjoint operator introduced by (3.13) and

$$v_{_Y} = \epsilon(\mathrm{Phase}\,\alpha)(\mathrm{Phase}\,\gamma), \tag{3.15}$$ vy

$$T_{\min} = q^{-\frac{1}{2}}(\mathrm{Phase}\,\gamma)\left[q^{\frac{1}{2}}\epsilon\gamma^* - \alpha\right], \tag{3.16}$$ TminY1

$$T = q^{-\frac{1}{2}}(\mathrm{Phase}\,\gamma)Y.$$

An elementary computation shows that

$$\begin{aligned} T_{\min} &= \epsilon|\gamma| - q^{\frac{1}{2}}\left(q^{-1}\epsilon|\alpha^*|\right)v_{_Y} \\ &= a - q^{\frac{1}{2}}g(a)v_{_Y}, \end{aligned} \tag{3.17}$$ TminY

where g is the function introduced by (3.14).

According to Conditions 5 and 6 of Definition 1.1, Y is a balanced extension of $q^{\frac{1}{2}}(\mathrm{Phase}\,\gamma)^* T_{\min}$ commuting with $\mathrm{Phase}\,\gamma$. Therefore T is a balanced extension of the operator $T_{\min}$.

Using the results of Proposition 3.1 one can easily show, that the operators a and $v = v_{_Y}$ introduced above satisfy the basic assumptions (2.1) – (2.3) of Section 2 (with μ replaced by q and n_0 replaced by -1). A simple analysis shows that, for $\lambda \to \infty$, the functions $f(\lambda) = \lambda$ and $g(\lambda)$ introduced by (3.14) exhibit the asymptotic behaviour (2.11) with constants $f_0 = g_0 = 0$. Summing up we see that the theory of balanced extensions developed in Section 2 may be applied to (3.17).

relkom1

Proposition 3.3 *Let $\pi = (\alpha, \gamma, \epsilon, Y)$ be a $S_q\tilde{U}(1,1)$-quadruple of unbounded type acting on a Hilbert space H and $v_{_Y}$ be the operator related to π by (3.15). We set: $P = \chi\left(|\gamma| = q^{-1}\right)$ and $K = H\left(|\gamma| = q^{-1}\right)$. Then there exists unitary involution:*

$$u : K \longrightarrow K$$

commuting with Phase γ *and anticommuting with* ϵ*, such that*

$$D(Y) = \left\{ x \in H : \quad \begin{matrix} \textit{The sequence } \left(q^{-\frac{k}{2}} P \left(v_{_Y}^*\right)^k x\right)_{k \in \mathbf{N}} \textit{ is} \\ q\textit{-converging to a vector } y \in K(u=1) \end{matrix} \right\}. \tag{3.18}$$ DY

Conversely, let α, γ *and* ϵ *be operators acting on a Hilbert space* H *satisfying the first four conditions of Definition 1.1. Assume that* $\ker\gamma = \{0\}$*. Then for any unitary involution* u *acting on* K*, commuting with* Phase γ *and anticommuting with* ϵ*, there exists unique closed operator* Y *with the domain given by* (3.18) *such that* $\pi = (\alpha, \gamma, \epsilon, Y)$ *is a* $S_q\tilde{U}(1,1)$*-quadruple.*

Remark: The operator u is called the linking operator associated with π. Its existence implies that $\dim K(\epsilon = -1) = \dim K(\epsilon = 1)$. It shows that the $S_q\tilde{U}(1,1)$-quadruple $(\alpha, \gamma, \epsilon, Y)$ of unbounded type must contain non-trivial bosonic and fermionic components.

Proof: We shall use Theorem 2.3. Let u be the linking operator associated with the balanced extension $T_{\min} \subset T$. Then $D(T) = D(Y)$ and (3.18) coincides with the second row of (2.21). Since Y commutes with Phase γ, $D(Y)$ must be (Phase γ)-invariant. Therefore $K(u = 1)$ is (Phase γ)-invariant and u commutes with (Phase γ).

Q.E.D.

Let R_Y be the reflection operator associated with the balanced extension $T_{\min} \subset T$:

$$R_Y = \sum_{l=0}^{\infty}{}^{\oplus} v_{_Y}^l u (v_{_Y}^*)^l, \tag{3.19}$$ RY

where u is the linking operator introduced above. We say that R_Y is the Y-reflection operator associated with π. Modifying Proposition 2.8 according to Remark 2.10 , we get in the present context:

DgammaY

Proposition 3.4 *Let* $\pi = (\alpha, \gamma, \epsilon, Y)$ *be a* $S_q\tilde{U}(1,1)$*-quadruple of unbounded type on a Hilbert space* H*,* $v_{_Y} = \epsilon(\text{Phase}\,\alpha)(\text{Phase}\,\gamma)$*,* R_Y *be the* Y*-reflection operator associated with* π *and* n *be an integer negative number. Then*

$$D(\gamma Y) = \left\{ x \in H : \quad \begin{matrix} \textit{The sequence } \left(q^{-\frac{k}{2}} \chi\left(|\gamma| = q^n\right) \left(v_{_Y}^*\right)^k x\right)_{k \in \mathbf{N}} \\ \textit{is } q^3\textit{-converging to a vector } y \in H\left(R_Y = 1\right) \end{matrix} \right\}. \tag{3.20}$$ dzinaaY

The set of y *appearing in the above formula coincides with* $H\left(|\gamma| = q^n, R_Y = 1\right)$*.*

We shall need the analogous result for the operator X. At first we shall prove:

gammaX

Proposition 3.5 *Let* $\pi = (\alpha, \gamma, \epsilon, Y)$ *be a* $S_q\tilde{U}(1,1)$*-quadruple of unbounded type acting on a Hilbert space* H *and* X *be the operator related to* π *via formula* (1.12)*. Then* X *is a balanced extension of* $\epsilon(q^{\frac{1}{2}}\gamma + \alpha)$*.*

Proof: Let V be the operator introduced by (1.13). Clearly V commutes with γ and ϵ. Formula (1.13) means that

$$Vx = \left(-\epsilon(\mathrm{Phase}\,\gamma)^2\right)^{k+1} x \tag{3.21}$$ vvv

for any $x \in H(|\gamma| = q^k)$. Unitarity of $\mathrm{Phase}\,\gamma$ implies the unitarity of V. Remembering that $\mathrm{Phase}\,\alpha$ divides the eigenvalues of $|\gamma|$ by q (cf Statement 6 of Theorem 3.1), we obtain:

$$V(\mathrm{Phase}\,\alpha) = -\epsilon(\mathrm{Phase}\,\gamma)^{-2}(\mathrm{Phase}\,\alpha)V. \tag{3.22}$$ vv

Unitary transform of a balanced operator is balanced. Hence VYV^* is balanced and remembering that $\mathrm{Phase}\,\gamma$ is central one can easily show, that $X = (\mathrm{Phase}\,\gamma)^2 VYV^*$ is balanced. To end the proof, we have to show, that X is an extension of $\epsilon(q^{\frac{1}{2}}\gamma + \alpha)$. By virtue of (3.22), $V\alpha = -\epsilon(\mathrm{Phase}\,\gamma)^{-2}\alpha V$. Therefore $(\mathrm{Phase}\,\gamma)^2 V\gamma^* V^* = \gamma$, $(\mathrm{Phase}\,\gamma)^2 V\alpha V^* = -\epsilon\alpha$ and (1.11) follows.

Q.E.D.

Let $\pi = (\alpha, \gamma, \epsilon, Y)$ be a $S_q\tilde{U}(1,1)$-quadruple of unbounded type acting on a Hilbert space H, u be the linking operator associated with π and V be the unitary introduced by (1.13). We set: $v_X = V v_Y V^*$ and $R_X = V R_Y V^*$. Then

$$v_X = -(\mathrm{Phase}\,\alpha)(\mathrm{Phase}\,\gamma)^*, \tag{3.23}$$ vX

$$R_X = \sum_{l=0}^{\infty}{}^{\oplus} v_X^l u (v_X^*)^l. \tag{3.24}$$ RX

$$H(R_X = 1) = VH(R_Y = 1) \tag{3.25}$$ RX1

Indeed, by virtue of (3.22) we have:

$$Vv_Y = V\epsilon(\mathrm{Phase}\,\alpha)(\mathrm{Phase}\,\gamma) = -(\mathrm{Phase}\,\alpha)(\mathrm{Phase}\,\gamma)^{-1}V$$

and (3.23) follows. Inserting in (3.21) $k = -1$ we see that $Vy = y$ for any $y \in K$. Therefore $VuV^* = u$ and (3.24) follows. (3.25) is obvious. We say that R_X is the X-reflection operator associated with π.

DgammaX

Proposition 3.6 *Let $\pi = (\alpha, \gamma, \epsilon, Y)$ be a $S_q\tilde{U}(1,1)$-quadruple of unbounded type on a Hilbert space H, X be the operator related to π by (1.12), v_X and R_X be the operators introduced above and let n be an integer negative number. Then*

$$D(\gamma X) = \left\{ x \in H : \begin{array}{c} \textit{The sequence } \left(q^{-\frac{k}{2}}\chi\left(|\gamma| = q^n\right)\left(v_X^*\right)^k x\right)_{k\in\mathbf{N}} \\ \textit{is } q^3\textit{-converging to a vector } y \in H\left(R_X = 1\right) \end{array} \right\}. \tag{3.26}$$ dzinaaX

The set of y appearing in the above formula coincides with $H\left(|\gamma| = q^n, R_X = 1\right)$.

Proof: Let $x \in H$. According to (1.12), $x \in D(X)$ iff $V^*x \in D(Y)$. Therefore $x \in D(\gamma X)$ iff $V^*x \in D(Y)$ and $YV^*x \in D(\gamma V)$. One can easily show, that $D(\gamma V) = D(\gamma)$. It proves that $x \in D(\gamma X)$ iff $V^*x \in D(\gamma Y)$. By the definition of v_X we get:

$$V\left(q^{-\frac{k}{2}}\chi\left(|\gamma| = q^n\right)\left(v_Y^*\right)^k V^*x\right) = q^{-\frac{k}{2}}\chi\left(|\gamma| = q^n\right)\left(v_X^*\right)^k x.$$

Taking into account (3.25), we see that (3.26) and (3.20) are equivalent. The last statement follows immediately from the corresponding statement of Proposition 3.4.

Q.E.D.

In what follows we shall need another formula relating R_X with R_Y. We know that R_Y commutes with γ and anticommutes with ϵ (cf (2.37)). Therefore $R_Y V = (-1)^{\log|q\gamma|} V R_Y$ and

$$R_X = (-1)^{\log|q\gamma|} R_Y. \tag{3.27}$$ RX2

Clearly the reflection operators R_Y with R_X satisfy the relations (2.37). More precisely we have:

$$\left.\begin{aligned} R_Y &= R_Y^*, & R_X &= R_X^*, \\ R_Y^2 &= \chi\left(|\gamma| \geq q^{-1}\right), & R_X^2 &= \chi\left(|\gamma| \geq q^{-1}\right), \\ R_Y\epsilon &= -\epsilon R_Y, & R_X\epsilon &= -\epsilon R_X, \\ R_Y|\gamma| &\subset |\gamma| R_Y, & R_X|\gamma| &\subset |\gamma| R_X, \\ v_{_Y} H(R_Y = \pm 1) &\subset H(R_Y = \pm 1), & v_{_X} H(R_X = \pm 1) &\subset H(R_X = \pm 1). \end{aligned}\right\} \tag{3.28}$$ reflYX

The following theorem reveals the structure of $S_q\tilde{U}(1,1)$-quadruples of unbounded type.

postac

Theorem 3.7 *Let $(\alpha_0, \gamma_0, \epsilon_0, Y_0)$ be the $S_q\tilde{U}(1,1)$-quadruple introduced by (1.9). Then for any $S_q\tilde{U}(1,1)$-quadruple π of unbounded type, there exists a unitary operator U acting on a Hilbert space K_+ such that π is unitarily equivalent to the quadruple*

$$\pi_{_U} = (U^* \otimes \alpha_0,\, U \otimes \gamma_0,\, I \otimes \epsilon_0,\, U^* \otimes Y_0).$$

Remark: Any decomposition of $K_+ = K'_+ \oplus K''_+$ respected by the operator U leads to the decomposition of $\pi_{_U}$ into direct sum of two $S_q\tilde{U}(1,1)$-quadruples $\pi_{_{U'}}$ and $\pi_{_{U''}}$, where U' and U'' are restrictions of U to K'_+ and K''_+ respectively. Therefore if π is irreducible, then $\dim K_+ = 1$, $U = e^{ir}I$ (where $r \in [0, 2\pi[$) and π is unitarily equivalent to the $S_q\tilde{U}(1,1)$-quadruple π_r introduced in Section 1. In the general case, U can be decomposed into a direct integral of one-dimensional operators. Therefore π is a direct integral of $S_q\tilde{U}(1,1)$-quadruples of the form π_r. Theorem 1.3 is a corollary of the present Theorem.

Proof: We shall use the notation introduced in Proposition 3.3. Let $a = \epsilon|\gamma|$. The orthogonal projections onto $K_\pm = H(a = \pm q^{-1})$ will be denoted by $P_\pm = \chi(a = \pm q^{-1})$. Moreover (Λ, m) will be the measure space introduced by (1.7) and (1.8). Using the Remark following Proposition 3.3 and Statements 3 and 4 of Proposition 3.1 we see that:

$$\operatorname{Sp} a = (-q^{-\mathbf{N}}) \cup q^{\mathbf{Z}} \cup \{0\} = \Lambda \cup \{0\}.$$

It means that a has pure point spectrum. Remembering that $\ker a = \ker\gamma = \{0\}$ we get

$$H = \sum_{\lambda \in \Lambda}^{\oplus} H(a = \lambda). \tag{3.29}$$ suma

The eigenspaces of a are all isomorphic. Indeed, Statement 6 of Theorem 3.1 shows that

$$\left.\begin{array}{rcll} \left(v_{_Y}^*\right)^{-k-1} & : & H(a=q^k) \longrightarrow K_+ & \text{for} \quad k=-1,-2,-3,\dots \\ \left(v_{_Y}\right)^{k+1} & : & H(a=q^k) \longrightarrow K_+ & \text{for} \quad k=0,1,2,3,\dots \\ u\left(v_{_Y}^*\right)^{-k-1} & : & H(a=-q^k) \longrightarrow K_+ & \text{for} \quad k=-1,-2,-3,\dots \end{array}\right\} \tag{3.30}$$

unitarne

are unitary mappings. Replacing on the right hand side of (3.29) all eigenspaces of a by K_+ we obtain the Hilbert space

$$\sum_{\lambda\in\Lambda}^{\oplus} K_+ = L^2(\Lambda, K_+) = K_+ \otimes L^2(\Lambda, m).$$

The direct sum of unitaries (3.30) defines a unitary operator

$$W : H \longrightarrow L^2(\Lambda, K_+).$$

If $x \in H$, then the corresponding element $\hat{x} = Wx \in L^2(\Lambda, K_+)$ is given by the formula:

$$\hat{x}(sq^k) = \begin{cases} q^{\frac{k}{2}} P_+ \left(v_{_Y}^*\right)^{-k-1} x & \text{for} \quad s=1 \quad \text{and} \quad k=-1,-2,-3,\dots \\ q^{\frac{k}{2}} P_+ \left(v_{_Y}\right)^{k+1} x & \text{for} \quad s=1 \quad \text{and} \quad k=0,1,2,3,\dots \\ q^{\frac{k}{2}} u^* P_- \left(v_{_Y}^*\right)^{-k-1} x & \text{for} \quad s=-1 \quad \text{and} \quad k=-1,-2,-3,\dots \end{cases} \tag{3.31}$$

Fourier

The appearance of powers of q on the right hand side of the above formula follows from (1.8): $q^k = m(\{sq^{-k}\})$. The inverse map is given by the formula

$$x = \sum_{k=-\infty}^{-1} q^{-\frac{k}{2}}\, v_{_Y}^{-k-1}\, \hat{x}(q^k) + \sum_{k=0}^{\infty} q^{-\frac{k}{2}} \left(v_{_Y}^*\right)^{k+1} \hat{x}(q^k) + \sum_{k=-\infty}^{-1} q^{-\frac{k}{2}}\, v_{_Y}^{-k-1}\, u\, \hat{x}(-q^k).$$

Transformation W diagonalizes the operator a:

$$(WaW^*\hat{x})(\lambda) = \lambda\hat{x}(\lambda).$$

Remembering that $|\gamma| = |a|$ and $\epsilon = \operatorname{sign}(a)$ we obtain:

$$\begin{aligned} (W|\gamma|W^*\hat{x})(\lambda) &= |\lambda|\hat{x}(\lambda), \\ (W\epsilon\, W^*\hat{x})(\lambda) &= \operatorname{sign}(\lambda)\hat{x}(\lambda). \end{aligned} \tag{3.32}$$

Wepsilon

Let U denote the restriction of $\operatorname{Phase}\gamma$ to K_+. Remembering that $\operatorname{Phase}\gamma$ commutes with $v_{_Y}$ and u we obtain:

$$(W \operatorname{Phase}\gamma\, W^*\hat{x})(\lambda) = U\hat{x}(\lambda).$$

Finally, using (3.31), one can easily verify, that

$$(W\, v_{Y}\, W^{*}\hat{x})(\lambda) = q^{-\frac{1}{2}}\hat{x}(q\lambda).$$

By definition $v_{Y} = \epsilon(\mathrm{Phase}\,\alpha)(\mathrm{Phase}\,\gamma)$. Therefore $\mathrm{Phase}\,\alpha = \epsilon(\mathrm{Phase}\,\gamma)^{*}v_{Y}$ and

$$(W\,\mathrm{Phase}\,\alpha\, W^{*}\hat{x})(\lambda) = q^{-\frac{1}{2}}\mathrm{sign}\,(\lambda)U^{*}\hat{x}(q\lambda). \tag{3.33}$$ WPhasea

Taking into account the polar decomposition of α given by Statement 7 of Theorem 3.1 and using the above formulae we get:

$$(W\,\gamma\, W^{*}\hat{x})(\lambda) = |\lambda|\, U\,\hat{x}(\lambda), \tag{3.34}$$ Wgamma

$$(W\,\alpha\, W^{*}\hat{x})(\lambda) = \lambda\,\sqrt{q + \frac{\mathrm{sign}\,(\lambda)}{q\lambda^{2}}}\; U^{*}\,\hat{x}(q\lambda). \tag{3.35}$$ Walpha

Comparing (3.35), (3.34) and (3.32) with (1.9) we see that $W\,\alpha\,W^{*} = U^{*}\otimes\alpha_{0}$, $W\,\gamma\,W^{*} = U\otimes\gamma_{0}$ and $W\,\epsilon\,W^{*} = I\otimes\epsilon_{0}$. To end the proof we have to show, that $W\,Y\,W^{*} = U^{*}\otimes Y_{0}$. Since both sides of the last relation are balanced extensions of $W(q^{\frac{1}{2}}\gamma^{*}-\alpha)W^{*}$, it is sufficient to prove the equality of domains.

Let $k\in\mathbf{N}$. By virtue of (3.31),

$$q^{-\frac{k}{2}}P\left(v_{Y}^{*}\right)^{k}x = q^{\frac{1}{2}}\left(\hat{x}(q^{-k-1}) + u\,\hat{x}(-q^{-k-1})\right).$$

Therefore (cf (3.18)) $x\in D(Y)$ if and only if the sequences $\left(\hat{x}(\pm q^{-k-1})\right)_{k\in\mathbf{N}}$ are q-convergent and $\lim\left(\hat{x}(q^{-k-1}) + u\,\hat{x}(-q^{-k-1})\right)\in K(u=1)$. The latter is equivalent to

$$\lim_{k\to\infty}\hat{x}(q^{-k-1}) = \lim_{k\to\infty}\hat{x}(-q^{-k-1}).$$

Comparing the last condition with (1.10) we get the Statement.

Q.E.D.

4 Tensor product I.

S4

In this section we shall construct the first three elements of the tensor product of two $S_{q}\tilde{U}(1,1)$-quadruples. Let $\pi_{1} = (\alpha_{1},\gamma_{1},\epsilon_{1},Y_{1})$ and $\pi_{2} = (\alpha_{2},\gamma_{2},\epsilon_{2},Y_{2})$ be $S_{q}\tilde{U}(1,1)$-quadruples acting on Hilbert spaces H_{1} and H_{2} respectively. We shall assume that π_{1} and π_{2} are of unbounded type. Let $\widetilde{H} = H_{1}\otimes H_{2}$. At first we have to find the suitable balanced extensions of operators:

$$\begin{aligned} \widetilde{\alpha}_{\min} &= \alpha_{1}\otimes\alpha_{2} + q\epsilon_{1}{\gamma_{1}}^{*}\otimes\gamma_{2}, \\ \widetilde{\gamma}_{\min} &= \gamma_{1}\otimes\alpha_{2} + \;\epsilon_{1}{\alpha_{1}}^{*}\otimes\gamma_{2}. \end{aligned} \tag{4.1}$$ minimal

To simplify the notation, we shall omit indexes writing α, γ and ϵ instead of α_{k}, γ_{l} and ϵ_{m}, where $k,l,m = 1,2$. The values of k,l and m will always be clear from the context.

For example, remembering that the operator $\epsilon\alpha^* \otimes \gamma$ acts on $\widetilde{H} = H_1 \otimes H_2$ we conclude that ϵ, α and γ stay for ϵ_1, α_1 and γ_2.

Let

$$\begin{aligned} \widetilde{\alpha}^+_{\min} &= \alpha^* \otimes \alpha^* + q\epsilon\gamma \otimes \gamma^*, \\ \widetilde{\gamma}^+_{\min} &= \gamma^* \otimes \alpha^* + \ \epsilon\alpha \otimes \gamma^*. \end{aligned} \tag{4.2}$$

minimal+

The operators (4.1) and (4.2) are originally defined on $D(\gamma) \otimes_{\text{alg}} D(\gamma)$. One can easily verify, that $\widetilde{\alpha}^+_{\min} \subset \widetilde{\alpha}^*_{\min}$ and $\widetilde{\gamma}^+_{\min} \subset \widetilde{\gamma}^*_{\min}$. Therefore $\widetilde{\alpha}_{\min}$, $\widetilde{\gamma}_{\min}$, $\widetilde{\alpha}^+_{\min}$ and $\widetilde{\gamma}^+_{\min}$ are closeable. Their closures will be denoted by the same symbols. By definition, $D(\gamma) \otimes_{\text{alg}} D(\gamma)$ is a core for these operators. We know (cf (3.29)) that the set Λ introduced by (1.7) coincides with $\mathrm{Sp}\,(\epsilon|\gamma|)$. Let $\lambda_1, \lambda_2 \in \Lambda$. One can easily verify that $\widetilde{\alpha}_{\min}$, $\widetilde{\gamma}_{\min}$, $\widetilde{\alpha}^+_{\min}$ and $\widetilde{\gamma}^+_{\min}$ are bounded on $H_1(\epsilon|\gamma| = \lambda_1) \otimes_{\text{alg}} H_2(\epsilon|\gamma| = \lambda_2)$. Therefore $H_1(\epsilon|\gamma| = \lambda_1) \otimes H_2(\epsilon|\gamma| = \lambda_2)$ is contained in the domains of these operators.

relkom2

Proposition 4.1 *Let $\widetilde{D}_{\text{comp}}$ be the linear span of all eigenspaces*

$$\widetilde{H}(\lambda_1, \lambda_2) = H_1(\epsilon|\gamma| = \lambda_1) \otimes H_2(\epsilon|\gamma| = \lambda_2), \tag{4.3}$$

h1h2

where $\lambda_1, \lambda_2 \in \Lambda$. Then

1. *$\widetilde{D}_{\text{comp}}$ is an invariant core for $\widetilde{\alpha}_{\min}$, $\widetilde{\gamma}_{\min}$, $\widetilde{\alpha}^+_{\min}$ and $\widetilde{\gamma}^+_{\min}$.*

2. *For any $x \in \widetilde{D}_{\text{comp}}$ we have:*

$$\widetilde{\alpha}_{\min}\widetilde{\gamma}_{\min}x = q\widetilde{\gamma}_{\min}\widetilde{\alpha}_{\min}x, \tag{4.4}$$

cr21

$$\widetilde{\alpha}_{\min}\widetilde{\gamma}^+_{\min}x = q\widetilde{\gamma}^+_{\min}\widetilde{\alpha}_{\min}x, \tag{4.5}$$

cr22

$$\widetilde{\gamma}_{\min}\widetilde{\gamma}^+_{\min}x = \widetilde{\gamma}^+_{\min}\widetilde{\gamma}_{\min}x, \tag{4.6}$$

cr23

$$\widetilde{\alpha}^+_{\min}\widetilde{\alpha}_{\min}x = \widetilde{\epsilon}x + \widetilde{\gamma}^+_{\min}\widetilde{\gamma}_{\min}x, \tag{4.7}$$

cr24

$$\widetilde{\alpha}_{\min}\widetilde{\alpha}^+_{\min}x = \widetilde{\epsilon}x + q^2\widetilde{\gamma}^+_{\min}\widetilde{\gamma}_{\min}x, \tag{4.8}$$

cr25

where $\widetilde{\epsilon} = \epsilon \otimes \epsilon$.

3. *$D(\widetilde{\alpha}_{\min}) = D(\widetilde{\gamma}_{\min}) = D(\widetilde{\alpha}^+_{\min}) = D(\widetilde{\gamma}^+_{\min})$. In what follows, this common domain will be denoted by $\widetilde{D}_{\min}$.*

4. *Each of the range $\widetilde{\gamma}_{\min}(\widetilde{D}_{\text{comp}})$ and $\widetilde{\gamma}^*_{\min}(\widetilde{D}_{\text{comp}})$ is dense in $\widetilde{H}$.*

Proof: In each of the spaces H_i $(i = 1, 2)$, we have the dense linear subset D^i_{comp} spanned by all eigenspaces $H_i(\epsilon|\gamma| = \lambda_i)$, where $\lambda_i \in \Lambda$. Clearly D^i_{comp} is a core for α_i, γ_i, α_i^* and γ_i^*. Therefore the set $\widetilde{D}_{\text{comp}}$ that contains $D^1_{\text{comp}} \otimes_{\text{alg}} D^2_{\text{comp}}$ is a core for $\widetilde{\alpha}_{\min}$, $\widetilde{\gamma}_{\min}$, $\widetilde{\alpha}^+_{\min}$ and $\widetilde{\gamma}^+_{\min}$. The operators γ_i and γ_i^* leave $H_i(\epsilon|\gamma| = \lambda_i)$ invariant, whereas, by virtue of Statement 6 of Proposition 3.1, the operators α_i and α_i^* map $H_i(\epsilon|\gamma| = \lambda_i)$ into $H_i(\epsilon|\gamma| = q^{-1}\lambda_i)$ and $H_i(\epsilon|\gamma| = q\lambda_i)$, respectively. Therefore

$$\begin{aligned} \widetilde{\alpha}_{\min}\left(\widetilde{H}(\lambda_1, \lambda_2)\right) &\subset \widetilde{H}(q^{-1}\lambda_1, q^{-1}\lambda_2) \oplus \widetilde{H}(\lambda_1, \lambda_2) \\ \widetilde{\gamma}_{\min}\left(\widetilde{H}(\lambda_1, \lambda_2)\right) &\subset \widetilde{H}(\lambda_1, q^{-1}\lambda_2) \oplus \widetilde{H}(q\lambda_1, \lambda_2) \\ \widetilde{\alpha}^+_{\min}\left(\widetilde{H}(\lambda_1, \lambda_2)\right) &\subset \widetilde{H}(q\lambda_1, q\lambda_2) \oplus \widetilde{H}(\lambda_1, \lambda_2) \\ \widetilde{\gamma}^+_{\min}\left(\widetilde{H}(\lambda_1, \lambda_2)\right) &\subset \widetilde{H}(\lambda_1, q\lambda_2) \oplus \widetilde{H}(q^{-1}\lambda_1, \lambda_2) \end{aligned} \tag{4.9}$$

kierunki

Therefore $\widetilde{\alpha}_{\rm min}$, $\widetilde{\gamma}_{\rm min}$, $\widetilde{\alpha}^+_{\rm min}$ and $\widetilde{\gamma}^+_{\rm min}$ leave $\widetilde{D}_{\rm comp}$ invariant and Statement 1 follows.

Let $\mathcal{A}$ be the Hopf *-algebra of polynomials on $S_q\tilde{U}(1,1)$. We know that (0.7) defines a *-algebra homomorphism $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$. Therefore, on the algebraic level (when no problems related with the domains of operators appears) the commutation relations (0.6) are satisfied by the triplet $(\widetilde{\alpha}, \widetilde{\gamma}, \widetilde{\epsilon})$. It shows that Statement 2 holds.

Statement 3 follows easily form the first two. Indeed, due to (4.6) – (4.8) we have: $\|\widetilde{\gamma}^+_{\rm min}x\|^2 = \|\widetilde{\gamma}_{\rm min}x\|^2$, $\|\widetilde{\alpha}_{\rm min}x\|^2 = (x|\epsilon x) + \|\widetilde{\gamma}_{\rm min}x\|^2$ and $\|\widetilde{\alpha}^+_{\rm min}x\|^2 = (x|\epsilon x) + q^2\|\widetilde{\gamma}_{\rm min}x\|^2$ for any $x \in \widetilde{D}_{\rm comp}$. It shows that the four graph norms on $\widetilde{D}_{\rm comp}$ related to operators $\widetilde{\alpha}_{\rm min}$, $\widetilde{\gamma}_{\rm min}$, $\widetilde{\alpha}^+_{\rm min}$ and $\widetilde{\gamma}^+_{\rm min}$ are equivalent. Therefore the corresponding four closures of $\widetilde{D}_{\rm comp}$ must coincide.

Q.E.D.

Using for π_1 and π_2 the decomposition (3.29) we get:

$$\widetilde{H} = \sum_{\lambda_1,\lambda_2\in\Lambda}^{\oplus} \widetilde{H}(\lambda_1,\lambda_2)$$

Rearranging this sum we obtain:

$$\widetilde{H} = \sum_{s=\pm1}^{\oplus}\sum_{m\in\mathbf{Z}}^{\oplus} H^{s,m}, \tag{4.10}$$ dsd

where

$$H^{s,m} = \sum_{\substack{\lambda_1,\lambda_2\in\Lambda,\\ \lambda_1\lambda_2^{-1}=sq^m}}^{\oplus} \widetilde{H}(\lambda_1,\lambda_2) \tag{4.11}$$ dsdsm

Clearly

$$H^{s,m} = \widetilde{H}\left(\begin{array}{c} \epsilon\otimes\epsilon = s \\ |\gamma\otimes\gamma^{-1}| = q^m \end{array}\right).$$

Till the end of this Section we shall use the following notation:

$$\begin{aligned} a &= \epsilon|\gamma|^2 \otimes I, \\ v &= v_{\scriptscriptstyle Y} \otimes v_{\scriptscriptstyle X} \\ &= -\epsilon(\mathrm{Phase}\,\alpha)(\mathrm{Phase}\,\gamma) \otimes (\mathrm{Phase}\,\alpha)(\mathrm{Phase}\,\gamma)^*, \\ P^{n_1,n_2} &= \chi(|\gamma| = q^{n_1}) \otimes \chi(|\gamma| = q^{n_2}). \end{aligned} \tag{4.12}$$ avi

If for some $x \in \widetilde{H}$, the sequence $\left(q^{-k}P^{n_1,n_2}\,(v^*)^k\,x\right)_{k\in\mathbf{N}}$ is converging, then we set

$$\theta^{n_1,n_2} = \lim_{k\to\infty} q^{-k}P^{n_1,n_2}\,(v^*)^k\,x. \tag{4.13}$$ thetadef

In the above formulae, n_1, n_2 are negative integers.

Proposition 4.2 *Let*

$$\begin{aligned} T_{\rm min} &= \left[\mathrm{Phase}\,\gamma \otimes (\mathrm{Phase}\,\gamma)^*\right]\widetilde{\alpha}_{\rm min}, \\ S_{\rm min} &= \left[\epsilon(\mathrm{Phase}\,\gamma) \otimes \mathrm{Phase}\,\alpha\right]\widetilde{\gamma}^+_{\rm min}. \end{aligned} \tag{4.14}$$ TS

Then $T_{\min}$ and $S_{\min}$ respect the direct sum decomposition (4.10) *and denoting by $T^{s,m}_{\min}$ and $S^{s,m}_{\min}$ the restrictions of $T_{\min}$ and $S_{\min}$ to $H^{s,m}$, we have:*

$$T^{s,m}_{\min} = q^{1-m}\left[a - a\left(I + \frac{sq^{2m}}{q^2 a}\right)^{\frac{1}{2}}\left(I + \frac{1}{q^2 a}\right)^{\frac{1}{2}} qv\right], \tag{4.15}$$ Tsmmin

$$S^{s,m}_{\min} = q^{1-m}\left[a\left(I + \frac{sq^{2m}}{q^2 a}\right)^{\frac{1}{2}} - a\left(I + \frac{1}{q^2 a}\right)^{\frac{1}{2}} qv\right]. \tag{4.16}$$ Ssmmin

Proof: The operators in square brackets standing in front of $\widetilde{\alpha}_{\min}$ and $\widetilde{\gamma}^{+}_{\min}$ are isometric. Therefore $T_{\min}$ and $S_{\min}$ are closed densely defined operators. We have

$$\begin{aligned} T_{\min} &= [\mathrm{Phase}\,\gamma \otimes (\mathrm{Phase}\,\gamma)^*]\,(\alpha \otimes \alpha + q\epsilon\gamma^* \otimes \gamma) \\ &= q\epsilon|\gamma| \otimes |\gamma| + |\alpha^*|(\mathrm{Phase}\,\gamma)(\mathrm{Phase}\,\alpha) \otimes |\alpha^*|(\mathrm{Phase}\,\gamma)^*(\mathrm{Phase}\,\alpha) \\ &= q\epsilon|\gamma| \otimes |\gamma| - (\epsilon|\alpha^*| \otimes |\alpha^*|)\,v, \end{aligned} \tag{4.17}$$ Tmin1

$$\begin{aligned} S_{\min} &= [\epsilon(\mathrm{Phase}\,\gamma) \otimes \mathrm{Phase}\,\alpha]\,(\gamma^* \otimes \alpha^* + \epsilon\alpha \otimes \gamma^*) \\ &= \epsilon|\gamma| \otimes |\alpha^*| + |\alpha^*|(\mathrm{Phase}\,\gamma)(\mathrm{Phase}\,\alpha) \otimes q|\gamma|(\mathrm{Phase}\,\alpha)(\mathrm{Phase}\,\gamma)^* \\ &= \epsilon|\gamma| \otimes |\alpha^*| - q\,(\epsilon|\alpha^*| \otimes |\gamma|)\,v. \end{aligned} \tag{4.18}$$ Smin1

We know that $\widetilde{D}_{\min}$ is a core for $T_{\min}$ and $S_{\min}$. Clearly operators $\epsilon|\gamma|\otimes|\gamma|$, $\epsilon|\alpha|\otimes|\alpha|$, $\epsilon|\gamma| \otimes |\alpha|$ and $\epsilon|\alpha| \otimes |\gamma|$ map (4.3) into itself, whereas (cf Statement 6 of Theorem 3.1) v divides the values of λ_1 and λ_2 by q. In all cases $\lambda_1\lambda_2^{-1}$ is left invariant. It shows that $T_{\min}$ and $S_{\min}$ respect the decomposition (4.10).

Let $m \in \mathbf{Z}$ and $s = \pm 1$. One can easily verify, that on the space $H^{s,m}$ we have:

$$\begin{aligned} \epsilon|\gamma| \otimes I &= |a|^{-\frac{1}{2}} a, \\ \epsilon|\alpha^*| \otimes I &= q|a|^{-\frac{1}{2}} a\left(I + \frac{1}{q^2 a}\right)^{\frac{1}{2}}, \\ I \otimes |\gamma| &= q^{-m}|a|^{\frac{1}{2}}, \\ I \otimes |\alpha^*| &= q^{1-m}|a|^{\frac{1}{2}}\left(I + \frac{sq^{2m}}{q^2 a}\right)^{\frac{1}{2}}. \end{aligned}$$

Inserting these data into (4.17) and (4.18) we obtain (4.15) and (4.16).

Q.E.D.

Let $s = \pm 1$ and $m \in \mathbf{Z}$. For the moment we restrict our considerations to the Hilbert space $H^{s,m}$. The Hilbert spaces on the right hand side of the decomposition (4.11) are eigenspaces of a: if $\lambda_1\lambda_2^{-1} = sq^m$, then

$$\widetilde{H}(\lambda_1, \lambda_2) = H^{s,m}\left(a = \lambda_1|\lambda_1|\right). \tag{4.19}$$ hsm

It shows that $|a|$ has pure point spectrum and the eigenvalues are of the form q^{2n}, where $n \in \mathbf{Z}$. The operator v is an isometry acting on $H^{s,m}$. It commutes with $\epsilon \otimes I = \operatorname{Phase} a$ and (by Statement 6 of Proposition 3.1) it maps eigenvectors of a onto eigenvectors of a dividing the eigenvalue by q^2. In other words $va \supset q^2\, av$.

We claim that

$$vv^* \geq \chi(|a| > q^{n_0}), \tag{4.20}$$

ess1

where $n_0 = \min(-1, m-1)$. Indeed, taking into account Statement 3 of Proposition 3.1 we see that $vv^* = \chi(\epsilon|\gamma| \neq -q^{-1}) \otimes \chi(\epsilon|\gamma| \neq -q^{-1}) \geq \chi(|\gamma| > q^{-1}) \otimes \chi(|\gamma| > q^{-1})$. Restricting the last formula to $H^{s,m}$ we obtain (4.20).

The reader should notice that the functions:

$$f_T(\lambda) = \lambda, \qquad g_T(\lambda) = \lambda \left(I + \frac{sq^{2m}}{q^2\lambda}\right)^{\frac{1}{2}} \left(I + \frac{1}{q^2\lambda}\right)^{\frac{1}{2}},$$

$$f_S(\lambda) = \lambda \left(I + \frac{sq^{2m}}{q^2\lambda}\right)^{\frac{1}{2}}, \qquad g_S(\lambda) = \lambda \left(I + \frac{1}{q^2\lambda}\right)^{\frac{1}{2}}$$

behaves for large λ in the way described by (2.11). Summing up we see that the theory of balanced extensions developed in Section 2 (with $\mu = q^2$) may be applied to operators (4.15) and (4.16).

Let R_Y be the Y-reflection operator associated with π_1 and R_X be the X-reflection operator associated with π_2. Remembering that reflection operators anticommute with ϵ one can easily show that $R_Y \otimes R_X$ acting on (4.3) changes the signs of λ_1 and λ_2:

$$(R_Y \otimes R_X)\widetilde{H}(\lambda_1, \lambda_2) \subset \widetilde{H}(-\lambda_1, -\lambda_2).$$

Therefore $\lambda_1\lambda_2^{-1}$ remains unchanged and (4.11) shows, that $H^{s,m}$ is $R_Y \otimes R_X$ - invariant: $(R_Y \otimes R_X)H^{s,m} \subset H^{s,m}$. Using (3.28) one can easily show, that $R_Y \otimes R_X$ is selfadjoint, it commutes with $|\gamma|^2 \otimes I = |a|$ and anticommutes with $\epsilon \otimes I = \operatorname{Phase} a$. By (3.28) $vH(R_Y \otimes R_X = \pm 1) = (v_{_Y} \otimes v_{_X})H(R_Y \otimes R_X = \pm 1) \subset H(R_Y \otimes R_X = \pm 1)$. Moreover $(R_Y \otimes R_X)^2 = \chi(|\gamma| \geq q^{-1}) \otimes \chi(|\gamma| \geq q^{-1})$. The latter operator restricted to $H^{s,m}$ coincides with $\chi(|a| \geq q^{n_0})$, where $n_0 = \min(-1, m-1)$. It shows (cf (2.37)), that the restriction of $(-1)^m R_Y \otimes R_X$ to $H^{s,m}$ is the reflection operator associated with some balanced extensions $T^{s,m}_{\min} \subset T^{s,m}$ and $S^{s,m}_{\min} \subset S^{s,m}$.

Let $T^{s,m}$ and $S^{s,m}$ be balanced extensions of $T^{s,m}_{\min}$ and $S^{s,m}_{\min}$ corresponding to the same reflection operator $(-1)^m R_Y \otimes R_X$. Then using Theorem 2.3 and Remark 2.10 we see that

$$D(T^{s,m}) = D(S^{s,m}) = \left\{ x \in H^{s,m} : \begin{matrix} \text{Sequence } \left(q^{-k} P^{n_1,n_2} \left(v^*\right)^k x\right)_{k\in\mathbf{N}} \\ \text{is } q^2\text{-converging and its limit} \\ \theta^{n_1,n_2}(x) \in H^{s,m}\left(R_Y \otimes R_X = (-1)^m\right) \end{matrix} \right\}. \tag{4.21}$$

Dtilde2

In this formula n_1, n_2 are negative integers such that $n_1 - n_2 = m$. The right hand side does not depend on the particular choice of n_1, n_2.

Let

$$T = \sum_{s=\pm 1}^{\oplus} \sum_{m \in \mathbf{Z}}^{\oplus} T^{s,m}, \tag{4.22}$$ dsdT

$$S = \sum_{s=\pm 1}^{\oplus} \sum_{m \in \mathbf{Z}}^{\oplus} S^{s,m}. \tag{4.23}$$ dsdS

Then T and S are extensions of $T_{\min}$ and $S_{\min}$ respectively.

DtildeP

Proposition 4.3 *Let*

$$\widetilde{D}_0 = \bigcup_{n \in \mathbf{Z}} \widetilde{H}\left(q^n \leq |\gamma \otimes \gamma^{-1}| \leq q^{-n}\right)$$

and (for any $p > 0$)

$$\widetilde{D}_p = \left\{ x \in \widetilde{D}_0 : \begin{array}{c} \textit{For any integers } n_1, n_2 < 0, \textit{ the sequence} \\ \left(q^{-k} P^{n_1,n_2} (v^*)^k x\right)_{k \in \mathbf{N}} \textit{ is } q^{2p}\textit{-converging and} \\ \textit{its limit } \theta^{n_1,n_2}(x) \in \widetilde{H}\left(R_Y \otimes R_X = (-1)^{n_1 - n_2}\right) \end{array} \right\} \tag{4.24}$$ Dtildep

Then:

1. *For any $p \geq 1$, $\widetilde{D}_p$ is a core for T, S, T^* and S^*.*
2. *T, S, T^* and S^* map $\widetilde{D}_3$ into $\widetilde{D}_1$.*
3. *For any $p \geq 0$, $(\mathrm{Phase}\,\gamma)^* \otimes \mathrm{Phase}\,\gamma$ maps $\widetilde{D}_p$ onto itself.*
4. *For any $p \geq 0$, $\epsilon(\mathrm{Phase}\,\gamma)^* \otimes (\mathrm{Phase}\,\alpha)^*$ maps $\widetilde{D}_p$ onto itself.*

Proof: Clearly each $H^{s,m}$ is contained in $\widetilde{D}_0$. A moment of reflection shows that $\widetilde{D}_0$ is the linear span of the union of all $H^{s,m}$. Moreover $\widetilde{D}_1$ is the linear span of all $D(T^{s,m})$ (compare (4.21) with (4.24)). Statement 1 follows now from Proposition 2.6. Similarly, Statement 2 follows from Proposition 2.7.

Statement 3 is trivial: the unitary operator $\mathrm{Phase}\,\gamma \otimes (\mathrm{Phase}\,\gamma)^*$ commutes with all operators appearing in the definition of $\widetilde{D}_p$.

Let $x \in \widetilde{H}$, $x' = \left[\epsilon(\mathrm{Phase}\,\gamma)^* \otimes (\mathrm{Phase}\,\alpha)^*\right] x$ and

$$y = \lim_{k \to \infty} q^{-k} P^{n_1,n_2} (v^*)^k x,$$

$$y' = \lim_{k \to \infty} q^{-k} P^{n_1,n_2+1} (v^*)^k x'.$$

The reader should notice, that the rate of convergence of the two sequences is the same and the limits are related by the formula

$$y' = \left[\epsilon(\mathrm{Phase}\,\gamma)^* \otimes (\mathrm{Phase}\,\alpha)^*\right] y. \tag{4.25}$$ vey1

If $x \in \widetilde{D}_p$, then

$$y \in \widetilde{H}\left(R_Y \otimes R_X = (-1)^{n_1 - n_2}\right). \tag{4.26}$$ vey

Using (4.25) and remembering, that $R_X\epsilon_1 = -\epsilon_1 R_X$ one can easily show that (4.26) is equivalent to

$$y' \in \widetilde{H}\left(R_Y \otimes R_X = (-1)^{n_1-n_2-1}\right).$$

The latter relation means that $x' \in \widetilde{D}_p$.

Q.E.D.

Proposition 4.4 *There exist unique closed operators $\widetilde{\alpha}$ and $\widetilde{\gamma}$ acting on $\widetilde{H}$ such that (cf (4.14))* 44

$$\begin{aligned} T &= \left[\operatorname{Phase}\gamma \otimes (\operatorname{Phase}\gamma)^*\right]\widetilde{\alpha}, \\ S &= \left[\epsilon(\operatorname{Phase}\gamma) \otimes \operatorname{Phase}\alpha\right]\widetilde{\gamma}^*. \end{aligned} \tag{4.27}$$ TS2

Moreover

1. *Operators $\widetilde{\alpha}$, $\widetilde{\gamma}$ and $\widetilde{\epsilon} = \epsilon \otimes \epsilon$ satisfy Conditions 1, 2 and 3 of Definition 1.1.*
2. *Operators $\widetilde{\alpha}$, $\widetilde{\gamma}$ and $\widetilde{\epsilon} = \epsilon \otimes \epsilon$ satisfy Statement 1 of Theorem 1.4.*

Proof: The uniqueness of $\widetilde{\alpha}$ and $\widetilde{\gamma}$ and the existence of $\widetilde{\alpha}$ are obvious. To prove the existence of $\widetilde{\gamma}$ it is sufficient to show, that

$$\operatorname{Range}(S) \subset H_1 \otimes \operatorname{Range}(\operatorname{Phase}\alpha). \tag{4.28}$$ egamma

Let $x \perp H_1 \otimes \operatorname{Range}(\operatorname{Phase}\alpha)$. Then

$$x = \sum_{s=\pm 1}\sum_{m\in\mathbf{Z}} x^{s,m},$$

where $x^{s,m} \in H_1(\epsilon|\gamma| = -sq^{m-1}) \otimes H_2(\epsilon|\gamma| = -q^{-1})$. Clearly $x^{s,m} \in D(S^{s,m}_{\min}) \subset D(S^{s,m}) \subset D(S) = D(S^*)$. Remembering that $S^* \subset (S_{\min})^*$ and using (4.14) we see that $S^*x^{s,m} = (S_{\min})^* x^{s,m} = 0$. Therefore $x^{s,m} \in \ker S^*$, $x \in \ker S^*$ and (4.28) follows.

By Propositions 4.3 and A.1, for any $p \geq 1$, the domain $\widetilde{D}_p$ introduced by (4.24) is a core for $\widetilde{\alpha}$, $\widetilde{\gamma}^*$, $\widetilde{\alpha}^*$ and $\widetilde{\gamma}$.

Comparing (4.14) with (4.27) we see that $\widetilde{\alpha}_{\min} \subset \widetilde{\alpha}$ and $\widetilde{\gamma}^+_{\min} \subset \widetilde{\gamma}^*$. Consequently $\widetilde{\alpha}^* \subset (\widetilde{\alpha}_{\min})^*$ and $\widetilde{\gamma} \subset \left(\widetilde{\gamma}^+_{\min}\right)^*$. We know that $(\widetilde{\alpha}_{\min})^*$ is an extension of $\widetilde{\alpha}^+_{\min}$ and $\left(\widetilde{\gamma}^+_{\min}\right)^*$ is an extension of $\widetilde{\gamma}_{\min}$. Therefore $\widetilde{\alpha}^*x = \widetilde{\alpha}^+_{\min}x$ and $\widetilde{\gamma}x = \widetilde{\gamma}_{\min}x$ for any x belonging to $D(\widetilde{\alpha}^*) \cap D(\widetilde{\alpha}^+_{\min}) \cap D(\widetilde{\gamma}) \cap D(\widetilde{\gamma}_{\min})$. In particular these relations hold for $x \in \widetilde{D}_{\min}$. Remembering that $\widetilde{D}_{\min}$ is a core for $\widetilde{\alpha}^+_{\min}$ and $\widetilde{\gamma}_{\min}$ we see that $\widetilde{\alpha}^+_{\min} \subset \widetilde{\alpha}^*$ and $\widetilde{\gamma}_{\min} \subset \widetilde{\gamma}$. Statement 1 of Theorem 1.4 is proven.

The operator $\widetilde{\epsilon} = \epsilon \otimes \epsilon$ is obviously unitary and selfadjoint. Moreover the spaces entering the direct sum decomposition (4.10) are eigenspaces of $\widetilde{\epsilon}$. Therefore $\widetilde{\epsilon}$ commutes with T and S. Consequently it commutes with $\widetilde{\alpha}$ and $\widetilde{\gamma}^*$ and Conditions 1 and 2 of Definition 1.1 are satisfied by our operators.

Let $x \in \widetilde{D}_{\min}$ and $y \in \widetilde{D}_3$. By virtue of (4.8),

$$\left(y\middle|\widetilde{\alpha}_{\min}\widetilde{\alpha}^+_{\min}x\right) = (y|\widetilde{\epsilon}x) + q^2\left(y\middle|\widetilde{\gamma}^+_{\min}\widetilde{\gamma}_{\min}x\right).$$

Remembering that $\widetilde{\alpha}^* \subset \widetilde{\alpha}^*_{\min}$ and $\widetilde{\gamma} \subset \left(\widetilde{\gamma}^+_{\min}\right)^*$ we obtain:

$$\left(\widetilde{\alpha}^* y \middle| \widetilde{\alpha}^+_{\min} x\right) = (\widetilde{\epsilon} y | x) + q^2 (\widetilde{\gamma} y | \widetilde{\gamma}_{\min} x) .$$

According to Proposition 4.3, $\widetilde{\alpha}^* y, \widetilde{\gamma} y \in \widetilde{D}_1$. The latter domain is contained in $D(\widetilde{\alpha})$ and $D(\widetilde{\gamma}^*)$. Therefore

$$(\widetilde{\alpha}\widetilde{\alpha}^* y | x) = (\widetilde{\epsilon} y | x) + q^2 (\widetilde{\gamma}^* \widetilde{\gamma} y | x) .$$

By density of $\widetilde{D}_{\min}$, this relation holds for all $x \in \widetilde{H}$. Assuming that $x \in \widetilde{D}_3$ we get:

$$(\widetilde{\alpha}^* y | \widetilde{\alpha}^* x) = (\widetilde{\epsilon} y | x) + q^2 (\widetilde{\gamma} y | \widetilde{\gamma} x) .$$

This relation holds for all $x, y \in \widetilde{D}_3$. Remembering that $\widetilde{D}_3$ is a core for $\widetilde{\alpha}^*$ and $\widetilde{\gamma}$ we see that $D(\widetilde{\alpha}^*) = D(\widetilde{\gamma})$ and that the above relation holds for all $x, y \in D(\widetilde{\gamma})$. In the same way, starting with relations (4.4) – (4.7) we can prove that operators $\widetilde{\alpha}$, $\widetilde{\gamma}$ and $\widetilde{\epsilon}$ satisfy the remaining requirements of Condition 3 of Definition 1.1.

Q.E.D.

By (4.27), $|\widetilde{\gamma}| = |\widetilde{\gamma}^*| = |S|$. Formula (4.23) shows that $|\widetilde{\gamma}|$ respects the direct sum decomposition (4.11). It means that $|\widetilde{\gamma}|$ strongly commutes with $\epsilon \otimes \epsilon$ and $|\gamma \otimes \gamma^{-1}|$.

We say that $x \in \widetilde{H}$ is homogeneous if it belongs to one of the subspace $H^{s,m}$.

core4

Proposition 4.5 *Let D' be a linear dense subset of $\widetilde{H}$. Assume that*

1. *D' contains $\widetilde{D}_{\mathrm{comp}}$,*
2. *Any element of D' is a finite sum of homogeneous elements belonging to D',*
3. *For any negative integers n_1, n_2 and $x \in D'$, the sequence $\left(q^{-k} P^{n_1,n_2} (v^*)^k x\right)_{k \in \mathbf{N}}$ is q^2-converging.*
4. *For any negative integers n_1, n_2 the set*

$$\left\{\theta^{n_1,n_2}(x) : x \in D'\right\}$$

is a dense subset of $\widetilde{H}\left(|\gamma| \otimes I = q^{n_1}, I \otimes |\gamma| = q^{n_2}, R_Y \otimes R_X = (-1)^{n_1 - n_2}\right)$.

Then D' is a core for $\widetilde{\alpha}$, $\widetilde{\gamma}$, $\widetilde{\alpha}^*$ and $\widetilde{\gamma}^*$.

Proof: Comparing Assumptions 3 and 4 with (4.21), we see that $D' \cap H^{s,m} \subset D(T^{s,m})$. Using Proposition 2.5 (modified in the way described in Remark 2.10) and Assumption 4 one can easily show, that $D' \cap H^{s,m}$ is a core of $T^{s,m}$. Therefore D' is a core of T. The first equation of (4.27) shows now, that D' is a core of $\widetilde{\alpha}$.

To end the proof we recall that $\widetilde{\alpha}$, $\widetilde{\gamma}$ and $\widetilde{\epsilon}$ satisfy Condition 3 of Definition 1.1. By Remark 1.2 any core of $\widetilde{\alpha}$ is a core for $\widetilde{\gamma}$, $\widetilde{\alpha}^*$ and $\widetilde{\gamma}^*$.

Q.E.D.

Now we are able to show that operators $\widetilde{\alpha}$ and $\widetilde{\gamma}$ satisfy the Statement 2 of Theorem 1.4.

ssss

Proposition 4.6 *Let D' be the linear span of $\widetilde{D}_{\mathrm{comp}}$ and all vectors x of the form*

$$x = E^{s,m}(\epsilon^m x_1 \otimes x_2), \tag{4.29}$$ Dalpha1

where $E^{s,m} = \chi\left(|\gamma \otimes \gamma^{-1}| = q^m, \epsilon \otimes \epsilon = s\right)$, $x_1 \in D(\gamma Y)$, $x_2 \in D(\gamma X)$, $s = \pm 1$ and $m \in \mathbf{Z}$. Then D' is a core for $\widetilde{\alpha}$, $\widetilde{\gamma}$, $\widetilde{\alpha}^*$ and $\widetilde{\gamma}^*$.

Proof: We have to verify that D' satisfies the assumptions of Proposition 4.5. Clearly Assumptions 1 and 2 are satisfied.

Let $x_1 \in D(\gamma_1 Y_1)$, $x_2 \in D(\gamma_2 X_2)$ and n_1, n_2 be negative integers. By Proposition 3.4, the sequence

$$\left(q^{-\frac{k}{2}}\chi\left(|\gamma_1| = q^{n_1}\right)\left(v_Y^*\right)^k x_1\right)_{k\in\mathbf{N}}$$

is q^3-converging to a vector $y_1 \in H_1(|\gamma| = q^{n_1}, R_Y = 1)$. Similarly, by Proposition 3.6, the sequence

$$\left(q^{-\frac{k}{2}}\chi\left(|\gamma_2| = q^{n_2}\right)\left(v_X^*\right)^k x_2\right)_{k\in\mathbf{N}}$$

is q^3-converging to a vector $y_2 \in H_2(|\gamma| = q^{n_2}, R_X = 1)$. We know, that the tensor product of q^3-converging sequences is q^3-converged (cf the paragraph preceding Proposition 2.1. Therefore the sequence

$$\left(q^{-k}P^{n_1,n_2}\left(v^*\right)^k\left(\epsilon^m x_1 \otimes x_2\right)\right)_{k\in\mathbf{N}}$$

is q^3-converging to $\epsilon^m y_1 \otimes y_2$. Let x be the element of D' introduced by (4.29). We know that the spaces $H^{s,m}$ are P^{n_1,n_2} and v -invariant. Therefore $E^{s,m}$ commutes with P^{n_1,n_2} and v and the sequence

$$\left(q^{-k}P^{n_1,n_2}\left(v^*\right)^k x\right)_{k\in\mathbf{N}}$$

is q^3-converging and its limit $\theta^{n_1,n_2}(x) = E^{s,m}(\epsilon^m y_1 \otimes y_2)$. Clearly q^3-convergence implies q^2-convergence. It shows that D' satisfies Assumption 3 of Proposition 4.5.

One may assume that $n_1 - n_2 = m$ (otherwise $\theta^{n_1,n_2}(x) = 0$). In this case $\theta^{n_1,n_2}(x) = \chi(\epsilon \otimes \epsilon = s)(\epsilon^m y_1 \otimes y_2)$.

By the last Statements of Propositions 3.4 and 3.6, the set $\left\{\theta^{n_1,n_2}(x) : x \in D'\right\}$ coincides with the linear span of

$$\left\{\chi(\epsilon \otimes \epsilon = s)(\epsilon^m y_1 \otimes y_2) : \begin{array}{c} y_1 \in H_1(|\gamma| = q^{n_1}, R_Y = 1) \\ y_2 \in H_2(|\gamma| = q^{n_2}, R_X = 1) \\ m = n_1 - n_2,\ s = \pm 1 \end{array}\right\}.$$

Therefore $\left\{\theta^{n_1,n_2}(x) : x \in D'\right\}$ is a dense subset of the linear span of

$$\bigcup_{s=\pm 1}\chi(\epsilon \otimes \epsilon = s)\left[\epsilon^{n_1-n_2}H_1(|\gamma| = q^{n_1}, R_Y = 1) \otimes H_2(|\gamma| = q^{n_2}, R_X = 1)\right]. \tag{4.30}$$ bigcup

The linear span of $\{\chi(\epsilon\otimes\epsilon=-1),\chi(\epsilon\otimes\epsilon=1)\}$ clearly coincides with the linear span of $\{I\otimes I,\epsilon\otimes\epsilon\}$. Moreover

$$\begin{aligned}\widetilde{H}(R_Y\otimes R_X=1) &= H_1(R_Y=1)\otimes H_2(R_X=1)\oplus H_1(R_Y=-1)\otimes H_2(R_X=-1)\\ &= H_1(R_Y=1)\otimes H_2(R_X=1)\oplus \epsilon H_1(R_Y=1)\otimes \epsilon H_2(R_X=1).\end{aligned}$$

It shows, that the linear span of (4.30) equals to

$$\begin{aligned}&(\epsilon^{n_1-n_2}\otimes I)\,\widetilde{H}\left(|\gamma|\otimes I=q^{n_1},I\otimes|\gamma|=q^{n_2},R_Y\otimes R_X=1\right)\\ &\qquad=\widetilde{H}\left(|\gamma|\otimes I=q^{n_1},I\otimes|\gamma|=q^{n_2},R_Y\otimes R_X=(-1)^{n_1-n_2}\right).\end{aligned}\tag{4.31}$$ bigcup1

We proved that $\left\{\theta^{n_1,n_2}(x):x\in D'\right\}$ is a dense subset of (4.31). It shows that D' satisfies Assumption 4 of Proposition 4.5. By virtue of the last Proposition, D' is a core of $\widetilde{\alpha}$, $\widetilde{\gamma}$, $\widetilde{\alpha}^*$ and $\widetilde{\gamma}^*$.

Q.E.D.

Let $\widetilde{\alpha}_{\max}=(\widetilde{\alpha}^+_{\max})^*$. Then $\widetilde{\alpha}_{\max}$ is an extension of $\widetilde{\alpha}$. A moment of reflection shows that

$$\widetilde{\alpha}_{\max}=[(\mathrm{Phase}\,\gamma)^*\otimes\mathrm{Phase}\,\gamma]\,T_{\max},$$

where

$$T_{\max}=\sum_{s=\pm1}{}^{\oplus}\sum_{m\in\mathbf{Z}}{}^{\oplus}\,T^{s,m}_{\max},$$

where operators $T^{s,m}_{\max}$ are related to the operators $T^{s,m}_{\min}$ in the way described in Section 2. Combining the Statement 1 of Theorem 2.2 and the last expression for $D(T)$ given by (2.21) with Remark 2.10 one can easily prove the following

kryterium

Proposition 4.7 *For any $x\in D(\widetilde{\alpha}_{\max})$ and any negative integers n_1,n_2 the sequence $\left(q^{-k}P^{n_1,n_2}\left(v^*\right)^k x\right)_{k\in\mathbf{N}}$ is q^2-converging. Denoting its limit by $\theta^{n_1,n_2}(x)$ we have:*

$$D(\widetilde{\alpha})=\left\{x\in D(\widetilde{\alpha}_{\max}):\begin{array}{c}\textit{For any negative integers } n_1,n_2,\\ \theta^{n_1,n_2}(x)\in\widetilde{H}(R_Y\otimes R_X=(-1)^{n_1-n_2})\end{array}\right\}.\tag{4.32}$$ Dtilde6

Taking into account (4.12) we have:

$$\theta^{n_1,n_2}(x)=\lim_{k\to\infty}q^{-k}\left[\chi(|\gamma|=q^{n_1})\left(v^*_{_Y}\right)^k\otimes\chi(|\gamma|=q^{n_2})\left(v^*_{_X}\right)^k\right]x\tag{4.33}$$ kryterium1

5 Dilations and wave operators.

S5

In many considerations, the non-unitarity of $\mathrm{Phase}\,\alpha$ causes some technical problems. To avoid them we shall use the theory of unitary dilations of isometric operators [4].

dilation

Proposition 5.1 *Let α,γ and ϵ be closed operators on a Hilbert space H satisfying the first three Conditions of Definition 1.1. Assume that $\ker\gamma=\{0\}$. Then there exist a Hilbert space H^{ex} and operators $\mathrm{Phase}^{\mathrm{ex}}\alpha$, γ^{ex}, ϵ^{ex} acting on H^{ex} such that:*

1. The space H is a subspace of H^{ex}. In what follows, $H^{\perp}$ will denote the orthogonal complement of H in H^{ex}.

2. Operator $\mathrm{Phase}^{\mathrm{ex}}\alpha$ *is unitary,* $\mathrm{Phase}^{\mathrm{ex}}\alpha$ *restricted to H coincides with* $\mathrm{Phase}\,\alpha$ *and*

$$\bigcup_{n\in\mathbf{N}} \left(\mathrm{Phase}^{\mathrm{ex}}\alpha\right)^{-n} H$$

is dense in H^{ex}.

3. Operators γ^{ex} and ϵ^{ex} respect the decomposition $H^{\mathrm{ex}} = H \oplus H^{\perp}$. Restrictions of γ^{ex} and ϵ^{ex} to H coincide with γ and ϵ. Moreover

$$H^{\perp} = H^{\mathrm{ex}} \left(\begin{array}{c} \epsilon^{\mathrm{ex}} = -1, \\ |\gamma^{\mathrm{ex}}| < q^{-1} \end{array} \right).$$

4. $\mathrm{Phase}^{\mathrm{ex}}\alpha$ *commutes with ϵ^{ex} and*

$$(\mathrm{Phase}^{\mathrm{ex}}\alpha)\,\gamma^{\mathrm{ex}}\,(\mathrm{Phase}^{\mathrm{ex}}\alpha)^{*} = q\gamma^{\mathrm{ex}}.$$

Proof: Clearly $\mathrm{Phase}^{\mathrm{ex}}\alpha$ is a unitary dilation of $\mathrm{Phase}\,\alpha$. Statements 1 and 2 belong to the general theory of unitary dilations. Let $K = H\,(\epsilon|\gamma| = -q^{-1})$. Then K is the orthogonal complement of $(\mathrm{Phase}\,\alpha)H$ in H and

$$H^{\perp} = \sum_{n=0}^{\infty}{}^{\oplus}\, H^{\perp}_{n},$$

where $H^{\perp}_{n} = (\mathrm{Phase}^{\mathrm{ex}}\alpha)^{-(n+1)}\,K$. We know that $\mathrm{Phase}\,\alpha$ commutes with unitary operator $\mathrm{Phase}\,\gamma$. Using the functorial properties of unitary dilations we see that there exists unique unitary operator $\mathrm{Phase}^{\mathrm{ex}}\gamma$ acting on H^{ex} such that

1. $\mathrm{Phase}^{\mathrm{ex}}\gamma$ commutes with $\mathrm{Phase}^{\mathrm{ex}}\alpha$,
2. The restriction of $\mathrm{Phase}^{\mathrm{ex}}\gamma$ to H coincides with $\mathrm{Phase}\,\gamma$.

Obviously $\mathrm{Phase}^{\mathrm{ex}}\gamma$ maps each $H^{\perp}_{n}$ onto itself. We set:

$$\epsilon^{\mathrm{ex}}x = \begin{cases} \epsilon x & \text{for} \quad x \in H, \\ -x & \text{for} \quad x \in H^{\perp}_{n}, \end{cases}$$

$$\gamma^{\mathrm{ex}}x = \begin{cases} \gamma x & \text{for} \quad x \in D(\gamma), \\ q^{n}(\mathrm{Phase}^{\mathrm{ex}}\gamma)x & \text{for} \quad x \in H^{\perp}_{n}. \end{cases} \tag{5.1}$$

gammaex

Using Statement 3 of Theorem 3.1 one can easily show that the operators γ^{ex} and ϵ^{ex} introduced above satisfy the requirements of our theorem. The reader should notice that $\mathrm{Phase}^{\mathrm{ex}}\gamma = \mathrm{Phase}\,\gamma^{\mathrm{ex}}$.

Q.E.D.

The reader should notice that only the fermionic part of H is subject to the extension ($\mathrm{Phase}\,\alpha$ is unitary on the bosonic sector). Therefore denoting by $\gamma^{\mathrm{ex}}_{\pm}$ the restriction of γ^{ex} to $H^{\mathrm{ex}}(\epsilon^{\mathrm{ex}} = \pm 1)$ we have:

$$\gamma^{\mathrm{ex}}_{+} = \gamma_{+}. \tag{5.2}$$

noex

Moreover taking into account Statement 3 of Proposition 3.1 and formula (5.1) we get

$$\operatorname{Sp}\gamma_-^{\mathrm{ex}} = q^{\mathbf{Z}} \cup \{0\}, \tag{5.3}$$
spex

provided the fermionic sector is not empty.

Many operators considered in Section 3 admit a natural extension to the space H^{ex}. In particular

$$\left.\begin{aligned}
v_{\scriptscriptstyle Y}^{\mathrm{ex}} &= \epsilon^{\mathrm{ex}}(\operatorname{Phase}{}^{\mathrm{ex}}\alpha)(\operatorname{Phase}\gamma^{\mathrm{ex}}),\\
v_{\scriptscriptstyle X}^{\mathrm{ex}} &= -(\operatorname{Phase}{}^{\mathrm{ex}}\alpha)(\operatorname{Phase}\gamma^{\mathrm{ex}})^*,\\
V^{\mathrm{ex}} &= \left[-\epsilon^{\mathrm{ex}}(\operatorname{Phase}\gamma^{\mathrm{ex}})^2\right]^{\log|q\gamma^{\mathrm{ex}}|},\\
R_{\scriptscriptstyle Y}^{\mathrm{ex}} &= \sum_{n\in\mathbf{Z}} \left(v_{\scriptscriptstyle Y}^{\mathrm{ex}}\right)^{-n} u \left(v_{\scriptscriptstyle Y}^{\mathrm{ex}}\right)^{n},\\
R_{\scriptscriptstyle X}^{\mathrm{ex}} &= \sum_{n\in\mathbf{Z}} \left(v_{\scriptscriptstyle X}^{\mathrm{ex}}\right)^{-n} u \left(v_{\scriptscriptstyle X}^{\mathrm{ex}}\right)^{n}
\end{aligned}\right\} \tag{5.4}$$
exwzory1

are unitary operators acting on H^{ex}. $v_{\scriptscriptstyle Y}^{\mathrm{ex}}$, $v_{\scriptscriptstyle X}^{\mathrm{ex}}$ and V^{ex} map H into itself and their restrictions to H coincide with $v_{\scriptscriptstyle Y}$, $v_{\scriptscriptstyle X}$ and V respectively. $R_{\scriptscriptstyle Y}^{\mathrm{ex}}$ and $R_{\scriptscriptstyle X}^{\mathrm{ex}}$ restricted to $K = H^{\mathrm{ex}}(|\gamma^{\mathrm{ex}}| = q^{-1})$ coincide with u. Clearly $R_{\scriptscriptstyle X}^{\mathrm{ex}} = V^{\mathrm{ex}} R_{\scriptscriptstyle Y}^{\mathrm{ex}} (V^{\mathrm{ex}})^*$. Repeating the derivation of (3.27) we obtain:

$$R_{\scriptscriptstyle X}^{\mathrm{ex}} = (-1)^{\log|q\gamma^{\mathrm{ex}}|} R_{\scriptscriptstyle Y}^{\mathrm{ex}}. \tag{5.5}$$
RX2ex

One can easily verify that γ^{ex} is a normal operator and

$$\left.\begin{aligned}
v_{\scriptscriptstyle Y}^{\mathrm{ex}}\,\gamma^{\mathrm{ex}} &= q\,\gamma^{\mathrm{ex}}\,v_{\scriptscriptstyle Y}^{\mathrm{ex}}, & \qquad v_{\scriptscriptstyle X}^{\mathrm{ex}}\,\gamma^{\mathrm{ex}} &= q\,\gamma^{\mathrm{ex}}\,v_{\scriptscriptstyle X}^{\mathrm{ex}},\\
v_{\scriptscriptstyle Y}^{\mathrm{ex}}\,R_{\scriptscriptstyle Y}^{\mathrm{ex}} &= R_{\scriptscriptstyle Y}^{\mathrm{ex}}\,v_{\scriptscriptstyle Y}^{\mathrm{ex}}, & \qquad v_{\scriptscriptstyle X}^{\mathrm{ex}}\,R_{\scriptscriptstyle X}^{\mathrm{ex}} &= R_{\scriptscriptstyle X}^{\mathrm{ex}}\,v_{\scriptscriptstyle X}^{\mathrm{ex}},\\
\epsilon^{\mathrm{ex}}\,R_{\scriptscriptstyle Y}^{\mathrm{ex}} &= -R_{\scriptscriptstyle Y}^{\mathrm{ex}}\,\epsilon^{\mathrm{ex}}, & \qquad \epsilon^{\mathrm{ex}}\,R_{\scriptscriptstyle X}^{\mathrm{ex}} &= -R_{\scriptscriptstyle X}^{\mathrm{ex}}\,\epsilon^{\mathrm{ex}}.
\end{aligned}\right\} \tag{5.6}$$
exwzory

Now let $\pi_1 = (\alpha_1, \gamma_1, \epsilon_1, Y_1)$ and $\pi_2 = (\alpha_2, \gamma_2, \epsilon_2, Y_2)$ be $S_q\tilde{U}(1,1)$-quadruples of unbounded type acting on Hilbert spaces H_1 and H_2 respectively. We shall use the notation introduced in Section 4. In particular $\widetilde{\alpha}$ and $\widetilde{\gamma}$ will denote operators on $\widetilde{H} = H_1 \otimes H_2$ introduced by (4.27). Copying (3.15) and (3.23) we set:

$$\widetilde{v}_{\scriptscriptstyle Y} = \widetilde{\epsilon}\,(\operatorname{Phase}\widetilde{\alpha})(\operatorname{Phase}\widetilde{\gamma}), \tag{5.7}$$
vytilde

$$\widetilde{v}_{\scriptscriptstyle X} = -(\operatorname{Phase}\widetilde{\alpha})(\operatorname{Phase}\widetilde{\gamma})^*. \tag{5.8}$$
vXtilde

With this notation we have:

Proposition 5.2 *There exists bounded operators $b_1, b_2 \in B(\widetilde{H})$ such that*

$$\left[(I \otimes v_{\scriptscriptstyle Y}) - \widetilde{v}_{\scriptscriptstyle Y}\right]\widetilde{\gamma}^* x = b_1|\gamma \otimes \gamma^{-1}|x + b_2 x \tag{5.9}$$
WO1

for any $x \in D(\widetilde{\gamma}) \cap D(\gamma \otimes \gamma^{-1})$. Operators b_1 and b_2 respect the direct sum decomposition (4.10).

Proof: Let

$$b_1 = \begin{cases} \epsilon \mathrm{Phase}\,\alpha \otimes \epsilon(\mathrm{Phase}\,\alpha)\left[|\gamma| - |\alpha|\right]|\gamma| \\ +(\mathrm{Phase}\,\gamma)^* \otimes \epsilon(\mathrm{Phase}\,\gamma)\left[|\alpha^*| - q|\gamma|\right]|\gamma|, \end{cases}$$

$$b_2 = \begin{cases} \widetilde{\epsilon}(\mathrm{Phase}\,\widetilde{\alpha})\left[|\widetilde{\alpha}| - |\widetilde{\gamma}|\right] \\ -\epsilon(\mathrm{Phase}\,\alpha)\left[|\alpha| - |\gamma|\right] \otimes \epsilon(\mathrm{Phase}\,\alpha)\left[|\alpha| - |\gamma|\right]. \end{cases}$$

By virtue of Proposition 3.2, $\|b_1\| \leq 1 + q^{-1}$ and $\|b_2\| \leq 2$. Clearly b_1 and b_2 respect the direct sum decomposition (4.10). Let $\widetilde{D}_{\min}$ be the domain introduced in Proposition 4.1 and $x \in \widetilde{D}_{\min}$. Performing simple computations we get:

$$b_1|\gamma \otimes \gamma^{-1}|x = \left[v_Y\gamma^* \otimes (v_Y\gamma^* - \epsilon\alpha)\right]x + \left[\gamma^* \otimes (v_Y\alpha^* - q\epsilon\gamma)\right]x,$$

$$b_2 x = \left[\widetilde{\epsilon}\widetilde{\alpha} - \widetilde{v}_Y\widetilde{\gamma}^*\right]x - \left[(\epsilon\alpha - v_Y\gamma^*) \otimes (\epsilon\alpha - v_Y\gamma^*)\right]x.$$

Moreover in this case, computing $\widetilde{\alpha}x$ and $\widetilde{\gamma}^*x$ we may replace $\widetilde{\alpha}$ by $\widetilde{\alpha}_{\min}$ given by (4.1) and $\widetilde{\gamma}^*$ by $\widetilde{\gamma}^+_{\min}$ given by (4.2). Using these data one can easily verify (5.9) for all $x \in \widetilde{D}_{\min}$.

To end the proof we notice, that the intersection of domains of $\widetilde{\gamma}(I \otimes v_Y^*)$, $\widetilde{\gamma}\widetilde{v}_Y^*$, $b_1^*|\gamma \otimes \gamma^{-1}|$ and b_2^* is dense in $\widetilde{H}$. Indeed, it contains $\widetilde{D}_{\min}$. Using now Proposition A.2 we get (5.9) in full generality.

Q.E.D.

Using the same technic one can prove

Proposition 5.3 *There exists bounded operators $b_1, b_2 \in B(\widetilde{H})$ such that*

$$\left[(v_X \otimes I) - \widetilde{v}_X\right]\widetilde{\gamma}x = b_1|\gamma^{-1} \otimes \gamma|x + b_2 x \qquad (5.10)$$ wo3

for any $x \in D(\widetilde{\gamma}) \cap D(\gamma^{-1} \otimes \gamma)$. Operators b_1 and b_2 respect the direct sum decomposition (4.10).

Proof: In this case

$$b_1 = \begin{cases} (\mathrm{Phase}\,\alpha)\left[|\alpha| - |\gamma|\right]|\gamma| \otimes \mathrm{Phase}\,\alpha \\ +\epsilon(\mathrm{Phase}\,\gamma)^*\left[q|\gamma| - |\alpha^*|\right]|\gamma| \otimes \mathrm{Phase}\,\gamma, \end{cases}$$

$$b_2 = \begin{cases} (\mathrm{Phase}\,\widetilde{\alpha})\left[|\widetilde{\alpha}| - |\widetilde{\gamma}|\right] \\ +(\mathrm{Phase}\,\alpha)\left[|\alpha| - |\gamma|\right] \otimes (\mathrm{Phase}\,\alpha)\left[|\alpha| - |\gamma|\right]. \end{cases}$$

The computational details are left to the reader.

Q.E.D.

We already know (cf Proposition 4.4) that operators $\widetilde{\alpha}$, $\widetilde{\gamma}$ and $\widetilde{\epsilon}$ satisfy the first three Conditions of definition 1.1. So we may use Proposition 5.1. This way we obtain a Hilbert $\widetilde{H}^{\rm ex} \supset \widetilde{H}$ and operators $\operatorname{Phase}^{\rm ex}\widetilde{\alpha}$, $\widetilde{\gamma}^{\rm ex}$ and $\widetilde{\epsilon}^{\rm ex}$. We shall also use operators:

$$\left.\begin{aligned} \widetilde{v}^{\rm ex}_{Y} &= \widetilde{\epsilon}^{\rm ex}(\operatorname{Phase}^{\rm ex}\widetilde{\alpha})(\operatorname{Phase}\widetilde{\gamma}^{\rm ex}), \\ \widetilde{v}^{\rm ex}_{X} &= -(\operatorname{Phase}^{\rm ex}\widetilde{\alpha})(\operatorname{Phase}\widetilde{\gamma}^{\rm ex})^*, \\ \widetilde{V}^{\rm ex} &= [-\widetilde{\epsilon}^{\rm ex}(\operatorname{Phase}\widetilde{\gamma}^{\rm ex})^2]^{\log|q\widetilde{\gamma}^{\rm ex}|}. \end{aligned}\right\} \tag{5.11}$$

exwzory1tilde

These are unitaries acting on $\widetilde{H}$.

$$\Gamma^1 = q\gamma \otimes |\gamma^{\rm ex}|(\operatorname{Phase}^{\rm ex}\alpha) + \epsilon\alpha^* \otimes \gamma^{\rm ex}$$

$$\Gamma^2 = \gamma^{\rm ex} \otimes \alpha + \epsilon^{\rm ex}|\gamma^{\rm ex}|\,(\operatorname{Phase}^{\rm ex}\alpha)^* \otimes \gamma$$

$$\Phi = \operatorname*{s-lim}_{r\to\infty} \left(I \otimes v^{\rm ex}_{Y}\right)^{-r} \left(\widetilde{v}^{\rm ex}_{Y}\right)^{r}$$

$$\Phi^* = \operatorname*{s-lim}_{r\to\infty} \left(\widetilde{v}^{\rm ex}_{Y}\right)^{-r} \left(I \otimes v^{\rm ex}_{Y}\right)^{r}$$

$$\Phi\, \widetilde{v}^{\rm ex}_{Y} = \left(I \otimes v^{\rm ex}_{Y}\right) \Phi \tag{5.12}$$

PvY

$$\Phi\, \widetilde{v}^{\rm ex}_{Y}\Phi^* = I \otimes v^{\rm ex}_{Y}$$

$$\Phi\, \widetilde{\gamma}^{\rm ex}\Phi^* = \Gamma^1 \tag{5.13}$$

PGamma1

$$\left(I \otimes R^{\rm ex}_{X}\right)^* \Gamma^1 \left(I \otimes R^{\rm ex}_{X}\right) = \Gamma^1 \tag{5.14}$$

RGamma1

$$\Psi = \operatorname*{s-lim}_{r\to\infty} \left(v^{\rm ex}_{X} \otimes I\right)^{-r} \left(\widetilde{v}^{\rm ex}_{X}\right)^{r}$$

$$\Psi^* = \operatorname*{s-lim}_{r\to\infty} \left(\widetilde{v}^{\rm ex}_{X}\right)^{-r} \left(v^{\rm ex}_{X} \otimes I\right)^{r}$$

$$\Psi\, \widetilde{v}^{\rm ex}_{X} = \left(v^{\rm ex}_{X} \otimes I\right) \Psi \tag{5.15}$$

PvX

$$\Psi\, \widetilde{v}^{\rm ex}_{X}\Psi^* = v^{\rm ex}_{X} \otimes I$$

$$\Psi\, \widetilde{\gamma}^{\rm ex}\Psi^* = \Gamma^2 \tag{5.16}$$

PGamma2

$$\left(R^{\rm ex}_{Y} \otimes I\right)^* \Gamma^2 \left(R^{\rm ex}_{Y} \otimes I\right) = \Gamma^2 \tag{5.17}$$

RGamma2

$$\Phi^* \left(I \otimes R^{\rm ex}_{X}\right) \Phi = (-1)^{\log|q\widetilde{\gamma}|}\Psi^* \left(R^{\rm ex}_{Y} \otimes I\right) \Psi \tag{5.18}$$

Fund

6 Tensor product II

S6

In this Section we achieve the construction of the tensor product of two $S_q\tilde{U}(1,1)$-quadruples. We shall use the notation introduced in Section 4. Let $\pi_1 = (\alpha_1, \gamma_1, \epsilon_1, Y_1)$ and $\pi_2 = (\alpha_2, \gamma_2, \epsilon_2, Y_2)$ be $S_q\tilde{U}(1,1)$-quadruples of unbounded type. In Section 4 we constructed the first three elements $\widetilde{\alpha}$, $\widetilde{\gamma}$ and $\widetilde{\epsilon}$ of the tensor product $\pi_1 \bigoplus\!\!\!\!\!\top\, \pi_2$. We already know, that $\widetilde{\alpha}$, $\widetilde{\gamma}$ and $\widetilde{\epsilon}$ satisfy the first three conditions of Definition 1.1. Now we shall show that $\widetilde{\gamma}$ satisfies the spectral condition (1.6):

SPCONtildeP

Proposition 6.1 *Let* $\widetilde{\gamma}_+$ *be the restriction of* $\widetilde{\gamma}$ *to* $\widetilde{H}(\widetilde{\epsilon}=1)$*. Then*

$$\mathrm{Sp}\,(|\widetilde{\gamma}_+|) \subset q^{\mathbf{Z}} \cup \{0\}. \tag{6.1}$$ SPCONtilde

Remark: This result (in a more general version) appeared first in the Korogodski paper (see Proposition A1 of [8]). The proof presented below seems to be different that the one of Korogodski. We include it to the paper, because it is not clear, to what extend our setting coincides with the one used by Korogodski.

Proof: Let $\widetilde{R}^{\mathrm{ex}}_{Y}$ be the unitary operator introduced by

$$\widetilde{R}^{\mathrm{ex}}_{Y} = \Psi^* \left(R^{\mathrm{ex}}_{Y} \otimes I\right) \Psi. \tag{6.2}$$ RYtilde

Combining (5.16) with (5.17) one can easily show that $\widetilde{\gamma}^{\mathrm{ex}}$ commutes with $\widetilde{R}^{\mathrm{ex}}_{Y}$. Therefore

$$\left(\widetilde{R}^{\mathrm{ex}}_{Y}\right)^* |\widetilde{\gamma}^{\mathrm{ex}}| \widetilde{R}^{\mathrm{ex}}_{Y} = |\widetilde{\gamma}^{\mathrm{ex}}|. \tag{6.3}$$ inv62

$$\left(\widetilde{R}^{\mathrm{ex}}_{Y}\right)^* \mathrm{Phase}\,(\widetilde{\gamma}^{\mathrm{ex}}) \widetilde{R}^{\mathrm{ex}}_{Y} = \mathrm{Phase}\,(\widetilde{\gamma}^{\mathrm{ex}}). \tag{6.4}$$ inv63

Clearly $\widetilde{R}^{\mathrm{ex}}_{Y}$ anticommutes with $\widetilde{\epsilon}^{\mathrm{ex}}$. Formula (6.3) shows that the restrictions $|\widetilde{\gamma}^{\mathrm{ex}}_{\pm}|$ of $|\widetilde{\gamma}^{\mathrm{ex}}|$ to $\widetilde{H}^{\mathrm{ex}}(\widetilde{\epsilon}^{\mathrm{ex}} = \pm 1)$ are unitarily equivalent. In particular

$$\mathrm{Sp}\,|\widetilde{\gamma}^{\mathrm{ex}}_{+}| = \mathrm{Sp}\,|\widetilde{\gamma}^{\mathrm{ex}}_{-}|.$$

Using now (5.2) and (5.3) (with γ replaced by $\widetilde{\gamma}$) we immediately obtain (6.1).

Q.E.D.

Now we shall introduce the forth element of the $S_q\tilde{U}(1,1)$-quadruple $\pi_1 \bigoplus\!\!\!\!\!\!\top\; \pi_2 = (\widetilde{\alpha}, \widetilde{\gamma}, \widetilde{\epsilon}, \widetilde{Y})$. According to Proposition 3.3, $\widetilde{Y}$ is determined by a unitary involution $\widetilde{u}$ acting on $\widetilde{K} = \widetilde{H}(|\widetilde{\gamma}| = q^{-1})$. $\widetilde{u}$ should commute with $\mathrm{Phase}\,\widetilde{\gamma}$ and anticommute with $\widetilde{\epsilon}$.

Let $\widetilde{R}^{\mathrm{ex}}_{Y}$ be the operator introduced by (6.2). By (6.3), subspace $\widetilde{H}^{\mathrm{ex}}(|\widetilde{\gamma}| = q^{-1})$ is $\widetilde{R}^{\mathrm{ex}}_{Y}$-invariant. Using Statement 3 of Proposition 5.1 one can easily show that $\widetilde{H}^{\mathrm{ex}}(|\widetilde{\gamma}| = q^{-1})$ coincides with $\widetilde{H}(|\widetilde{\gamma}| = q^{-1}) = \widetilde{K}$. By definition $\widetilde{u}$ is the restriction of $\widetilde{R}^{\mathrm{ex}}_{Y}$ to $\widetilde{K}$. Relation (6.4) shows that $\widetilde{u}$ commutes with $\mathrm{Phase}\,\widetilde{\gamma}$. We know that Ψ commutes with $\widetilde{\epsilon}$. Remembering that R^{ex}_{Y} anticommutes with ϵ, one can easily show that $\widetilde{R}^{\mathrm{ex}}_{Y}$ anticommutes with $\widetilde{\epsilon}$. So does $\widetilde{u}$. The construction of $\pi_1 \bigoplus\!\!\!\!\!\!\top\; \pi_2 = (\widetilde{\alpha}, \widetilde{\gamma}, \widetilde{\epsilon}, \widetilde{Y})$ has came to the end.

Combining (5.5) with (5.18) we may rewrite (6.2) in the following way:

$$\widetilde{R}^{\mathrm{ex}}_{Y} = (-1)^{\log|q\widetilde{\gamma}^{\mathrm{ex}}|}\; \Phi^* \left(I \otimes (-1)^{\log|q\gamma^{\mathrm{ex}}|} R^{\mathrm{ex}}_{Y}\right) \Phi. \tag{6.5}$$ RYtilde1

Using the first relation of (5.6) one can easily show that v_Y anticommutes with $(-1)^{\log|q\gamma^{\mathrm{ex}}|}$. Consequently $\widetilde{v}_Y$ anticommutes with $(-1)^{\log|q\widetilde{\gamma}^{\mathrm{ex}}|}$. Taking into account

(5.12) and remembering that $v_{_Y}$ commutes with $R^{\rm ex}_{_Y}$ (cf (5.6)), we see that $\widetilde{v}_{_Y}$ commutes $\widetilde{R}^{\rm ex}_{_Y}$. Therefore

$$\widetilde{R}^{\rm ex}_{_Y} = \sum_{n\in\mathbf{Z}} \left(\widetilde{v}^{\rm ex}_{_Y}\right)^{-n} \widetilde{u} \left(\widetilde{v}^{\rm ex}_{_Y}\right)^{n}.$$

It shows that $\widetilde{R}^{\rm ex}_{_Y}$ is the extended Y-reflection operator associated with the $S_q\tilde{U}(1,1)$-quadruple $(\widetilde{\alpha},\widetilde{\gamma},\widetilde{\epsilon},\widetilde{Y})$.

We shall also need the formulae for the extended X-reflection operator $\widetilde{R}^{\rm ex}_{_X}$ associated with this quadruple, parallel to (6.2) and (6.5). According to (5.5), $\widetilde{R}^{\rm ex}_{_X} = (-1)^{\log|q\widetilde{\gamma}^{\rm ex}|}\widetilde{R}^{\rm ex}_{_Y}$. In particular $\widetilde{R}^{\rm ex}_{_X}$ restricted to $\widetilde{K}$ coincides with $\widetilde{u}$. Taking into account (5.18) we obtain:

$$\begin{aligned}\widetilde{R}^{\rm ex}_{_X} &= \Phi^*\left(I\otimes R^{\rm ex}_{_X}\right)\Phi\\ &= (-1)^{\log|q\widetilde{\gamma}^{\rm ex}|}\ \Psi^*\left((-1)^{\log|q\gamma^{\rm ex}|}R^{\rm ex}_{_X}\otimes I\right)\Psi.\end{aligned}\tag{6.6}$$
RXtilde

The rest of the Section is devoted to the proof of Statement 3 of Theorem 1.4.

Statem3

Proposition 6.2 *Let $\widetilde{K}=\widetilde{H}\left(|\widetilde{\gamma}|=q^{-1}\right)$ and*

$$z=\chi\left(|\widetilde{\gamma}|\,|\gamma\otimes\gamma^{-1}|=q^{n}\right)(x\otimes\epsilon^{n-r}y),\tag{6.7}$$
DYtdow

where $n,r\in\mathbf{Z}$, $x\in H_1(|\gamma|=q^r)$ and $y\in D(Y_2)$. Then:

1. *The sequence $\left(q^{-\frac{k}{2}}\chi(|\widetilde{\gamma}|=q^{-1})\left(\widetilde{v}^*_{_Y}\right)^k z\right)_{k\in\mathbf{N}}$ is q-convergent and its limit*

$$\widetilde{\theta}_{_Y}(z)=q^{\frac{r-n}{2}}\chi\left(|\widetilde{\gamma}^{\rm ex}|=q^{-1}\right)\ \Phi^*\left[x\otimes\left(\epsilon\, v^{\rm ex}_{_Y}\right)^{n-r}\theta_{_Y}(y)\right],\tag{6.8}$$
theta6'

where

$$\theta_{_Y}(y)=\lim_{k\to\infty}\left(q^{-\frac{k}{2}}\chi(|\gamma|=q^{-1})\left(v^*_{_Y}\right)^k y\right).\tag{6.9}$$
theta6

2. *$\widetilde{\theta}_{_Y}(z)\in\widetilde{K}(\widetilde{u}=1)$.*
3. *The set*

$$\left\{\widetilde{\theta}_{_Y}(z):\ \begin{array}{c} n,r\in\mathbf{Z},\ y\in D(Y_2)\\ x\in H_1(|\gamma|=q^r)\end{array}\right\}$$

is linearly dense in $\widetilde{K}(\widetilde{u}=1)$.

Proof: Let

$$\zeta_k=q^{-\frac{k}{2}}\chi(|\widetilde{\gamma}|=q^{-1})\left(\widetilde{v}^*_{_Y}\right)^k z.\tag{6.10}$$
dzetak

We know (cf Statement 6 of Proposition 3.1) that Phase $\widetilde{\alpha}^*$ multiplies the eigenvalues of $|\widetilde{\gamma}|$ by q. So does $\widetilde{v}^*_{_Y}$. Therefore

$$\chi(|\widetilde{\gamma}|=q^{-1})\left(\widetilde{v}^*_{_Y}\right)^k\ =\ \chi(|\widetilde{\gamma}|=q^{-1})\left(\widetilde{v}^*_{_Y}\right)^k\chi(|\widetilde{\gamma}|=q^{-1-k}).$$

Remembering that $|\widetilde{\gamma}|$ and $|\gamma \otimes \gamma^{-1}|$ strongly commute we get:

$$\chi\left(|\widetilde{\gamma}| = q^{-1-k}\right)\chi\left(|\widetilde{\gamma}|\,|\gamma \otimes \gamma^{-1}| = q^{n}\right) \;=\; \chi\left(|\widetilde{\gamma}| = q^{-1-k}\right)\chi\left(|\gamma \otimes \gamma^{-1}| = q^{n+k+1}\right)$$

We assumed that $x \in H_1(|\gamma| = q^r)$. Therefore

$$\chi\left(|\gamma \otimes \gamma^{-1}| = q^{n+k+1}\right)\left(x \otimes \epsilon^{n-r}\, y\right) = x \otimes \epsilon^{n-r}\chi\left(|\gamma| = q^{r-n-k-1}\right) y.$$

Finally, by virtue of (5.6),

$$\chi\left(|\gamma| = q^{r-n-k-1}\right) = \left(v^{\mathrm{ex}}_{Y}\right)^{k+n-r}\chi\left(|\gamma| = q^{-1}\right)\left(v^{\mathrm{ex}}_{Y}\right)^{-k-n+r}.$$

Inserting these data into (6.10) we get:

$$\zeta_k = \chi\left(|\widetilde{\gamma}| = q^{-1}\right)\left(\widetilde{v}^{\mathrm{ex}}_{Y}\right)^{-k}\left(I \otimes v^{\mathrm{ex}}_{Y}\right)^{k+n-r}\left[x \otimes \epsilon^{n-r} q^{-\frac{k}{2}}\chi\left(|\gamma| = q^{-1}\right)\left(v^{*}_{Y}\right)^{k+n-r}\, y\right].$$

We have to show that the sequence $(\zeta_k)_{k\in\mathbf{N}}$ is q-convergent. The vector y in the above formula belongs to $D(Y_2)$. By (3.18), the sequence in (6.9) is q-convergent. Therefore it is sufficient to show that the sequence

$$\left(q^{\frac{r-n}{2}}\chi\left(|\widetilde{\gamma}| = q^{-1}\right)\left(\widetilde{v}^{\mathrm{ex}}_{Y}\right)^{-k}\left(I \otimes v^{\mathrm{ex}}_{Y}\right)^{k}\left[x \otimes \left(v^{\mathrm{ex}}_{Y}\,\epsilon\right)^{n-r}\theta_{Y}(y)\right]\right)_{k\in\mathbf{N}}$$

is q-convergent. This fact follows immediately from (**??**). Clearly the limit coincides with (6.8). Statement 1 is proved.

We shall prove Statement 2. Comparing the first two formulae of (5.4) we see that $\epsilon^{\mathrm{ex}}\, v^{\mathrm{ex}}_{Y} = -(\mathrm{Phase}\,\gamma)^2\, v^{\mathrm{ex}}_{X}$. Therefore (cf. (5.6)) $\epsilon^{\mathrm{ex}}\, v^{\mathrm{ex}}_{Y}$ commutes with R^{ex}_{X}. We know that $y \in D(Y)$. According to (3.18), $\theta_{Y}(y) \in K(u = 1)$. Remembering that R^{ex}_{X} restricted to K coincides with u we get

$$\left(\epsilon^{\mathrm{ex}}\, v^{\mathrm{ex}}_{Y}\right)^{n-r}\theta_{Y}(y) \in H^{\mathrm{ex}}(R^{\mathrm{ex}}_{X} = 1). \tag{6.11}$$ 001

Therefore

$$x \otimes \left(\epsilon^{\mathrm{ex}}\, v^{\mathrm{ex}}_{Y}\right)^{n-r}\theta_{Y}(y) \in (H_1 \otimes H_2^{\mathrm{ex}})((I \otimes R^{\mathrm{ex}}_{X}) = 1) \tag{6.12}$$ 002

and

$$\Phi^{*}\left[x \otimes \left(\epsilon^{\mathrm{ex}}\, v^{\mathrm{ex}}_{Y}\right)^{n-r}\theta_{Y}(y)\right] \in \widetilde{H}^{\mathrm{ex}}\left(\Phi^{*}(I \otimes R^{\mathrm{ex}}_{X})\Phi = 1\right). \tag{6.13}$$ 003

Using (6.6) and remembering that $\widetilde{R}^{\mathrm{ex}}_{X}$ restricted to $\widetilde{K}$ coincides with $\widetilde{u}$ we get

$$\chi\left(|\widetilde{\gamma}^{\mathrm{ex}}| = q^{-1}\right)\Phi^{*}\left[x \otimes \left(\epsilon^{\mathrm{ex}}\, v^{\mathrm{ex}}_{Y}\right)^{n-r}\theta_{Y}(y)\right] \in \widetilde{K}(\widetilde{u} = 1). \tag{6.14}$$ 004

Statement 2 is proved.

To prove Statement 3 we notice, that due to Theorem 2.3, the set $\{\theta_{Y}(y) : y \in D(Y)\}$ coincides with $K(u = 1)$. Therefore the set of elements of the form (6.11) coincides with $H^{\mathrm{ex}}\left(|\gamma^{\mathrm{ex}}| = q^{r-n-1}, R^{\mathrm{ex}}_{X} = 1\right)$ and the set of elements of the form (6.12) is (for fixed

$r,n\in\mathbf{Z}$) linearly dense in $H_1(|\gamma|=q^r)\otimes H_2^{\mathrm{ex}}\left(|\gamma^{\mathrm{ex}}|=q^{r-n-1}, R_X^{\mathrm{ex}}=1\right)$. Considering all possible $r,n\in\mathbf{Z}$ we see that the set of elements of the form (6.12) is linearly dense in $(H_1\otimes H_2^{\mathrm{ex}})((I\otimes R_X^{\mathrm{ex}})=1)$. Consequently the set of elements of the form (6.13) is linearly dense in $\widetilde{H}^{\mathrm{ex}}\left(\Phi^*(I\otimes R_X^{\mathrm{ex}})\Phi=1\right)$ and the set of elements of the form (6.14) is linearly dense in $\widetilde{K}(\widetilde{u}=1)$.

Q.E.D.

Statement 3 of Theorem 1.4 follows immediately from Proposition 6.2 combined with Theorem 2.3 and Proposition 2.5. The proof of Theorem 1.4 is complete.

7 The associativity of the tensor product

S7

This Section is devoted to the proof of Theorem 1.5. Let $\pi_1=(\alpha_1,\gamma_1,\epsilon_1,Y_1)$, $\pi_2=(\alpha_2,\gamma_2,\epsilon_2,Y_2)$ and $\pi_3=(\alpha_3,\gamma_3,\epsilon_3,Y_3)$ be $S_q\tilde{U}(1,1)$-quadruples of unbounded type acting on Hilbert spaces H_1,H_2 and H_3 respectively. We have to show that

$$\pi_1\mathbin{\textcircled{$\top$}}\widetilde{\pi}_{23}=\widetilde{\pi}_{12}\mathbin{\textcircled{$\top$}}\pi_3,$$

where $\widetilde{\pi}_{12}=\pi_1\mathbin{\textcircled{$\top$}}\pi_2$ and $\widetilde{\pi}_{23}=\pi_2\mathbin{\textcircled{$\top$}}\pi_3$. To make temporary distinction between the two threefold tensor products we shall use the following notation. Operators $\widetilde{\alpha}_{\min}$, $\widetilde{\gamma}_{\min}$, $\widetilde{\alpha}^+_{\min}$, $\widetilde{\gamma}^+_{\min}$, $\widetilde{\alpha}_{\max}$, $\widetilde{\gamma}_{\max}$, $\widetilde{\alpha}^+_{\max}$, $\widetilde{\gamma}^+_{\max}$, $\widetilde{\alpha}$, $\widetilde{\gamma}$, $\widetilde{\epsilon}$ and $\widetilde{Y}$ related to the tensor product $\pi_1\mathbin{\textcircled{$\top$}}\widetilde{\pi}_{23}$ will be denoted by $\overset{\scriptscriptstyle R}{\alpha}_{\min}$, $\overset{\scriptscriptstyle R}{\gamma}_{\min}$, $\overset{\scriptscriptstyle R+}{\alpha}_{\min}$, $\overset{\scriptscriptstyle R+}{\gamma}_{\min}$, $\overset{\scriptscriptstyle R}{\alpha}_{\max}$, $\overset{\scriptscriptstyle R}{\gamma}_{\max}$, $\overset{\scriptscriptstyle R+}{\alpha}_{\max}$, $\overset{\scriptscriptstyle R}{\gamma}_{\max}$, $\overset{\scriptscriptstyle R}{\alpha}$, $\overset{\scriptscriptstyle R}{\gamma}$, $\overset{\scriptscriptstyle R}{\epsilon}$ and $\overset{\scriptscriptstyle R}{Y}$ ('tilde' is replaced by R). The corresponding operators related to $\widetilde{\pi}_{12}\mathbin{\textcircled{$\top$}}\pi_3$ will be denoted by $\overset{\scriptscriptstyle L}{\alpha}_{\min}$, $\overset{\scriptscriptstyle L}{\gamma}_{\min}$, $\overset{\scriptscriptstyle L+}{\alpha}_{\min}$, $\overset{\scriptscriptstyle L+}{\gamma}_{\min}$, $\overset{\scriptscriptstyle L}{\alpha}_{\max}$, $\overset{\scriptscriptstyle L}{\gamma}_{\max}$, $\overset{\scriptscriptstyle L+}{\alpha}_{\max}$, $\overset{\scriptscriptstyle L}{\gamma}_{\max}$, $\overset{\scriptscriptstyle L}{\alpha}$, $\overset{\scriptscriptstyle L}{\gamma}$, $\overset{\scriptscriptstyle L}{\epsilon}$ and $\overset{\scriptscriptstyle L}{Y}$ ('tilde' is replaced by L). We have:

$$\begin{aligned}\pi_1\mathbin{\textcircled{$\top$}}(\pi_2\mathbin{\textcircled{$\top$}}\pi_3)&=(\overset{\scriptscriptstyle R}{\alpha},\overset{\scriptscriptstyle R}{\gamma},\overset{\scriptscriptstyle R}{\epsilon},\overset{\scriptscriptstyle R}{Y}),\\(\pi_1\mathbin{\textcircled{$\top$}}\pi_2)\mathbin{\textcircled{$\top$}}\pi_3&=(\overset{\scriptscriptstyle L}{\alpha},\overset{\scriptscriptstyle L}{\gamma},\overset{\scriptscriptstyle L}{\epsilon},\overset{\scriptscriptstyle L}{Y}).\end{aligned}$$

We shall keep 'tilde' to denote operators related to the tensor products $\pi_i\mathbin{\textcircled{$\top$}}\pi_{i+1}$ $(i=1,2)$. The value of i will be clear from the context. For example in (**??**), $\widetilde{\alpha}$ denotes the first component of π_{12}. Like in Section 4 the components of π_i we be simply denoted by α, γ, ϵ and Y. The value of the omitted index i follows from the position of the considered operator in the tensor product.

Iterating the third formula of Statement 1 of Theorem 1.4 we get:

$$\begin{aligned}\overset{\scriptscriptstyle R}{\epsilon}&=\epsilon\otimes\widetilde{\epsilon}=\epsilon\otimes\epsilon\otimes\epsilon,\\\overset{\scriptscriptstyle L}{\epsilon}&=\widetilde{\epsilon}\otimes\epsilon=\epsilon\otimes\epsilon\otimes\epsilon.\end{aligned}$$

Therefore

$$\overset{\scriptscriptstyle R}{\epsilon}=\overset{\scriptscriptstyle L}{\epsilon}.$$

We know, that the comultiplication (0.7) is coassociative on the Hopf *-algebra level. Therefore $\overset{\scriptscriptstyle R}{\alpha}$ and $\overset{\scriptscriptstyle L}{\alpha}$ coincide on $D(\alpha)\otimes_{\mathrm{alg}}D(\alpha)\otimes_{\mathrm{alg}}D(\alpha)$. It shows that $\overset{\scriptscriptstyle R}{\alpha}$ and $\overset{\scriptscriptstyle L}{\alpha}$ are balanced extensions of one operator. The same statement holds for $\overset{\mathrm{R}}{\gamma}$ and $\overset{\mathrm{L}}{\gamma}$. We have to show that $\overset{\mathrm{R}}{\alpha}=\overset{\mathrm{L}}{\alpha}$ and $\overset{\mathrm{R}}{\gamma}=\overset{\mathrm{L}}{\gamma}$. By Propositions 0.2 and Remark 1.2 it is sufficient to prove

prfinal

Proposition 7.1 *$D(\overset{\mathrm{R}}{\alpha})$ contains a core of $\overset{\mathrm{L}}{\alpha}$.*

Let Λ be the set introduced by (1.7). For any $\lambda=(\lambda_1,\lambda_2,\lambda_3)\in\Lambda^3$ we set

$$\tilde{\tilde{H}}(\lambda)=H_1(\epsilon|\gamma|=\lambda_1)\otimes H_2(\epsilon|\gamma|=\lambda_2)\otimes H_3(\epsilon|\gamma|=\lambda_3).$$

Any vector $x\in\tilde{\tilde{H}}=H_1\otimes H_2\otimes H_3$ is of the form: $x=\sum x(\lambda)$, where $x(\lambda)\in\tilde{\tilde{H}}(\lambda)$. By definition the support of x is the set $\operatorname{supp} x=\{\lambda\in\Lambda^3 : x(\lambda)\neq 0\}$. Remembering that $\overset{\mathrm{R}}{\alpha}$ and $\overset{\mathrm{L}}{\alpha}$ coincide on $D(\alpha)\otimes_{\mathrm{alg}} D(\alpha)\otimes_{\mathrm{alg}} D(\alpha)$ one can easily show that

$$\left(x\middle|\ \overset{\mathrm{R}}{\alpha}\ y\right)=\left(\overset{\mathrm{L}*}{\alpha}x\middle|y\right)\quad\text{and}\quad\left(x\middle|\ \overset{\mathrm{R}*}{\alpha}y\right)=\left(\overset{\mathrm{L}}{\alpha}\ x\middle|y\right)\tag{7.1}$$ finsup

for any vectors $x,y\in\tilde{\tilde{H}}$ with finite supports. A moment of reflection shows that (7.1) holds for any $x\in D(\overset{\mathrm{L}}{\alpha})$ and $y\in D(\overset{\mathrm{R}}{\alpha})$ provided the intersection $\operatorname{supp} x\cap\operatorname{supp} y$ is finite.

Let

$$x=\chi\begin{pmatrix}|\gamma_1\otimes\gamma_2^{-1}|=q^m\\ \epsilon_1\otimes\epsilon_2=s\end{pmatrix}(\epsilon_1^m\,x_1\otimes x_2)\otimes x_3,\tag{7.2}$$ npx

$$y=y_1\otimes\chi\begin{pmatrix}|\gamma_2\otimes\gamma_3^{-1}|=q^{m'}\\ \epsilon_2\otimes\epsilon_3=s'\end{pmatrix}(\epsilon_2^{m'}\,y_2\otimes y_3),\tag{7.3}$$ npy

where $x_1\in H_1$, $x_2\in H_2$, $x_3\in H_3(|\gamma_3|=q^n)$, $y_1\in H_1(|\gamma_1|=q^{n'})$, $y_2\in H_2$, $y_3\in H_3$, $n,m,n',m'\in\mathbf{Z}$ and $s,s'=\pm1$. One can easily verify that the intersection of the supports of x and y is empty if $n'-m\neq m'+n$. If $n'-m=m'+n$ then any point $\lambda\in\operatorname{supp} x\cap\operatorname{supp} y$ is of the form $\lambda=\left(\pm q^{n'},\pm q^{m'+n},\pm q^n\right)$. In any case the intersection is finite. According to Statement 2 of Theorem 1.4, operator $\overset{\mathrm{L}}{\alpha}_{\min}$ has a core consisting of linear combinations of vectors of the form (7.2) and vectors with finite supports. Similarly operator $\overset{\mathrm{R}}{\alpha}_{\min}$ has a core consisting of linear combinations of vectors of the form (7.3) and vectors with finite supports. Therefore (7.1) holds for any $x\in D(\overset{\mathrm{L}}{\alpha}_{\min})$ and $y\in D(\overset{\mathrm{R}}{\alpha}_{\min})$. It shows that

$$\begin{aligned}\overset{\mathrm{L}}{\alpha}_{\min}&\subset\overset{\mathrm{R}}{\alpha}_{\max},\\ \overset{\mathrm{R}}{\alpha}_{\min}&\subset\overset{\mathrm{L}}{\alpha}_{\max}\ .\end{aligned}\tag{7.4}$$ porownanie

Appendices

A Unbounded operators

Proposition A.1 *Let T be a closed operator acting on H and w be a bounded operator such that* $\inf \mathrm{Sp}\,(w^*w) > 0$. *Then operators wT, T^*w^* are closed, densely defined and $(wT)^* = T^*w^*$. Operators wT and T have the same cores. If D' is a core for $(wT)^*$, then w^*D' is a core for T^*.* A1 mnozenie

Proof: Let c be the strictly positive number such that $c^2 = \inf \mathrm{Sp}\,(w^*w)$. Then $\|w\|\,\|x\| \geq \|wx\| \geq c\|x\|$ for any $x \in H$. Therefore for any $x_n \in H$ $(n = 1, 2, \ldots)$, the convergence of $(x_n)_{n\in\mathbf{N}}$ is equivalent to the convergence of $(wx_n)_{n\in\mathbf{N}}$. Using this fact one can easily show that wT is closed and that wT and T have the same cores.

Let $x \in D\left((wT)^*\right)$. Then for any $y \in D(T)$ we have: $((wT)^*x|y) = (x|wTy) = (w^*x|Ty)$. It shows that $w^*x \in D(T^*)$ and $T^*w^*x = (wT)^*x$. Therefore $(wT)^* \subset T^*w^*$. Conversely if $x \in H$ and $w^*x \in D(T^*)$, then for any $y \in D(T)$ we have: $(x|wTy) = (w^*x|Ty) = (T^*w^*x|y)$. It shows that $x \in D((wT)^*)$ and $(wT)^*x = T^*w^*x$. Therefore $T^*w^* \subset (wT)^*$ and $T^*w^* = (wT)^*$. In particular T^*w^* is densely defined and closed.

Let D' be a core for $(wT)^* = T^*w^*$. Strict positivity of w^*w implies that w^* is surjective. Therefore w^*D' is dense in H. Clearly $w^*D' \subset D(T^*)$. Let T^+ be the restriction of T^* to w^*D'. Then $T^+ \subset T^*$ and $T \subset (T^+)^*$.

Let $x \in D((T^+)^*)$. Then for any $y \in D'$ we have: $(w(T^+)^*x|y) = ((T^+)^*x|w^*y) = (x|T^+w^*y) = (x|T^*w^*y) = (x|(wT)^*y)$. Remembering that D' be a core for $(wT)^*$ we conclude that $x \in D(wT) = D(T)$. Therefore $D((T^+)^*) \subset D(T)$ and $(T^+)^* = T$. We showed that T^+ and T^* have the same adjoint. Therefore the closure of T^+ coincides with T^*. It means that w^*D' is a core for T^*.

Q.E.D.

Proposition A.2 *Let T_1, T_2 be a closed operators acting on a Hilbert space H and D_0 be a dense subset of H such that $D_0 \subset D(T_i)$ $(i = 1, 2)$ and* WO2

$$T_1x = T_2x \tag{A.1}$$

sumaa

for any $x \in D_0$. Assume that $D(T_1^*) \cap D(T_2^*)$ is dense in H. Then (A.1) *holds for any $x \in D(T_1) \cap D(T_2)$.*

Proof: Let $y \in D(T_1^*) \cap D(T_2^*)$. Then for any $x \in D_0$ we have:

$$(T_1^*y|x) = (y|T_1x) = (y|T_2x) = (T_2^*y|x)\,.$$

Remembering that D_0 is dense in H we see that $T_1^*y = T_2^*y$.

Now, let $x \in D(T_1) \cap D(T_2)$. Then for any $y \in D(T_1^*) \cap D(T_2^*)$ we have:

$$(y|T_1x) = (T_1^*y|x) = (T_2^*y|x) = (y|T_2x)$$

and (A.1) follows.

Q.E.D.

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