QUANTUM '$ax + b$' GROUP.

S.L. WORONOWICZ AND S. ZAKRZEWSKI

Abstract. '$ax + b$' is the group of affine transformations of real line $\mathbf{R}$. In quantum version $ab = q^2 ba$, where $q^2 = e^{-i\hbar}$ is a number of modulus 1. The main problem of constructing quantum deformation of this group on the C*-level consists in non-selfadjointness of $\Delta(b) = a \otimes b + b \otimes I$. This problem is overcome by introducing (in addition to a and b) a new generator β commuting with a and anticommuting with b. β (or more precisely $\beta \otimes \beta$) is used to select a suitable selfadjoint extension of $a \otimes b + b \otimes I$. Furthermore we have to assume that $\hbar = \pm\frac{\pi}{2k+3}$, where $k = 0, 1, 2, \ldots$. In this case q is a root of 1.

To construct the group we write an explicit formula for the Kac–Takesaki operator W. It is shown that W is a manageable multiplicative unitary in the sense of [3, 19]. Then using the general theory we construct a C*-algebra A and a comultiplication $\Delta \in \mathrm{Mor}(A, A \otimes A)$. A should be interpreted as the algebra of all continuous functions vanishing at infinity on quantum '$ax + b$' - group. The group structure of is encoded by Δ. The existence of coinverse also follows from the general theory [19].

0. Introduction (written by SLW)

This research was proposed and originated by S. Zakrzewski at the end of 1997. Working within the semiclassical framework (Poisson-Lie groups, simplectic leaves, Manin pairs, simplectic groupoids) he gained a deep understanding, how certain incompletenesses on the semiclassical level are reflected in an attempt to construct the corresponding quantum group on the C*-level. This paper was supposed to contain a number of Sections devoted to this framework. We planned to explain in detail, how the semiclassical considerations lead in a natural way to a concept of reflection operator used on the C*-level. It was Stanisław Zakrzewski, who was supposed to write these Sections. Unfortunately after his sudden death in April 1998, the first author was unable to reconstruct this part of the paper.

In the construction of the quantum deformation of the '$ax + b$' group on the Hilbert space level one meets the following two problems. First one has to give meaning to the relation '$ab = q^2 ba$', where a, b are selfadjoint operators acting on a Hilbert space and q^2 is a number of modulus 1. This problem was considered by many authors. Assume for the moment that a and b are strictly positive. In [10, 11] K. Schmüdgen proposed to rewrite '$ab = q^2 ba$' in the Weyl form: $a^{it} b^{i\tau} = e^{i\hbar t\tau} b^{i\tau} a^{it}$, where $\hbar$ is a real number such that $q^2 = e^{-i\hbar}$ and t, τ are variable running over $\mathbf{R}$. We shall use this formula in the the form: $a^{it} b a^{-it} = e^{\hbar t} b$ which is meaningful for any selfadjoint b. The original relation '$ab = q^2 ba$' is recovered by analytic continuation up to the point $t = -i$.

The second problem is related to the formula $\Delta(b) = a \otimes b + b \otimes I$. Since the comultiplication Δ is a C*-algebra morphism, we expect that $\Delta(b)$ has the same analytical properties as b. In particular $a \otimes b + b \otimes I$ should be selfadjoint. However this is not guaranteed and we have to use the theory of selfadjoint extension developed in [20].

We would like to make a short comment on the quantization of $\hbar$. It comes from the formula (6.5) of [20], where the constant $\alpha = ie^{\frac{i\pi^2}{2\hbar}}$ enters in an implicit way. The point is that this formula essentially simplifies, when $\alpha = \overline{\alpha}$. Solving this condition we obtain $\hbar = \frac{\pi}{2k+3}$, where k is an integer. The theory presented in this paper works only for these values of $\hbar$. It follows that q^2 is a root of unity: $q^{2(2k+3)} = -1$.

There is another version of quantum '$ax + b$' group for which $\hbar = \frac{\pi}{2k}$ (k - integer). It will be described in a separate paper [8].

The quantization of $\hbar$ seems to be of analytical nature. In particular the semiclassical theory developed by the second author does not imply any limitation of this sort.

A few words about the content of the paper. In Section 1 we present the quantum '$ax+b$' group on the Hopf *-algebra level. Next we outline the passage to the Hilbert space and C*-levels. To solve the selfadjointness problem arising on the way we have to extend our group by adding a new *generator* β called the reflection operator. The three operators a, b, β are subject to suitable commutation relations. The Section ends with a short description of the quantum '$ax + b$' group on the Hilbert space and C*-levels.

To construct '$ax + b$' we shall use the theory of multiplicative unitaries of Baaj and Skandalis [3, 19]. In Section 2 we consider a unitary operator W acting on the tensor square of a Hilbert space H: $W \in B(H \otimes H)$. It is introduced by an explicit formula containing four selfadjoint operators: a, b, β, s acting on H. The three first operators are subject to the commutation relations introduced in Section 1. The main result of Section 2 is contained in Theorem 2.1. It states that W is a manageable multiplicative unitary. The proof is based on the Fourier transform formula (1.41) of [20].

Once we have a manageable multiplicative unitary W we apply the theory developed in [3, 19] to construct a quantum group. This is done in Sections 3 and 4. In Section 3 we introduce the C*-algebra $A_{\rm cp}$ generated by three elements a, b, β subject to the commutation relations considered in Section 1. By definition $A_{\rm cp}$ is the crossed product: $A_{\rm cp} = B_0 \times_\sigma \mathbf{R}$, where B_0 is an algebra of continuous $M_{2\times 2}(\mathbf{C})$-valued functions on $\overline{\mathbf{R}_+}$ and σ is a natural action of $\mathbf{R}$ on B_0. We investigate in detail properties of $A_{\rm cp}$. In particular an interesting action ϕ of $\mathbf{Z}_4$ on $A_{\rm cp}$ is described at the end of this Section.

In Section 4 we show that the crossed product algebra $A_{\rm cp}$ coincides with the Baaj-Skandalis left-slice algebra related to the multiplicative unitary W considered in Section 2. We compute, that the comultiplication acts on generators a, b, β in the way described in Section 1. This way the construction of the quantum '$ax + b$' group on the C*-algebra level is completed. At the end of Section 4 we show that the action ϕ preserves the group structure of quantum '$ax + b$' .

Section 5 is devoted to the dual of the quantum '$ax+b$' group. By the definition the regular dual is the quantum group related to the multiplicative unitary $\widehat{W} = \Sigma W^* \Sigma$. We show that the regular dual of the quantum '$ax + b$' group is isomorphic to the same group, provided we reverse the order of the group rule. The same result holds for the universal (Pontryagin) dual. It is shown in [9] that in this case the regular dual and universal dual coincide.

This paper heavily depends on the results of [20]. In particular we shall use the quantum exponential function

$$F_\hbar(r,\varrho) = \begin{cases} V_\theta(\log r) & \text{for} \quad r > 0 \text{ and } \varrho = 0 \\ \left[1 + i\varrho|r|^{\frac{\pi}{\hbar}}\right] V_\theta\left(\log|r| - \pi i\right) & \text{for} \quad r < 0 \text{ and } \varrho = \pm 1. \end{cases} \tag{0.1}$$

where $\theta = \frac{2\pi}{\hbar}$ and V_θ is a meromorphic function on $\mathbf{C}$ such that

$$V_\theta(x) = \exp\left\{\frac{1}{2\pi i}\int_0^\infty \log(1 + a^{-\theta})\,\frac{da}{a + e^{-x}}\right\} \tag{0.2}$$

for all $x \in \mathbf{C}$ such that $|\Im x| < \pi$.

We shall also use the theory of selfadjoint operators on Hilbert spaces [1, 6], in particular the theory of selfadjoint extensions and functional calculus of many strongly commuting selfadjoint operators. Throughout the paper the symbol $\chi(\mathcal{R})$ denotes the logical evaluation of sentence $\mathcal{R}$: $\chi(\mathcal{R}) = 1$ for true $\mathcal{R}$ and $\chi(\mathcal{R}) = 0$ for $\mathcal{R}$. The sentence $\mathcal{R}$ may depend on a selfadjoint operator (or a pair of strongly commuting selfadjoint operators). Then $\chi(\mathcal{R})$ is the corresponding spectral projection. The range of $\chi(\mathcal{R})$ will be denoted by $H(\mathcal{R})$, where H stands for the Hilbert space, where the operator acts (for details, see the last part of Section 0 of [20]).

We refer to [2, 7] for the theory of C*-algebras. We shall freely use such notions as: multiplier algebra M(A) of a C*-algebra A, unbounded elements affiliated with a C*-algebra A, the set $\mathrm{Mor}(A, B)$

of all morphisms from A into B, a C*-algebra generated by a set of affiliated elements and so on. All these notions are presented in [16, 18, 17].

In this paper we use the physicists' conventions concerning Hilbert spaces. In particular the scalar product $(x|y)$ is by definition linear with respect to y. We shall also use the triple product $(x|a|y)$ to denote $(x|ay)$. When vectors x, y and operator a are themselves complicated expressions, then $(x|a|y)$ is more readable then $(x|ay)$. Formula (2.26) is a good example of this situation.

We would like to point out the further development of the subject. In what follows, G denotes the quantum '$ax + b$'-group constructed in this paper. A. Van Daele [15] has found left and right invariant Haar weights on G. He has shown that G is a locally compact quantum group in the sense of Kustermans and Vaes [4]. It turned out that the Haar weights are scaled by the scaling group in a non-trivial way. This is one of the first examples of this phenomenon. It was foreseen by the theory of Kustermans and Vaes, however some of the experts believed that in the proper theory the Haar weights should be invariant with respect to the scaling group. Using the nontrivial scaling of the Haar weights, S. Vaes and L. Vainerman have shown [14] that G is essentially different from the quantum deformation of classical '$ax + b$' proposed by Baaj and Skandalis [13]. In [9], M. Rowicka has shown that all unitary representations of G acting on a Hilbert space K are described by the formula (2.6).

The quantum group '$ax + b$' described in the present paper will be used as a building block in future constructions of higher dimensional quantum groups. We refer to [22], where quantum deformations of $SL(2, \mathbf{R})$ are presented.

For a long time quantum groups at roots of unity seemed to be inaccessible for the C*-approach. The present paper is one of the first successful attempts to include these groups into the theory of locally compact quantum groups. Another example of this kind is given in [21].

Acknowledgement A large part of the paper was written during the stays of the first author at the Institute of Mathematics of Trondheim University. Numerous discussions with members and visitors of the Institute helped me fix many points of this work. Among persons that contributed in this way were: Fons Van Daele, Magnus Landstad, Johan Kustermans, Stefaan Vaes, Yoshiomi Nakagami and many others. Special thanks are due to Magnus Landstad, Christian Skau and other members of the Institute for their exceptional hospitality and creation of an excellent atmosphere for the work.

Finally I would like to thank my close collaborators in Warsaw: Wiesław Pusz and Piotr Sołtan. The paper owes a lot to them. They read the entire manuscript making many important remarks and pointing out numerous errors and misprints.

The authors are grateful to Komitet Badań Naukowych (grant No 2 P0A3 030 14) and to The Foundation for Polish Science for the financial support.

1. First encounter with '$ax + b$' -group.

The group '$ax + b$' considered in this paper is the group of affine transformations of real line $\mathbf{R}$ preserving the orientation (in the transformation formula $x' = ax + b$ the coefficient a is strictly positive). The group will be denoted by G. The *-algebra $\mathcal{A}$ of polynomial functions on G is generated by three hermitian commuting elements a, a^{-1}, b subject to the one relation: $a^{-1}a = I$. The comultiplication Δ encoding the group structure is the *-algebra homomorphism from $\mathcal{A}$ into $\mathcal{A} \otimes \mathcal{A}$ such that

$$\begin{aligned} \Delta(a) &= \quad a \otimes a, \\ \Delta(b) &= a \otimes b + b \otimes I. \end{aligned} \tag{1.1}$$

One can easily verify that $(\mathcal{A}, \Delta)$ is a Hopf *-algebra. In particular counit e and coinverse κ are given by the formulae:

$$\begin{aligned} e(a) &= 1, \qquad & \kappa(a) &= a^{-1}, \\ e(b) &= 0, \qquad & \kappa(b) &= -a^{-1}b. \end{aligned} \tag{1.2}$$

Now we perform quantum deformation of G. The quantum '$ax + b$' -group on the level of Hopf *-algebra is an object with no problems. The deformation parameter q is a complex number of modulus 1. We shall assume that $q^2 \neq -1$. Then

$$q^2 = e^{-i\hbar}, \tag{1.3}$$

where $\hbar$ is a real number such that $|\hbar| < \pi$. The change of sign of $\hbar$ is equivalent to the passage to the opposite algebra. Therefore we shall assume that $\hbar > 0$.

The Hopf *-algebra $\mathcal{A}$ of polynomial functions on quantum '$ax+b$' is generated by three hermitian elements a, a^{-1}, b subject to the following relations:

$$\begin{aligned} a^{-1}a = aa^{-1} = I, \\ ab = q^2 ba \end{aligned} \tag{1.4}$$

The comultiplication $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is the *-algebra homomorphism acting on generators in the way described in (1.1). One can easily verify that the object $(\mathcal{A}, \Delta)$ described above is a Hopf *-algebra. The counit e and coinverse κ are given by the same formulae (1.2) as in the classical case. Moreover the matrix

$$u = \begin{pmatrix} a \ , \ b \\ 0 \ , \ I \end{pmatrix}$$

is a corepresentation of $(\mathcal{A}, \Delta)$. In other words, u is a two dimensional representation of the quantum '$ax + b$' - group.

On the Hilbert space level generators a, a^{-1} and b should be treated as unbounded[1] selfadjoint operators acting on a Hilbert space. Since for unbounded operators the algebraic operations are often ill defined, one has to give a more precise meaning to the formulae (1.4). In the operator setting equation $a^{-1}a = aa^{-1} = I$ simply means that a^{-1} is the inverse of a. Furthermore we shall assume that a is positive. This condition is obviously related to the fact that the corresponding classical group consists of transformations preserving the orientation of $\mathbf{R}$. Let $\hbar$ be the number related to deformation parameter q via formula (1.3). To give the precise meaning to the second relation of (1.4), we shall use the following definition:

Definition 1.1. *Let a and b be selfadjoint operators acting on a Hilbert space H. Assume that a is strictly positive. We write $a \multimap b$ if*

$$a^{it} b\, a^{-it} = e^{\hbar t} b \tag{1.5}$$

for any $t \in \mathbf{R}$.

More general definition of the relation $a \multimap b$ is given in [20]. It does not require any additional assumption on a. Inserting $t = -i$ in (1.5) and using (1.3) we obtain the second relation of (1.4). The reader should notice that the condition (1.4) is much weaker then the relation $a \multimap b$. For example (1.4) remains unchanged, when $\hbar$ is replaced by $\hbar + 2\pi$, whereas (1.5) is very sensitive to the choice of $\hbar$ solving the equation (1.3). Recall that we chose $\hbar$ such that $|\hbar| < \pi$.

Let H be a Hilbert space and a, b be operators acting on H. We say that (a, b) is a G-pair if

$$\left.\begin{array}{c} a, b \text{ are selfadjoint operators on } H, \\ a \text{ is strictly positive, } a \multimap b \end{array}\right\} \tag{1.6}$$

We shall use the terminology introduced in [18]. By the procedure described in Section 7 of [18], relations (1.6) give rise to a C*-algebra A. This C*-algebra is generated by two unbounded elements $\log a$, b affiliated with it and

$$\pi \longleftrightarrow \Big(\pi(a), \pi(b)\Big) \tag{1.7}$$

defines continuous one to one correspondence between the set $\mathrm{Rep}\,(A, H)$ of all representation of A acting on a Hilbert space H and the set of all G-pairs acting on H.

Assume now, that A is equipped with a comultiplication $\Delta \in \mathrm{Mor}(A, A \otimes A)$ such that (1.1) holds. Then for any $\pi_1 \in \mathrm{Rep}\,(A, H_1)$ and $\pi_2 \in \mathrm{Rep}\,(A, H_2)$ one may consider the tensor product:

$$\pi_1 \textcircled{\scriptsize T} \pi_2 = (\pi_1 \otimes \pi_2) \circ \Delta.$$

---

[1] one can easily check that relations (1.4) cannot be satisfied by bounded operators a and $b \neq 0$.

Clearly $\pi_1 \textcircled{\top} \pi_2 \in \mathrm{Rep}\,(A, H_1 \otimes H_2)$. Using the one to one correspondence (1.7) we may define the tensor product for G-pairs. If (a_1, b_1) is a G-pair acting on a Hilbert space H_1 and (a_2, b_2) is a G-pair acting on a Hilbert space H_2, then by virtue of (1.1):

$$(a_1, b_1) \textcircled{\top} (a_2, b_2) = (\widetilde{a}, \widetilde{b}),$$

where

$$\begin{array}{rcl} \widetilde{a} & = & a_1 \otimes a_2, \\ \widetilde{b} & = & a_1 \otimes b_2 + b_1 \otimes I. \end{array} \tag{1.8}$$

One expects that $(\widetilde{a}, \widetilde{b})$ is a G-pair acting on $H_1 \otimes H_2$. Unfortunately this is not always the case. It turns out that the operator $\widetilde{b}$ is symmetric but not selfadjoint in general (cf [20, Theorem 5.4]). This is a serious obstacle in constructing the quantum '$ax+b$' -group on C^*-level.

One may try to overcome this problem by extending $\widetilde{b}$ to a larger domain. Let $R = a_1 \otimes b_2$ and $S = b_1 \otimes I$. Then $R \multimap S$ and $\widetilde{b} = R + S$. By the theory developed in [20], selfadjoint extensions of $R+S$ are determined by reflection operators τ such that $\tau^* = \tau$, τ anticommutes with R and S and $\tau^2 = \chi(e^{i\hbar/2}RS < 0)$. Selfadjoint extension of $R+S$ corresponding to a reflection operator τ will be denoted by $\Big[R+S\Big]_\tau$. By definition $\Big[R+S\Big]_\tau$ is the restriction of $(R+S)^*$ to the domain $D(R+S) + \Big\{x \in D\Big((R+S)^*\Big) : \tau x = x\Big\}$.

For given R and S, the existence of a reflection operator is not guaranteed ($R+S$ may have no selfadjoint extensions). To assure the existence of τ in our setting we have to extend our scheme. Instead of G-pairs we have to consider G-triples. Let a, b, β be operators acting on a Hilbert space H. We say that (a, b, β) is a G-triple if

$$\left.\begin{array}{c} a, b, \beta \text{ are selfadjoint operators on } H, \\ a \text{ is strictly positive, } a \multimap b, \\ \beta^2 = \chi(b \neq 0), \beta a = a\beta \text{ and } \beta b = -b\beta. \end{array}\right\} \tag{1.9}$$

The set of all G-triples acting on a Hilbert space H will be denoted by G_H. Passing from G-pairs to G-triples means that we extended our group by adding a new element β to the set of generators of the algebra of functions on G. The extended group is in a sense two times bigger then the original one. In what follows, the term "quantum '$ax+b$' -group" will refer to the extended group (we shall skip the word 'extended' as we did in the title of this paper).

Let H_1 and H_2 be Hilbert spaces. We would like to introduce the '$\textcircled{\top}$' product of G-triples: for any $(a_1, b_1, \beta_1) \in G_{H_1}$ and $(a_2, b_2, \beta_2) \in G_{H_2}$,

$$(a_1, b_1, \beta_1) \textcircled{\top} (a_2, b_2, \beta_2) = (\widetilde{a}, \widetilde{b}, \widetilde{\beta}) \in G_{H_1 \otimes H_2}. \tag{1.10}$$

The first formula of (1.8) may be kept unchanged. To modify the second formula we chose $\alpha = \pm 1$. Then the operator $\tau = \alpha(\beta_1 \otimes \beta_2)\chi(b_1 \otimes b_2 < 0)$ is a reflection operator defining a selfadjoint extension of $a_1 \otimes b_2 + b_1 \otimes I$. Let

$$\widetilde{b} = \Big[a_1 \otimes b_2 + b_1 \otimes I\Big]_\tau. \tag{1.11}$$

To end the definition of the '$\textcircled{\top}$' product (1.10) we have to write a formula for $\widetilde{\beta}$. The simplest proposal is:

$$\widetilde{\beta} = \beta_1 \otimes \beta_2. \tag{1.12}$$

However this formula is not the right one. It leads to the tensor product (1.10) which is not associative, which contradicts the coassociativity of comultiplication Δ. Also the computations performed by the second author within the theory of Poisson – Lie grupoids indicated that the correct formula for $\widetilde{\beta}$ should be rather linear than quadratic with respect to β.

To find the correct replacement for (1.12) we shall use the theory of quantum exponential function developed in [20]. In particular the exponential equality (cf. [20, formula (6.5)]) will play an essential role. It implicitly contains a phase factor α related to the deformation parameter $\hbar$ by the formula:

$$\alpha = i\, e^{\frac{i\pi^2}{2\hbar}}. \tag{1.13}$$

The theory presented in this paper works only if this number coincides with the one used in the definition of the reflection operator appearing in (1.11). Now the condition $\alpha = \pm 1$ selects a discrete set of admissible values of deformation parameter $\hbar = \frac{\pi}{2k+3}$, where $k = 0, 1, 2, \dots$. Clearly $\alpha = (-1)^k$.

The correct formula replacing (1.12) is rather complicated:

$$\widetilde{\beta} = w\left(e^{i\hbar/2} b_1^{-1} a_1 \otimes b_2\right)^{-1} \left(\beta_1 \otimes I\right) + \left(I \otimes \beta_2\right) w\left(e^{i\hbar/2} b_1 a_1^{-1} \otimes b_2^{-1}\right)^{-1}, \tag{1.14}$$

where w is the polynomial of order $(2k+3)$ introduced by the formula:

$$w(t) = \prod_{\ell=1}^{2k+3} \left(1 + e^{i\left(\frac{1}{2}-\ell\right)\hbar} t\right). \tag{1.15}$$

This way we completed the definition of the tensor product (1.10). It will be shown that the triple on the right hand side of (1.10) really belongs to $G_{H_1 \otimes H_2}$ and that the '$\textcircled{\scriptsize T}$' product is associative.

We end this Section with a short description of quantum '$ax+b$' group on the C^*-level. The C^*-algebra A of all 'continuous functions vanishing at infinity on G' is generated (in the sense explained in [18]) by three selfadjoint affiliated elements: $\log a$, b and $i\beta b$. Element β is not affiliated with A. It corresponds to a 'non-continuous function' on the group. It becomes continuous when we remove the manifold $b = 0$ out of G. More precisely $\beta \in M(A_{b=0})$, where $A_{b=0}$ is the ideal of A generated by b. The comultiplication $\Delta \in \mathrm{Mor}(A, A \otimes A)$ is associative. On generators it acts in the following way:

$$\begin{aligned}
\Delta(a) &= a \otimes a, \\
\Delta(b) &= \Big[a \otimes b + b \otimes I\Big]_{\alpha(\beta\otimes\beta)\chi(b\otimes b<0)}, \\
\Delta(i\beta b) &= i\left\{w\left(e^{i\hbar/2} b^{-1} a \otimes b\right)^{-1} \left(\beta \otimes I\right) + \left(I \otimes \beta\right) w\left(e^{i\hbar/2} b a^{-1} \otimes b^{-1}\right)^{-1}\right\} \Delta(b).
\end{aligned}$$

It would be interesting to see, how the above objects and formulae behave when $\hbar \to 0$. This subject that is not discussed in the present paper.

2. The Kac–Takesaki operator.

The theory of Baaj and Skandalis provides us with a powerful method of constructing quantum groups on the C*-level. Let H be a Hilbert space and $W \in B(H \otimes H)$ be a unitary operator. We shall use the leg numbering notation: W_{kl} is a copy of W acting on $H \otimes H \otimes H$, affecting only k-th and l-th copy of H in $H \otimes H \otimes H$. According to [3], W is called a multiplicative unitary if it satisfies the pentagon equation:

$$W_{23} W_{12} = W_{12} W_{13} W_{23}. \tag{2.1}$$

Let $\overline{H}$ be the complex conjugate of H. For any $x \in H$, the corresponding element of $\overline{H}$ will be denoted by $\overline{x}$. Then $H \ni x \to \overline{x} \in \overline{H}$ is an antiunitary map. We say that a multiplicative unitary W is manageable [19] if there exist a positive selfadjoint operator Q acting on H and a unitary operator $\widetilde{W}$ acting on $\overline{H} \otimes H$ such that $\ker(Q) = \{0\}$,

$$W^*(Q \otimes Q) W = Q \otimes Q \tag{2.2}$$

and

$$(x \otimes u | W | z \otimes y) = (\overline{z} \otimes Qu | \widetilde{W} | \overline{x} \otimes Q^{-1} y) \tag{2.3}$$

for any $x, z \in H$, $y \in D(Q^{-1})$ and $u \in D(Q)$. As it is shown in [19], any manageable multiplicative unitary gives rise to a quantum group on the C*-level.

Throughout this Section we assume that the deformation parameter $q^2 = e^{-i\hbar}$, where $\hbar = \pm\frac{\pi}{2k+3}$, where $k = 0, 1, 2, \dots$. Then the constant

$$\alpha = i\, e^{\frac{i\pi^2}{2\hbar}} = (-1)^k = \pm 1. \tag{2.4}$$

The main result of this Section is contained in the following

Theorem 2.1. *Let H be a Hilbert space, $(a,b,\beta)\in G_H$ and r,s be strictly positive selfadjoint operators acting on H. Assume that $\ker b=\{0\}$, r and s strongly commute with a,b and β and r—∘ s. Then the operator*

$$W = F_\hbar\left(e^{i\hbar/2}b^{-1}a\otimes b, \alpha(\beta\otimes\beta)\chi(b\otimes b<0)\right)^* e^{\frac{i}{\hbar}\log(s|b|^{-1})\otimes\log a} \tag{2.5}$$

is a manageable multiplicative unitary.

The pentagon equation for (2.5) will follow from

Proposition 2.2. *Let H and K be Hilbert spaces, $(a,b,\beta)\in G_H$, $(\hat a,\hat b,\hat\beta)\in G_K$ and s be a strictly positive selfadjoint operators acting on H. Assume that $\ker b=\{0\}$ and s strongly commutes with a,b and β. Then the operators* (2.5) *and*

$$V = F_\hbar\left(\hat b\otimes b, \alpha(\hat\beta\otimes\beta)\chi(\hat b\otimes b<0)\right)^* e^{\frac{i}{\hbar}\log\hat a\otimes\log a} \tag{2.6}$$

satisfy the pentagon equation:

$$W_{23}V_{12} = V_{12}V_{13}W_{23}. \tag{2.7}$$

Proof: We shall consider the following selfadjoint operators acting on $K\otimes H\otimes H$:

$$\begin{aligned}
R &= \hat b\otimes a\otimes b, & \rho &= \alpha(\hat\beta\otimes I\otimes\beta)\chi(\hat b\otimes I\otimes b<0),\\
S &= \hat b\otimes b\otimes I, & \sigma &= \alpha(\hat\beta\otimes\beta\otimes I)\chi(\hat b\otimes b\otimes I<0),\\
T &= I\otimes e^{i\hbar/2}b^{-1}a\otimes b, & \tau &= \alpha(I\otimes\beta\otimes\beta)\chi(I\otimes b\otimes b<0).
\end{aligned}$$

One can easily verify that these operators satisfy the assumptions of Theorem 6.1 of [20]. Therefore

$$F_\hbar(R,\rho)F_\hbar(S,\sigma) = F_\hbar(T,\tau)^*F_\hbar(S,\sigma)F_\hbar(T,\tau).$$

Rearranging this formula we obtain:

$$F_\hbar(T,\tau)^*F_\hbar(S,\sigma)^* = F_\hbar(S,\sigma)^*F_\hbar(R,\rho)^*F_\hbar(T,\tau)^*. \tag{2.8}$$

Using the leg numbering notation we get:

$$X_{23}Y_{12} = Y_{12}\widetilde{Y}X_{23}, \tag{2.9}$$

where

$$\begin{aligned}
X &= F_\hbar\left(e^{i\hbar/2}b^{-1}a\otimes b, \alpha(\beta\otimes\beta)\chi(b\otimes b<0)\right)^*,\\
Y &= F_\hbar\left(\hat b\otimes b, \alpha(\hat\beta\otimes\beta)\chi(\hat b\otimes b<0)\right)^*,\\
\widetilde{Y} &= F_\hbar\left(\hat b\otimes a\otimes b, \alpha(\hat\beta\otimes I\otimes\beta)\chi(\hat b\otimes I\otimes b<0)\right)^*.
\end{aligned}$$

The reader should notice that replacing a by I in the right hand side of the third formula we obtain Y_{13}. With this small modification, (2.9) coincides with the pentagon equation of Baaj and Skandalis.

The operators X,Y are the first factors appearing on the right hand side of definitions (2.5) and (2.6). Now we shall investigate the second factors:

$$U = e^{\frac{i}{\hbar}\log\hat a\otimes\log a}, \qquad Z = e^{\frac{i}{\hbar}\log(s|b|^{-1})\otimes\log a}.$$

Using the relations $s|b|^{-1}$—∘ a (which follows immediately from a—∘ b) and $\hat a$—∘ $\hat b$ one can easily verify that

$$\begin{aligned}
Z(a\otimes I)Z^* &= a\otimes a,\\
U(\hat b\otimes I)U^* &= \hat b\otimes a.
\end{aligned} \tag{2.10}$$

The first relation implies that

$$\begin{aligned}
Z_{23}U_{12}Z_{23}^* &= e^{\frac{i}{\hbar}\log\hat a\otimes\log(a\otimes a)}\\
&= e^{\frac{i}{\hbar}\log\hat a\otimes(\log a\otimes I+I\otimes\log a)} = U_{12}U_{13}.
\end{aligned} \tag{2.11}$$

The reader should notice that U_{12} commutes with $(\hat{\beta}\otimes I\otimes\beta)\chi(\hat{b}\otimes I\otimes b<0)$. Therefore the second relation of (2.10) implies that

$$U_{12}Y_{13}U_{12}^{*}=\widetilde{Y}. \tag{2.12}$$

One can easily verify that b and β commute with $s|b|^{-1}$. Therefore Y_{12} commutes with Z_{23}:

$$Y_{12}Z_{23}=Z_{23}Y_{12}. \tag{2.13}$$

Our assumptions imply that $e^{i\hbar/2}b^{-1}a\otimes b$, $\beta\otimes\beta$ and $\chi(b\otimes b<0)$ commute with $a\otimes a$. Therefore X commutes with $a\otimes a$. Taking into account (2.11) we obtain:

$$X_{23}U_{12}U_{13}=U_{12}U_{13}X_{23}. \tag{2.14}$$

Now the proof of (2.7) is a matter of elementary computations. Remembering that $W=XZ$ and $V=YU$ and using (2.13), (2.9), (2.11), (2.14) and (2.12) we obtain:

$$\begin{aligned}W_{23}V_{12}&=X_{23}Z_{23}Y_{12}U_{12}=X_{23}Y_{12}Z_{23}U_{12}=Y_{12}\widetilde{Y}X_{23}U_{12}U_{13}Z_{23}\\&=Y_{12}\widetilde{Y}U_{12}U_{13}X_{23}Z_{23}=Y_{12}U_{12}Y_{13}U_{13}X_{23}Z_{23}=V_{12}V_{13}W_{23}\end{aligned}$$

Q.E.D.

Let H be a Hilbert space, $(a,b,\beta)\in G_H$ and s be a strictly positive selfadjoint operator acting on H. Assume that $\ker b=\{0\}$ and s strongly commutes with a,b and β. Then one can easily verify that $(s|b|^{-1},e^{i\hbar/2}b^{-1}a,\beta)\in G_H$. For $K=H$, $\hat{a}=s|b|^{-1}$, $\hat{b}=e^{i\hbar/2}b^{-1}a$ and $\hat{\beta}=\beta$, the operator (2.6) coincides with (2.5) and using (2.7) we obtain (2.1). It shows that the operator W introduced by (2.5) is a multiplicative unitary.

For any Hilbert space K we denote by $\overline{K}$ the complex conjugate Hilbert space. Then we have canonical antiunitary bijection:

$$K\ni x\longrightarrow\overline{x}\in\overline{K}. \tag{2.15}$$

If m is a closed operator acting on K, then its transpose $m^{\top}$ is introduced by the formula

$$m^{\top}\overline{x}=\overline{m^{*}x}$$

for any $x\in D(m^{*})$. Clearly $m^{\top}$ is a closed operator acting on $\overline{K}$ with the domain $D(m^{\top})=\{\overline{x}:x\in D(m^{*})\}$. If $x\in D(m^{*})$ and $z\in D(m)$, then

$$\left(\overline{z}\middle|m^{\top}\middle|\overline{x}\right)=(x|m|z)\,. \tag{2.16}$$

Indeed: $\left(\overline{z}\middle|m^{\top}\overline{x}\right)=\left(\overline{z}\middle|\overline{m^{*}x}\right)=(m^{*}x|z)=(x|mz)$.

One can easily verify that the transposition commutes with the adjoint operation: $(m^{*})^{\top}=\left(m^{\top}\right)^{*}$, so $m^{\top}$ is selfadjoint for selfadjoint m. Moreover the transposition inverses the order of multiplication: $(ab)^{\top}=b^{\top}a^{\top}$. Therefore $a\multimap b$ implies $b^{\top}\multimap a^{\top}$. If $\hat{a}$ is a selfadjoint operator on K and f is a bounded measurable function on $\operatorname{Sp}\hat{a}$ then $\operatorname{Sp}\hat{a}=\operatorname{Sp}\hat{a}^{\top}$ and $f(\hat{a})^{\top}=f(\hat{a}^{\top})$.

Let $\hat{a}$ and a be selfadjoint operators acting on K and H respectively. Then $\hat{a}\otimes I$ and $I\otimes a$ are strongly commuting selfadjoint operators acting on $K\otimes H$. Their joint spectrum coincides with $\operatorname{Sp}\hat{a}\times\operatorname{Sp}a$. We have the following 'partial transposition' formula

$$\left(\overline{z}\otimes u\middle|f(\hat{a}^{\top}\otimes I,I\otimes a)\middle|\overline{x}\otimes y\right)=(x\otimes u|f(\hat{a}\otimes I,I\otimes a)|z\otimes y)\,.$$

In this formula $x,z\in K$, $u,y\in H$ and $f(\cdot,\cdot)$ is a bounded measurable function on $\operatorname{Sp}\hat{a}\times\operatorname{Sp}a$. By linearity and continuity it is sufficient to prove this formula for functions of the form $f=f_1\otimes f_2$, where f_1 and f_2 are functions of one variable. In this case the formula follows immediately from (2.16). We shall use the following particular case of the partial transposition formula:

$$\left(\overline{z}\otimes u\middle|e^{\frac{i}{\hbar}\hat{a}^{\top}\otimes a}\middle|\overline{x}\otimes y\right)=\left(x\otimes u\middle|e^{\frac{i}{\hbar}\hat{a}\otimes a}\middle|z\otimes y\right). \tag{2.17}$$

To prove the manageability of the multiplicative unitary (2.5) we shall use the following

Proposition 2.3. *Let H and K be Hilbert spaces, $(a,b,\beta)\in G_H$ and $(\hat{a},\hat{b},\hat{\beta})\in G_K$ and let V be the unitary operator introduced by (2.6). Moreover let Q be a strictly positive selfadjoint operator acting on H such that Q strongly commutes with a and β and $Q^2 \multimap b$. We set:*

$$\widetilde{V} = F_\hbar\left(-\hat{b}^\top \otimes e^{i\hbar/2} b a^{-1}, -\left(\hat{\beta}^\top \otimes \beta\right)\chi\left(\hat{b}^\top \otimes b > 0\right)\right) e^{\frac{i}{\hbar}\log \hat{a}^\top \otimes \log a}. \tag{2.18}$$

Then $\widetilde{V}$ is unitary and for any $x,z\in K$, $y\in D(Q^{-1})$, $u\in D(Q)$ we have:

$$(x\otimes u|V|z\otimes y) = \left(\overline{z}\otimes Qu\middle|\widetilde{V}\middle|\overline{x}\otimes Q^{-1}y\right). \tag{2.19}$$

Remark: Formula (2.7) shows that (2.6) is an adapted operator in the sense of [19, Definition 1.3]. Comparing (2.18) with Statement 5 of Theorem 1.6 of [19] one can easily find the unitary antipode R of our quantum group. It acts on a, b, β as follows:

$$\begin{aligned} a^R &= a^{-1},\\ b^R &= -e^{i\hbar/2} b a^{-1},\\ \beta^R &= -\alpha\beta. \end{aligned}$$

Proof: To make our formulae shorter we set:

$$\begin{aligned} U &= e^{\frac{i}{\hbar}\log \hat{a}\otimes \log a}, & \widetilde{U} &= e^{\frac{i}{\hbar}\log \hat{a}^\top\otimes \log a},\\ B &= \left|\hat{b}\otimes b\right|, & \widetilde{B} &= \left|\hat{b}^\top \otimes e^{i\hbar/2} b a^{-1}\right|. \end{aligned} \tag{2.20}$$

If either $\hat{b}=0$ or $b=0$, then $V=U$, $\widetilde{V}=\widetilde{U}$ and (2.19) follows immediately from (2.17) (the reader should remember that Q commutes with a). Therefore we may assume that $\ker\hat{b}=\{0\}$ and $\ker b=\{0\}$. In this case, by the spectral theorem

$$\begin{aligned} K &= K_+ \oplus K_-,\\ H &= H_+ \oplus H_-, \end{aligned}$$

where

$$\begin{aligned} K_+ &= K(\hat{b}>0), & K_- &= K(\hat{b}<0),\\ H_+ &= H(b>0), & H_- &= H(b<0). \end{aligned}$$

For tensor products we have the decompositions:

$$K\otimes H = K_+\otimes H_+ \oplus K_+\otimes H_- \oplus K_-\otimes H_+ \oplus K_-\otimes H_- \tag{2.21}$$

$$\overline{K}\otimes H = \overline{K}_+\otimes H_+ \oplus \overline{K}_+\otimes H_- \oplus \overline{K}_-\otimes H_+ \oplus \overline{K}_-\otimes H_- \tag{2.22}$$

Operators B and U respect the decomposition (2.21), whereas $\left(\hat{\beta}\otimes\beta\right)\chi\left(\hat{b}\otimes b<0\right)$ interchanges $K_+\otimes H_-$ with $K_-\otimes H_+$ and kills $K_+\otimes H_+$ and $K_-\otimes H_-$. For the same reason, $\widetilde{B}$ and $\widetilde{U}$ respect the decomposition (2.22), whereas $\left(\hat{\beta}^\top\otimes\beta\right)\chi\left(\hat{b}^\top\otimes b>0\right)$ interchanges $\overline{K}_+\otimes H_+$ with $\overline{K}_-\otimes H_-$ and kills $\overline{K}_+\otimes H_-$ and $\overline{K}_-\otimes H_+$.

We may assume that $x\in K_{s_x}$, $u\in H_{s_u}$, $z\in K_{s_z}$, $y\in H_{s_y}$, where $s_x,s_u,s_z,s_y=+,-$. There are $2^4=16$ possible combinations of the signs. However a moment of reflection shows that for 10 combinations both sides of (2.19) vanish. The remaining combinations are:

$$(s_x,s_u,s_z,s_y) = \begin{cases} \left.\begin{array}{l}(+,+,+,+)\\(-,-,-,-)\end{array}\right\} & \text{– case 1}\\[2ex] \left.\begin{array}{l}(+,-,+,-)\\(-,+,-,+)\end{array}\right\} & \text{– case 2}\\[2ex] \left.\begin{array}{l}(+,-,-,+)\\(-,+,+,-)\end{array}\right\} & \text{– case 3}\end{cases}$$

We have divided the six possibilities into three cases. Using formula (0.1) it is not difficult to show that equation (2.19) reduces to

$$(x\otimes u|V_\theta(\log B)^*U|z\otimes y)=\left(\overline{z}\otimes Qu\middle|V_\theta(\log\widetilde{B}-\pi i)\widetilde{U}\middle|\overline{x}\otimes Q^{-1}y\right) \tag{2.23}$$

and

$$(x\otimes u|V_\theta(\log B-\pi i)^*U|z\otimes y)=\left(\overline{z}\otimes Qu\middle|V_\theta(\log\widetilde{B})\widetilde{U}\middle|\overline{x}\otimes Q^{-1}y\right) \tag{2.24}$$

in case 1 and 2 respectively. In case 3 we obtain a more complicated formula:

$$\begin{aligned}&\left(x\otimes u\middle|\left[i\alpha\left(\hat\beta\otimes\beta\right)B^{\frac{\pi}{\hbar}}V_\theta(\log B-\pi i)\right]^*U\middle|z\otimes y\right)\\&\qquad=\left(\overline{z}\otimes Qu\middle|\left[-i\left(\hat\beta^\top\otimes\beta\right)\widetilde{B}^{\frac{\pi}{\hbar}}V_\theta(\log\widetilde{B}-\pi i)\right]\widetilde{U}\middle|\overline{x}\otimes Q^{-1}y\right).\end{aligned} \tag{2.25}$$

Remembering that $\hat\beta$, β are selfadjoint, $I\otimes\beta$ commutes with B and using formula $\left(\hat\beta^\top\right)^*\overline{z}=\overline{\hat\beta z}$ we may rewrite the above equation in the following equivalent form:

$$\begin{aligned}&\left(x\otimes u'\middle|\alpha B^{\frac{\pi}{\hbar}}V_\theta(\log B-\pi i)^*U\middle|z'\otimes y\right)\\&\qquad=\left(\overline{z'}\otimes Qu'\middle|\widetilde{B}^{\frac{\pi}{\hbar}}V_\theta(\log\widetilde{B}-\pi i)\widetilde{U}\middle|\overline{x}\otimes Q^{-1}y\right),\end{aligned} \tag{2.26}$$

where $u'=\beta u$ and $z'=\hat\beta z$. To prove formulae (2.23), (2.24) and (2.26) we shall use the following

Proposition 2.4. *Let a,b,Q be selfadjoint operators acting on a Hilbert space H and $\hat a,\hat b$ be selfadjoint operators acting on a Hilbert space K. Assume that: a and Q are strictly positive,* $\ker b=\{0\}$, *a—∘ b, Q strongly commutes with a and Q^2—∘ b. Assume also that $\hat a$ is strictly positive,* $\ker\hat b=\{0\}$ *and $\hat a$—∘ $\hat b$. Moreover, let $x,z\in H$, $y\in D(Q^{-1})$, $u\in D(Q)$ and for any $k\in\mathbf{R}$,*

$$\begin{aligned}\varphi(k)&=\left(x\otimes u\middle|B^{ik}U\middle|z\otimes y\right),\\ \psi(k)&=\left(\overline{z}\otimes Qu\middle|\widetilde{B}^{ik}\widetilde{U}\middle|\overline{x}\otimes Q^{-1}y\right),\end{aligned} \tag{2.27}$$

where B, U, $\widetilde{B}$ and $\widetilde{U}$ are operators introduced by (2.20). *Then*

$$\psi(k)=e^{\hbar k/2}e^{-\frac{i\hbar}{2}k^2}\,\varphi(k) \tag{2.28}$$

for any $k\in\mathbf{R}$.

Proof: Relation a—∘ b implies that $e^{i\hbar/2}|b|a^{-1}$is selfadjoint and that

$$\left(e^{i\hbar/2}|b|a^{-1}\right)^{ik}=e^{-\frac{i\hbar}{2}k^2}|b|^{ik}a^{-ik} \tag{2.29}$$

for any $k\in\mathbf{R}$ (cf [20, Formula (3.8)]).

Remembering that Q strongly commutes with a and Q^2—∘ b, one can easily show that $I\otimes Q^2$ commutes with $\widetilde{U}$ and $I\otimes Q^2$—∘ $\widetilde{B}$. Therefore $(I\otimes Q)\widetilde{B}^{ik}\widetilde{U}=e^{\hbar k/2}\,\widetilde{B}^{ik}\widetilde{U}(I\otimes Q)$ and

$$\psi(k)=e^{\hbar k/2}\,\left(\overline{z}\otimes u\middle|\widetilde{B}^{ik}\widetilde{U}\middle|\overline{x}\otimes y\right).$$

Taking into account (2.29) and using in the third step (2.17) we obtain:

$$\begin{aligned}\psi(k)&=e^{\hbar k/2}e^{-\frac{i\hbar}{2}k^2}\,\left(\overline{z}\otimes u\middle|\left(\left|\hat b^\top\right|^{ik}\otimes|b|^{ik}a^{-ik}\right)e^{\frac{i}{\hbar}\log\hat a^\top\otimes\log a}\middle|\overline{x}\otimes y\right)\\&=e^{\hbar k/2}e^{-\frac{i\hbar}{2}k^2}\,\left(\overline{|\hat b|^{ik}z}\otimes a^{ik}|b|^{-ik}u\middle|e^{\frac{i}{\hbar}\log\hat a^\top\otimes\log a}\middle|\overline{x}\otimes y\right)\\&=e^{\hbar k/2}e^{-\frac{i\hbar}{2}k^2}\,\left(x\otimes a^{ik}|b|^{-ik}u\middle|e^{\frac{i}{\hbar}\log\hat a\otimes\log a}\middle||\hat b|^{ik}z\otimes y\right)\\&=e^{\hbar k/2}e^{-\frac{i\hbar}{2}k^2}\,\left(x\otimes u\middle|\left(I\otimes|b|^{ik}a^{-ik}\right)e^{\frac{i}{\hbar}\log\hat a\otimes\log a}\left(|\hat b|^{ik}\otimes I\right)\middle|z\otimes y\right).\end{aligned}$$

Now, to prove (2.28) it is sufficient to show that

$$e^{\frac{i}{\hbar}\log\hat{a}\otimes\log a}\left(|\hat{b}|^{ik}\otimes I\right)=\left(|\hat{b}|^{ik}\otimes a^{ik}\right)e^{\frac{i}{\hbar}\log\hat{a}\otimes\log a}. \tag{2.30}$$

If a is a multiple of I: $a=e^{\hbar l}I$, then $e^{\frac{i}{\hbar}\log\hat{a}\otimes\log a}=\hat{a}^{il}$ and the above formula reduces to the equality

$$\hat{a}^{il}|\hat{b}|^{ik}=e^{i\hbar kl}|\hat{b}|^{ik}\hat{a}^{il}$$

equivalent to the assumed relation $\hat{a}\multimap|\hat{b}|$. By spectral decomposition, (2.30) holds for any strictly positive operator a.

Q.E.D.

We have to investigate the regularity properties of functions φ and ψ introduced by (2.27). If

$$x\in D(\hat{b}^{\pm1}),\qquad u\in D(b^{\pm1}Q^{\pm2}),\qquad y\in D(Q^{\pm2}) \tag{2.31}$$

for all possible combinations of signs, then the functions φ and ψ belong to the Schwartz space $\mathcal{S}(\mathbf{R})$. Indeed using the relation $Q^2\multimap b$ one can easily show that $(I\otimes Q^2)\multimap B$ and

$$e^{\pm\hbar k}\,\varphi(k)=\left(x\otimes Q^{\pm2}u\middle|B^{ik}U\middle|z\otimes Q^{\mp2}y\right).$$

By (2.31), $x\otimes Q^{\pm2}u\in D\left(B^{\pm1}\right)$. Therefore the functions $e^{\pm\hbar k}\,\varphi(k)$ admit holomorphic continuation to functions bounded on the strip $\{k\in\mathbf{C}:-1<\Im k<1\}$. It implies that $\varphi\in\mathcal{S}(\mathbf{R})$. Moreover using (2.28) we see that the functions $e^{\pm\hbar k/4}\,\psi(k)$ admit holomorphic continuation to functions bounded on the strip $\{k\in\mathbf{C}:-1/4<\Im k<1/4\}$. It shows that $\psi\in\mathcal{S}(\mathbf{R})$.

In the following we shall use the language of distribution theory. Let f and g be measurable bounded functions on $\mathbf{R}_+$. Then the functions $\mathbf{R}\ni t\longrightarrow f(e^t)\in\mathbf{C}$ and $\mathbf{R}\ni t\longrightarrow g(e^t)\in\mathbf{C}$ are bounded and may be considered as a tempered distributions on $\mathbf{R}$. We denote by $\hat{f}$ and $\hat{g}$ the inverse Fourier transforms of these distributions. Then

$$\begin{aligned}f(t)&=\int_{\mathbf{R}}\hat{f}(k)\,t^{ik}dk\\ g(t)&=\int_{\mathbf{R}}\hat{g}(k)\,t^{ik}dk\end{aligned} \tag{2.32}$$

for almost all $t\in\mathbf{R}_+$.

Proposition 2.5. *Let f,g be bounded measurable functions on $\mathbf{R}_+$ and $\hat{f}$ and $\hat{g}$ be tempered distributions related to f and g via formulae* (2.32). *Assume that*

$$\hat{f}(k)=e^{\hbar k/2}e^{\frac{-i\hbar}{2}k^2}\,\hat{g}(k). \tag{2.33}$$

Then, using the notation and assumptions of Proposition 2.4 we have:

$$(x\otimes u|f(B)U|z\otimes y)=\left(\overline{z}\otimes Qu\middle|g(\widetilde{B})\widetilde{U}\middle|\overline{x}\otimes Q^{-1}y\right). \tag{2.34}$$

Proof: Assume for the moment that vectors x,y,z,u satisfy conditions (2.31). Then the functions (2.27) belong to $\mathcal{S}(\mathbf{R})$. Comparing (2.32) with (2.27) we obtain:

$$\begin{aligned}(x\otimes u|f\,(B)\,U|z\otimes y)\quad&=\int_{\mathbf{R}}\hat{f}(k)\varphi(k)dk,\\ \left(\overline{z}\otimes Qu\middle|g\left(\widetilde{B}\right)\widetilde{U}\middle|\overline{x}\otimes Q^{-1}y\right)\;&=\int_{\mathbf{R}}\hat{g}(k)\psi(k)dk,\end{aligned}$$

Using now (2.33) and (2.28) we see that the right hand sides of the above formulae coincide and (2.34) follows. To end the proof we notice that the conditions (2.31) select sufficiently large sets of vectors: $D(\hat{b})\cap D(\hat{b}^{-1})$ is dense in H, $D(bQ^2)\cap D(bQ^{-2})\cap D(b^{-1}Q^2)\cap D(b^{-1}Q^{-2})$ is a core for Q and $D(Q^2)\cap D(Q^{-2})$ is a core for Q^{-1}.

Q.E.D.

We continue the proof of Proposition 2.3. For any $t \in \mathbf{R}_+$ we set

$$\begin{aligned} f_1(t) &= \overline{V_\theta(\log t)} , \quad & g_1(t) &= V_\theta(\log(t)-\pi i),\\ f_2(t) &= \overline{V_\theta(\log(t)-\pi i)} , \quad & g_2(t) &= V_\theta(\log t),\\ f_3(t) &= \alpha t^{\frac{\pi}{\hbar}}\overline{V_\theta(\log(t)-\pi i)} , \quad & g_3(t) &= t^{\frac{\pi}{\hbar}}V_\theta(\log(t)-\pi i). \end{aligned} \tag{2.35}$$

Let $\hat{f}_i$ and $\hat{g}_i$ be the tempered distributions related to the above functions via formulae (2.32). We already know that (2.19) resolves into (2.23), (2.24) and (2.26). By virtue of Proposition 2.5, in order to prove this relation it is sufficient to verify that

$$\hat{f}_i(k) = e^{\hbar k/2} e^{\frac{-i\hbar}{2}k^2}\, \hat{g}_i(k) \tag{2.36}$$

for $i = 1, 2, 3$. Let us notice that $f_1(t) = \overline{g_2(t)}$, $f_2(t) = \overline{g_1(t)}$ and $f_3(t) = -\alpha\overline{g_3(t)}$. Therefore

$$\hat{f}_1(k) = \overline{\hat{g}_2(-k)}, \qquad \hat{f}_2(k) = \overline{\hat{g}_1(-k)}, \qquad \hat{f}_3(k) = \alpha\, \overline{\hat{g}_3(-k)}. \tag{2.37}$$

To verify relations (2.36) we shall use the formulae (cf formulae (1.36) and (1.41) of [20]):

$$C\,\overline{V_\theta(x)} = \exp\left\{\frac{i\overline{x}^2}{2\hbar}\right\} V_\theta\left(-\overline{x}\right), \tag{2.38}$$

$$\frac{1}{\sqrt{2\pi\hbar}}\int_{\mathbf{R}} V_\theta\left(y + i\varepsilon - \frac{i\hbar}{2} - i\pi\right) e^{\frac{iy^2}{2\hbar}}\, e^{\frac{ixy}{\hbar}}\, dy = C'\, V_\theta(x). \tag{2.39}$$

where $C = \exp\left\{\left(\frac{2\pi}{\hbar} + \frac{\hbar}{2\pi}\right)\frac{\pi}{12\,i}\right\}$ and $C' = \exp\left\{i\left(\frac{\pi}{4} + \frac{\hbar}{24} + \frac{\pi^2}{6\hbar}\right)\right\}$ are phase factors and ε is a small positive number indicating that the integration path is rounding the pole of the integrand at the point $y = 0$ from above. Inserting in (2.39), $x = \log t$ and $y = \hbar k$ we obtain:

$$V_\theta(\log t) = \frac{\hbar}{C'\sqrt{2\pi\hbar}}\int_{\mathbf{R}} V_\theta\left(\hbar k + i\varepsilon - \frac{i\hbar}{2} - i\pi\right) e^{\frac{i\hbar k^2}{2}}\, t^{ik}\, dk. \tag{2.40}$$

The left hand side coincides with $g_2(t)$. Therefore:

$$\hat{g}_2(k) = \frac{\hbar}{C'\sqrt{2\pi\hbar}} V_\theta\left(\hbar k + i\varepsilon - \frac{i\hbar}{2} - i\pi\right) e^{\frac{i\hbar k^2}{2}}. \tag{2.41}$$

Now, using (2.37) and (2.38) we obtain:

$$\begin{aligned} \hat{f}_1(k) &= \frac{\hbar C'}{\sqrt{2\pi\hbar}}\overline{V_\theta\left(-\hbar k + i\varepsilon - \frac{i\hbar}{2} - i\pi\right)}\, e^{-\frac{i\hbar k^2}{2}} \\ &= \frac{\hbar C'}{C\sqrt{2\pi\hbar}} V_\theta\left(\hbar k + i\varepsilon - \frac{i\hbar}{2} - i\pi\right) e^{\frac{i}{2\hbar}\left(\hbar k + i\varepsilon - \frac{i\hbar}{2} - i\pi\right)^2 - \frac{i\hbar k^2}{2}} \\ &= \frac{\hbar C''}{\sqrt{2\pi\hbar}} V_\theta\left(\hbar k + i\varepsilon - \frac{i\hbar}{2} - i\pi\right) e^{\left(\frac{\hbar}{2}+\pi\right)k}, \end{aligned} \tag{2.42}$$

where $C'' = \dfrac{C'}{C} e^{-\frac{i}{2\hbar}\left(\frac{\hbar}{2}+\pi\right)^2} = \dfrac{1}{C'}$.

Function g_1 is related to g_2 by imaginary shift: replacing '$\log t$' by '$\log t - i\pi$' in the formula for $g_2(t)$ we obtain $g_1(t)$. Using this fact one can easily show that $\hat{g}_1(k) = \hat{g}_2(k)e^{\pi k}$. Taking into account (2.41) we obtain:

$$\hat{g}_1(k) = \frac{\hbar}{C'\sqrt{2\pi\hbar}} V_\theta\left(\hbar k + i\varepsilon - \frac{i\hbar}{2} - i\pi\right) e^{\pi k}\, e^{\frac{i\hbar k^2}{2}}. \tag{2.43}$$

Comparing now (2.42) with (2.43) one can easily verify formula (2.36) for $i = 1$.

Using (2.37) one can easily show that formulae (2.36) for $i = 1$ and $i = 2$ are equivalent. To end the proof we have to verify (2.36) for $i = 3$. According to (2.35), $g_3(t) = g_1(t)t^{\frac{\pi}{\hbar}}$. Therefore $\hat{g}_3$ is related to $\hat{g}_1$ by imaginary shift: $\hat{g}_3(k) = \hat{g}_1(k + \frac{i\pi}{\hbar})$. Taking into account (2.43) we get:

$$\hat{g}_3(k) = \frac{\hbar}{C'\sqrt{2\pi\hbar}} V_\theta\left(\hbar k - \frac{i\hbar}{2}\right) e^{\frac{i\pi^2}{2\hbar}}\, e^{\frac{i\hbar k^2}{2}}. \tag{2.44}$$

Now, using (2.37) and (2.38) we obtain:

$$
\begin{aligned}
\hat{f}_3(k) = & \frac{\alpha\hbar C'}{\sqrt{2\pi\hbar}}\overline{V_\theta\left(-\hbar k - \frac{i\hbar}{2}\right)}\, e^{-\frac{i\pi^2}{2\hbar}}\, e^{-\frac{i\hbar k^2}{2}} \\
= & \frac{\alpha\hbar C'}{C\sqrt{2\pi\hbar}} V_\theta\left(\hbar k - \frac{i\hbar}{2}\right) e^{\frac{i}{2\hbar}\left(\hbar k - \frac{i\hbar}{2}\right)^2 - \frac{i\pi^2}{2\hbar} - \frac{i\hbar k^2}{2}} \\
= & \frac{\hbar C''}{\sqrt{2\pi\hbar}} V_\theta\left(\hbar k - \frac{i\hbar}{2}\right) e^{\hbar k/2},
\end{aligned}
\tag{2.45}
$$

where $C''' = \dfrac{\alpha C'}{C} e^{-\frac{i\hbar}{8} - \frac{i\pi^2}{2\hbar}}$. Comparing now (2.45) with (2.44) one can easily verify formula (2.36) for $i = 3$.

This ends the proof of formula (2.19) and of Proposition 2.3.

Q.E.D.

Now we are able to prove Theorem 2.1. Let a, b, r, s, β be selfadjoint operators acting on a Hilbert space H, satisfying the assumptions of Theorem 2.1. Setting $K = H$, $\hat{a} = s|b|^{-1}$, $\hat{b} = e^{i\hbar/2}b^{-1}a$, $\hat{\beta} = \beta$ and $Q = \sqrt{ra}$ we satisfy all the assumptions of Propositions 2.2 and 2.3. Introducing the above data into (2.6) and (2.18) we obtain unitary operators $W \in B(H \otimes H)$ and $\widetilde{W} \in B(\overline{H} \otimes H)$. Clearly W is given by (2.5). Proposition 2.2 shows that the operator W satisfies the pentagon equation (2.7). In the present setting formula (2.19) coincides with (2.3).

To finish the proof of manageability of W we have to show that W commutes with $Q \otimes Q$. To this end we notice that $ra \multimap b$ and $e^{i\hbar/2}b^{-1}a \multimap ra$. Therefore $Q^2 \otimes Q^2 = ra \otimes ra$ strongly commutes with $e^{i\hbar/2}b^{-1}a \otimes b$. Moreover remembering that $a \multimap b$ and $r \multimap s$ one can easily show that $Q^2 = ra$ strongly commutes with $s|b|^{-1}$. Clearly Q commutes with a. Therefore $Q^2 \otimes Q^2$ strongly commutes with $\log\left(s|b|^{-1}\right) \otimes \log a$. Using this information we see that $Q \otimes Q$ commutes with (2.5) and manageability of W follows. This is the end of the proof of Theorem 2.1.

Remark: Operators r, s appearing in this Section play an auxiliary role and may be removed from the considerations. Let H be a Hilbert space and $(a, b, \beta) \in G_H$. Assume that $\ker b = \{0\}$. Setting $K = H$, $\hat{a} = |b|^{-1}$, $\hat{b} = e^{i\hbar/2}b^{-1}a$, $\hat{\beta} = \beta$, $s = I$ and $Q = \sqrt{a}$ we satisfy all the assumptions of Propositions 2.2 and 2.3. Introducing the above data into (2.6) and (2.18) we obtain unitary operators $W \in B(H \otimes H)$ and $\widetilde{W} \in B(\overline{H} \otimes H)$. Now W is given by the simpler formula

$$
W = F_\hbar\left(e^{i\hbar/2}b^{-1}a \otimes b, \alpha(\beta \otimes \beta)\chi(b \otimes b < 0)\right)^* e^{\frac{i}{\hbar}\log(|b|^{-1}) \otimes \log a}. \tag{2.46}
$$

By Proposition 2.2, the above operator satisfies the pentagon equation (2.1). As before, formula (2.19) coincides with (2.3). However now W does not commute with $Q \otimes Q$. It means that operator (2.46) is not manageable in the sense of [19]. Instead of (2.2) we have:

$$
W(\hat{Q} \otimes Q)W^* = \hat{Q} \otimes Q, \tag{2.47}
$$

where $\hat{Q} = \sqrt{\hat{a}} = |b|^{-\frac{1}{2}}$ is a strictly positive selfadjoint operator.

It turns out that all the results of [19] remain valid, when (2.2) is replaced (2.47) (cf [12]).

3. Crossed product algebra

In this Section we construct the C^*-algebra related to the commutation relations (1.9). Let $C_\infty\left(\overline{R_+}\right)$ be the C^*-algebra of all continuous functions vanishing at infinity on the closed halfline $\overline{R_+} = [0, +\infty[$, M_2 be the algebra of all 2×2 matrices with complex entries and $B = C_\infty\left(\overline{R_+}\right) \otimes M_2$. Elements of B are continuous mappings $f : \overline{R_+} \longrightarrow M_2$ such that $\lim_{\tau \to \infty} f(\tau) = 0$. Imposing an additional condition saying that $f(0)$ is a multiple of I we select a non-degenerate C^*-subalgebra $B_0 \subset B$. The matrix elements of any $f \in B$ we be denoted by $f_{kl} \in C_\infty\left(\overline{R_+}\right)$ $(k, l = 1, 2)$:

$$
f = \begin{pmatrix} f_{11} & , f_{12} \\ f_{21} & , f_{22} \end{pmatrix}. \tag{3.1}
$$

Then

$$B_0 = \left\{ f \in B : \begin{array}{c} f_{11}(0) = f_{22}(0), \\ f_{12}(0) = f_{21}(0) = 0 \end{array} \right\}.$$

For any $\tau \in \overline{\mathbf{R}_+}$ we set:

$$b(\tau) = \begin{pmatrix} \tau \ , & 0 \\ 0 \ , & -\tau \end{pmatrix}, \qquad \beta(\tau) = \begin{pmatrix} 0 & , \ \chi(\tau \neq 0) \\ \chi(\tau \neq 0) \ , & 0 \end{pmatrix}. \tag{3.2}$$

Then $b\beta = -\beta b$ and

$$(ib\beta)(\tau) = \begin{pmatrix} 0 & , \ i\tau \\ -i\tau \ , & 0 \end{pmatrix}. \tag{3.3}$$

The reader should notice that $b(\tau)$ and $(ib\beta)(\tau)$ depend continuously on τ and that $b(0)$ and $(ib\beta)(0)$ are multiple of $I \in M_2$. According to [18, formula (2.6)], b and $ib\beta$ are elements affiliated with B_0. Clearly these elements are selfadjoint. On the other hand $\beta(\tau)$ is not continuous with respect to τ. Therefore β is not affiliated with B_0. Instead, it belongs to the W^*-envelope of B_0.

Let $t \in \mathbf{R}$ and $f \in B_0$. For any $\tau \in \mathbf{R}$ we set:

$$(\sigma_t f)(\tau) = f\left(e^{\hbar t}\tau\right).$$

Then $\sigma_t f \in B_0$, $\sigma_t \in \mathrm{Aut}\,(B_0)$ and $(\sigma_t)_{t\in\mathbf{R}}$ is a pointwise continuous one parameter group of automorphisms of B_0. In other words, $\left(B_0, (\sigma_\tau)_{\tau\in\mathbf{R}}\right)$ is a C^*-dynamical system. Let

$$A_{\mathrm{cp}} = B_0 \times_\sigma \mathbf{R} \tag{3.4}$$

be the corresponding C^*-crossed product algebra [5]. The canonical embedding $B_0 \hookrightarrow M(A_{\mathrm{cp}})$ is a morphism from B_0 into A_{cp}. Therefore the elements affiliated with B_0 are affiliated with A_{cp}. In particular $b, ib\beta \; \eta \; A_{\mathrm{cp}}$. The similar conclusion holds for β. It belongs to W^*-envelope of A_{cp}. By the definition of crossed product, $M(A_{\mathrm{cp}})$ contains a strictly continuous one parameter group of unitaries implementing the action σ of $\mathbf{R}$ on B_0. The infinitesimal generator of this group will be denoted by $\log a$. Then a is a strictly positive selfadjoint element affiliated with A_{cp}. For any $f \in B_0$ we have:

$$a^{it} f a^{-it} = \sigma_t f$$

One can easily verify that $\sigma_t b = e^{\hbar t} b$ and $\sigma_t(ib\beta) = e^{\hbar t} ib\beta$. Therefore $a^{it} b = e^{\hbar t} b a^{it}$ and $a^{it} ib\beta = e^{\hbar t} ib\beta a^{it}$ for any $t \in \mathbf{R}$. It means that

$$a \multimap b \qquad \text{and} \qquad a\beta = \beta a.$$

By construction, the set

$$\Big\{ f g(\log a) : f \in B_0,\ g \in C_\infty(\mathbf{R}) \Big\}^{\text{linear envelope}} \tag{3.5}$$

is a dense subset of the C^*-crossed product $A_{\mathrm{cp}} = B_0 \times_\sigma \mathbf{R}$.

Proposition 3.1. *The C^*-algebra A_{cp} is generated (in the sense explained in [18]) by the three affiliated elements $\log a, b, ib\beta \; \eta \; A_{\mathrm{cp}}$.*

Proof: We shall use Theorem 3.3 of [18]. One can easily verify that

$$\Big(g_1(b) + g_2(b) ib\beta\Big)(\tau) = \begin{pmatrix} g_1(\tau) & , \ i\tau g_2(\tau) \\ -i\tau g_2(-\tau) \ , & g_1(-\tau) \end{pmatrix}$$

for any $g_1, g_2 \in C_{\mathrm{compact}}(\mathbf{R})$ and that the set of elements of the above form is dense in B_0. Therefore elements b and $ib\beta$ separate representations of B_0. Remembering that (3.5) is dense in A_{cp} we see that elements $\log a$, b and $ib\beta$ separate representations of A_{cp}. This way we verified Assumption 1 of Theorem 3.3 of [18].

Let $r_1 = (I + b^* b)^{-1}$ and $r_2 = (I + (\log a)^*(\log a))^{-1}$. To end the proof it is sufficient to notice, that

$$r_1 r_2 = f g(\log a),$$

where $f=(I+b^2)^{-1}$ and $g(\lambda)=\left(1+\lambda^2\right)^{-1}$. Clearly $f\in B_0$ and $g\in C_\infty(\mathbf{R})$. Therefore r_1r_2 belongs to (3.5) and consequently $r_1r_2\in A_{\rm cp}$. It shows that assumption 2 of Theorem 3.3 of [18] holds. Now this Theorem says that $A_{\rm cp}$ is generated by $\log a$, b and $ib\beta$.

Q.E.D.

Let H be a Hilbert space and π be a non-degenerate representation of $A_{\rm cp}$ acting on H: $\pi\in{\rm Rep}\,(A_{\rm cp},H)$. According to the general theory, π admits a natural extension to the set of affiliated elements $A_{\rm cp}^\eta$ and to the W^*-envelope of $A_{\rm cp}$. Clearly $\pi(a)$, $\pi(b)$ and $\pi(\beta)$ are selfadjoint operators. Moreover $\pi(a)$ is strictly positive, $\pi(a)$—∘ $\pi(b)$, $\pi(\beta)^2=\chi(\pi(b)\neq 0)$, $\pi(\beta)$ commutes with $\pi(a)$ and anticommutes with $\pi(b)$. It means that $(\pi(a),\pi(b),\pi(\beta))$ is a G-triple. It turns out that any G-triple is of this form.

Proposition 3.2. *Let H be a Hilbert space and $(a_o,b_o,\beta_o)\in G_H$. Then there exists unique representation $\pi\in{\rm Rep}\,(A_{\rm cp},H)$ such that $a_o=\pi(a)$, $b_o=\pi(b)$ and $\beta_o=\pi(\beta)$. If $A\in C^*(H)$ and $\log a_o, b_o, ib_o\beta_o\,\eta\, A$, then $\pi\in{\rm Mor}(A_{\rm cp},A)$.*

Proof: For any $f\in B_0$ of the form (3.1) we set:

$$\pi_o(f)=\left\{\begin{array}{c} f_{11}(b_o)\chi(b_o\geq 0)+f_{12}(b_o)\chi(b_o>0)\beta_o \\ +\,\beta_o f_{21}(b_o)\chi(b_o>0)+\beta_o f_{22}(b_o)\chi(b_o\geq 0)\beta_o.\end{array}\right. \tag{3.6}$$

Elementary computations show that π_o is a non-degenerate representation of B_0. The action of π_o on elements affiliated with B_0 is described by the same formula (3.6). In particular for elements (3.2) we have: $\pi_o(b)=b_o$ and $\pi_o(\beta)=\beta_o$. Indeed

$$\begin{aligned}\pi_o(b)&=b_o\chi(b_o\geq 0)-\beta_o b_o\chi(b_o\geq 0)\beta_o\\ &=b_o\chi(b_o\geq 0)+b_o\chi(-b_o\geq 0)\beta_o^2\\ &=b_o\chi(b_o\geq 0)+b_o\chi(b_o<0)=b_o\end{aligned}$$

and similarly

$$\begin{aligned}\pi_o(\beta)&=\chi(b_o>0)\beta_o+\beta_o\chi(b_o>0)\\ &=\chi(b_o>0)\beta_o+\chi(b_o<0)\beta_o=\beta_o.\end{aligned}$$

Now we shall use the relation a_o—∘ b_o. It means that $a_o^{it}b_oa_o^{-it}=e^{\hbar t}b_o$. We also know that a_o commutes with β_o. Therefore

$$\begin{aligned}a_o^{it}\pi_o(f)a_o^{-it}&=\left\{\begin{array}{c} f_{11}(e^{\hbar t}b_o)\chi(b_o\geq 0)+f_{12}(e^{\hbar t}b_o)\chi(b_o>0)\beta_o \\ +\,\beta_o f_{21}(e^{\hbar t}b_o)\chi(b_o>0)+\beta_o f_{22}(e^{\hbar t}b_o)\chi(b_o\geq 0)\beta_o\end{array}\right.\\ &=\pi_o(\sigma_t f).\end{aligned} \tag{3.7}$$

It shows that the pair $\left(\pi_o,\left(a_o^{it}\right)_{t\in\mathbf{R}}\right)$ is a covariant representation of the C^*-dynamical system $\left(B_0,(\sigma_t)_{t\in\mathbf{R}}\right)$. Let π be the corresponding representation of the crossed product algebra $A_{\rm cp}$. Then $\pi(a^{it})=a_o^{it}$ and $\pi(a)=a_o$. Moreover π restricted to $B_o\subset M(A_{\rm cp})$ coincides with π_o. In particular $\pi(b)=b_o$ and $\pi(\beta)=\beta_o$.

This way we constructed representation $\pi\in{\rm Rep}\,(A_{\rm cp},H)$ having desired properties. The uniqueness of π and the last Statement of the Proposition follows immediately from Proposition 3.1 (cf Definition 3.1 and Theorem 6.2 of [18]).

Q.E.D.

We shall use the above results to show the following

Proposition 3.3. *There exists unique automorphism ϕ of the C^*-algebra A_{cp} such that*

$$\begin{aligned}\phi(a)&=a,\\ \phi(b)&=b,\\ \phi(i\beta b)&=\beta|b|.\end{aligned} \tag{3.8}$$

This automorphism is of order 4: $\phi^4={\rm id}$.

Proof: We may assume that A_{cp} is a non-degenerate C*-algebra of operators acting on a Hilbert space H. Then $(a, b, \beta) \in G_H$. Let $a_o = a$, $b_o = b$ and $\beta_o = -i\beta \operatorname{sign} b$. One can easily verify that $(a_o, b_o, \beta_o) \in G_H$ and that a_o, b_o and $i\beta_o b_o = \beta|b|$ are affiliated with A_{cp}. By Proposition 3.2, there exists unique $\phi \in \operatorname{Mor}(A_{cp}, A_{cp})$ satisfying relations (3.8).

Let f be a continuous function on $\mathbf{R}$ vanishing at 0 and at infinity. Then $\beta f(b) \in A_{cp}$. The second and third formulae of (3.8) show that $\phi(\beta f(b)) = -i\beta \operatorname{sign}(b) f(b)$. Iterating this formula we obtain: $\phi^2(\beta f(b)) = -\beta f(b)$ and $\phi^4(\beta f(b)) = \beta f(b)$. It shows that $\phi^4 = \mathrm{id}$.

Q.E.D.

4. From multiplicative unitary to quantum group

Let G be the quantum space corresponding to the C^*-algebra $A_{\rm cp}$. In other words, elements of $A_{\rm cp}$ are interpreted as continuous functions vanishing at infinity on G. In this Section we endow G with a group structure introducing a comultiplication $\Delta \in \operatorname{Mor}(A_{\rm cp}, A_{\rm cp} \otimes A_{\rm cp})$. It will be shown that the quantum group G coincides with the (extended) '$ax + b$' -group introduced in Section 1.

From now until the end of the paper we assume that the deformation parameter

$$\hbar = \frac{\pi}{2k+3},$$

where $k = 0, 1, 2, \ldots$. Then $\alpha = i \exp \frac{i\pi^2}{2\hbar} = (-1)^k$.

Any C^*-algebra may be embedded in a non-degenerate way into $B(H)$, where H is a Hilbert space. Then affiliated elements become closed operators acting on H. Let

$$\jmath : A_{\rm cp} \hookrightarrow B(H) \tag{4.1}$$

be a non-degenerate embedding. Then $\jmath \in \operatorname{Rep}(A_{\rm cp}, H)$ and $\jmath(a)$, $\jmath(b)$ and $\jmath(\beta)$ are selfadjoint operators acting on H. To simplify the notation we will drop the embedding symbol '$\jmath$' writing a, b, β instead of $\jmath(a), \jmath(b), \jmath(\beta)$. With this notation $A_{\rm cp} \subset B(H)$.

One can check that the subspace $\ker b$ is $A_{\rm cp}$-invariant. Replacing if necessary H by $(\ker b)^\perp$ we may assume that $\ker b = \{0\}$. We may also assume that the commutant $A'_{\rm cp} = \{a' \in B(H) : a'c = ca' \text{ for any } c \in A_{\rm cp}\}$ contains a W^*-algebra isomorphic to $B(K)$, where K is an infinite-dimensional Hilbert space. If this is not the case, then we replace (4.1) by $\jmath' : A_{\rm cp} \hookrightarrow B(K \otimes H)$ introduced by the formula $\jmath'(c) = I_{B(K)} \otimes \jmath(c)$ for any $c \in A_{\rm cp}$. Since the commutant $A'_{\rm cp}$ is large enough, there exist strictly positive selfadjoint operators r, s acting on H such that r, s strongly commute with a, b, β and $r \multimap s$.

Let $\hat{b} = e^{i\hbar/2} b^{-1} a$, $\hat{\beta} = \beta$ and $\hat{a} = s|b|^{-1}$. Then $\operatorname{sign} b = \operatorname{sign} \hat{b}$. Therefore $\chi(b \otimes b < 0) = \chi(\hat{b} \otimes b < 0)$ and the operator (2.5) equals

$$W = F_\hbar\left(\hat{b} \otimes b, \alpha(\hat{\beta} \otimes \beta)\chi(\hat{b} \otimes b < 0)\right)^* e^{\frac{i}{\hbar} \log \hat{a} \otimes \log a}. \tag{4.2}$$

This operator acts on $H \otimes H$. By Theorem 2.1 W is a manageable multiplicative unitary. Corresponding operators Q and $\widetilde{W}$ are given by: $Q = \sqrt{ra}$ and

$$\widetilde{W} = F_\hbar\left(-\hat{b}^\top \otimes e^{i\hbar/2} b a^{-1}, -\left(\hat{\beta}^\top \otimes \beta\right) \chi\left(\hat{b}^\top \otimes b > 0\right)\right) e^{\frac{i}{\hbar} \log \hat{a}^\top \otimes \log a}. \tag{4.3}$$

We shall use the theory developed in [3, 19]. Let $B(H)_*$ be the set of all normal linear functionals defined on $B(H)$ and

$$A = \left\{(\omega \otimes \mathrm{id})W : \omega \in B(H)_*\right\}^{\text{norm closure}}. \tag{4.4}$$

According to the general theory [3, 19], A is a C^*-algebra and $W \in M(CB(H) \otimes A)$, where $CB(H)$ the C^*-algebra of all compact operators acting on H. The algebra A is interpreted as the algebra of all 'continuous functions vanishing at infinity on the quantum group'. The corresponding comultiplication Δ is introduced by the formula:

$$\Delta(c) = W(c \otimes I)W^*. \tag{4.5}$$

It is known that $\Delta(c)\in M(A\otimes A)$ for any $c\in A$ and that $\Delta\in \mathrm{Mor}(A,A\otimes A)$. By the pentagon equation we have

$$(\mathrm{id}\otimes\Delta)W = W_{12}W_{13}.$$

Using this formula one can easily show that Δ is coassociative. The main result of this Section is contained in the following

Theorem 4.1.

1. The Baaj-Skandalis algebra (4.4) *coincides with the crossed product algebra* A_{cp}*:*

$$A = A_{\mathrm{cp}}. \tag{4.6}$$

2. The comultiplication Δ *acts on distinguished elements affiliated with* A_{cp} *in the following way:*

- $\Delta(a) = a\otimes a$,
- $\Delta(b)$ *is the selfadjoint extension of* $a\otimes b + b\otimes I$ *corresponding to the reflection operator* $\tau = \alpha(\beta\otimes\beta)\chi(b\otimes b<0)$. *In short:* $\Delta(b) = [a\otimes b + b\otimes I]_\tau$.
- $\Delta(i\beta b) = i\left\{w\left(e^{i\hbar/2}b^{-1}a\otimes b\right)^{-1}(\beta\otimes I) + (I\otimes\beta)\,w\left(e^{i\hbar/2}ba^{-1}\otimes b^{-1}\right)^{-1}\right\}\Delta(b)$,
  where w *is the polynomial introduced by* (1.15).

Proof:

Ad 1. Any closed operator acting on H is affiliated with $CB(H)$. In particular $\hat b, i\hat b\hat\beta,\ \log\hat a\in CB(H)^\eta$. Remembering that $b,\ ib\beta,\ \log a\in A_{\mathrm{cp}}^\eta$ we obtain: $\hat b\otimes b,\ \hat b\hat\beta\otimes b\beta, \log\hat a\otimes\log a\in (CB(H)\otimes A_{\mathrm{cp}})^\eta$. Therefore

$$\begin{aligned} e^{\frac{i}{\hbar}\log\hat a\otimes\log a} &\in M\left(CB(H)\otimes A_{\mathrm{cp}}\right),\\ F_\hbar\left(\hat b\otimes b, \alpha(\hat\beta\otimes\beta)\chi(\hat b\otimes b<0)\right) &\in M\left(CB(H)\otimes A_{\mathrm{cp}}\right). \end{aligned}$$

To obtain the second relation we used [20, Theorem 8.1]). Consequently $W\in M\left(CB(H)\otimes A_{\mathrm{cp}}\right)$. Now using (4.4) we obtain $A\subset M(A_{\mathrm{cp}})$ and $AA_{\mathrm{cp}}\subset A_{\mathrm{cp}}$.

W is a unitary element of the multiplier algebra. Therefore $W\left(CB(H)\otimes A_{\mathrm{cp}}\right) = CB(H)\otimes A_{\mathrm{cp}}$ and the set

$$\left\{W(m\otimes c) : m\in CB(H),\ c\in A_{\mathrm{cp}}\right\} \tag{4.7}$$

is linearly dense in $CB(H)\otimes A_{\mathrm{cp}}$. For any $\omega\in B(H)_*$, $m\in CB(H)$ and $c\in A_{\mathrm{cp}}$ we have: $(\omega\otimes\mathrm{id})\left(W(m\otimes c)\right) = \left((m\omega\otimes\mathrm{id})W\right)c\in AA_{\mathrm{cp}}$. Applying $\omega\otimes\mathrm{id}$ to all elements of (4.7) we see that

$$AA_{\mathrm{cp}} \text{ is a linearly dense subset of } A_{\mathrm{cp}}. \tag{4.8}$$

We shall prove that

$$\log a,\ b,\ ib\beta\ \eta\, A. \tag{4.9}$$

For all $t\in\mathbf{R}$ we set

$$V(t) = F_\hbar\left(t\hat b\otimes b, \alpha(\hat\beta\otimes\beta)\chi(t\hat b\otimes b<0)\right)^* e^{\frac{i}{\hbar}\log\hat a\otimes\log a}. \tag{4.10}$$

Then $V(t)\in B(H\otimes H) = M\Big(CB(H)\otimes CB(H)\Big)$. In what follows we endow multiplier algebras with the strict topology. Using Theorem 8.1 of [20] one can easily show that $\Big(V(t)\Big)_{t\in\mathbf{R}}$ is a continuous family of elements of $M\Big(CB(H)\otimes CB(H)\Big)$. Tensoring by $I\in M(A)$ and using the leg numbering notation $V_{12}(t) = V(t)\otimes I$ we obtain a continuous family $\Big(V_{12}(t)\Big)_{t\in\mathbf{R}}$ of elements of $M\Big(CB(H)\otimes CB(H)\otimes A\Big)$.

By Proposition 2.2, operators (4.10) satisfy the pentagon equation (2.7). Therefore

$$V_{13}(t) = V_{12}(t)^* W_{23} V_{12}(t) W_{23}^*. \tag{4.11}$$

Using this formula and remembering that $W \in M(CB(H) \otimes A)$ we see that $\Big(V_{13}(t)\Big)_{t\in\mathbf{R}}$ is a continuous family of elements of $M\Big(CB(H) \otimes CB(H) \otimes A\Big)$. It implies that $\Big(V(t)\Big)_{t\in\mathbf{R}}$ is a continuous family of elements of $M\Big(CB(H) \otimes A\Big)$.

Therefore $F_\hbar\Big(t\hat{b} \otimes b, \alpha(\hat{\beta} \otimes \beta)\chi(t\hat{b} \otimes b < 0)\Big) = V(0)V(t)^* \in M(CB(H) \otimes A)$ depends continuously on $t \in \mathbf{R}$. Now, Theorem 8.1 of [20] shows that $\hat{b} \otimes b$ and $\hat{b}\hat{\beta} \otimes b\beta$ are affiliated with $CB(H) \otimes A$. Taking into account (A.1) we get $b,\ ib\beta\ \eta\, A$.

Let $t \in \mathbf{R}$. Inserting $\hat{b} = 0$ and $\hat{a} = e^{\hbar t} I$ in Proposition 2.2 we see that the operator

$$V(t) = I \otimes e^{it \log a} = I \otimes a^{it}. \tag{4.12}$$

satisfies the pentagon equation (2.7). In the present case equation (4.11) takes the form

$$I \otimes a^{it} = \left(a^{-it} \otimes I\right) W \left(a^{it} \otimes I\right) W^*. \tag{4.13}$$

It shows that $\Big(I \otimes a^{it}\Big)_{t\in\mathbf{R}}$ is a continuous one parameter group of unitary elements of the multiplier algebra $M\Big(CB(H) \otimes A\Big)$. Consequently $\Big(a^{it}\Big)_{t\in\mathbf{R}}$ is a continuous one parameter group of unitary elements of $M(A)$. Therefore the infinitesimal generator $\log a$ is affiliated with A. This way (4.9) is shown.

Now we combine Proposition 3.1 with (4.9). By Definition 3.1 of [18], the embedding (4.1) belongs to $\mathrm{Mor}(A_{\mathrm{cp}}, A)$. It means that $A_{\mathrm{cp}}A$ is a linearly dense subset of A. Comparing this result with (4.8) we obtain (4.6). This way we revealed the structure of the algebra of 'continuous functions vanishing at infinity on G'.

Ad 2. Let Δ be the comultiplication introduced by (4.5). Clearly the action of Δ on elements affiliated with A is described by the same formula. We have to compute the action of Δ on generators $a,\ b,\ ib\beta$ of A. Formula (4.13) shows that $\Delta(a^{it}) = a^{it} \otimes a^{it}$ for any $t \in \mathbf{R}$. Therefore

$$\Delta(a) = a \otimes a.$$

One can easily verify that b strongly commutes with $\hat{a} = s|b|^{-1}$. Therefore $b \otimes I$ commutes with $e^{\frac{i}{\hbar}\log \hat{a} \otimes a}$ and

$$\Delta(b) = F_\hbar\Big(\hat{b} \otimes b, \alpha(\hat{\beta} \otimes \beta)\chi(\hat{b} \otimes b < 0)\Big)^* (b \otimes I) F_\hbar\Big(\hat{b} \otimes b, \alpha(\hat{\beta} \otimes \beta)\chi(\hat{b} \otimes b < 0)\Big).$$

We know that $a \multimap b$. Therefore $b^{-1} \multimap a$, $\hat{b} = e^{i\hbar/2} b^{-1} a \multimap a$ and $\hat{b} \otimes b \multimap b \otimes I$. The reader should notice that $b \otimes I$ anticommutes with the operator $\tau = \alpha(\hat{\beta} \otimes \beta)\chi(\hat{b} \otimes b < 0) = \alpha(\beta \otimes \beta)\chi(b \otimes b < 0)$. Using Theorem 5.3 of [20] we see that $\Delta(b)$ is the selfadjoint extension of $e^{i\hbar/2}\hat{b}b \otimes b + b \otimes I = a \otimes b + b \otimes I$ corresponding to the reflection operator τ:

$$\Delta(b) = \Big[a \otimes b + b \otimes I\Big]_\tau .$$

More explicitly $\Delta(b)$ is the restriction of $(a \otimes b + b \otimes I)^*$ to the domain

$$D(\Delta(b)) = D\Big(a \otimes b + b \otimes I\Big) + D\Big((a \otimes b + b \otimes I)^*\Big) \cap (H \otimes H)(\tau = 1),$$

where $(H \otimes H)(\tau = 1)$ is the eigenspace of τ corresponding to the eigenvalue 1.

The action of Δ on the third generator is given by the formula

$$\Delta(ib\beta) = i\widetilde{\beta}\Delta(b),$$

where

$$\widetilde{\beta} = W(\beta \otimes I)W^*.$$

We have to find a formula for $\widetilde{\beta}$. Remembering that β commutes with $\hat{a}$ and anticommutes with $\hat{b}$ we see that $\beta \otimes I$ commutes with $e^{\frac{i}{\hbar}\log \hat{a} \otimes a}$ and anticommutes with $\hat{b} \otimes b$. Therefore

$$\begin{aligned}\widetilde{\beta} &= F_\hbar\Big(\hat{b} \otimes b, \alpha(\hat{\beta} \otimes \beta)\chi(\hat{b} \otimes b < 0)\Big)^* (\beta \otimes I) F_\hbar\Big(\hat{b} \otimes b, \alpha(\hat{\beta} \otimes \beta)\chi(\hat{b} \otimes b < 0)\Big) \\ &= F_\hbar\Big(\hat{b} \otimes b, \alpha(\hat{\beta} \otimes \beta)\chi(\hat{b} \otimes b < 0)\Big)^* F_\hbar\Big(-\hat{b} \otimes b, \alpha(\hat{\beta} \otimes \beta)\chi(\hat{b} \otimes b > 0)\Big)(\beta \otimes I).\end{aligned}$$

Taking into account formula (B.5) of Appendix B we obtain:

$$\widetilde{\beta}=\left\{w\left(\hat{b}\otimes b\right)^{-1}+\left(\hat{\beta}\otimes\beta\right)w\left(-(\hat{b}\otimes b)^{-1}\right)^{-1}\right\}(\beta\otimes I).$$

Remembering that $\hat{\beta}=\beta$ anticommutes with $\hat{b}=e^{i\hbar/2}b^{-1}a$ we finally obtain:

$$\widetilde{\beta}=w\left(e^{i\hbar/2}b^{-1}a\otimes b\right)^{-1}(\beta\otimes I)+(I\otimes\beta)\,w\left(e^{i\hbar/2}ba^{-1}\otimes b^{-1}\right)^{-1}. \tag{4.14}$$

This formula proves the last point of Statement 2 of our Theorem.

Q.E.D.

Remark: Using (1.4), (1.3) and (1.15) one can verify that on the Hopf *-algebra level the product $\left(b^{2k+3}\otimes I\right)w\left(e^{i\hbar/2}b^{-1}a\otimes b\right)=(\Delta b)^{2k+3}=\left(a^{2k+3}\otimes b^{2k+3}\right)w\left(-e^{i\hbar/2}ba^{-1}\otimes b^{-1}\right)$. Combining this formula with (4.14) we get

$$\Delta\left(ib^{2k+3}\beta\right)=ib^{2k+3}\beta\otimes I+a^{2k+3}\otimes ib^{2k+3}\beta. \tag{4.15}$$

On the Hilbert space and C*-levels, instead of equality we have inclusion: operator on the left hand side of (4.15) is a selfadjoint extension of the symmetric operator appearing on the right hand side. This extension is determined by reflection operator $-\mathrm{sign}\,(b\otimes b)$:

$$\Delta\left(ib^{2k+3}\beta\right)=\left[ib^{2k+3}\beta\otimes I+a^{2k+3}\otimes ib^{2k+3}\beta\right]_{-\mathrm{sign}\,(b\otimes b)}. \tag{4.16}$$

See [8] for details. The formula (4.16) seems to be very interesting. It encodes in a simple form the complicated formula (4.14) describing the action of Δ on β. Moreover it shows that in a certain sense

$$u'=\begin{pmatrix} a^{2k+3} & , & ib^{2k+3}\beta \\ 0 & , & I \end{pmatrix}$$

is a two-dimensional representation of quantum '$ax+b$' group.

Now we shall discuss the coinverse map κ. According to [19] we have the polar decomposition

$$\kappa(c)=\left(\tau_{i/2}(c)\right)^R, \tag{4.17}$$

where $\tau_{i/2}$ is the analytic generator of the scaling group and the map $A\ni c\mapsto c^R$ is the unitary antipode. The action of the scaling group is described by the formula: $\tau_t(c)=Q^{2it}cQ^{-2it}$. Remembering that $Q^2=ra$ commutes with a and β and that $Q^2 \multimap b$ we obtain:

$$\tau_t(a)=a,\qquad \tau_t(b)=e^{\hbar t}b\qquad\text{and}\qquad \tau_t(\beta)=\beta.$$

Consequently: $\tau_{i/2}(a)=a$, $\tau_{i/2}(b)=e^{i\hbar/2}b$ and $\tau_{i/2}(\beta)=\beta$. The unitary antipode is defined by the relation $W^{\top\otimes R}=\widetilde{W}^*$ (cf [19, Formula (1.14)]). Comparing (4.2) with (4.3) and remembering that $\top\otimes R$ is antimultiplicative we obtain:

$$a^R=a^{-1},\qquad b^R=-e^{i\hbar/2}ba^{-1}\qquad\text{and}\qquad \beta^R=-\alpha\beta.$$

Now, formula (4.17) shows that:

$$\kappa(a)=a^{-1},\qquad \kappa(b)=-a^{-1}b\qquad\text{and}\qquad \kappa(\beta)=-\alpha\beta.$$

It turns out that the automorphism ϕ introduced in Proposition 3.3 preserves the group structure of our quantum group. We have:

Proposition 4.2. *For any $c\in A$:*

$$\Delta(\phi(c))=(\phi\otimes\phi)\Delta(c) \tag{4.18}$$

Proof: We recall that we use the embedding $A\hookrightarrow B(H)$ such that b is represented by an operator with trivial kernel. Therefore $\beta^2=\chi(b\neq 0)=I$ and the operator

$$w=\chi(b>0)+i\chi(b<0)$$

is unitary. Clearly w commutes with a and b. We compute:

$$\begin{aligned} w^*\beta w &= \Big(\chi(b>0)-i\chi(b<0)\Big)\,\beta\,\Big(\chi(b>0)+i\chi(b<0)\Big)\\ &= \beta\Big(\chi(b<0)-i\chi(b>0)\Big)\Big(\chi(b>0)+i\chi(b<0)\Big)\\ &= -i\beta\Big(\chi(b>0)-\chi(b<0)\Big)=-i\beta\,\mathrm{sign}\,b. \end{aligned}$$

Therefore $w^*(i\beta b)w = \beta|b|$. It shows (cf (3.8)) that w implements the action of ϕ:

$$\phi(c) = w^* c w$$

for any $c \in A$. We claim that $w \otimes w$ commutes with τ. Indeed

$$\begin{aligned}(w \otimes w)\tau(w \otimes w)^* &= \alpha(w\beta w^* \otimes w\beta w^*)\chi(wbw^* \otimes wbw^* < 0) \\ &= \alpha\left((-i\beta \operatorname{sign} b) \otimes (-i\beta \operatorname{sign} b)\right)\chi(b \otimes b < 0) \\ &= -\alpha(\beta \otimes \beta)(\operatorname{sign} b \otimes \operatorname{sign} b)\chi(b \otimes b < 0) \\ &= \alpha(\beta \otimes \beta)\chi(b \otimes b < 0) = \tau\end{aligned}$$

Using this formula one can easily show that $w \otimes w$ commutes with W (cf (4.2)). Now, for any $c \in A$ we have

$$\begin{aligned}(\phi \otimes \phi)\Delta(c) &= (w \otimes w)^* W(c \otimes I)W^*(w \otimes w) \\ &= W(w \otimes w)^*(c \otimes I)(w \otimes w)W^* \\ &= W(\phi(c) \otimes I)W^* = \Delta(\phi(c)).\end{aligned}$$

Q.E.D.

We end this Section with a short discussion showing that manageability is the condition distinguishing groups from semigroups. We recall that classical '$ax+b$' group $G_{\text{classical}}$ consists of all affine transformations $\mathbf{R} \ni x \longmapsto ax + b \in \mathbf{R}$ with $a > 0$. Assuming in addition that $b > 0$ we define a subsemigroup $G^+_{\text{classical}} \subset G_{\text{classical}}$. In the quantum setting, the condition $b > 0$ selects a subspace of H. Let $H_+ = H(b > 0)$ and $x \in H_+ \otimes H_+$. On this subspace operator $\hat{b} \otimes b$ is strictly positive and computing $F_\hbar\left(\hat{b} \otimes b, \alpha(\hat{\beta} \otimes \beta)\chi(\hat{b} \otimes b < 0)\right)^*$ we have to use the first version of formula (0.1). Therefore

$$Wx = V_\theta\left(\log\left(e^{i\hbar/2} b^{-1} a \otimes b\right)\right)^* e^{\frac{i}{\hbar}\log(sb^{-1}) \otimes \log a} x$$

All operators appearing in this formula leave H_+ invariant. Therefore $H_+ \otimes H_+$ is W-invariant. The restriction of W to this invariant subspace will be denoted by W_+:

$$W_+ = V_\theta\left(\log\left(e^{i\hbar/2} b_+^{-1} a_+ \otimes b_+\right)\right)^* e^{\frac{i}{\hbar}\log(s_+ b_+^{-1}) \otimes \log a_+},$$

where a_+, b_+, s_+ are restrictions of a, b, s to H_+. Restricting both sides of (2.1) to the subspace $H_+ \otimes H_+ \otimes H_+$ we see that W_+ is a multiplicative unitary. This multiplicative unitary is not manageable. Indeed $\overline{H_+} \otimes H_+$ is not $\widetilde{W}$-invariant and the operator $\widetilde{W_+} = \chi\left(b^\top \otimes b > 0\right)\widetilde{W}\chi\left(b^\top \otimes b > 0\right)$ is not unitary. To obtain a C^*-algebra we have to replace (4.4) by the formula

$$A_+ = \left\{(\omega \otimes \mathrm{id})W_+ + (\omega' \otimes \mathrm{id})W_+^* : \omega, \omega' \in B(H_+)_*\right\}^{\text{norm closure}} \tag{4.19}$$

One can show that $\log a_+, b_+ \,\eta\, A_+$ and that A_+ is generated by these two elements. Let G^+ be the quantum space corresponding to the C^*-algebra A_+. The formula

$$\Delta_+(c) = W_+(c \otimes I)\, W_+^*$$

defines coassociative comultiplication $\Delta_+ \in \operatorname{Mor}(A_+, A_+ \otimes A_+)$. One can verify that

$$\Delta_+(a_+) = a_+ \otimes a_+, \qquad \Delta_+(b_+) = a_+ \otimes b_+ + b_+ \otimes I.$$

In the second formula $a_+ \otimes b_+ + b_+ \otimes I$ is essentially selfadjoint and has unique selfadjoint extension. Δ_+ introduces a semigroup structure on G_+. Clearly G_+ is a quantum deformation of $G^+_{\text{classical}}$. In this way we constructed an example of a quantum semigroup coming from a non-manageable multiplicative unitary. This is rather surprising: Kac-Takesaki operators corresponding to semi-subgroups of locally compact groups are non-unitary coisometries. It shows that manageability (rather than unitarity) is the condition distinguishing groups from semigroups.

5. The dual of '$ax+b$' quantum group.

Let W be the multiplicative unitary introduced by (4.2). The theory of multiplicative unitaries provide a simple method of constructing group duals. Following Baaj and Skandalis we denote by $\Sigma : H \otimes H \longrightarrow H \otimes H$ the flip operator: $\Sigma(x \otimes y) = y \otimes x$ for any $x, y \in H$. The corresponding flip acting on operators will be denoted by σ:

$$\sigma(c \otimes c') = \Sigma(c \otimes c')\Sigma = c' \otimes c$$

for any $c, c' \in B(H)$. It is well known that for any manageable multiplicative unitary W, the operator $\widehat{W} = \Sigma W^* \Sigma$ is also a manageable multiplicative unitary. By definition the regular dual of the quantum group related to a multiplicative unitary W is the quantum group related to $\widehat{W}$. The algebra of 'continuous functions vanishing at infinity' on the dual of the group is introduced by the formula:

$$\hat{A} = \left\{ (\mathrm{id} \otimes \omega) W^* : \omega \in B(H)_* \right\}^{\text{norm closure}}$$

The dual group structure is given by the comultiplication $\widehat{\Delta} \in \mathrm{Mor}(\hat{A}, \hat{A} \otimes \hat{A})$ such that

$$(\widehat{\Delta} \otimes \mathrm{id}) W = W_{23} W_{13}. \tag{5.1}$$

The following theorem reduces the description of the dual of '$ax+b$' group to the original group.

Theorem 5.1. *Operators $\hat{a}$, $\hat{b}$, $i\hat{b}\hat{\beta}$ are affiliated with $\hat{A}$. There exists a C^*-isomorphism $\psi : A \longrightarrow \hat{A}$ such that $\psi(a) = \hat{a}$, $\psi(b) = \hat{b}$ and $\psi(ib\beta) = i\hat{b}\hat{\beta}$. This isomorphism reverses order of the group operation:*

$$\widehat{\Delta}(\psi(c)) = \sigma(\psi \otimes \psi)\Delta(c) \tag{5.2}$$

for any $c \in A$.

We shall use the following

Proposition 5.2. *Let H be a Hilbert space and $(a, b, \beta) \in G_H$. Assume that $\ker b = \{0\}$. Then the triples (a, b, β), $(a, e^{i\hbar/2} ab, \beta)$ and $(e^{i\hbar/2}|b|^{-1} a, b, \beta)$ are unitarily equivalent. In particular the triple $(a, e^{i\hbar/2} ab, \beta) \in G_H$ and $(e^{i\hbar/2}|b|^{-1} a, b, \beta) \in G_H$. Moreover if s is a strictly positive selfadjoint operator acting on H such that s commutes with a, b, β, then the triple (sa, b, β) is unitarily equivalent to $(a, b, \beta) \in G_H$.*

Proof: Let

$$U_1 = e^{\frac{i}{2\hbar}(\log a)^2}, \qquad U_2 = e^{\frac{i}{2\hbar}(\log|b|)^2} \qquad \text{and} \qquad U_3 = |b|^{-\frac{1}{\hbar}\log s}.$$

Clearly β commutes with U_1, U_2, U_3, a commutes with U_1 and b commutes with U_1, U_3. Using [20, Statement 3 of Theorem 3.3] we check that $U_1 b U_1^* = e^{i\hbar/2} ab$, $U_2 a U_2^* = e^{i\hbar/2}|b|^{-1} a$ and $U_3 a U_3^* = sa$. Therefore

$$\begin{aligned} U_1(a, b, \beta)U_1^* &= \quad (a, e^{i\hbar/2} ab, \beta) \\ U_2(a, b, \beta)U_2^* &= (e^{i\hbar/2}|b|^{-1} a, b, \beta) \\ U_3(a, b, \beta)U_3^* &= \qquad (sa, b, \beta) \end{aligned}$$

Q.E.D.

Proof of Theorem 5.1:

We set $(a_1, b_1) = (e^{i\hbar} b^{-2} a, b)$, $(a_2, b_2) = (a_1, e^{i\hbar/2} a_1 b_1)$, $(a_3, b_3) = (e^{i\hbar/2}|b_2|^{-1} a, b_2)$ and $(a_4, b_4) = (sa_3, b_3)$. One can easily verify that $a_4 = s|b|^{-1} = \hat{a}$ and $b_4 = e^{i\hbar/2} b^{-1} a = \hat{b}$. By Proposition 5.2 the triples (a, b, β), (a_1, b_1, β), (a_2, b_2, β), (a_3, b_3, β) and $(a_4, b_4, \beta) = (\hat{a}, \hat{b}, \hat{\beta})$ are unitarily equivalent.

Let $Z \in B(H)$ be a unitary operator such that $\hat{a} = Z^* a Z$, $\hat{b} = Z^* b Z$, $\hat{\beta} = Z^* \beta Z$ and ψ be the automorphism of $B(H)$ implemented by Z:

$$\psi(c) = Z^* c Z$$

Then $\psi(a) = \hat{a}$, $\psi(b) = \hat{b} =$, $\psi(\beta) = \hat{\beta}$ and taking into account definition (4.2) we see that operator $(\mathrm{id} \otimes \psi) W$ is invariant with respect to the flip:

$$\sigma(\mathrm{id} \otimes \psi) W = (\mathrm{id} \otimes \psi) W.$$

Therefore, for any $\omega \in B(H)_*$ we have:

$$\begin{aligned} \psi\Big((\omega \otimes \mathrm{id}) W\Big) &= (\omega \otimes \mathrm{id})(\mathrm{id} \otimes \psi) W \\ &= (\mathrm{id} \otimes \omega)(\mathrm{id} \otimes \psi) W = (\mathrm{id} \otimes \omega_Z) W, \end{aligned} \tag{5.3}$$

where $\omega_Z = \omega \circ \psi \in B(H)_*$. Formula (5.3) shows that $\psi(A) = \hat{A}$. We shall verify (5.2). Let $c = (\omega \otimes \mathrm{id}) W \in A$. Then by the above formula $\psi(c) = (\mathrm{id} \otimes \omega_Z) W$ and using (5.1) we obtain

$$\widehat{\Delta}(\psi(c)) = (\mathrm{id} \otimes \mathrm{id} \otimes \omega_Z)(\widehat{\Delta} \otimes \mathrm{id}) W = (\mathrm{id} \otimes \mathrm{id} \otimes \omega_Z) W_{23} W_{13}.$$

Therefore

$$\sigma\widehat{\Delta}(\psi(c)) = (\mathrm{id}\otimes\mathrm{id}\otimes\omega_Z)W_{13}W_{23}. \tag{5.4}$$

On the other hand

$$\begin{aligned}(\psi\otimes\psi)\Delta(c) &= (\psi\otimes\psi)\Delta\Big((\omega\otimes\mathrm{id})W\Big) = (\omega\otimes\psi\otimes\psi)W_{12}W_{13}\\ &= (\omega\otimes\mathrm{id}\otimes\mathrm{id})\Big[(\mathrm{id}\otimes\psi)W\Big]_{12}\Big[(\mathrm{id}\otimes\psi)W\Big]_{13}.\end{aligned}$$

Remembering that $(\mathrm{id}\otimes\psi)W$ is flip-invariant and that ψ is multiplicative we obtain:

$$\begin{aligned}(\psi\otimes\psi)\Delta(c) &= (\mathrm{id}\otimes\mathrm{id}\otimes\omega)\Big[(\mathrm{id}\otimes\psi)W\Big]_{13}\Big[(\mathrm{id}\otimes\psi)W\Big]_{23}\\ &= (\mathrm{id}\otimes\mathrm{id}\otimes\omega)(\mathrm{id}\otimes\mathrm{id}\otimes\psi)W_{13}W_{23} = (\mathrm{id}\otimes\mathrm{id}\otimes\omega_Z)W_{13}W_{23}.\end{aligned}$$

Comparing this formula with (5.4) we obtain (5.2)

Q.E.D.

Appendices

A. Affiliation relation and tensor product.

For any Hilbert space H we denote by $C^*(H)$ the set of all non-degenerate separable C*-algebras of operators acting on H.

Proposition A.1. *Let T_1, T_2 be non-zero normal operators acting on Hilbert spaces H_1, H_2 respectively and let $A_1 \in C^*(H_1)$ and $A_2 \in C^*(H_2)$. Then*

$$\Big(T_1\otimes T_2\ \eta\ A_1\otimes A_2\Big) \Longleftrightarrow \Big(T_1\ \eta\ A_1 \ \ \textit{and}\ \ T_2\ \eta\ A_2\Big). \tag{A.1}$$

Proof: The implication '$\Longleftarrow$' follows from [17, Theorem 6.1]. We shall prove the converse. Multiplying if necessary T_2 by a complex number, we may assume that $1 \in \mathrm{Sp}\, T_2$. Then for any $r > 0$ the spectral subspace $H_2\Big(|T_2 - 1| < r\Big) \neq \{0\}$. Let Ω_r be a norm 1 vector belonging to this subspace and ω_r be the state of A_2 corresponding to this vector:

$$\omega_r(c) = (\Omega_r|c|\Omega_r)$$

for any $c \in A_2$. For any $f \in C_\infty(\mathbf{C})$ and any $t \in \mathbf{C}$ we set

$$f_r(t) = (\Omega_r|f(tT_2)|\Omega_r).$$

Clearly

$$f_r(t) = \int_{\mathbf{R}} f(t\tau)d\mu_r(\tau),$$

where μ_r is a probability measure on $\mathbf{C}$ such that $\mu_r(\Lambda) = (\Omega_r|\chi(T_2\in\Lambda)|\Omega_r)$ for any measurable subset $\Lambda \subset \mathbf{R}$. Condition $\Omega_r \in H_2\Big(|T_2 - 1| < r\Big)$ implies that the support of μ_r is contained in the ball $\{t \in \mathbf{C} : |t-1| < r\}$. Using this result one can easily show that $f_r \in C_\infty(\mathbf{C})$ and that f_r converges uniformly to f, when $r \to 0$:

$$\lim_{r\to 0} f_r = f. \tag{A.2}$$

A moment of reflection shows that

$$(\mathrm{id}\otimes\omega_r)\, f(T_1\otimes T_2) = f_r(T_1). \tag{A.3}$$

If $T_1\otimes T_2\ \eta\ A_1\otimes A_2$, then $f(T_1\otimes T_2) \in M(A_1\otimes A_2)$ and the above formula shows that $f_r(T_1) \in M(A_1)$. Taking into account (A.2) we obtain: $f(T_1) \in M(A_1)$. Clearly the mapping

$$C_\infty(\mathbf{C}) \ni f \longrightarrow f(T_1) \in M(A_1) \tag{A.4}$$

is a *-algebra homomorphism. Assume for the moment that $f(t) > 0$ for all $t \in \mathbf{C}$. Then (cf [18, formula 1.8]) $f\,(T_1\otimes T_2) > 0$ on Sp $(A_1\otimes A_2)$ and by (A.3), $f_r(T_1) > 0$ on $\mathrm{Sp}\, A_1$. It means that $f_r(T_1)A_1$ is dense in A_1. Therefore (A.4) is a morphism from $C_\infty(\mathbf{C})$ into A_1. The function

$f(t)=t$ is an element affiliated with $C_\infty(\mathbf{C})$. Applying the morphism (A.4) to this element we obtain $f(T_1)=T_1$. Therefore $T_1\,\eta\,A_1$. In the same way one can show that $T_2\,\eta\,A_2$.

Q.E.D.

Remark: We are strongly convinced that the equivalence (A.1) holds for any non-zero closed operators T_1 and T_2. However we were unable to find a proof working for operators that are not normal.

B. A QEF equality

This Appendix may be treated as a supplement to [20]. We shall prove an equality satisfied by the quantum exponential function $F_\hbar$ with $\hbar=\frac{\pi}{2k+3}$, where $k=0,1,2,\ldots$. We start with some simple properties of the polynomial $w(t)$ of the order $2k+3$ introduced by (1.15). Let

$$\Phi=\left\{-e^{-i\left(\frac{1}{2}-\ell\right)\hbar}:\ell=1,2,\ldots,2k+3\right\} \tag{B.1}$$

be the set of all zeroes of w. The reader should notice that Φ is contained in the upper half plane. One can easily verify that the set $\Phi\cup(-\Phi)=\left\{t:1+t^{2(2k+3)}=0\right\}$. Therefore

$$w(t)w(-t)=1+t^{2(2k+3)}. \tag{B.2}$$

Moreover $\Phi^{-1}=-\Phi$ and the product of all elements of Φ equals $-i(-1)^k=-i\alpha$. Therefore

$$w(t)=i\alpha t^{2k+3}w\left(-t^{-1}\right). \tag{B.3}$$

Finally Φ is symmetric with respect to the imaginary axis. Therefore

$$\overline{w(t)}=w(-\bar{t}). \tag{B.4}$$

Let $r\in\mathbf{R}$ and $\varrho=\pm1$. We claim that

$$\begin{aligned}\overline{F_\hbar(r,\varrho\chi(r<0))}F_\hbar(-r,\varrho\chi(r>0)) &= w\left(r\right)^{-1}\left[1+i\varrho r^{2k+3}\right]\\ &= w(r)^{-1}+\alpha\varrho w\left(-r^{-1}\right)^{-1}.\end{aligned} \tag{B.5}$$

The last equality follows immediately from (B.3). Applying complex conjugation to all parts of (B.5) we obtain the same formula with r replaced by $-r$. Therefore it is sufficient to prove (B.5) for $r>0$. In this case computing $F_\hbar(r,\varrho\chi(r<0))$ ($F_\hbar(-r,\varrho\chi(r>0))$ respectively) we have to use the first (the second respectively) version of formula (0.1). We obtain:

$$\text{LHS}=\overline{V_\theta\left(\log r\right)}V_\theta\left(\log r-\pi i\right)\left[1+i\varrho r^{\frac{\pi}{\hbar}}\right]. \tag{B.6}$$

We recall that $\hbar=\frac{\pi}{2k+3}$, where $k=0,1,2,\ldots$. Therefore $\pi=(2k+3)\hbar$. We know (cf [20, Formula 1.31]) that

$$V_\theta(x+i\hbar)=\left(1+e^{i\hbar/2}e^x\right)V_\theta(x)$$

for any $x\in\mathbf{C}$. Using this formula $(2k+3)$-times with $x=\log r-i\ell\hbar$ ($\ell=1,2,\ldots,2k+3$) we obtain

$$V_\theta(\log r)=\prod_{\ell=1}^{2k+3}\left(1+e^{i\left(\frac{1}{2}-\ell\right)\hbar}r\right)V_\theta(\log r-\pi i).$$

For real r, $|V_\theta(\log r)|=1$. Therefore

$$\overline{V_\theta(\log r)}V_\theta(\log r-\pi i)=\prod_{\ell=1}^{2k+3}\left(1+e^{i\left(\frac{1}{2}-\ell\right)\hbar}r\right)^{-1}=w(r)^{-1}.$$

Formula (B.6) shows now, that

$$\text{LHS}=w\left(r\right)^{-1}\left[1+i\varrho r^{\frac{\pi}{\hbar}}\right]$$

and (B.5) follows.

References

[1] N.I. Achiezer and I.M. Glazman: Theory of linear operators in Hilbert space. Pitman Publishing, Boston, London, Melbourne 1981

[2] W. Arveson: An invitation to C^*-algebra, Springer-Verlag New York, Heidelberg, Berlin 1976.

[3] S. Baaj and G. Skandalis: Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres. *Ann. scient. Éc. Norm. Sup.*, 4^e série, t. 26, 1993, pp 425 – 488.

[4] J. Kustermans and S. Vaes: Locally compact quantum groups. *Ann. Scient. Ec. Norm. Sup.*, to appear, see also A simple definition for locally compact quantum groups. *C.R. Acad. Sci.*, Paris, sér. I **328** (10) (1999) 871-876

[5] M.B. Landstad: Duality theory of covariant systems. *Trans. Amer. Math. Soc.* 248 (1979) no 2, pp 223–267.

[6] K. Maurin: Methods of Hilbert spaces, Warszawa 1967.

[7] G.K. Pedersen: C^*-algebras and their automorphism groups. Academic Press, London, New York, San Francisco 1979.

[8] W. Pusz and S.L. Woronowicz: A new quantum deformation of '$ax + b$' group. In preparation.

[9] M. Rowicka-Kudlicka: Unitary representations of the quantum "ax+b" group, math.QA/0102151.

[10] K. Schmüdgen: Operator representations of $\mathbf{R}^2_q$. *Publications of RIMS Kyoto University* 29, 1030–1061, (1993)

[11] K. Schmüdgen: Integral operator representations of $\mathbf{R}^2_q, X_{q,\gamma}$ and $SL_q(2,\mathbf{R})$. *Commun. Math. Phys.* 159, pp 217 – 237 (1994).

[12] P.M. Sołtan and S.L. Woronowicz: A remark on manageable multiplicative unitaries. In preparation.

[13] G. Skandalis: Duality for locally compact 'quantum groups'(joint work with S. Baaj). Mathematische Forschungsinstitut Obervolfach, Tagungsbericht 46/1991, C*-algebren, 20.10 – 26.10.1991, p. 20.

[14] S. Vaes and L. Vainerman: Extensions of locally compact quantum groups and bicrossed product construction, Preprint of the K.U. Lueven (2001).

[15] A. Van Daele: The Haar measure on some locally compact groups, Preprint of the K.U. Leuven (2001).

[16] S.L. Woronowicz: Unbounded elements affiliated with C^*-algebras and non-compact quantum groups. *Commun. Math. Phys.* 136, 399–432 (1991).

[17] S.L. Woronowicz and K. Napiórkowski: Operator theory in the C^*-algebra framework *Reports on Math. Phys.* Vol 31 (1992) 353–371.

[18] S.L. Woronowicz: C*-algebras generated by unbounded elements. Reviews on Mathematical Physics, Vol. 7, No. 3, 481 – 521 (1995)

[19] S.L. Woronowicz: From multiplicative unitaries to quantum groups. *International Journal of Mathematics*, Vol. 7, No. 1, pp 127 – 149 (1996)

[20] S.L. Woronowicz: Quantum exponential function. *Reviews in Mathematical Physics*, Vol. 12, No 6 (2000) pp 873 – 920

[21] S.L. Woronowicz: Quantum '$az + b$' group on complex plane. Preprint of KMMF No XX/2000, University of Warsaw, 44 pages, will appear in *International Journal of Mathematics*

[22] S.L. Woronowicz: Quantum $SL(2,\mathbf{R})$ group on the C*-algebra level. In preparation.

Department of Mathematical Methods in Physics,, Faculty of Physics, University of Warsaw,, Hoża 74, 00-682 Warszawa, Poland

E-mail address, S. L. WORONOWICZ: slworono@fuw.edu.pl