

A NEW QUANTUM DEFORMATION OF ‘ $ax + b$ ’ GROUP.

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ABSTRACT. The paper is devoted to locally compact quantum groups that are related to classical ‘ $ax + b$ ’ group. We discuss in detail the quantization of the deformation parameter assumed with no justification in the previous paper. Next we construct (on the C^* -level) a larger family of quantum deformations of ‘ $ax + b$ ’ group corresponding to the deformation parameter q^2 running over an interval in unit circle. To this end, beside the reflection operator β known from the previous paper we use a new unitary generator w . It commutes with a , b and $\beta w \beta = s^{\text{sgn } b} w$, where $s \in S^1$ is a new deformation parameter related to q^2 . At the end we discuss the groups at roots of unity.

0. INTRODUCTION

In last years a lot of effort was devoted to constructing explicit examples of (non-compact) locally compact quantum groups. The present paper inscribes into this line of research. It is devoted to quantum deformations of the group ‘ $ax + b$ ’ of affine transformations of real line. Such quantum ‘ $ax + b$ ’ groups were presented first in [19]. We shall use the adjective *old* to distinguish them from the *new* ‘ $ax + b$ ’ groups constructed in section 4 of present paper.

We go back to the subject for the following reasons. At first the quantizations of the deformation parameter \hbar introduced in the previous paper was not discussed in detail. Now we give strong arguments that the values of $\hbar = \frac{\pi}{2k+3}$ are the only ones allowed within the setting considered in [19]. Secondly one of the important formulae in [19] was not proven. We fill this gap. Third, the old quantum ‘ $ax + b$ ’ admits a large set of automorphisms. In [19] we identified only four of them. Now we show that the group of automorphisms is as large as S^1 . These automorphisms play an important role in constructing the new quantum ‘ $ax + b$ ’ groups. The new groups do exist for \hbar running over an interval in \mathbb{R} (no more quantization of deformation parameter).

The new quantum groups constructed in the present paper seem to be very important. They will serve as *building blocks* in construction of quantum $SL(2, \mathbb{R})$ group. This is our next target.

Let G be the ‘ $ax + b$ ’ group. On the classical level G consists of all transformations of the form

$$(0.1) \quad \mathbb{R} \ni x \longmapsto ax + b \in \mathbb{R},$$

where a and b are real parameters labeling the elements of the group. We shall assume that $a > 0$. Assigning to each element of the group the values of the parameters we define two unbounded continuous real functions on G . To denote the functions we shall use the same letters: $a, b \in C(G)$. Then the C^* -algebra $C_\infty(G)$ of all continuous functions vanishing at infinity on G is generated by $\log a$ and b :

$$C_\infty(G) = \left\{ f(\log a)g(b) : f, g \in C_\infty(\mathbb{R}) \right\} \Big|_{\substack{\text{uniformly closed} \\ \text{linear envelope}}}$$

Functions a and b may be considered as elements affiliated with $C_\infty(G)$. Composing two transformations of the form (0.1) with parameters (a_1, b_1) and (a_2, b_2) one obtains the transformation with parameters $(a_1 a_2, a_1 b_2 + b_1)$. This result leads to the following formulae describing the multiplication:

$$(0.2) \quad \begin{aligned} \Delta(a) &= a \otimes a, \\ \Delta(b) &= a \otimes b + b \otimes I. \end{aligned}$$

At the moment the elements of G are considered as affine transformations of \mathbb{R} . However one may realize them as unitary operators acting on a Hilbert space. To this end, to any transformation of the form (0.1) we assign unitary operator $V_{(a,b)} \in B(L^2(\mathbb{R}))$ introduced by the formula:

$$(V_{(a,b)} f)(x) = a^{-1/2} f(a^{-1}(x - b))$$

for any $f \in L^2(\mathbb{R})$. Then G may be identified with the set of unitary operators:

$$(0.3) \quad G = \{V_{(a,b)} : a, b \in \mathbb{R}; a > 0\}.$$

This identification preserves the group structure and the topology. More precisely $V_{(1,0)} = I$ and

$$(0.4) \quad V_{(a_1, b_1)} V_{(a_2, b_2)} = V_{(a_1 a_2, a_1 b_2 + b_1)}$$

for any $a_1, a_2 \in]0, \infty[$ and $b_1, b_2 \in \mathbb{R}$. Moreover a sequence $V_{(a_n, b_n)}$ converges to $V_{(a_\infty, b_\infty)}$ in strong topology if and only if $a_n \rightarrow a_\infty > 0$ and $b_n \rightarrow b_\infty$. In particular (0.3) with the strong operator topology is a locally compact space. One can also show that (0.3) is a closed subset of $B(L^2(\mathbb{R}))$ (in strong operator topology).

For any Hilbert space H we denote by $\mathcal{K}(H)$ the C^* -algebra of all compact operators acting on H . According to the general theory [13] the strongly continuous family of unitaries (0.3) is described by a single unitary $V \in M(\mathcal{K}(L^2(\mathbb{R})) \otimes C_\infty(G))$. The C^* -algebra $C_\infty(G)$ is generated (in the sense of [13]) by V . Formula (0.4) means that

$$(\text{id} \otimes \Delta)V = V_{12}V_{13}.$$

This way we arrive to the notion of a (quantum) group of unitary operators. Let H be a Hilbert space. We shall consider pairs (A, V) , where A is a C^* -algebra and V is a unitary element of the multiplier algebra $M(\mathcal{K}(H) \otimes A)$. If A is generated by $V \in M(\mathcal{K}(H) \otimes A)$ then (A, V) is called a quantum family of unitary operators. We say that the family is closed with respect to operator multiplication if there exists a morphism $\Delta \in \text{Mor}(A, A \otimes A)$ such that

$$(0.5) \quad (\text{id} \otimes \Delta)V = V_{12}V_{13}.$$

Then Δ is unique (because A is generated by V). Finally (A, V) is said to be a quantum group of unitary operators, if it is closed with respect to the operator multiplication and if (A, Δ) is a locally quantum group in the sense of Kustermans and Vaes [3]. It should be possible to formulate the last condition directly in terms of (A, V) . However this is not the subject of the present paper.

Let us go back to the ‘ $ax + b$ ’ group. In the quantum setting functions a and b are replaced by selfadjoint elements $a = a^* > 0$ and $b = b^*$ that no longer commute. Instead they satisfy the relation

$$(0.6) \quad ab = q^2 ba,$$

where the deformation parameter q^2 is a number of modulus 1. Unfortunately in our case elements a and b are represented by unbounded operators and the products ab and ba may not be well defined because of the domain problem. For this reason we replace (0.6) by the so called Zakrzewski relation. It says that for any $\tau \in \mathbb{R}$:

$$a^{i\tau} b a^{-i\tau} = e^{\hbar\tau} b.$$

In this formula \hbar is a real constant such that $q^2 = e^{-i\hbar}$. For technical reasons we shall assume that $0 < \hbar < \frac{\pi}{2}$. The reader should notice that for $\tau = -i$ the above relation reduces to (0.6).

The second problem is related to the comultiplication. We would like to keep formulae (0.2). However in general $a \otimes b + b \otimes I$ is not selfadjoint and in the best case we may expect that $\Delta(b)$ is a selfadjoint extension of $a \otimes b + b \otimes I$:

$$a \otimes b + b \otimes I \subset \Delta(b).$$

To choose the extension in a well defined way we have to use additional operators independent of a and b . For old quantum ‘ $ax + b$ ’ groups we use a selfadjoint unitary β commuting with a and anticommuting with b . For new groups the situation is even more complicated. It means that the algebra A is no longer generated by $\log a$ and b .

It is not obvious, how to present quantum ‘ $ax + b$ ’ group as a quantum group of unitary operators (A, V) . The crucial point is the formula $V = V(a, b, \dots)$ expressing V in terms of a, b and perhaps some other elements related to A . The equation (0.5) takes the form

$$V(a \otimes a, [a \otimes b + b \otimes I], \dots) = V(a \otimes I, b \otimes I, \dots) V(I \otimes a, I \otimes b, \dots),$$

where $[a \otimes b + b \otimes I]$ is a suitable selfadjoint extension of $a \otimes b + b \otimes I$. To find solutions of this equation we spent a lot of time making use of our experience in the area of quantum exponential functions and quantum groups (cf. [15, 14, 11, 18, 17, 19, 7, 9]). As a result we got formulae (3.8) and (4.8) that are starting points in our presentation.

Let us briefly discuss the content of the paper. Sections 1 and 2 are devoted to mathematical tools used in the paper. In the first one we recall the Zakrzewski commutation relation and related quantum exponential function (with a slightly modified notation). Most of the results presented in that section come from [17]; the essentially new result is contained in Proposition 1.4. The second

section deals with the notion of a C^* -algebra generated by affiliated elements. We prove a number of results used in the main part of the paper.

Section 3 is devoted to the quantum 'ax + b' groups introduced in [19]. These groups exist only for special values of deformation parameter $q^2 = e^{-i\hbar}$ with $\hbar = \frac{\pi}{2k+3}$, where $k = 0, 1, 2, \dots$. This fact was not really shown in [19]. The special values of the deformation parameter were chosen to proceed with some computations. It was not clear that (at the expense of some complications) one is not able to construct quantum 'ax + b' group for larger set of values of the deformation parameter. Now, presenting the 'ax + b' group as a quantum group of unitary operators we obtain the quantization of the deformation parameter as a precise mathematical statement (cf. Theorem 3.3). More precisely for $q^2 = e^{-i\hbar}$ we shall construct a C^* -algebra A with distinguished selfadjoint elements a, b and $i\beta b$ affiliated with it (the so called reflection operator β is a unitary involution which is not affiliated with A). These elements satisfy (in a well defined sense) the relations $ab = q^2ba$, $a\beta = \beta a$ and $b\beta = -\beta b$. The algebra A is generated by a unitary element $V \in M(\mathcal{K}(L^2(\mathbb{R})) \otimes A)$. The pair (A, V) is defined for all $0 < \hbar < \pi/2$. However, the existence of Δ satisfying the condition (0.5) selects much smaller subset of admissible \hbar 's. We shall prove that Δ exists if and only if \hbar is of the form indicated above. Next we derive formulae showing how Δ acts on generators of A . In particular we prove an elegant formula describing the action of the comultiplication on the reflection operator. This formula appeared (with no proof) in the previous paper (cf. formula (4.16) of [19]). At the end of section 3 we find an interesting action of S^1 on the algebra A : for any $s \in S^1$ we have an automorphism ϕ_s of A and $\phi_{ss'} = \phi_s \circ \phi_{s'}$. If Δ exists (i.e. if $\hbar = \frac{\pi}{2k+3}$, where $k = 0, 1, 2, \dots$) then $\Delta \circ \phi_s = (\phi_s \otimes \phi_s) \circ \Delta$.

New quantum groups related to classical 'ax + b' group are constructed in section 4. Using one of the automorphism ϕ_s described at the end of section 3 we consider the corresponding crossed product. In other words we extend the algebra A by adding a new unitary generator w implementing ϕ_s . This enlargement of the algebra opens new possibilities. In particular we obtain a new admissible values of the deformation parameter. Now $\hbar = \frac{\pi}{p}$, where p is a number larger than 2 such that $-e^{i\pi p} = s$. The latter relation distinguishes a discrete set of possible p . However the quantization of \hbar disappears because changing s we may cover the whole interval $\hbar \in]0, \frac{\pi}{2}[$. For $s = 1$ we obtain $p = 2k + 3$, where $k = 0, 1, 2, \dots$. In this case the new quantum 'ax + b' group reduces to a semidirect product of the old one by S^1 . For $s \neq 1$ we get essentially new examples of locally compact quantum groups.

In the next section (section 5) we investigate the multiplicative unitaries for the quantum groups constructed in section 4. We prove their modularity and find the unitary antipode and scaling group. In particular the objects constructed in section 4 satisfy all the axioms of Kustermans and Vaes [3] and the ones of Masuda, Nakagami and Woronowicz [4]. At the end of the section we briefly discuss the duals of the new quantum 'ax + b' groups.

The last section is devoted to new quantum 'ax + b' groups with q^2 being a root of unity. In this case we may pass to groups with *smaller size*. To this end we have to assume that the unitary generator w satisfies the additional relation of the form $w^N = I$.

The word *size* used in the previous paragraph has a precise meaning. It is based on the Stone - von Neumann theorem. Let (A, Δ) be one of the quantum 'ax + b' group considered in this paper and π be a representation of A acting on a Hilbert space H_π such that $\ker(\pi(b)) = \{0\}$. Then operators $\log(\pi(a))$ and $\log(|\pi(b)|)$ satisfy the same commutation relations as position x_{QM} and momentum p_{QM} in quantum mechanics. By the Stone - von Neumann theorem the pair $(\log(\pi(a)), \log(|\pi(b)|))$ is unitarily equivalent to a direct sum of k copies of (x_{QM}, p_{QM}) . The number k will be called the multiplicity of π . We say that the size of the group (A, Δ) is equal k if there exists a faithful representation of A with multiplicity k and if k is the smallest number with this property.

In the classical situation the algebra of functions A is generated by $\log a$ and b . This is not the case when we consider quantum 'ax + b' groups: we use additional generators such as β and w . It means that together with the quantum deformation we pass to a sort of extension of the group. The size says us, how large the extension is.

The old quantum 'ax + b' groups are of size 2. This is the minimal value. One can show that the old quantum 'ax + b' groups are the only ones with size 2. The new groups introduced in section 4 are of infinite size. On the other hand the groups at roots of unity considered in the last section are of size $2N$, where N is the number appearing in the relation $w^N = I$.

Our approach extensively uses the C^* -algebra language and the theory of selfadjoint operators on Hilbert space. For the basic facts concerning the general C^* -algebra theory we refer to [1, 6]. The notation used in the paper follows the one explained in [13, 12]. In particular $M(A)$ is the multiplier algebra of a C^* -algebra A . The affiliation relation in the sense of C^* -algebra theory is denoted by “ η ” and A^η is the set of all affiliated elements (“unbounded multipliers”). It is known that $M(A) \subset A^\eta$. A morphism from A to a C^* -algebra B is by definition any $*$ -homomorphism $\pi : A \rightarrow M(B)$ such that $\pi(A)B$ is dense in B . Let us recall that any such π has the unique extension to a unital $*$ -homomorphism $\pi : M(A) \rightarrow M(B)$ and to $*$ -preserving map $\pi : A^\eta \rightarrow B^\eta$ respectively (both denoted by the same symbol). The set of all morphisms from A to B is denoted by $\text{Mor}(A, B)$.

With some abuse of notation, the symbol $\text{Rep}(A)$ will stay for the “set” of all non-degenerate representations of a C^* -algebra A . For any $\pi \in \text{Rep}(A)$, we denote by H_π the carrier Hilbert space of π . Then $\pi \in \text{Mor}(A, \mathcal{K}(H_\pi))$.

In the paper we mostly deal with concrete C^* -algebras. By definition they are norm closed $*$ -subalgebras of the algebra $B(H)$ of all bounded operators acting on some (separable) Hilbert space H . As a rule, C^* -algebras we deal with are separable. Non separable ones will appear only as a multiplier algebras. In particular $B(H) = M(\mathcal{K}(H))$. We shall denote by $C^*(H)$ the set of all non-degenerate separable C^* -algebras of operators acting on a Hilbert space H . We recall that an algebra $A \subset B(H)$ is non-degenerate if AH is dense in H .

We shall use functional calculus for strongly commuting selfadjoint operators. If T and β are selfadjoint operators acting on a Hilbert space H and T and β strongly commute then

$$T = \int_{\Lambda}^{\oplus} r dE(r, \varrho), \quad \beta = \int_{\Lambda}^{\oplus} \varrho dE(r, \varrho),$$

where $dE(r, \varrho)$ is the common spectral measure supported by the joint spectrum $\Lambda \subset \mathbb{R}^2$ of (T, β) . Moreover for any measurable complex valued function on Λ we have

$$f(T, \beta) = \int_{\Lambda} f(r, \varrho) dE(r, \varrho).$$

In this context the characteristic function χ will appear quite often. By definition for any sentence \mathcal{R} , we have

$$\chi(\mathcal{R}) = \begin{cases} 0 & \text{if } \mathcal{R} \text{ is false,} \\ 1 & \text{if } \mathcal{R} \text{ is true.} \end{cases}$$

Typically \mathcal{R} is a formula involving (in)equality sign. For example $\chi(r \leq 0)$ is equal 0 for positive r and 1 for $r = 0$ or negative. Consequently $\chi(T \leq 0)$ is the spectral projection assigned to the negative part of the spectrum of a selfadjoint operator T . The corresponding spectral subspace will be denoted by $H(T \leq 0)$: $H(T \leq 0) = \chi(T \leq 0)H$. Similarly $\chi(T = \lambda)$ is the orthogonal projection on the eigenspace $H(T = \lambda)$ of T corresponding to the eigenvalue $\lambda \in \mathbb{R}$. We refer to [17] for more detailed explanation of this notation.

Let $\text{Np}(r) = r\chi(r < 0)$. Then for any selfadjoint T ,

$$(0.7) \quad \text{Np}(T) = T\chi(T < 0)$$

is a selfadjoint operator acting on H . This is the negative part of the operator T . Another function frequently used in the paper is the one that returns the sign of the argument:

$$\text{sgn } r = \chi(r > 0) - \chi(r < 0).$$

Then $\text{sgn } T$ is the partial isometry that appears in the polar decomposition: $T = (\text{sgn } T)|T|$.

1. A SPECIAL FUNCTION AND SELFADJOINT EXTENSIONS.

In this section we recall (in a slightly modified version with a certain loss of generality) the basic definitions and statements of [17]. The only essentially new result is contained in formula (1.11). Later on it will help us to prove the formula announced in [19, formula (4.16)]. We start with a modified version the quantum exponential function introduced in [17]. Let $\hbar \in \mathbb{R}$ and $0 < \hbar < \frac{\pi}{2}$. Instead of function F_{\hbar} defined on the set $\mathbb{R}_- \times \{-1, 1\} \cup \mathbb{R}_+ \times \{0\}$ we shall use function G_{\hbar} defined on $\mathbb{R} \times \{-1, 1\}$. It is related to the function F_{\hbar} by the formula

$$(1.1) \quad G_{\hbar}(r, \varrho) = F_{\hbar}(r, \varrho\chi(r < 0))$$

for any $r \in \mathbb{R}$ and $\varrho = \pm 1$. Taking into account definition [17, formula (1.19)] we obtain

$$(1.2) \quad G_{\hbar}(r, \varrho) = \begin{cases} V_{\theta}(\log r) & \text{for } r > 0 \\ [1 + i\varrho|r|^{\frac{\pi}{\hbar}}] V_{\theta}(\log|r| - \pi i) & \text{for } r < 0, \end{cases}$$

where $\theta = \frac{2\pi}{\hbar}$ and V_{θ} is the meromorphic function on \mathbb{C} such that

$$V_{\theta}(x) = \exp \left\{ \frac{1}{2\pi i} \int_0^{\infty} \log(1 + t^{-\theta}) \frac{dt}{t + e^{-x}} \right\}$$

for all $x \in \mathbb{C}$ such that $|\Im x| < \pi$. In addition $G_{\hbar}(0, \pm 1) = 1$. Then $G_{\hbar}(r, \varrho)$ is a continuous function on $\mathbb{R} \times \{-1, 1\}$ and

$$(1.3) \quad \left(G_{\hbar}(r, \varrho) = G_{\hbar}(r, \varrho') \right) \iff \left(\varrho \chi(r < 0) = \varrho' \chi(r < 0) \right).$$

The asymptotic behavior of $G_{\hbar}(r, \varrho)$ for large r is described by the formula

$$(1.4) \quad G_{\hbar}(r, \varrho) \approx C \exp \left\{ \frac{(\log|r|)^2}{2i\hbar} \right\},$$

where C is a phase factor depending only on $\text{sgn } r$ and ρ and ' \approx ' means that the difference goes to 0 when $r \rightarrow \pm\infty$ (see Statements 9 and 10 of [17, Theorem 1.1]).

It is known that the quantum exponential function assumes values of modulus 1. Therefore if T and β are operators acting on a Hilbert space H , T is selfadjoint and β is unitary selfadjoint commuting with T then $G_{\hbar}(T, \beta)$ is unitary.

Now we recall the concept of selfadjoint extension of a symmetric operator defined by a reflection operator. Let Q be a symmetric operator acting on a Hilbert space H and ρ be a unitary selfadjoint operator ($\rho^* = \rho$, $\rho^2 = I$) anticommuting with Q . Then we denote by $[Q]_{\rho}$ the restriction of Q^* to the domain $\{x \in \mathcal{D}(Q^*) : (\rho - I)x \in \mathcal{D}(Q)\}$. It is known (cf. [17, Proposition 5.1]) that $[Q]_{\rho}$ is a selfadjoint extension of Q . We shall use the following simple

Proposition 1.1. *Let Q , X and ρ be operators acting on a Hilbert space H such that Q is symmetric, X is selfadjoint, ρ is unitary selfadjoint, $\rho Q = -Q\rho$ and $\rho X = -X\rho$. Assume that the restrictions of Q and X to $H(\rho = -1)$ coincide:*

$$(1.5) \quad Q|_{H(\rho=-1)} = X|_{H(\rho=-1)}.$$

Then $X = [Q]_{\rho}$.

Proof. Let $H_1 = H(\rho = -1)$ and $H_2 = H(\rho = 1)$. Then $H = H_1 \oplus H_2$ and (all) bounded and (some) unbounded operators may be represented by 2×2 matrices. In particular

$$\rho = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}.$$

Remembering that Q and X anticommute with ρ we obtain:

$$Q = \begin{pmatrix} 0 & Q_- \\ Q_+ & 0 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} 0 & X_- \\ X_+ & 0 \end{pmatrix},$$

where Q_+ and X_+ are operators acting from H_1 to H_2 and Q_- and X_- are operators acting from H_2 to H_1 . Clearly $Q_+ \subset Q_-^*$ (Q is symmetric) and $X_- = X_+^*$ (X is selfadjoint). Assumption (1.5) means that $Q_+ = X_+$. Therefore

$$X = \begin{pmatrix} 0 & Q_+^* \\ Q_+ & 0 \end{pmatrix}.$$

On the other hand

$$Q^* = \begin{pmatrix} 0 & Q_+^* \\ Q_-^* & 0 \end{pmatrix}.$$

It shows that $X \subset Q^*$ and $\mathcal{D}(X) = \{x \in \mathcal{D}(Q^*) : (\rho - I)x \in \mathcal{D}(Q)\}$. □

Let $\hbar \in \mathbb{R}$. We shall use the Zakrzewski relation $\xrightarrow{\hbar}$ (cf. [17]). Let R and S be selfadjoint operators acting on a Hilbert space H with the polar decompositions $R = \text{sgn } R |R|$ and $S =$

$\text{sgn } S |S|$. For simplicity we shall assume that one of the operators R and S has trivial kernel. If $\ker S = \{0\}$, then $\text{sgn } S$ is unitary selfadjoint and

$$\left(R \overset{\hbar}{\circ} S \right) \iff \begin{pmatrix} \text{sgn } S \text{ commutes with } R \\ \text{and } |S|^{-i\lambda} R |S|^{i\lambda} = e^{\hbar\lambda} R \\ \text{for any } \lambda \in \mathbb{R}. \end{pmatrix}.$$

If $\ker R = \{0\}$, then $\text{sgn } R$ is unitary selfadjoint and

$$\left(R \overset{\hbar}{\circ} S \right) \iff \begin{pmatrix} \text{sgn } R \text{ commutes with } S \\ \text{and } |R|^{i\lambda} S |R|^{-i\lambda} = e^{\hbar\lambda} S \\ \text{for any } \lambda \in \mathbb{R}. \end{pmatrix}.$$

If $\ker R = \ker S = \{0\}$, then the two above conditions are equivalent.

One can easily show that antiunitary operators reverse the direction of Zakrzewski relation:

$$(1.6) \quad \left(\begin{array}{l} R \overset{\hbar}{\circ} S \text{ and } J \text{ is an} \\ \text{antiunitary involution} \end{array} \right) \implies \left(JSJ \overset{\hbar}{\circ} JRJ \right)$$

Let R and S be selfadjoint operators with trivial kernels and $R \overset{\hbar}{\circ} S$. It is known [17, Example 3.1] that in this case, the operators $e^{i\hbar/2} S^{-1} R$ and $e^{i\hbar/2} S R^{-1}$ are selfadjoint and

$$\text{sgn} \left(e^{i\hbar/2} S^{-1} R \right) = \text{sgn} \left(e^{i\hbar/2} S R^{-1} \right) = (\text{sgn } R)(\text{sgn } S).$$

We shall use the following result (cf. [17, Theorem 5.2]):

Proposition 1.2. *Let R, S and τ be operators acting on a Hilbert space H . Assume that R and S are selfadjoint with trivial kernels, $R \overset{\hbar}{\circ} S$, and that τ is unitary, selfadjoint anticommuting with R and S . We set $T = e^{i\hbar/2} S^{-1} R$. Then T is a selfadjoint operator with trivial kernel, T commutes with τ , $R + S$ is a closed symmetric operator and the selfadjoint extension*

$$(1.7) \quad \begin{aligned} [R + S]_{\tau} &= G_{\hbar}(T, \tau)^* S G_{\hbar}(T, \tau) \\ &= G_{\hbar}(T^{-1}, \tau) R G_{\hbar}(T^{-1}, \tau)^* \end{aligned}$$

Remark 1.3. If τ' is another unitary, selfadjoint operator anticommuting with R and S and if in addition there exists a unitary selfadjoint operator ρ that commutes with τ, τ' and S and anticommutes with R then

$$(1.8) \quad \left([R + S]_{\tau} = [R + S]_{\tau'} \right) \implies \left(\tau = \tau' \right).$$

Indeed if $[R + S]_{\tau} = [R + S]_{\tau'}$, then (cf. (1.7))

$$G_{\hbar}(T, \tau)^* S G_{\hbar}(T, \tau) = G_{\hbar}(T, \tau')^* S G_{\hbar}(T, \tau').$$

It shows that the unitary operator $U = G_{\hbar}(T, \tau') G_{\hbar}(T, \tau)^*$ commutes with S and hence with $|S|$. Clearly

$$(1.9) \quad G_{\hbar}(T, \tau)^* = G_{\hbar}(T, \tau')^* U.$$

Moreover $T \overset{\hbar}{\circ} S$ due to Zakrzewski relation $R \overset{\hbar}{\circ} S$ and ρ anticommutes with T . As we know τ and τ' anticommute with S , hence they commute with $|S|$. We shall use Proposition 2.4 (see the next section). Setting $R_1 = R_2 = T$, $\rho_1 = \tau$, $\rho_2 = \tau'$, $U_1 = I$, $U_2 = U$ and replacing S by $|S|$ we have all the assumptions of that proposition satisfied. Therefore (1.9) implies the equality $\tau \text{Np}(T) = \tau' \text{Np}(T)$. It means that τ and τ' coincide on $H(T < 0)$. Then τ and τ' coincide on $\rho H(T < 0)$ for any operator ρ commuting with τ and τ' . If ρ commutes with S and anticommutes with R then it anticommutes with T and $\rho H(T < 0) = H(T > 0)$. In this case τ and τ' coincide on $H(T < 0) \oplus H(T > 0) = H$ (this is because $\ker T$ is trivial). Hence $\tau = \tau'$.

We shall prove a result of the same flavor as (1.7):

Proposition 1.4. *Let R and S be strictly positive selfadjoint operators acting on a Hilbert space H such that $R \overset{\hbar}{\circ} S$ and let τ, ρ, σ and ξ be unitary selfadjoint operators commuting with R and S . Assume that τ commutes with ξ and anticommutes with ρ and σ and*

$$(1.10) \quad \xi \chi(\tau = -1) = \alpha \rho \sigma \chi(\tau = -1),$$

where $\alpha = i e^{\frac{i\pi^2}{2\hbar}}$. We set: $T = e^{i\hbar/2} S^{-1} R$. Then T is a positive selfadjoint operator with trivial kernel, $\sigma S^{\frac{\pi}{\hbar}} + \rho R^{\frac{\pi}{\hbar}}$ is a closed symmetric operator anticommuting with τ and the selfadjoint extension

$$(1.11) \quad \begin{aligned} [\sigma S^{\frac{\pi}{\hbar}} + \rho R^{\frac{\pi}{\hbar}}]_{-\tau} &= G_{\hbar}(\tau T, \xi)^* \sigma S^{\frac{\pi}{\hbar}} G_{\hbar}(\tau T, \xi) \\ &= G_{\hbar}(\tau T^{-1}, \xi) \rho R^{\frac{\pi}{\hbar}} G_{\hbar}(\tau T^{-1}, \xi)^* \end{aligned}$$

Proof. At first we shall prove the first equality of (1.11). Inserting S^{-1} instead of R and R instead of S in [17, Example 3.1] we see that T is a positive selfadjoint operator with trivial kernel and

$$(1.12) \quad T^{ik} = e^{-\frac{i\hbar}{2} k^2} S^{-ik} R^{ik} = e^{\frac{i\hbar}{2} k^2} R^{ik} S^{-ik}$$

for any $k \in \mathbb{R}$.

Denote by X the right hand side of the first equality in (1.11). We know that $G_{\hbar}(\tau T, \xi)$ is unitary (in what follows we write $G_{\hbar}(\tau T, \xi)^{-1}$ instead of $G_{\hbar}(\tau T, \xi)^*$). Operator $S^{\frac{\pi}{\hbar}}$ commutes with σ and τ whereas σ and τ anticommute. Therefore $\sigma S^{\frac{\pi}{\hbar}}$ is a selfadjoint operator anticommuting with τ . So is X .

Let $Q = \sigma S^{\frac{\pi}{\hbar}} + \rho R^{\frac{\pi}{\hbar}}$. Clearly Q is a symmetric operator anticommuting with τ . By virtue of Proposition 1.1 it is sufficient to show that

$$(1.13) \quad Q|_{H(\tau=1)} = X|_{H(\tau=1)}.$$

Restricting $G_{\hbar}(\tau T, \xi)^* \sigma S^{\frac{\pi}{\hbar}} G_{\hbar}(\tau T, \xi)$ to $H(\tau = 1)$ we may replace the second τ by 1 and the first τ by -1 (this is because σ maps $H(\tau = 1)$ onto $H(\tau = -1)$):

$$X|_{H(\tau=1)} = G_{\hbar}(-T, \xi)^{-1} \sigma S^{\frac{\pi}{\hbar}} G_{\hbar}(T, \xi)|_{H(\tau=1)}$$

and using (1.2) we obtain

$$(1.14) \quad X|_{H(\tau=1)} = [1 + i\xi T^{\frac{\pi}{\hbar}}]^{-1} V_{\theta} \left(\log T - \pi i \right)^{-1} \sigma S^{\frac{\pi}{\hbar}} V_{\theta}(\log T)|_{H(\tau=1)}$$

Now we shall move $\sigma S^{\frac{\pi}{\hbar}}$ to the right end of (1.14). It is known (cf. [17, relation (1.30)]) that the function $V_{\theta}(x)$ has no poles and no zeroes in strip $\Sigma = \{x \in \mathbb{C} : 0 \leq \Im x \leq \pi\}$. Therefore functions $V_{\theta}(x)$ and $V_{\theta}(x)^{-1}$ are continuous on Σ and holomorphic inside Σ . Moreover (cf. [17, the asymptotic formula (1.37)]), $V_{\theta}(x) \rightarrow 1$ when $\Re x \rightarrow -\infty$ whereas $\Im x$ stays bounded and using formula (1.32) of [17] one can easily show that for any $\lambda > 0$, functions $e^{-\lambda x^2} V_{\theta}(x)$ and $e^{-\lambda x^2} V_{\theta}(x)^{-1}$ are bounded on Σ . Furthermore T is a strictly positive selfadjoint operator and $T \xrightarrow{\hbar} S$. Therefore $T \xrightarrow{\hbar} S^{\frac{\pi}{\hbar}}$ and using Statement (3) of Theorem 3.1 of [17] we obtain

$$S^{\frac{\pi}{\hbar}} V_{\theta}(\log T) = V_{\theta}(\log T + i\pi) S^{\frac{\pi}{\hbar}}.$$

Inserting this formula into (1.14) and using in the second step formula (1.28) of [17] we get:

$$\begin{aligned} X|_{H(\tau=1)} &= [1 + i\xi T^{\frac{\pi}{\hbar}}]^{-1} V_{\theta}(\log T - \pi i)^{-1} V_{\theta}(\log T + \pi i) \sigma S^{\frac{\pi}{\hbar}}|_{H(\tau=1)} \\ &= [1 + i\xi T^{\frac{\pi}{\hbar}}]^{-1} [1 + T^{\frac{2\pi}{\hbar}}] \sigma S^{\frac{\pi}{\hbar}}|_{H(\tau=1)} \\ &= [1 - i\xi T^{\frac{\pi}{\hbar}}] S^{\frac{\pi}{\hbar}} \sigma|_{H(\tau=1)}. \end{aligned}$$

On the other hand multiplying both sides of (1.10) by σ from the right we obtain $\xi \sigma \chi(\tau = 1) = \alpha \rho \chi(\tau = 1)$. Therefore $\rho|_{H(\tau=1)} = \bar{\alpha} \xi \sigma|_{H(\tau=1)}$ and

$$Q|_{H(\tau=1)} = (S^{\frac{\pi}{\hbar}} + \bar{\alpha} \xi R^{\frac{\pi}{\hbar}}) \sigma|_{H(\tau=1)}.$$

To end this part of the proof it is sufficient to show that

$$(1.15) \quad S^{\frac{\pi}{\hbar}} + \bar{\alpha} \xi R^{\frac{\pi}{\hbar}} = [1 - i\xi T^{\frac{\pi}{\hbar}}] S^{\frac{\pi}{\hbar}}.$$

We shall use (1.12). It shows that for any $x, y \in H$ and any $k \in \mathbb{R}$ we have

$$(y|S^{ik}x) - i e^{\frac{i\hbar}{2} k^2} (y|\xi R^{ik}x) = \left((I + i\xi T^{-ik}) y | S^{ik}x \right)$$

Let $x \in \mathcal{D}(S^{\frac{\pi}{\hbar}}) \cap \mathcal{D}(R^{\frac{\pi}{\hbar}})$. If $y \in \mathcal{D}(T^{\frac{\pi}{\hbar}})$ then both sides of the above formula have continuous holomorphic continuation to the strip $-\frac{\pi}{\hbar} \leq \Im k \leq 0$. Inserting $k = -i\frac{\pi}{\hbar}$ we obtain

$$(y|S^{\frac{\pi}{\hbar}}x) + \bar{\alpha} (y|\xi R^{\frac{\pi}{\hbar}}x) = ((I + i\xi T^{\frac{\pi}{\hbar}}) y | S^{\frac{\pi}{\hbar}}x).$$

This formula holds for any y in the domain of $I + i\xi T^{\frac{\pi}{\hbar}}$. Therefore $S^{\frac{\pi}{\hbar}}x \in \mathcal{D}(I - i\xi T^{\frac{\pi}{\hbar}})$ and $S^{\frac{\pi}{\hbar}}x + \bar{\alpha}\xi R^{\frac{\pi}{\hbar}}x = (I - i\xi T^{\frac{\pi}{\hbar}})S^{\frac{\pi}{\hbar}}x$. This way we showed that

$$(1.16) \quad S^{\frac{\pi}{\hbar}} + \bar{\alpha}\xi R^{\frac{\pi}{\hbar}} \subset (I - i\xi T^{\frac{\pi}{\hbar}})S^{\frac{\pi}{\hbar}}.$$

To prove the converse inclusion we use again (1.12). Let $x \in \mathcal{D}(S^{\frac{\pi}{\hbar}})$ and $S^{\frac{\pi}{\hbar}}x \in \mathcal{D}(T^{\frac{\pi}{\hbar}})$. Then for any $y \in H$ and $k \in \mathbb{R}$:

$$e^{-\frac{i\hbar}{2}k^2} \left(R^{-ik}y \middle| S^{\frac{\pi}{\hbar}-ik}x \right) = (y \middle| T^{ik}S^{\frac{\pi}{\hbar}}x).$$

If $y \in \mathcal{D}(R^{\frac{\pi}{\hbar}})$ then both sides of the above formula have continuous holomorphic continuation to the strip $-\frac{\pi}{\hbar} \leq \Im k \leq 0$. Inserting $k = -i\frac{\pi}{\hbar}$ we obtain

$$i\bar{\alpha} \left(R^{\frac{\pi}{\hbar}}y \middle| x \right) = (y \middle| T^{\frac{\pi}{\hbar}}S^{\frac{\pi}{\hbar}}x).$$

This formula holds for any $y \in \mathcal{D}(R^{\frac{\pi}{\hbar}})$. Therefore $x \in \mathcal{D}(R^{\frac{\pi}{\hbar}})$. This way we showed the inclusion $\mathcal{D}(T^{\frac{\pi}{\hbar}}S^{\frac{\pi}{\hbar}}) \subset \mathcal{D}(R^{\frac{\pi}{\hbar}})$. Consequently $\mathcal{D}((I - i\xi T^{\frac{\pi}{\hbar}})S^{\frac{\pi}{\hbar}}) \subset \mathcal{D}(S^{\frac{\pi}{\hbar}} + \bar{\alpha}\xi R^{\frac{\pi}{\hbar}})$. Combining this result with (1.16) we get (1.15) and (1.13). This way the first equality of (1.11) is shown.

The second equality may be shown in the same manner. However it is simpler to use the following trick based on (1.6). Let J be an antiunitary involutive operator acting on H and

$$\begin{aligned} R_n &= JSJ, & S_n &= JRJ, & \tau_n &= J\tau J, \\ \rho_n &= J\sigma J, & \sigma_n &= J\rho J, & \xi_n &= J\xi J. \end{aligned}$$

The subscript ‘ n ’ stands for ‘new’. One can easily show that the new operators satisfy all the assumptions of our theorem. In particular $\xi_n\chi(\tau_n = -1) = J\xi\chi(\tau = -1)J = J\alpha\rho\sigma\chi(\tau = -1)J = \bar{\alpha}\sigma_n\rho_n\chi(\tau_n = -1) = (\bar{\alpha}\sigma_n\rho_n\chi(\tau_n = -1))^* = \alpha\rho_n\sigma_n\chi(\tau_n = -1)$. In the present case

$$T_n = e^{i\hbar/2}S_n^{-1}R_n = Je^{-i\hbar/2}R^{-1}SJ = JT^{-1}J$$

and the first equality of (1.11) takes the form:

$$\begin{aligned} \left[\sigma_n S_n^{\frac{\pi}{\hbar}} + \rho_n R_n^{\frac{\pi}{\hbar}} \right]_{-\tau_n} &= G_{\hbar}(J\tau T^{-1}J, J\xi J)^* \sigma_n S_n^{\frac{\pi}{\hbar}} G_{\hbar}(J\tau T^{-1}J, J\xi J) \\ &= JG_{\hbar}(\tau T^{-1}, \xi)\rho R^{\frac{\pi}{\hbar}}G_{\hbar}(\tau T^{-1}, \xi)^* J. \end{aligned}$$

A moment of reflection shows that the left hand side of this formula equals $J[\sigma S^{\frac{\pi}{\hbar}} + \rho R^{\frac{\pi}{\hbar}}]_{-\tau}J$ and the second equality of (1.11) follows immediately.

To end the proof we have to show that the operator $Q = \sigma S^{\frac{\pi}{\hbar}} + \rho R^{\frac{\pi}{\hbar}}$ is closed. Operator $\xi T^{\frac{\pi}{\hbar}}$ is selfadjoint. Therefore operator $I - i\xi T^{\frac{\pi}{\hbar}}$ is invertible with the inverse $(I - i\xi T^{\frac{\pi}{\hbar}})^{-1} \in B(H)$. Using this fact one can easily show that the composition $(I - i\xi T^{\frac{\pi}{\hbar}})S^{\frac{\pi}{\hbar}}\sigma$ is a closed operator. Restricting this operator to $H(\tau = 1)$ we obtain $Q|_{H(\tau=1)}$. Hence $Q|_{H(\tau=1)}$ is closed. Remembering that Q anticommute with τ we conclude that Q is closed. \square

Remark 1.5. According to (1.10) operator $\alpha\rho\sigma\chi(\tau = -1)$ is selfadjoint. Using this fact and remembering that ρ and σ anticommute with τ one can show that

$$\rho\sigma = \alpha^2\tau\sigma\rho.$$

Conversely let τ, ρ, σ be unitary selfadjoint operators commuting with R, S and let τ anticommutes with ρ and σ . If the above relation is satisfied, then using (1.10) to define ξ on $H(\tau = -1)$ and extending it in an arbitrary way to a unitary selfadjoint operator defined on the whole space we obtain the quadruple $(\rho, \sigma, \tau, \xi)$ of operators satisfying the assumptions of Proposition 1.4.

We end this section with the reformulation of Theorem 6.1 of [17].

Theorem 1.6. *Let (R, S) be a pair of selfadjoint operators acting on a Hilbert space H such that $\ker R = \ker S = \{0\}$ and $R \xrightarrow{\hbar} S$ and let ρ, σ be unitary selfadjoint operators on H . Assume that ρ commutes with R , ρ anticommutes with S , σ commutes with S and σ anticommutes with R . We set:*

$$\begin{aligned} T &= e^{i\hbar/2}S^{-1}R, \\ \tau &= \alpha\rho\sigma\chi(S < 0) + \bar{\alpha}\sigma\rho\chi(S > 0), \end{aligned}$$

where $\alpha = ie^{\frac{i\pi^2}{2\hbar}}$. Then

1. T is selfadjoint, $\operatorname{sgn} T = (\operatorname{sgn} R)(\operatorname{sgn} S)$, $T \xrightarrow{\hbar} R$ and $T \xrightarrow{\hbar} S$
2. τ is unitary selfadjoint, τ commutes with T and τ anticommutes with R and S .

3. G_{\hbar} satisfies the following exponential function equality:

$$(1.17) \quad \begin{aligned} G_{\hbar}(R, \rho)G_{\hbar}(S, \sigma) &= G_{\hbar}(T, \tau)^*G_{\hbar}(S, \sigma)G_{\hbar}(T, \tau) \\ &= G_{\hbar}([R + S]_{\tau}, \tilde{\sigma}), \end{aligned}$$

where $[R + S]_{\tau}$ is the selfadjoint extension of $R + S$ corresponding to the reflection operator τ and $\tilde{\sigma} = G_{\hbar}(T, \tau)^*\sigma G_{\hbar}(T, \tau)$.

Proof. By direct computation one can easily show that $\tau^2 = I$, $\tau^* = \tau$ and

$$\tau\chi(T < 0) = \alpha\rho\chi(R < 0)\sigma\chi(S < 0) + \bar{\alpha}\sigma\chi(S < 0)\rho\chi(R < 0).$$

Now, our theorem follows immediately from [17, Theorem 6.1]. \square

Remark 1.7. In Theorem 1.6, operator τ may be replaced by $\tau' = \alpha\rho\sigma\chi(R > 0) + \bar{\alpha}\sigma\rho\chi(R < 0)$. Operator $\tilde{\sigma}$ is not affected by this change.

Indeed, using the formula $\text{sgn } T = \text{sgn } R \text{sgn } S$, one can verify that $\tau'\chi(T < 0) = \tau\chi(T < 0)$. It shows that $G_{\hbar}(T, \tau') = G_{\hbar}(T, \tau)$.

2. THE SPECIAL FUNCTIONS AND AFFILIATION RELATION.

In this section we shall use the concept of a C^* -algebra generated by a set of affiliated elements [13, Definition 4.1, page 501]. Let C , A be C^* -algebras and V be an element affiliated with $C \otimes A$. We say that A is generated by an element $V \eta(C \otimes A)$ if and only if for any $\pi \in \text{Rep}(A)$ and any $B \in C^*(H_{\pi})$ we have:

$$(2.1) \quad \left((\text{id} \otimes \pi)V \eta(C \otimes B) \right) \implies \left(\pi \in \text{Mor}(A, B) \right)$$

In general the above condition is not easy to verify. We shall use the following criterion (cf. [13, Example 10, page 507]):

Proposition 2.1. *Let C , A be C^* -algebras and V be a unitary element of $M(C \otimes A)$. Assume that there exists a faithful representation ϕ of C such that:*

1. For any ϕ -normal linear functional ω on C we have $(\omega \otimes \text{id})V \in A$
2. The smallest $*$ -subalgebra of A containing $\{(\omega \otimes \text{id})V : \omega \text{ is } \phi\text{-normal}\}$ is dense in A .

Then A is generated by $V \in M(C \otimes A)$.

We recall that a linear functional ω on C is said to be ϕ -normal if there exists a trace-class operator ρ acting on H_{ϕ} such that $\omega(c) = \text{Tr}(\rho\phi(c))$ for all $c \in C$.

Let Λ be the locally compact space obtained from $\mathbb{R} \times \{-1, 1\}$ by gluing points $(r, -1)$ and $(r, 1)$ for all $r \geq 0$. Then:

$$C_{\infty}(\Lambda) = \left\{ f \in C_{\infty}(\mathbb{R} \times \{-1, 1\}) : \begin{array}{l} f(r, -1) = f(r, 1) \\ \text{for all } r \geq 0 \end{array} \right\}.$$

If R , ρ are operators acting on a Hilbert space H , R is selfadjoint, ρ is unitary selfadjoint and ρ commutes with R then the mapping

$$(2.2) \quad C_{\infty}(\Lambda) \ni f \longmapsto \pi(f) = f(R, \rho) \in B(H)$$

is a representation of $C_{\infty}(\Lambda)$ acting on H . Operators R and $\rho \text{Np}(R)$ are determined by π . Indeed $R = \pi(f_1)$ and $\rho \text{Np}(R) = \pi(f_2)$, where f_1, f_2 are elements of $C_{\infty}(\Lambda)^{\eta} = C(\Lambda)$ introduced by the formulae

$$(2.3) \quad f_1(r, \varrho) = r, \quad f_2(r, \varrho) = \varrho \text{Np}(r)$$

for any $r \in \mathbb{R}$ and $\varrho = \pm 1$. Using [13, Example 2, page 497] we see that f_1, f_2 generate $C_{\infty}(\Lambda)$. Therefore for any $\pi \in \text{Rep}(C_{\infty}(\Lambda))$ and any $B \in C^*(H_{\pi})$ we have:

$$\left(\pi(f_1), \pi(f_2) \eta B \right) \implies \left(\pi \in \text{Mor}(C_{\infty}(\Lambda), B) \right) \implies \left(\pi(f) \eta B \text{ for any } f \in C(\Lambda) \right)$$

In particular for π introduced by (2.2) we obtain the following result:

$$(2.4) \quad \left(\begin{array}{l} R, \rho \text{Np}(R) \eta B \\ f \in C(\Lambda) \end{array} \right) \implies \left(f(R, \rho) \eta B \right).$$

Our special function G_{\hbar} is continuous and satisfies the relation $G_{\hbar}(r, -1) = G_{\hbar}(r, 1)$ for all $r \geq 0$. In other words $G_{\hbar} \in C(\Lambda)$. For any $r \in \mathbb{R}$, $\varrho = \pm 1$ and $t > 0$ we set:

$$(2.5) \quad F(t; r, \varrho) = \overline{G_{\hbar}(r, \varrho)} G_{\hbar}(tr, \varrho).$$

Let $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$. Then F is a continuous function on $\mathbb{R}_+ \times \Lambda$ with values of modulus 1 and we may treat F as unitary element of $M(C_{\infty}(\mathbb{R}_+) \otimes C_{\infty}(\Lambda))$. We shall prove the following

Proposition 2.2. *The C^* -algebra $C_{\infty}(\Lambda)$ is generated by $F \in M(C_{\infty}(\mathbb{R}_+) \otimes C_{\infty}(\Lambda))$.*

Proof. We shall use Proposition 2.1 with $C = C_{\infty}(\mathbb{R}_+)$, $A = C_{\infty}(\Lambda)$ and $V = F$. Let ϕ be the natural representation of $C_{\infty}(\mathbb{R}_+)$ acting on $L^2(\mathbb{R}_+)$. For any $g \in C_{\infty}(\mathbb{R}_+)$, $\phi(g)$ is the multiplication by g . Then ϕ is faithful and a linear functional ω on $C_{\infty}(\mathbb{R}_+)$ is ϕ -normal if and only if it is of the form

$$\omega(g) = \int_{\mathbb{R}_+} g(t) \varphi(t) dt,$$

where $\varphi \in L^1(\mathbb{R}_+)$. Applying $\omega \otimes \text{id}$ to $F \in M(C_{\infty}(\mathbb{R}_+) \otimes C_{\infty}(\Lambda))$ we obtain an element of $M(C_{\infty}(\Lambda))$ i.e. a bounded continuous function on Λ . Clearly for any $r \in \mathbb{R}$ and $\varrho = \pm 1$ we have

$$(2.6) \quad \begin{aligned} (\omega \otimes \text{id})F(r, \varrho) &= \int_{\mathbb{R}_+} F(t; r, \varrho) \varphi(t) dt \\ &= \overline{G_{\hbar}(r, \varrho)} \int_{\mathbb{R}_+} G_{\hbar}(tr, \varrho) \varphi(t) dt. \end{aligned}$$

Taking into account the asymptotic behavior (1.4) and using the Riemann–Lebesgue lemma one can verify that the integral on right hand side tends to 0 when $r \rightarrow \pm\infty$. In other words, $(\omega \otimes \text{id})F \in C_{\infty}(\Lambda)$.

Using Statement 7 of Theorem 1.1 of [17] one can easily show that

$$(2.7) \quad \lim_{t \rightarrow 0^+} \frac{1}{t} \left[G_{\hbar}(tr, \varrho) - 1 \right] = \frac{r}{2i \sin(\hbar/2)}$$

for all $r \in \mathbb{R}$ and $\varrho = \pm 1$.

Let $r, r' \in \mathbb{R}$ and $\varrho, \varrho' = \pm 1$. Assume for the moment that $(\omega \otimes \text{id})F(r, \varrho) = (\omega \otimes \text{id})F(r', \varrho')$ for all ϕ -normal functionals ω . Then $\overline{G_{\hbar}(r, \varrho)} G_{\hbar}(tr, \varrho) = \overline{G_{\hbar}(r', \varrho')} G_{\hbar}(tr', \varrho')$ for all $t > 0$. Going to the limit when $t \rightarrow +0$ we get $\overline{G_{\hbar}(r, \varrho)} = \overline{G_{\hbar}(r', \varrho')}$. Comparing this formula with the previous one we see that $G_{\hbar}(tr, \varrho) = G_{\hbar}(tr', \varrho')$ for all $t > 0$. Formula (2.7) shows now that $r = r'$ and by (1.3) $\varrho \chi(r < 0) = \varrho' \chi(r' < 0)$. This way we have shown that the functions (2.6) separate points of Λ . Now, using the Stone - Weierstrass theorem (applied to the one point compactification of Λ) we conclude that the smallest $*$ -algebra containing all functions (2.6) is dense in $C_{\infty}(\Lambda)$. \square

The following Proposition will be very useful in proving many technical details important in the future considerations.

Proposition 2.3. *Let R, ρ, U, S be operators acting on a Hilbert space H and $C \in C^*(H)$. Assume that:*

1. R is selfadjoint and ρ is unitary selfadjoint commuting with R ,
2. U is unitary,
3. S is positive selfadjoint, $\ker S = \{0\}$, S commutes with ρ and U and $R \xrightarrow{\hbar} S$,
4. Operators $R, \rho \text{Np}(R), U$ and $\log S$ are affiliated with C .

Then $G_{\hbar}(R, \rho) \in M(C)$ and

1. For any $\phi \in \text{Rep}(C)$ and any $B \in C^*(H_{\phi})$ we have:

$$\left(\begin{array}{l} \phi(\log S), \phi(G_{\hbar}(R, \rho)^* U) \\ \text{are affiliated with } B \end{array} \right) \implies \left(\begin{array}{l} \phi(R), \phi(\rho \text{Np}(R)), \phi(U) \\ \text{are affiliated with } B \end{array} \right)$$

2. For any $\phi_1, \phi_2 \in \text{Rep}(C)$ such that $H_{\phi_1} = H_{\phi_2}$ we have:

$$\left(\begin{array}{l} \phi_1(S) = \phi_2(S), \\ \phi_1(G_{\hbar}(R, \rho)^*U) = \phi_2(G_{\hbar}(R, \rho)^*U) \end{array} \right) \implies \left(\begin{array}{l} \phi_1(R) = \phi_2(R), \\ \phi_1(\rho \text{Np}(R)) = \phi_2(\rho \text{Np}(R)), \\ \phi_1(U) = \phi_2(U) \end{array} \right)$$

Proof. Relation $G_{\hbar}(R, \rho) \in M(C)$ follows immediately from (2.4).

Ad 1. Let $\lambda \in \mathbb{R}$. Using the commutation relations satisfied by operators R, ρ, U, S we have:

$$S^{-i\lambda} G_{\hbar}(R, \rho)^* U S^{i\lambda} = G_{\hbar}(tR, \rho)^* U,$$

where $t = e^{\hbar\lambda} > 0$. Applying a representation ϕ of C to both sides of the above relation we get

$$\phi(S)^{-i\lambda} \phi(G_{\hbar}(R, \rho)^*U) \phi(S)^{i\lambda} = \phi(G_{\hbar}(tR, \rho)^*U).$$

If $\phi(\log S), \phi(G_{\hbar}(R, \rho)^*U) \eta B$, then all factors on the left hand side of the above equation belong to $M(B)$ and depend continuously on λ (we use strict topology on $M(B)$). Therefore $\phi(G_{\hbar}(tR, \rho)^*U) \in M(B)$ for any $t \in \mathbb{R}_+$ and the mapping

$$\mathbb{R}_+ \ni t \longmapsto \phi(G_{\hbar}(tR, \rho)^*U) \in M(B)$$

is strictly continuous. Applying the hermitian conjugation and multiplying from the left by $\phi(G_{\hbar}(R, \rho)^*U) \in M(B)$ we see that $\phi(G_{\hbar}(R, \rho)^*G_{\hbar}(tR, \rho)) = \phi(F(t; R, \rho)) \in M(B)$ and the mapping

$$(2.8) \quad \mathbb{R}_+ \ni t \longmapsto \phi(F(t; R, \rho)) \in M(B)$$

is strictly continuous. In the above relations F is the function introduced by (2.5). According to the general theory [13], strictly continuous bounded mappings from \mathbb{R}_+ into $M(B)$ correspond to elements of $M(C_{\infty}(\mathbb{R}_+) \otimes B)$. A moment of reflection shows that the mapping (2.8) corresponds to the element $(\text{id} \otimes \phi \circ \pi)F$, where π is the representation of $C_{\infty}(\Lambda)$ introduced by (2.2).

This way we have shown that $(\text{id} \otimes \phi \circ \pi)F \in M(C_{\infty}(\mathbb{R}_+) \otimes B)$. Using now Proposition 2.2 we conclude that $\phi \circ \pi \in \text{Mor}(C_{\infty}(\Lambda), B)$. Therefore $\phi \circ \pi$ maps continuous functions on Λ into elements affiliated with B . Applying this rule to functions f_1, f_2 (cf. (2.3)) and G_{\hbar} we obtain: $\phi(R), \phi(\rho \text{Np}(R)) \eta B$ and $\phi(G_{\hbar}(R, \rho)) \in M(B)$. Comparing the last relation with the assumed one $\phi(G_{\hbar}(R, \rho)^*U) \in M(B)$ we see that $\phi(U) \in M(B)$. Statement 1 is shown.

Ad 2. Let $\phi = \phi_1 \oplus \phi_2$. Then $H_{\phi} = H_{\phi_1} \oplus H_{\phi_2}$ and $\phi(c) = \phi_1(c) \oplus \phi_2(c)$. In our case $H_{\phi_1} = H_{\phi_2}$. We set: $B = \{m \oplus m : m \in \mathcal{K}(H_{\phi_1})\}$. Then $B \in C^*(H_{\phi})$. One can easily verify that for any $c \eta C$ we have:

$$\left(\phi(c) \eta B \right) \iff \left(\phi_1(c) = \phi_2(c) \right).$$

Now Statement 2 follows immediately from Statement 1. \square

We shall use slightly different version of Statement 2 of the above proposition.

Proposition 2.4. *Let $R_1, \rho_1, U_1, R_2, \rho_2, U_2, S$ be operators acting on a Hilbert space H . Assume that for each $k = 1, 2$ the operators R_k, ρ_k, U_k, S satisfy the assumptions 1-3 of the previous Proposition. Then*

$$(2.9) \quad \left(G_{\hbar}(R_1, \rho_1)^*U_1 = G_{\hbar}(R_2, \rho_2)^*U_2 \right) \implies \left(\begin{array}{l} R_1 = R_2, \\ \rho_1 \text{Np}(R_1) = \rho_2 \text{Np}(R_2), \\ U_1 = U_2. \end{array} \right)$$

Proof. Let $C = \mathcal{K}(H) \oplus \mathcal{K}(H)$ and for any $m_1, m_2 \in \mathcal{K}(H)$ we set $\phi_k(m_1 \oplus m_2) = m_k$ ($k = 1, 2$). We use Proposition 2.3 with R, ρ, U and S replaced by $R_1 \oplus R_2, \rho_1 \oplus \rho_2, U_1 \oplus U_2$ and $S \oplus S$, Now (2.9) follows immediately from Statement 2 of Proposition 2.3. \square

Proposition 2.5. *Let X and Y be selfadjoint operators acting on Hilbert spaces K and H respectively. Assume that the spectral measure of X is absolutely continuous with respect to the Lebesgue measure. Then for any $A \in C^*(H)$ we have:*

$$\left(\begin{array}{l} e^{iX \otimes Y} \text{ is affiliated} \\ \text{with } \mathcal{K}(K) \otimes A \end{array} \right) \implies \left(Y \text{ is affiliated with } A \right)$$

Proof. For any normal linear functional ω on $B(K)$ and $t \in \mathbb{R}$ we set

$$f_\omega(t) = \omega(e^{itX}).$$

Then f_ω is a continuous function on \mathbb{R} . Remembering that the spectral measure of X is absolutely continuous with respect to the Lebesgue measure and using the Riemann-Lebesgue lemma one can easily show that $f_\omega(t) \rightarrow 0$ when $t \rightarrow \pm\infty$. Therefore $f_\omega \in C_\infty(\mathbb{R})$.

Let $t, t' \in \mathbb{R}$, $t \neq t'$. Assume for the moment that $f_\omega(t) = f_\omega(t')$ for all ω . Then $e^{itX} = e^{it'X}$ and $e^{i(t-t')X} = I$. It shows that the spectral measure of X is supported by the set $\frac{2\pi}{t-t'}\mathbb{Z}$, which is in contradiction with the assumption saying that the spectral measure of X is absolutely continuous with respect to the Lebesgue measure. This way we showed that functions f_ω separate points of \mathbb{R} . By the Stone – Weierstrass theorem, the smallest *-subalgebra of $C_\infty(\mathbb{R})$ containing all f_ω is dense in $C_\infty(\mathbb{R})$.

By the general theory strongly continuous mappings from \mathbb{R} into the set of unitary operators acting on K correspond to unitary multipliers of $\mathcal{K}(K) \otimes C_\infty(\mathbb{R})$. Let $\mathfrak{X} \in M(\mathcal{K}(K) \otimes C_\infty(\mathbb{R}))$ be the unitary corresponding to the mapping

$$\mathbb{R} \ni t \longmapsto e^{itX} \in B(K).$$

Then for any normal linear functional ω on $B(K)$ we have

$$(\omega \otimes \text{id})\mathfrak{X} = f_\omega.$$

Using Proposition 2.1 we see that $C_\infty(\mathbb{R})$ is generated by $\mathfrak{X} \in M(\mathcal{K}(K) \otimes C_\infty(\mathbb{R}))$. For any $f \in C_\infty(\mathbb{R})$ we set:

$$\pi(f) = f(Y).$$

Then π is a representation of $C_\infty(\mathbb{R})$ acting on the Hilbert space $H_\pi = H$. A moment of reflection shows that $(\text{id} \otimes \pi)\mathfrak{X} = e^{iX \otimes Y}$. If $e^{iX \otimes Y}$ is affiliated with $\mathcal{K}(K) \otimes A$ then $\pi \in \text{Mor}(C_\infty(\mathbb{R}), A)$ and π maps continuous functions on \mathbb{R} into elements affiliated with A . Applying this rule to the coordinate function $f(t) = t$ we obtain $Y = \pi(f) \eta A$. \square

3. CONSTRUCTIONS RELATED TO OLD QUANTUM ‘ $ax + b$ ’ GROUPS.

In this section we recall the main results of [19]. The quantum ‘ $ax + b$ ’ group will be presented as a quantum group of unitary operators. We shall construct a pair (A, V) , where A is a C^* -algebra and V is a unitary element of $M(\mathcal{K}(K) \otimes A)$, where K is a Hilbert space endowed with a certain structure and $\mathcal{K}(K)$ denotes the algebra of all compact operators acting on K . (A, V) may be treated as a quantum family of unitary operators acting on K ‘labeled by elements’ of quantum space related to the C^* -algebra A . Our construction will depend on a real parameter \hbar . We shall assume that $0 < \hbar < \pi/2$. Negative value of \hbar leads to the C^* -algebra anti-isomorphic to that with positive \hbar . On the other hand the restriction $\hbar < \pi/2$ is related to the technical assumption used in the theory of the quantum exponential function [17].

The main result of this section is contained in Theorem 3.2. It states that (A, V) is a quantum group if and only if $\hbar = \frac{\pi}{2k+3}$ with $k = 0, 1, 2, \dots$.

To define A we consider three operators a , b and β acting on the Hilbert space $L^2(\mathbb{R})$. Operator a is strictly positive selfadjoint and such that for any $\tau \in \mathbb{R}$ and any $x \in L^2(\mathbb{R})$ we have

$$(a^{i\tau}x)(t) = e^{\hbar\tau/2}x(e^{\hbar\tau}t)$$

In other words a is the analytic generator of one-parameter group of unitaries corresponding to the homotheties of \mathbb{R} . Operator b is the multiplication operator:

$$(bx)(t) = tx(t).$$

By definition domain $\mathcal{D}(b)$ consists of all $x \in L^2(\mathbb{R})$ such that the right hand side of the above equation is square integrable. Finally, β is the reflection: for any $x \in L^2(\mathbb{R})$ we have:

$$(\beta x)(t) = x(-t)$$

Clearly β is unitary selfadjoint. One can easily verify that $a\beta = \beta a$ and $b\beta = -\beta b$. By the last relation $ib\beta$ is selfadjoint. Moreover

$$(3.1) \quad a^{i\tau}ba^{-i\tau} = e^{\hbar\tau}b$$

for any $\tau \in \mathbb{R}$. This relation means that $a \xrightarrow{\hbar} b$.

Theorem 3.1. *Let*

$$(3.2) \quad A = \left\{ \left(f_1(b) + \beta f_2(b) \right) g(\log a) : \begin{array}{l} f_1, f_2, g \in C_\infty(\mathbb{R}) \\ f_2(0) = 0 \end{array} \right\}^{\text{norm closed linear envelope}}.$$

Then: 1. A is a nondegenerate C^* -algebra of operators acting on $L^2(\mathbb{R})$,

2. $\log a$, b and $i\beta$ are affiliated with A : $\log a$, b , $i\beta \eta A$,

3. $\log a$, b and $i\beta$ generate A .

Proof.

Ad 1. Using the relation $b\beta = -\beta b$ one can easily show that

$$(3.3) \quad B = \left\{ f_1(b) + \beta f_2(b) : \begin{array}{l} f_1, f_2 \in C_\infty(\mathbb{R}) \\ f_2(0) = 0 \end{array} \right\}^{\text{norm closed linear envelope}}$$

is a non-degenerate C^* -algebra of operators acting on $L^2(\mathbb{R})$. Let $C_0(\mathbb{R}, B)$ denote the set of all continuous mappings from \mathbb{R} into B with compact support. Then

$$(3.4) \quad A = \left\{ \int_{\mathbb{R}} f(t) a^{it} dt : f \in C_0(\mathbb{R}, B) \right\}^{\text{norm closure}}.$$

To prove this formula it is sufficient to notice that for $f(t) = (f_1(b) + \beta f_2(b)) \varphi(t)$, where $t \in \mathbb{R}$ and $\varphi \in C_0(\mathbb{R})$ we have

$$\int_{\mathbb{R}} f(t) a^{it} dt = (f_1(b) + \beta f_2(b)) g(\log a),$$

where $g(\lambda) = \int_{\mathbb{R}} \varphi(t) e^{i\lambda t} dt$ ($\lambda \in \mathbb{R}$) and by the Riemann-Lebesgue Lemma, $g \in C_\infty(\mathbb{R})$. On the other hand (3.1) shows that the unitaries a^{it} ($t \in \mathbb{R}$) implement a one parameter group of automorphisms of B . Using now the standard technique of the theory of crossed products (cf. [6, Section 7.6]) one can easily show that (3.4) is a non-degenerate C^* -algebra of operators acting on $L^2(\mathbb{R})$. Statement 1 is proven.

Ad 2. We recall (cf. [5, 13]) that a closed operator T is affiliated with a C^* -algebra A if the z -transform $z_T = T(I + T^*T)^{-\frac{1}{2}} \in M(A)$ and if $(I + T^*T)^{-\frac{1}{2}} A$ is dense in A . Inspecting definition (3.2) one can easily show that $z_{\log a} = (\log a) [I + (\log a)^2]^{-\frac{1}{2}}$ is a right multiplier of A and that $A [I + (\log a)^2]^{-\frac{1}{2}}$ is dense in A . Passing to adjoint operators we see that $z_{\log a}^* = z_{\log a}$ is a left multiplier (hence $z_{\log a} \in M(A)$) and that $[I + (\log a)^2]^{-\frac{1}{2}} A$ is dense in A . It shows that $\log a$ is affiliated with A .

For $T = b$ and $T = i\beta b$ we have $z_T = b(I + b^2)^{-\frac{1}{2}}$ and $z_T = i\beta b(I + b^2)^{-\frac{1}{2}}$ respectively. In both cases $(I + T^*T)^{-\frac{1}{2}} = (I + b^2)^{-\frac{1}{2}}$. Taking into account definition (3.2) one can easily show that $(I + T^*T)^{-\frac{1}{2}} A$ is dense in A and that z_T is a left multiplier of A . However in both cases z_T is selfadjoint. Therefore z_T is also a right multiplier and $z_T \in M(A)$. It shows that b and $i\beta b$ are affiliated with A .

Ad 3. We shall use Theorem 3.3 of [13]. By definition (3.2), $(I + b^2)^{-1} (I + (\log a)^2)^{-1} \in A$. To end the proof it is sufficient to show that a , b , $i\beta b$ separate representations of A . If $c \in A$ is of the form

$$(3.5) \quad c = (f_1(b) + \beta f_2(b)) g(\log a),$$

where $f_1, f_2, g \in C_\infty(\mathbb{R})$, $f_2(0) = 0$ and f_2 is differentiable at point $0 \in \mathbb{R}$, then $f_2(t) = ith(t)$, where $t \in \mathbb{R}$ and $h \in C_\infty(\mathbb{R})$ and

$$(3.6) \quad \pi(c) = (f_1(\pi(b)) + \pi(i\beta b)h(\pi(b))) g(\pi(\log a))$$

for any representation π of A . One can easily see that elements of the form (3.5) form a dense subset of A . Formula (3.6) shows now that π is determined uniquely by $\pi(\log a)$, $\pi(b)$ and $\pi(i\beta b)$. \square

Now we pass to the description of the Hilbert space K (cf. the first paragraph of this section). The structure of K is determined by a triple of selfadjoint operators $(\widehat{a}, \widehat{b}, \widehat{\beta})$ acting on K and having the following properties:

1. $\widehat{a} > 0$, $\ker \widehat{a} = \ker \widehat{b} = \{0\}$ and $\widehat{a} \xrightarrow{\hbar} \widehat{b}$,

2. $\widehat{\beta}$ is a unitary involution, $\widehat{\beta}$ commutes with \widehat{a} and anticommutes with \widehat{b} .

One of the possible choices is: $K = L^2(\mathbb{R})$ and $(\widehat{a}, \widehat{b}, \widehat{\beta}) = (a, b, \beta)$. However there is another possibility that is even more interesting:

$$(3.7) \quad (\widehat{a}, \widehat{b}, \widehat{\beta}) = (|b|^{-1}, e^{i\hbar/2} b^{-1} a, \alpha \beta),$$

where $\alpha = \pm 1$. The reader easily verifies that these operators possess the required properties.

Zakrzewski relation $\widehat{a} \xrightarrow{\hbar} \widehat{b}$ implies that the spectral measures of \widehat{a} and \widehat{b} are absolutely continuous with respect to the Lebesgue measure. Moreover $\text{Sp}(\widehat{a}) = \mathbb{R}_+$ and $\text{Sp}(\widehat{b}) = \mathbb{R}$. The latter fact follows from the relation $\widehat{\beta} \widehat{b} = -\widehat{b} \widehat{\beta}$.

Let

$$(3.8) \quad V = G_{\hbar}(\widehat{b} \otimes b, \widehat{\beta} \otimes \beta)^* e^{\frac{i}{\hbar} \log \widehat{a} \otimes \log a}.$$

This is the basic object considered in this section. We shall prove

Theorem 3.2.

1. V is a unitary operator and $V \in M(\mathcal{K}(K) \otimes A)$,
2. A is generated by $V \in M(\mathcal{K}(K) \otimes A)$.

Proof. Let $R = \widehat{b} \otimes b$, $\rho = \widehat{\beta} \otimes \beta$, $U = e^{\frac{i}{\hbar} \log \widehat{a} \otimes \log a}$, $S = \widehat{a}^{-1} \otimes I$ and $C = \mathcal{K}(K) \otimes A$. Then all the assumptions of Proposition 2.3 are satisfied. Clearly $V = G_{\hbar}(R, \rho)^* U \in M(C)$ and Statement 1 is proved.

Let $\pi \in \text{Rep}(A)$ and $B \in C^*(H_{\pi})$. Then $\text{id} \otimes \pi$ is a representation of C acting on $K \otimes H_{\pi}$. The reader should notice that $(\text{id} \otimes \pi)S = \widehat{a}^{-1} \otimes I$ is affiliated with $\mathcal{K}(K) \otimes B$. Assume that $(\text{id} \otimes \pi)V \in M(\mathcal{K}(K) \otimes B)$. By Statement 1 of Proposition 2.3, operators: $(\text{id} \otimes \pi)R = \widehat{b} \otimes \pi(b)$, $(\text{id} \otimes \pi)(\rho \text{Np}(R))$ and $(\text{id} \otimes \pi)U = e^{\frac{i}{\hbar} \log \widehat{a} \otimes \pi(\log a)}$ are affiliated with $\mathcal{K}(K) \otimes B$. Using now Proposition A.1 of [19] we see that $\pi(b)$ is affiliated with B . One can easily verify that $\widehat{\beta} \otimes I$ commutes with ρ and anticommutes with R . Therefore

$$\rho \text{Np}(R) - (\widehat{\beta} \otimes I) \rho \text{Np}(R) (\widehat{\beta} \otimes I) = \rho (\text{Np}(R) - \text{Np}(-R)) = \rho R$$

and applying $\text{id} \otimes \pi$ to both sides we get

$$\begin{aligned} (\text{id} \otimes \pi)(\rho \text{Np}(R)) - (\widehat{\beta} \otimes I)(\text{id} \otimes \pi)(\rho \text{Np}(R))(\widehat{\beta} \otimes I) &= (\text{id} \otimes \pi)(\rho R) \\ &= -i \widehat{\beta} \otimes \pi(i \beta b) \end{aligned}$$

The operators $\widehat{\beta} \otimes I$ and $(\text{id} \otimes \pi)(\rho \text{Np}(R))$ appearing on the left hand side are affiliated with $\mathcal{K}(K) \otimes B$. Therefore $i \widehat{\beta} \otimes \pi(i \beta b) \in \mathcal{K}(K) \otimes B$ and using again Proposition A.1 of [19] we see that $\pi(i \beta b)$ is affiliated with B . Moreover, remembering that $e^{\frac{i}{\hbar} \log \widehat{a} \otimes \pi(\log a)} \in \mathcal{K}(K) \otimes B$ and using Proposition 2.5 we see that $\pi(\log a)$ is affiliated with B . According to Statement 3 of Theorem 3.1, b , $i \beta b$ and $\log a$ generate A . Therefore $\pi \in \text{Mor}(A, B)$. We showed that $(\text{id} \otimes \pi)V \in M(\mathcal{K}(K) \otimes B)$ implies $\pi \in \text{Mor}(A, B)$. It means that A is generated by $V \in M(\mathcal{K}(K) \otimes A)$. \square

Now we are able to formulate the main result of this section:

Theorem 3.3.

$$\left(\begin{array}{l} \text{There exists } \Delta \in \text{Mor}(A, A \otimes A) \\ \text{such that} \\ (\text{id} \otimes \Delta)V = V_{12} V_{13} \end{array} \right) \iff \left(\hbar = \frac{\pi}{2k+3}, \quad k = 0, 1, 2, \dots \right).$$

Proof. Let

$$(3.9) \quad \begin{aligned} \alpha &= i e^{\frac{i\pi^2}{2\hbar}}, \\ T &= I \otimes e^{i\hbar/2} b^{-1} a \otimes b, \\ \tau &= (I \otimes \beta \otimes \beta) \left[\alpha \chi(\widehat{b} \otimes b \otimes I < 0) + \bar{\alpha} \chi(\widehat{b} \otimes b \otimes I > 0) \right]. \end{aligned}$$

and

$$(3.10) \quad W' = G_{\hbar}(T, \tau)^* e^{-\frac{i}{\hbar} [I \otimes \log |b| \otimes \log a]}.$$

Clearly W' is a unitary operator acting on $K \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$. We shall prove that

$$(3.11) \quad V_{12}V_{13} = W'V_{12}W'^*.$$

To make our formulae shorter we set

$$U = e^{\frac{i}{\hbar} \log \widehat{a} \otimes \log a}, \quad Z = e^{-\frac{i}{\hbar} \log |b| \otimes \log a}.$$

Using the relations $\widehat{a} \xrightarrow{\hbar} \widehat{b}$, $\widehat{a}\widehat{\beta} = \widehat{\beta}\widehat{a}$ and $a \xrightarrow{\hbar} b$ one can easily verify that

$$(3.12) \quad U(\widehat{b} \otimes I)U^* = \widehat{b} \otimes a, \quad U(\widehat{\beta} \otimes I)U^* = \widehat{\beta} \otimes I,$$

$$(3.13) \quad Z(a \otimes I)Z^* = a \otimes a.$$

With the above notation $V = G_{\hbar}(\widehat{b} \otimes b, \widehat{\beta} \otimes \beta)^*U$ and

$$V_{12}V_{13} = G_{\hbar}(\widehat{b} \otimes b \otimes I, \widehat{\beta} \otimes \beta \otimes I)^* U_{12} G_{\hbar}(\widehat{b} \otimes I \otimes b, \widehat{\beta} \otimes I \otimes \beta)^* U_{13}.$$

Using (3.12) we get

$$U_{12} G_{\hbar}(\widehat{b} \otimes I \otimes b, \widehat{\beta} \otimes I \otimes \beta)^* = G_{\hbar}(\widehat{b} \otimes a \otimes b, \widehat{\beta} \otimes I \otimes \beta)^* U_{12}$$

and

$$(3.14) \quad V_{12}V_{13} = \left[G_{\hbar}(\widehat{b} \otimes a \otimes b, \widehat{\beta} \otimes I \otimes \beta) G_{\hbar}(\widehat{b} \otimes b \otimes I, \widehat{\beta} \otimes \beta \otimes I) \right]^* U_{12}U_{13}.$$

Let us consider the first factor in (3.14). We apply Theorem 1.6 with

$$(3.15) \quad \begin{aligned} R &= \widehat{b} \otimes a \otimes b, & \rho &= \widehat{\beta} \otimes I \otimes \beta, \\ S &= \widehat{b} \otimes b \otimes I, & \sigma &= \widehat{\beta} \otimes \beta \otimes I, \end{aligned}$$

Then T and τ are given by (3.9) and

$$G_{\hbar}(\widehat{b} \otimes a \otimes b, \widehat{\beta} \otimes I \otimes \beta) G_{\hbar}(\widehat{b} \otimes b \otimes I, \widehat{\beta} \otimes \beta \otimes I) = G_{\hbar}(T, \tau)^* G_{\hbar}(\widehat{b} \otimes b \otimes I, \widehat{\beta} \otimes \beta \otimes I) G_{\hbar}(T, \tau).$$

Now (3.14) takes the form

$$(3.16) \quad V_{12}V_{13} = G_{\hbar}(T, \tau)^* G_{\hbar}(\widehat{b} \otimes b \otimes I, \widehat{\beta} \otimes \beta \otimes I)^* G_{\hbar}(T, \tau) U_{12}U_{13}.$$

We shall move $G_{\hbar}(T, \tau)$ to the end of the right hand side of this formula. Performing simple computations and using (3.13) we obtain:

$$\begin{aligned} U_{12}U_{13} &= e^{\frac{i}{\hbar} \log \widehat{a} \otimes \log(a \otimes a)} \\ &= Z_{23}U_{12}Z_{23}^*. \end{aligned}$$

It turns out that

$$(3.17) \quad \log \widehat{a} \otimes \log(a \otimes a) \text{ commutes with } T,$$

$$(3.18) \quad \log \widehat{a} \otimes \log(a \otimes a) \text{ commutes with } \tau.$$

Indeed Zakrzewski relation $a \xrightarrow{\hbar} b$ implies $b^{-1} \xrightarrow{\hbar} a$. Using the both relations we see that $a \otimes a$ commutes with $e^{i\hbar/2} b^{-1} a \otimes b$. Therefore $\log(a \otimes a)$ commutes with $e^{i\hbar/2} b^{-1} a \otimes b$ and $\log \widehat{a} \otimes \log(a \otimes a)$ commutes with $T = I \otimes e^{i\hbar/2} b^{-1} a \otimes b$. Relation (3.17) is shown.

To prove (3.18) we use Zakrzewski relations $a \xrightarrow{\hbar} b$ and $\widehat{a} \xrightarrow{\hbar} \widehat{b}$. They show that a commutes with $\text{sgn } b$ and \widehat{a} commutes with $\text{sgn } \widehat{b}$. Therefore $\log \widehat{a} \otimes \log(a \otimes a)$ commutes with $\text{sgn}(\widehat{b} \otimes b \otimes I) = \text{sgn } \widehat{b} \otimes \text{sgn } b \otimes I$ and (3.18) follows.

Taking into account (3.17) and (3.18) we see that $G_{\hbar}(T, \tau)$ commutes with $U_{12}U_{13}$. Now relation (3.16) takes the form:

$$(3.19) \quad V_{12}V_{13} = G_{\hbar}(T, \tau)^* G_{\hbar}(\widehat{b} \otimes b \otimes I, \widehat{\beta} \otimes \beta \otimes I)^* Z_{23}U_{12}Z_{23}^* G_{\hbar}(T, \tau).$$

Finally $b \otimes I$ and $\beta \otimes I$ commute with $\log |b| \otimes \log a$. Therefore $G_{\hbar}(\widehat{b} \otimes b \otimes I, \widehat{\beta} \otimes \beta \otimes I)$ commutes with Z_{23} . Clearly $G_{\hbar}(\widehat{b} \otimes b \otimes I, \widehat{\beta} \otimes \beta \otimes I)^* U_{12} = V_{12}$ and $W' = G_{\hbar}(T, \tau)^* Z_{23}$. Now (3.11) follows immediately from (3.19).

By the Zakrzewski relation $\widehat{a}^{i\lambda} \widehat{b} \widehat{a}^{-i\lambda} = e^{i\hbar\lambda} \widehat{b}$ for all $\lambda \in \mathbb{R}$. Multiplication by a strictly positive number does not change the sign of an operator. Using this fact one can easily show that τ commutes with $\widehat{a}^{i\lambda} \otimes I \otimes I$. Consequently τ commutes with $\widehat{a} \otimes I \otimes I$. Since $T = I \otimes e^{i\hbar/2} b^{-1} a \otimes b$

and $I \otimes \log |b| \otimes \log a$ obviously commute with $\widehat{a} \otimes I \otimes I$, we conclude that W' commutes with $\widehat{a} \otimes I \otimes I$.

Now we are ready to prove the main statement.

\implies . Let $\Delta \in \text{Mor}(A, A \otimes A)$ and $(\text{id} \otimes \Delta)V = V_{12}V_{13}$. We go back to the notation used in the proof of Theorem 3.2. In particular $C = \mathcal{K}(K) \otimes A$. For any $c \in C$ we set:

$$\begin{aligned}\phi_1(c) &= (\text{id} \otimes \Delta)(c), \\ \phi_2(c) &= W'(c \otimes I)W'^*.\end{aligned}$$

Then ϕ_1 and ϕ_2 are representations of C acting on the same Hilbert space $K \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$. One can easily verify that $\phi_1(\widehat{a} \otimes I) = \widehat{a} \otimes I \otimes I = \phi_2(\widehat{a} \otimes I)$. Formula (3.11) shows that $\phi_1(V) = \phi_2(V)$. In our notation (cf. the beginning of the proof of Theorem 3.2), $\widehat{a} \otimes I = S$ and $V = G_{\hbar}(R, \rho)^*U$, where in particular $R = \widehat{b} \otimes b$. Statement 2 of Theorem 2.3 shows now that $\phi_1(R) = \phi_2(R)$. It means that

$$\widehat{b} \otimes \Delta(b) = W'(\widehat{b} \otimes b \otimes I)W'^*.$$

Taking into account (3.10) and using Proposition 1.2 we get:

$$(3.20) \quad \begin{aligned}\widehat{b} \otimes \Delta(b) &= G_{\hbar}(T, \tau)(\widehat{b} \otimes b \otimes I)G_{\hbar}(T, \tau)^* \\ &= \left[\widehat{b} \otimes a \otimes b + \widehat{b} \otimes b \otimes I \right]_{\tau}.\end{aligned}$$

We recall that

$$\tau = (I \otimes \beta \otimes \beta) \left[\alpha \chi(\widehat{b} \otimes b \otimes I < 0) + \bar{\alpha} \chi(\widehat{b} \otimes b \otimes I > 0) \right].$$

Inspecting last two formulae we observe that \widehat{b} is the only operator appearing in the first leg position. We know that \widehat{b} is selfadjoint. Therefore replacing in both sides of (3.20) operator \widehat{b} by a real number λ we obtain a formula that should hold for almost all $\lambda \in \text{Sp } \widehat{b}$. For positive λ we get

$$(3.21) \quad \Delta(b) = \left[a \otimes b + b \otimes I \right]_{\tau_+}$$

where

$$(3.22) \quad \tau_+ = (\beta \otimes \beta) \left[\alpha \chi(b \otimes I < 0) + \bar{\alpha} \chi(b \otimes I > 0) \right].$$

On the other hand for negative λ we have

$$(3.23) \quad \Delta(b) = \left[a \otimes b + b \otimes I \right]_{\tau_-}$$

where

$$(3.24) \quad \tau_- = (\beta \otimes \beta) \left[\alpha \chi(b \otimes I > 0) + \bar{\alpha} \chi(b \otimes I < 0) \right].$$

Clearly the two expressions for $\Delta(b)$ must coincide. Let us notice that the operator $I \otimes \beta$ commutes with τ_+ , τ_- and $b \otimes I$ and anticommutes with $a \otimes b$. Therefore $\tau_+ = \tau_-$ by Remark 1.3. Comparing (3.22) and (3.24) we get $\alpha = \bar{\alpha}$. Remembering that $\alpha = i e^{\frac{i\pi^2}{2\hbar}}$ and $0 < \hbar < \frac{\pi}{2}$ we conclude that $\hbar = \frac{\pi}{2k+3}$ ($k = 0, 1, 2, \dots$).

\Leftarrow . Assume that $\hbar = \frac{\pi}{2k+3}$ for some $k = 0, 1, 2, \dots$. Then formula (3.10) essentially simplifies. In this case $\alpha = (-1)^k$, $\tau = (-1)^k(I \otimes \beta \otimes \beta)$ and $W' = W_{23} = I \otimes W$, where

$$(3.25) \quad W = G_{\hbar} \left(e^{i\hbar/2} b^{-1} a \otimes b, (-1)^k \beta \otimes \beta \right)^* e^{-\frac{i}{\hbar} \log |b| \otimes \log a}.$$

Formula (3.11) takes the form

$$(3.26) \quad V_{12}V_{13} = W_{23}V_{12}W_{23}^*.$$

For any $c \in A$ we set

$$(3.27) \quad \Delta(c) = W(c \otimes I)W^*.$$

Then Δ is a representation of A acting on $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$. We know that $V \in \text{M}(\mathcal{K}(K) \otimes A)$. Formula (3.26) shows that

$$(\text{id} \otimes \Delta)V = V_{12}V_{13}.$$

Clearly $V_{12}, V_{13} \in \text{M}(\mathcal{K}(K) \otimes A \otimes A)$. Therefore $(\text{id} \otimes \Delta)V = V_{12}V_{13} \in \text{M}(\mathcal{K}(K) \otimes A \otimes A)$. Remembering that A is generated by V we conclude that $\Delta \in \text{Mor}(A, A \otimes A)$. \square

Let $\hbar = \frac{\pi}{2k+3}$ ($k = 0, 1, 2, \dots$). Then formula (3.27) makes it possible to calculate $\Delta(c)$ for any $c \in A$. The same holds for any c affiliated with A . We shall show that

$$(3.28) \quad \begin{aligned} \Delta(a) &= a \otimes a, \\ \Delta(b) &= [a \otimes b + b \otimes I]_{(-1)^k \beta \otimes \beta}, \\ \Delta(ib^{2k+3}\beta) &= [a^{2k+3} \otimes ib^{2k+3}\beta + ib^{2k+3}\beta \otimes I]_{-\text{sgn}(b \otimes b)}. \end{aligned}$$

Formula for $\Delta(a)$ follows immediately from (3.13); the reader should notice that operators $e^{i\hbar/2}b^{-1}a \otimes b$ and $\beta \otimes \beta$ commute with $a \otimes a$. The formula for $\Delta(b)$ was in fact shown in the proof of Theorem 3.3; in the present case $\tau_+ = \tau_- = (-1)^k \beta \otimes \beta$ and the second formula of (3.28) coincides with (3.21) (and with (3.23) as well).

It remains to prove the third formula. We know that $|b|$ commutes with $ib^{2k+3}\beta$. Taking into account (3.25) we obtain

$$(3.29) \quad \begin{aligned} \Delta(ib^{2k+3}\beta) &= W(ib^{2k+3}\beta \otimes I)W^* \\ &= G_{\hbar}(e^{i\hbar/2}b^{-1}a \otimes b, (-1)^k \beta \otimes \beta)^* (ib^{2k+3}\beta \otimes I) G_{\hbar}(e^{i\hbar/2}b^{-1}a \otimes b, (-1)^k \beta \otimes \beta). \end{aligned}$$

To compute the right hand side we use Proposition 1.4 with

$$\begin{aligned} R &= a \otimes |b|, & S &= |b| \otimes I, \\ \tau &= \text{sgn}(b \otimes b), & \xi &= (-1)^k \beta \otimes \beta, \\ \rho &= I \otimes i(\text{sgn } b)\beta, & \sigma &= i(\text{sgn } b)\beta \otimes I. \end{aligned}$$

Remembering that $\beta^2 = I$ and β anticommutes with b and hence commutes with $|b|$ one can easily check that these operators fulfill all assumption of Proposition 1.4. In this case we have $T = (e^{i\hbar/2}|b|^{-1}a) \otimes |b|$ and $\tau T = e^{i\hbar/2}b^{-1}a \otimes b$. According to our assumption $\frac{\pi}{\hbar} = 2k + 3$ is an odd positive integer. Therefore

$$\begin{aligned} \sigma S^{\frac{\pi}{\hbar}} &= [i(\text{sgn } b)\beta \otimes I] [|b|^{2k+3} \otimes I] = ib^{2k+3}\beta \otimes I, \\ \rho R^{\frac{\pi}{\hbar}} &= [I \otimes i(\text{sgn } b)\beta] [a^{2k+3} \otimes |b|^{2k+3}] = a^{2k+3} \otimes ib^{2k+3}\beta \end{aligned}$$

and formula (1.11) takes the form

$$(3.30) \quad \begin{aligned} & [ib^{2k+3}\beta \otimes I + a^{2k+3} \otimes ib^{2k+3}\beta]_{-\text{sgn}(b \otimes b)} \\ &= G_{\hbar}(e^{i\hbar/2}b^{-1}a \otimes b, (-1)^k \beta \otimes \beta)^* (ib^{2k+3}\beta \otimes I) G_{\hbar}(e^{i\hbar/2}b^{-1}a \otimes b, (-1)^k \beta \otimes \beta). \end{aligned}$$

Comparing (3.29) with (3.30) we get the last formula of (3.28). This formula appeared without proof in [19].

Remark 3.4. Let $s \in S^1$ be a number of modulus 1. Replacing in the above computations $\sigma = i(\text{sgn } b)\beta \otimes I$ and $\rho = i(\text{sgn } b)\beta \otimes I$ by $\sigma = s^{\text{sgn } b}\beta \otimes I$ and $\rho = s^{\text{sgn } b}\beta \otimes I$ respectively, one can prove that

$$(3.31) \quad \Delta(s^{\text{sgn } b} |b|^{2k+3} \beta) = [a^{2k+3} \otimes s^{\text{sgn } b} |b|^{2k+3} \beta + s^{\text{sgn } b} |b|^{2k+3} \beta \otimes I]_{-\text{sgn}(b \otimes b)}.$$

If $s = i$ then $s^{\text{sgn } b} = i \text{sgn } b$ and (3.31) reduces to the previous formula. For $s = 1$ we get

$$(3.32) \quad \Delta(|b|^{2k+3} \beta) = [a^{2k+3} \otimes |b|^{2k+3} \beta + |b|^{2k+3} \beta \otimes I]_{-\text{sgn}(b \otimes b)}.$$

Assume now that $K = L^2(\mathbb{R})$ and that the operators $\widehat{a}, \widehat{b}, \widehat{\beta}$ are given by (3.7). Then operator (3.8) coincides with (3.25): $V = W$. Relation (3.26) takes the form:

$$W_{23}W_{12} = W_{12}W_{13}W_{23}.$$

This is the famous pentagon equation of Baa' and Skandalis [2]. It means that W is a multiplicative unitary. It is known that W is modular [8]. This property enables us to introduce unitary antipode, scaling group and Haar weight (see [8, 16, 19, 10, 20] for details).

In [19] we discussed the cyclic group of four elements acting on a quantum 'ax + b' group. In fact this action may be extended to an action of S^1 . At the beginning we set no condition for $\hbar \in \mathbb{R}$. For any $s \in S^1$ and any closed operator c we set

$$\phi_s(c) = w_s^* c w_s,$$

where w_s is unitary operator introduced by $w_s = s^{\chi(b<0)}$. Obviously w_s commutes with a and b . Moreover $w_s^* \beta w_s = s^{-\chi(b<0)} \beta s^{\chi(b<0)} = s^{-\chi(b<0)} s^{\chi(b>0)} \beta = s^{\text{sgn } b} \beta$. These facts show that the algebra A introduced by (3.2) is invariant under ϕ_s (for all real \hbar). For special values $h = \frac{\pi}{2k+3}$ the algebra A is equipped with the comultiplication Δ introduced by Theorem 3.3. Using (1.3) one can easily check that the multiplicative unitary (3.25) commutes with $w_s \otimes w_s$. Formula (3.27) shows now that the comultiplication is preserved by the automorphisms ϕ_s :

$$\Delta(\phi_s(c)) = (\phi_s \otimes \phi_s)\Delta(c)$$

for all $s \in S^1$ and $c \in A$.

4. NEW QUANTUM DEFORMATIONS OF 'ax + b' GROUP.

In this section we shall show how to enlarge the set of admissible values of the deformation parameter \hbar beyond the one described in Theorem 3.3. To this end one has to add a new element to the set of generators of the C*-algebra A . This new element denoted by w is a unitary operator commuting with a and b such that

$$(4.1) \quad w^* \beta w = s^{\text{sgn } b} \beta.$$

In this formula $s \in S^1$ is a new deformation parameter. We shall see later that s is related to \hbar .

To define the new C*-algebra A we consider four operators a , b , β and w acting on the Hilbert space $L^2(\mathbb{R} \times S^1)$ introduced in the following way: for any $\tau \in \mathbb{R}$ and any $x \in L^2(\mathbb{R} \times S^1)$ we set:

$$(4.2) \quad \begin{aligned} (a^{i\tau} x)(t, z) &= e^{\hbar\tau/2} x(e^{\hbar\tau} t, z), & (bx)(t, z) &= tx(t, z), \\ (\beta x)(t, z) &= x(-t, z), & (wx)(t, z) &= s^{\chi(t<0)} z x(t, z). \end{aligned}$$

As in the previous section a is the analytic generator of the group of unitaries defined by the first formula. Operator b is selfadjoint. Its domain consists of all x such that $|tx(t, z)|^2$ is integrable over $\mathbb{R} \times S^1$. Clearly β and w are unitary and $\beta^* = \beta$. By simple computations, $a\beta = \beta a$, $b\beta = -\beta b$, $aw = wa$, $bw = wb$, $w^* \beta w = s^{\text{sgn } b} \beta$ and $a \overset{\hbar}{\circ} b$. Furthermore $\beta w \beta = s^{\text{sgn } b} w$ and $\beta w^{\text{sgn } b} \beta = (\beta w \beta)^{-\text{sgn } b} = (s^{\text{sgn } b} w)^{-\text{sgn } b} = s^{-1} w^{-\text{sgn } b}$. Hence $\beta w^{\text{sgn } b} \beta = s^{-1} w^{-\text{sgn } b}$.

In what follows we shall use operator L introduced by the formula

$$(4.3) \quad (Lx)(t, z) = z \frac{\partial x(t, z)}{\partial z}.$$

One can easily verify that L is a selfadjoint operator with integer spectrum, it commutes with a , b and β and $w^* L w = L + I$. Using the last relation we get

$$(4.4) \quad (I \otimes w)^{L \otimes I} (w \otimes I) (I \otimes w)^{-L \otimes I} = w \otimes w.$$

Using essentially the same method as in the proof of Theorem 3.1 one can easily show

Theorem 4.1. *Let*

$$(4.5) \quad A = \left\{ (f_1(b) + \beta f_2(b)) g(\log a) w^k : \begin{array}{l} f_1, f_2, g \in C_\infty(\mathbb{R}) \\ f_2(0) = 0, k \in \mathbb{Z} \end{array} \right\}^{\text{norm closed linear envelope}}.$$

Then: 1. A is a nondegenerate C*-algebra of operators acting on $L^2(\mathbb{R} \times S^1)$,

2. $\log a$, b , $ib\beta$ and w are affiliated with A : $\log a$, b , $ib\beta$, $w \eta A$,

3. $\log a$, b , $ib\beta$ and w generate A ,

The reader should notice that the C*-algebra A introduced by (4.5) coincides with the crossed product of the C*-algebra A considered in the previous section (cf. (3.2)) by the automorphism that leaves a and b invariant and maps β into $s^{\text{sgn } b} \beta$.

Now we pass to the description of the Hilbert space K . The structure of K is determined by a quadruple of selfadjoint operators $(\widehat{a}, \widehat{b}, \widehat{\beta}, \widehat{L})$ acting on K and having the following properties:

$$(4.6) \quad \begin{aligned} & 1. \widehat{a} > 0, \ker \widehat{a} = \ker \widehat{b} = \{0\} \text{ and } \widehat{a} \xrightarrow{\hbar} \widehat{b}, \\ & 2. \widehat{\beta} \text{ is a unitary involution, } \widehat{\beta} \text{ commutes with } \widehat{a} \text{ and anticommutes with } \widehat{b}, \\ & 3. \widehat{L} \text{ is of integer spectrum, } \widehat{L} \text{ strongly commutes with } \widehat{a} \text{ and } \widehat{b}, \\ & 4. \widehat{\beta} \widehat{L} \widehat{\beta} = \widehat{L} - \text{sgn } \widehat{b}. \end{aligned}$$

One of the possible choices is: $K = L^2(\mathbb{R} \times S^1)$ and $(\widehat{a}, \widehat{b}, \widehat{\beta}, \widehat{L}) = (a, b, \beta, L + \chi(b > 0))$. However there is another possibility that is even more interesting:

$$(4.7) \quad (\widehat{a}, \widehat{b}, \widehat{\beta}, \widehat{L}) = (|b|^{-1}, e^{i\hbar/2} b^{-1} a, \alpha w^{\text{sgn } b} \beta, L),$$

where $\alpha \in S^1$ and $\alpha^2 = s$. The reader easily verifies that these operators possess the properties (4.6).

Let

$$(4.8) \quad V = G_{\hbar}(\widehat{b} \otimes b, \widehat{\beta} \otimes \beta)^* e^{\frac{i}{\hbar} \log \widehat{a} \otimes \log a} (I \otimes w)^{\widehat{L} \otimes I}.$$

This is the basic object considered in this section. We shall prove

Theorem 4.2.

1. V is a unitary operator and $V \in M(\mathcal{K}(K) \otimes A)$,
2. A is generated by $V \in M(\mathcal{K}(K) \otimes A)$.

Proof. Let $R = \widehat{b} \otimes b$, $\rho = \widehat{\beta} \otimes \beta$, $U = e^{\frac{i}{\hbar} \log \widehat{a} \otimes \log a} (I \otimes w)^{\widehat{L} \otimes I}$, $S = \widehat{a}^{-1} \otimes I$ and $C = \mathcal{K}(K) \otimes A$. Then all the assumptions of Proposition 2.3 are satisfied. Hence $V = G_{\hbar}(R, \rho)^* U \in M(C)$ and Statement 1 is proved.

Let π be a representation of A and $B \in C^*(H_{\pi})$. Then $\text{id} \otimes \pi$ is a representation of C acting on $K \otimes H_{\pi}$. Assume that $(\text{id} \otimes \pi)V \in M(\mathcal{K}(K) \otimes B)$. Repeating the reasoning used in the proof of Theorem 3.2 we see that $\pi(b)$ and $\pi(i\beta b)$ are affiliated with B . Furthermore $(\text{id} \otimes \pi)U = e^{\frac{i}{\hbar} \log \widehat{a} \otimes \pi(\log a)} (I \otimes \pi(w))^{\widehat{L} \otimes I}$ is affiliated with $\mathcal{K}(K) \otimes B$.

We know that \widehat{a} commutes with \widehat{L} . Therefore \widehat{a} respects the decomposition of K into direct sum of eigenspaces of \widehat{L} . Let $K_{\ell} = K(\widehat{L} = \ell)$. Then

$$K = \bigoplus_{\ell \in \mathbb{Z}} K_{\ell}, \quad \widehat{a} = \bigoplus_{\ell \in \mathbb{Z}} \widehat{a}_{\ell} \quad \text{and} \quad \widehat{L} = \bigoplus_{\ell \in \mathbb{Z}} \ell I.$$

With this notation

$$(\text{id} \otimes \pi)U = \bigoplus_{\ell \in \mathbb{Z}} e^{\frac{i}{\hbar} \log \widehat{a}_{\ell} \otimes \pi(\log a)} (I \otimes \pi(w))^{\ell}.$$

Let $\ell = 0, 1$. Remembering that $(\text{id} \otimes \pi)U$ is affiliated with $\mathcal{K}(K) \otimes B$ we see that $e^{\frac{i}{\hbar} \log \widehat{a}_0 \otimes \pi(\log a)}$ is affiliated with $\mathcal{K}(K_0) \otimes B$ and $e^{\frac{i}{\hbar} \log \widehat{a}_1 \otimes \pi(\log a)} (I \otimes \pi(w))$ is affiliated with $\mathcal{K}(K_1) \otimes B$. Proposition 2.5 shows now that $\pi(\log a)$ is affiliated with B . Using this fact one can easily show that $\pi(w)$ is also affiliated with B . According to Statement 3 of Theorem 4.1, b , $i\beta b$, $\log a$ and w generate A . Therefore $\pi \in \text{Mor}(A, B)$. We showed that $(\text{id} \otimes \pi)V \in M(\mathcal{K}(K) \otimes B)$ implies $\pi \in \text{Mor}(A, B)$. It means that A is generated by $V \in M(\mathcal{K}(K) \otimes A)$. \square

Now we are able to formulate the main result of this section:

Theorem 4.3.

$$\left(\begin{array}{l} \text{There exists } \Delta \in \text{Mor}(A, A \otimes A) \\ \text{such that} \\ (\text{id} \otimes \Delta)V = V_{12}V_{13} \end{array} \right) \iff \left(\begin{array}{l} \hbar = \frac{\pi}{p}, \text{ where } p \in \mathbb{R}, p > 2 \\ \text{and } e^{i\pi p} = -s \end{array} \right).$$

Proof. We essentially repeat the proof of Theorem 3.3. Since in the great part calculations are very similar, we sketch the main steps only and point out necessary modifications. We shall use the operator L introduced in (4.3).

Let

$$(4.9) \quad \begin{aligned} \alpha &= ie^{\frac{i\pi^2}{2\hbar}}, \\ T &= I \otimes e^{i\hbar/2} b^{-1} a \otimes b, \\ \tau &= (I \otimes \beta w^{-\text{sgn } b} \otimes \beta) \left[\alpha s^{-1} \chi(\widehat{b} \otimes b \otimes I < 0) + \bar{\alpha} \chi(\widehat{b} \otimes b \otimes I > 0) \right]. \end{aligned}$$

and

$$(4.10) \quad W' = G_{\hbar}(T, \tau)^* e^{-\frac{i}{\hbar} I \otimes \log |b| \otimes \log a} (I \otimes I \otimes w)^{I \otimes L \otimes I}.$$

Clearly W' is a unitary operator acting on $K \otimes L^2(\mathbb{R} \times S^1) \otimes L^2(\mathbb{R} \times S^1)$. We shall prove that

$$(4.11) \quad V_{12} V_{13} = W' V_{12} W'^*.$$

In order to make our formulae shorter we set

$$\begin{aligned} U &= e^{\frac{i}{\hbar} \log \widehat{a} \otimes \log a} (I \otimes w)^{\widehat{L} \otimes I}, \\ Z &= e^{-\frac{i}{\hbar} \log |b| \otimes \log a} (I \otimes w)^{L \otimes I}. \end{aligned}$$

Using the commutation relations, one can easily verify that

$$(4.12) \quad U(\widehat{b} \otimes I)U^* = \widehat{b} \otimes a, \quad U(\widehat{\beta} \otimes I)U^* = (\widehat{\beta} \otimes I)(I \otimes w)^{-\text{sgn } \widehat{b} \otimes I},$$

$$(4.13) \quad Z(a \otimes I)Z^* = a \otimes a, \quad Z(w \otimes I)Z^* = w \otimes w.$$

The last formula follows from (4.4). With the above notation $V = G_{\hbar}(\widehat{b} \otimes b, \widehat{\beta} \otimes \beta)^* U$ and

$$V_{12} V_{13} = G_{\hbar}(\widehat{b} \otimes b \otimes I, \widehat{\beta} \otimes \beta \otimes I)^* U_{12} G_{\hbar}(\widehat{b} \otimes I \otimes b, \widehat{\beta} \otimes I \otimes \beta)^* U_{13}.$$

Taking into account (4.12) we get

$$(4.14) \quad \begin{aligned} U_{12} G_{\hbar}(\widehat{b} \otimes I \otimes b, \widehat{\beta} \otimes I \otimes \beta)^* &= G_{\hbar}(\widehat{b} \otimes a \otimes b, (\widehat{\beta} \otimes I \otimes \beta)(I \otimes w \otimes I)^{-\text{sgn } \widehat{b} \otimes I \otimes I})^* U_{12} \\ &= G_{\hbar}(\widehat{b} \otimes a \otimes b, \widehat{\beta} \otimes [(I \otimes \beta)(w \otimes I)^{I \otimes \text{sgn } b}])^* U_{12}. \end{aligned}$$

The second equality follows from (1.3). Therefore

$$(4.15) \quad V_{12} V_{13} = [G_{\hbar}(R, \rho) G_{\hbar}(S, \sigma)]^* U_{12} U_{13}.$$

where

$$(4.16) \quad \begin{aligned} R &= \widehat{b} \otimes a \otimes b, \quad \rho = \widehat{\beta} \otimes [(I \otimes \beta)(w \otimes I)^{I \otimes \text{sgn } b}], \\ S &= \widehat{b} \otimes b \otimes I, \quad \sigma = \widehat{\beta} \otimes \beta \otimes I, \end{aligned}$$

One can easily verify that R, S are selfadjoint, ρ, σ are unitary selfadjoint, R commutes with ρ and anticommutes with σ and S anticommutes with ρ and commutes with σ . Operator $T = e^{i\hbar/2} S^{-1} R = I \otimes e^{i\hbar/2} b^{-1} a \otimes b$ coincides with the operator T introduced by (4.9). Moreover

$$\begin{aligned} \sigma \rho &= I \otimes (\beta \otimes \beta)(w \otimes I)^{I \otimes \text{sgn } b} \equiv I \otimes \beta w^{-\text{sgn } b} \otimes \beta, \\ \rho \sigma &= I \otimes (I \otimes \beta)(w \otimes I)^{I \otimes \text{sgn } b} (\beta \otimes I) = I \otimes (\beta \otimes \beta)(\beta w \beta \otimes I)^{I \otimes \text{sgn } b} \\ &= I \otimes (\beta \otimes \beta)(s^{\text{sgn } b} w \otimes I)^{I \otimes \text{sgn } b} \equiv s^{-1} I \otimes \beta w^{-\text{sgn } b} \otimes \beta, \end{aligned}$$

where ‘ \equiv ’ denotes the equivalence relation: $x \equiv y$ if and only if $x\chi(T < 0) = y\chi(T < 0)$. Consequently

$$\alpha \rho \sigma \chi(S < 0) + \bar{\alpha} \sigma \rho \chi(S > 0) \equiv \tau,$$

where τ is given by (4.9). Theorem 1.6 shows now that

$$G_{\hbar}(R, \rho) G_{\hbar}(S, \sigma) = G_{\hbar}(T, \tau)^* G_{\hbar}(S, \sigma) G_{\hbar}(T, \tau)$$

and (4.15) takes the form

$$(4.17) \quad V_{12} V_{13} = G_{\hbar}(T, \tau)^* G_{\hbar}(\widehat{b} \otimes b \otimes I, \widehat{\beta} \otimes \beta \otimes I)^* G_{\hbar}(T, \tau) U_{12} U_{13}.$$

Performing simple computations and using (4.13) we obtain

$$\begin{aligned} U_{12} U_{13} &= e^{\frac{i}{\hbar} \log \widehat{a} \otimes \log(a \otimes a)} (I \otimes w \otimes w)^{\widehat{L} \otimes I \otimes I} \\ &= Z_{23} U_{12} Z_{23}^*. \end{aligned}$$

Repeating the arguments used in the proof of Theorem 3.3 we see that $G_{\hbar}(T, \tau)$ commutes with $e^{\frac{i}{\hbar} \log \widehat{a} \otimes \log(a \otimes a)}$. One can easily check that T commutes with $\widehat{L} \otimes I \otimes I$, $I \otimes w \otimes w$ and τ commutes with $\widehat{L} \otimes I \otimes I$. Moreover

$$(I \otimes w \otimes w)^* \tau(I \otimes w \otimes w) = \tau(I \otimes s^{\text{sgn } b} \otimes s^{\text{sgn } b}) \equiv \tau.$$

Therefore $G_{\hbar}(T, \tau)$ commutes with $(I \otimes w \otimes w)^{\widehat{L} \otimes I \otimes I}$ and in (4.17) we may move $G_{\hbar}(T, \tau)$ to the most right position:

$$(4.18) \quad V_{12}V_{13} = G_{\hbar}(T, \tau)^* G_{\hbar}(\widehat{b} \otimes b \otimes I, \widehat{\beta} \otimes \beta \otimes I)^* Z_{23}U_{12}Z_{23}^* G_{\hbar}(T, \tau)$$

Finally one easily verifies that $b \otimes I$ and $\beta \otimes I$ commute with $\log |b| \otimes \log a$, $L \otimes I$ and $I \otimes w$. Therefore $G_{\hbar}(\widehat{b} \otimes b \otimes I, \widehat{\beta} \otimes \beta \otimes I)$ commutes with Z_{23} . Clearly $G_{\hbar}(\widehat{b} \otimes b \otimes I, \widehat{\beta} \otimes \beta \otimes I)^* U_{12} = V_{12}$ and $W' = G_{\hbar}(T, \tau)^* Z_{23}$. Now (4.11) follows immediately from (4.18).

Also in the present case W' commutes with $\widehat{a} \otimes I \otimes I$. The same proof applies.

Now we are ready to prove the main statement.

\implies . Let $\Delta \in \text{Mor}(A, A \otimes A)$ and $(\text{id} \otimes \Delta)V = V_{12}V_{13}$. Repeating the reasoning used in the proof of Theorem 3.3 we easily arrive to the formula

$$(4.19) \quad \Delta(b) = [a \otimes b + b \otimes I]_{\tau_+} = [a \otimes b + b \otimes I]_{\tau_-}$$

where

$$(4.20) \quad \begin{aligned} \tau_+ &= (\beta w^{-\text{sgn } b} \otimes \beta) [\alpha s^{-1} \chi(b \otimes I < 0) + \bar{\alpha} \chi(b \otimes I > 0)], \\ \tau_- &= (\beta w^{-\text{sgn } b} \otimes \beta) [\alpha s^{-1} \chi(b \otimes I > 0) + \bar{\alpha} \chi(b \otimes I < 0)]. \end{aligned}$$

Clearly the two expressions for $\Delta(b)$ must coincide. Let us notice that the operator $I \otimes \beta$ commutes with τ_+ , τ_- and $b \otimes I$ and anticommutes with $a \otimes b$. Therefore $\tau_+ = \tau_-$ by Remark 1.3. Using (4.20) we get $s = \alpha^2$. Remembering that $\alpha = i e^{\frac{i\pi^2}{2\hbar}}$ and $0 < \hbar < \frac{\pi}{2}$ we conclude that $\hbar = \frac{\pi}{p}$, where $p \in \mathbb{R}$, $p > 2$ and $e^{i\pi p} = -s$.

\Leftarrow . Assume that $\hbar = \frac{\pi}{p}$, for some $p \in \mathbb{R}$ such that $p > 2$ and $e^{i\pi p} = -s$. Then formula (4.10) essentially simplifies. In this case $\alpha s^{-1} = \bar{\alpha}$, $\tau = I \otimes \bar{\alpha} \beta w^{-\text{sgn } b} \otimes \beta = I \otimes \alpha w^{\text{sgn } b} \beta \otimes \beta$ and $W' = W_{23} = I \otimes W$, where

$$(4.21) \quad W = G_{\hbar} \left(e^{i\hbar/2} b^{-1} a \otimes b, \alpha w^{\text{sgn } b} \beta \otimes \beta \right)^* e^{-\frac{i}{\hbar} \log |b| \otimes \log a} (I \otimes w)^{L \otimes I}.$$

Formula (4.11) takes the form

$$(4.22) \quad V_{12}V_{13} = W_{23}V_{12}W_{23}^*.$$

For any $c \in A$ we set

$$(4.23) \quad \Delta(c) = W(c \otimes I)W^*.$$

Then Δ is a representation of A acting on $L^2(\mathbb{R} \times S^1) \otimes L^2(\mathbb{R} \times S^1)$. We know that $V \in \text{M}(\mathcal{K}(K) \otimes A)$. Formula (4.22) shows that

$$(\text{id} \otimes \Delta)V = V_{12}V_{13}.$$

Clearly $V_{12}, V_{13} \in \text{M}(\mathcal{K}(K) \otimes A \otimes A)$. Therefore $(\text{id} \otimes \Delta)V = V_{12}V_{13} \in \text{M}(\mathcal{K}(K) \otimes A \otimes A)$. Remembering that A is generated by V we conclude that $\Delta \in \text{Mor}(A, A \otimes A)$. \square

Let $s = -e^{i\pi p}$ and $\hbar = \frac{\pi}{p}$ for some $p > 2$. Formula (4.23) enables us to calculate $\Delta(c)$ for any $c \in A$. The same holds for any c affiliated with A . We shall show that

$$(4.24) \quad \begin{aligned} \Delta(a) &= a \otimes a, \\ \Delta(b) &= \left[a \otimes b + b \otimes I \right]_{\alpha w^{\text{sgn } b} \beta \otimes \beta}, \\ \Delta(\beta |b|^p) &= \left[(w \otimes I)^{-I \otimes \text{sgn } b} (a^p \otimes \beta |b|^p) + \beta |b|^p \otimes I \right]_{-\text{sgn}(b \otimes b)}, \\ \Delta(w) &= w \otimes w. \end{aligned}$$

Repeating the reasoning preceding the formula (4.18) one can show that $a \otimes a$ and $w \otimes w$ commute with $G_{\hbar} \left(e^{i\hbar/2} b^{-1} a \otimes b, \alpha w^{\text{sgn } b} \beta \otimes \beta \right)$ and formulae for $\Delta(a)$ and $\Delta(w)$ follow immediately from (4.13). The formula for $\Delta(b)$ coincides with (4.19).

It remains to prove the third formula. According to (4.21) operator W is the composition of two unitaries: $G_{\hbar}(e^{i\hbar/2}b^{-1}a \otimes b, \alpha w^{\text{sgn } b} \beta \otimes \beta)^*$ and $e^{-\frac{i}{\hbar} \log |b| \otimes \log a} (I \otimes w)^{L \otimes I}$. Formula (4.23) shows now that $\Delta = \psi \circ \varphi$, where

$$\begin{aligned} \varphi(c) &= e^{-\frac{i}{\hbar} \log |b| \otimes \log a} (I \otimes w)^{L \otimes I} (c \otimes I) (I \otimes w)^{-L \otimes I} e^{\frac{i}{\hbar} \log |b| \otimes \log a}, \\ \psi(d) &= G_{\hbar}(e^{i\hbar/2}b^{-1}a \otimes b, \alpha w^{\text{sgn } b} \beta \otimes \beta)^* d G_{\hbar}(e^{i\hbar/2}b^{-1}a \otimes b, \alpha w^{\text{sgn } b} \beta \otimes \beta). \end{aligned}$$

One can easily verify that $\varphi(b) = b \otimes I$ and $\varphi(\beta) = \beta \otimes I$. Therefore

$$\varphi(\beta |b|^p) = \beta |b|^p \otimes I$$

and

$$(4.25) \quad \Delta(\beta |b|^p) = \psi(\beta |b|^p \otimes I).$$

To compute the right hand side we use Proposition 1.4 with

$$\begin{aligned} R &= a \otimes |b|, & S &= |b| \otimes I, \\ \tau &= \text{sgn}(b \otimes b), & \xi &= \alpha w^{\text{sgn } b} \beta \otimes \beta, \\ \rho &= (w \otimes I)^{-I \otimes \text{sgn } b} (I \otimes \beta), & \sigma &= \beta \otimes I. \end{aligned}$$

One can easily check that these operators fulfill all assumptions of Proposition 1.4. In this case we have $T = (e^{i\hbar/2} |b|^{-1} a) \otimes |b|$ and $\tau T = e^{i\hbar/2} b^{-1} a \otimes b$. According to our assumption $\frac{\pi}{\hbar} = p$. Therefore

$$\begin{aligned} \sigma S^{\frac{\pi}{\hbar}} &= \beta |b|^p \otimes I, \\ \rho R^{\frac{\pi}{\hbar}} &= (w \otimes I)^{-I \otimes \text{sgn } b} (a^p \otimes \beta |b|^p) \end{aligned}$$

and formula (1.11) takes the form

$$(4.26) \quad \left[\beta |b|^p \otimes I + (w \otimes I)^{-I \otimes \text{sgn } b} (a^p \otimes \beta |b|^p) \right]_{-\text{sgn}(b \otimes b)} = \psi(\beta |b|^p \otimes I).$$

Comparing (4.25) with (4.26) we get the third formula of (4.24).

5. MODULARITY AND ALL THAT

Now we shall investigate the unitary W introduced by (4.21). We shall prove that W is a modular multiplicative unitary. Throughout this section $s = -e^{i\pi p}$ and $\hbar = \frac{\pi}{p}$ where $p > 2$.

Let \bar{K} be the Hilbert space complex conjugate to K . The structure of \bar{K} is established by an antiunitary mapping $K \ni x \longmapsto \bar{x} \in \bar{K}$. For any closed operator c acting on K , we denote by c^\top the transpose of c . By definition c^\top is an operator acting on \bar{K} with domain $\mathcal{D}(c^\top) = \{\bar{x} : x \in \mathcal{D}(c^*)\}$ such that

$$c^\top \bar{x} = \overline{c^* x}$$

for any $x \in \mathcal{D}(c^*)$. In what follows $Q = a^{\frac{1}{2}}$.

Proposition 5.1. *Let V be the unitary operator introduced by (4.8) and*

$$(5.1) \quad \tilde{V} = G_{\hbar} \left(-\widehat{b}^\top \otimes e^{i\hbar/2} b a^{-1}, -\widehat{\beta}^\top \otimes \alpha w^{\text{sgn } b} \beta \right) e^{\frac{i}{\hbar} \log \widehat{a}^\top \otimes \log a} (I \otimes w)^{\widehat{L}^\top \otimes I}.$$

Then \tilde{V} is unitary and for any $x, z \in K$, $y \in D(Q^{-1})$, $u \in D(Q)$ we have:

$$(5.2) \quad (x \otimes u | V | z \otimes y) = \left(\bar{z} \otimes Q u \left| \tilde{V} \right| \bar{x} \otimes Q^{-1} y \right).$$

Proof. Let us notice that $\chi(-\widehat{b}^\top \otimes e^{i\hbar/2} b a^{-1} < 0) = \chi(\widehat{b}^\top \otimes b > 0)$. By virtue of (1.3), we may replace $\widehat{\beta}^\top \otimes \alpha w^{\text{sgn } b} \beta = \widehat{\beta}^\top \otimes \bar{\alpha} \beta w^{-\text{sgn } b}$ by $\bar{\alpha} (I \otimes \beta) \tilde{\tau}$, where

$$(5.3) \quad \tilde{\tau} = (\widehat{\beta}^\top \otimes I) (I \otimes w)^{-\text{sgn } \widehat{b}^\top \otimes I}.$$

Therefore

$$(5.4) \quad \tilde{V} = G_{\hbar} \left(-\widehat{b}^\top \otimes e^{i\hbar/2} b a^{-1}, -\bar{\alpha} (I \otimes \beta) \tilde{\tau} \right) e^{\frac{i}{\hbar} \log \widehat{a}^\top \otimes \log a} (I \otimes w)^{\widehat{L}^\top \otimes I}.$$

We know that $\widehat{\beta} \widehat{L} = (\widehat{L} - \text{sgn } \widehat{b}) \widehat{\beta}$. Therefore $\widehat{L}^\top \widehat{\beta}^\top = \widehat{\beta}^\top (\widehat{L}^\top - \text{sgn } \widehat{b}^\top)$ and

$$(I \otimes w)^{\widehat{L}^\top \otimes I} (\widehat{\beta}^\top \otimes I) = (\widehat{\beta}^\top \otimes I) (I \otimes w)^{\widehat{L}^\top \otimes I - \text{sgn } \widehat{b}^\top \otimes I}.$$

It shows that

$$(5.5) \quad \tilde{\tau}(I \otimes w)^{\widehat{L}^\top \otimes I} = (I \otimes w)^{\widehat{L}^\top \otimes I} (\widehat{\beta}^\top \otimes I).$$

We shall follow the proof of Proposition 2.3 of [19]. The reader should notice that in large part that proof is independent of the particular value of \hbar . To make our formulae shorter we set:

$$(5.6) \quad \begin{aligned} U' &= e^{\frac{i}{\hbar} \log \widehat{a} \otimes \log a}, & \widetilde{U}' &= e^{\frac{i}{\hbar} \log \widehat{a}^\top \otimes \log a}, \\ U &= U'(I \otimes w)^{\widehat{L} \otimes I}, & \widetilde{U} &= \widetilde{U}'(I \otimes w)^{\widehat{L}^\top \otimes I}, \\ B &= \widehat{b} \otimes b, & \widetilde{B} &= \widehat{b}^\top \otimes e^{i\hbar/2} b a^{-1}. \end{aligned}$$

We know that $\text{sgn } b$ and Q commute. Therefore we may assume that u and y are eigenvectors of $\text{sgn } b$. Similarly we may assume that x and z are common eigenvectors of $\text{sgn } \widehat{b}$. Proceeding in the same way as in [19] we reduce (5.2) to the following three equations (cf. [19, formula (2.23) and next two]):

$$(5.7) \quad (x \otimes u | V_\theta(\log B)^* U | z \otimes y) = (\bar{z} \otimes Qu | V_\theta(\log \widetilde{B} - \pi i) \widetilde{U} | \bar{x} \otimes Q^{-1} y),$$

$$(5.8) \quad (x \otimes u | V_\theta(\log B - \pi i)^* U | z \otimes y) = (\bar{z} \otimes Qu | V_\theta(\log \widetilde{B}) \widetilde{U} | \bar{x} \otimes Q^{-1} y),$$

$$(5.9) \quad \begin{aligned} &(x \otimes u | [i(\widehat{\beta} \otimes \beta) B^{\frac{\pi}{\hbar}} V_\theta(\log B - \pi i)]^* U | z \otimes y) \\ &= (\bar{z} \otimes Qu | -i\bar{\alpha}(I \otimes \beta) \tilde{\tau} \widetilde{B}^{\frac{\pi}{\hbar}} V_\theta(\log \widetilde{B} - \pi i) \widetilde{U} | \bar{x} \otimes Q^{-1} y). \end{aligned}$$

In these formulae $\theta = \frac{2\pi}{\hbar} = 2p$, The left hand side of the last formula

$$\text{LHS of (5.9)} = -i (\widehat{\beta} x \otimes \beta u | B^{\frac{\pi}{\hbar}} V_\theta(\log B - \pi i)^* U | z \otimes y).$$

Similarly remembering that β commutes with Q we have:

$$\text{RHS of (5.9)} = -i\bar{\alpha} (\bar{z} \otimes Q\beta u | \tilde{\tau} \widetilde{B}^{\frac{\pi}{\hbar}} V_\theta(\log \widetilde{B} - \pi i) \widetilde{U} | \bar{x} \otimes Q^{-1} y).$$

We shall move $\tilde{\tau}$ to the most right position. Clearly (cf. (5.3)) this operator commutes with \widetilde{B} and $\widetilde{U}' = e^{\frac{i}{\hbar} \log \widehat{a}^\top \otimes \log a}$. Taking into account (5.5) we obtain

$$\text{RHS of (5.9)} = -i\bar{\alpha} (\bar{z} \otimes Q\beta u | \widetilde{B}^{\frac{\pi}{\hbar}} V_\theta(\log \widetilde{B} - \pi i) \widetilde{U} | \overline{\widehat{\beta} x} \otimes Q^{-1} y).$$

Replacing $\widehat{\beta} x$ and βu by x and u respectively we see that (5.9) is equivalent to the equation

$$(5.10) \quad (x \otimes u | \alpha B^{\frac{\pi}{\hbar}} V_\theta(\log B - \pi i)^* U | z \otimes y) = (\bar{z} \otimes Qu | \widetilde{B}^{\frac{\pi}{\hbar}} V_\theta(\log \widetilde{B} - \pi i) \widetilde{U} | \bar{x} \otimes Q^{-1} y)$$

Let us notice that our crucial formulae (5.7), (5.8) and (5.10) fit the same pattern:

$$(5.11) \quad (x \otimes u | f_i(B) U | z \otimes y) = (\bar{z} \otimes Qu | g_i(\widetilde{B}) \widetilde{U} | \bar{x} \otimes Q^{-1} y),$$

where f_i and g_i ($i = 1, 2, 3$) are functions on positive reals:

$$\begin{aligned} f_1(t) &= \overline{V_\theta(\log t)}, & g_1(t) &= V_\theta(\log t - \pi i), \\ f_2(t) &= \overline{V_\theta(\log t - \pi i)}, & g_2(t) &= V_\theta(\log t), \\ f_3(t) &= \alpha t^{\frac{\pi}{\hbar}} \overline{V_\theta(\log t - \pi i)}, & g_3(t) &= t^{\frac{\pi}{\hbar}} V_\theta(\log t - \pi i). \end{aligned}$$

for all $t > 0$.

Replacing operators U and \widetilde{U} by U' and \widetilde{U}' (cf. (5.6)) we obtain a simplified version of (5.11):

$$(5.12) \quad (x \otimes u | f_i(B) U' | z \otimes y) = (\bar{z} \otimes Qu | g_i(\widetilde{B}) \widetilde{U}' | \bar{x} \otimes Q^{-1} y).$$

It is known that the last equality holds in all three cases $i = 1, 2, 3$ (cf. [19, proof of Proposition 2.3]). We shall show that (5.11) follows from (5.12). We shall use the expansion

$$(5.13) \quad (I \otimes w)^{\widehat{L} \otimes I} = \sum_{m=-\infty}^{\infty} \chi(\widehat{L} = m) \otimes w^m.$$

Inserting in (5.12), $\chi(\widehat{L} = m)x$ and $w^m y$ instead of x and y we obtain:

$$\left(x \otimes u \left| f_k(B)U' \left(\chi(\widehat{L} = m) \otimes w^m \right) \right| z \otimes y \right) = \left(\bar{z} \otimes Qu \left| g_k(\widetilde{B})\widetilde{U}' \left(\chi(\widehat{L}^\top = m) \otimes w^m \right) \right| \bar{x} \otimes Q^{-1}y \right).$$

Summing over m and using (5.13) we obtain (5.11). The proof is complete. \square

We recall the basic definitions [2, 16, 8]. Let H be a Hilbert space and W be a unitary operator acting on $H \otimes H$. We say that W is multiplicative unitary if it satisfies the pentagonal equation

$$W_{23}W_{12} = W_{12}W_{13}W_{23}.$$

A multiplicative unitary W is said to be modular if there exist strictly positive selfadjoint operators \widehat{Q} and Q acting on H and a unitary operator \widetilde{W} acting on $\overline{H} \otimes H$ such that $\widehat{Q} \otimes Q$ commutes with W and

$$(5.14) \quad (x \otimes u | W | z \otimes y) = \left(\bar{z} \otimes Qu \left| \widetilde{W} \right| \bar{x} \otimes Q^{-1}y \right)$$

for any $x, z \in H$, $u \in \mathcal{D}(Q)$ and $y \in \mathcal{D}(Q^{-1})$. In this definition \overline{H} is the complex conjugate Hilbert space related to H by an antiunitary mapping $H \ni x \longleftrightarrow \bar{x} \in \overline{H}$. The main result of this section is contained in the following

Theorem 5.2. *The operator W introduced by (4.21) is a modular multiplicative unitary acting on $L^2(\mathbb{R} \times S^1) \otimes L^2(\mathbb{R} \times S^1)$.*

Proof. Assume that $K = L^2(\mathbb{R} \times S^1)$. One can easily verify that operators

$$(5.15) \quad \begin{aligned} \widehat{a} &= |b|^{-1}, & \widehat{\beta} &= \alpha w^{\text{sgn } b} \beta, \\ \widehat{b} &= e^{i\hbar/2} b^{-1} a, & \widehat{L} &= L \end{aligned}$$

obey the properties listed in (4.6). In particular $\widehat{\beta}^2 = I$, $\widehat{\beta}^* = \widehat{\beta}$ and $\widehat{\beta} \widehat{L} \widehat{\beta} = w^{\text{sgn } b} \beta L \beta w^{-\text{sgn } b} = w^{\text{sgn } b} L w^{-\text{sgn } b} = L - \text{sgn } b = \widehat{L} - \text{sgn } \widehat{b}$. With this choice, right hand side of (4.8) coincides with that of (4.21): $V = W$ and relation (4.22) takes the form:

$$W_{23}W_{12} = W_{12}W_{13}W_{23}.$$

Hence W is a multiplicative unitary operator.

Let $Q = a^{1/2}$ and $\widehat{Q} = |b|^{1/2}$. Inserting in (5.1) operators (5.15) we obtain a unitary operator \widetilde{W} satisfying formula (5.14). To end the proof we have to show that W commutes with $\widehat{Q} \otimes Q$.

We know that a commutes with β and w . One can easily check that $|b|$ commutes with $\alpha w^{\text{sgn } b} \beta$ and L . Therefore $\widehat{Q} \otimes Q = |b|^{1/2} \otimes a^{1/2}$ commutes with $\alpha w^{\text{sgn } b} \beta \otimes \beta$, $L \otimes I$ and $I \otimes w$. Clearly it commutes with $\log |b| \otimes \log a$. Moreover due to the Zakrzewski relation $a \xrightarrow{\hbar} b$, $\widehat{Q} \otimes Q$ commutes with $e^{i\hbar/2} b^{-1} a \otimes b$. Inspecting formula (4.21) we see that $\widehat{Q} \otimes Q$ commutes with W . \square

Now we can use the full power of the theory of multiplicative unitaries [2, 16, 8]. Denoting by $B(L^2(\mathbb{R} \times S^1))_*$ the set of all normal functionals on $B(L^2(\mathbb{R} \times S^1))$ we have:

$$A = \left\{ (\omega \otimes \text{id})W : \omega \in B(L^2(\mathbb{R} \times S^1))_* \right\}^{\text{norm closure}}.$$

Indeed according to Theorem 1.5 of [16], the set on the right hand side is a C^* -algebra generated by W and above equality follows immediately from Theorem 4.2 (in the present setting $V = W$).

Formula (4.22) shows that (4.8) is an adapted operator in the sense of [16, Definition 1.3]. Comparing (5.1) with Statement 5 of Theorem 1.6 of [16] one can easily find the unitary antipode R of our quantum group. It acts on a, b, β, w as follows:

$$\begin{aligned} a^R &= a^{-1}, & \beta^R &= -\alpha w^{\text{sgn } b} \beta, \\ b^R &= -e^{i\hbar/2} b a^{-1}, & w^R &= w^*. \end{aligned}$$

The action of the scaling group is described by the formulae:

$$\begin{aligned} \tau_t(a) &= a, & \tau_t(\beta) &= \beta, \\ \tau_t(b) &= e^{\hbar t} b, & \tau_t(w) &= w. \end{aligned}$$

In the following, Tr denotes the trace of operators acting on $L^2(\mathbb{R} \times S^1)$ and E_0 denotes the orthogonal projection onto kernel of $\widehat{L} - \chi(\widehat{b} > 0)$:

$$\begin{aligned} E_0 &= \chi\left(\widehat{L} - \chi(\widehat{b} > 0) = 0\right) \\ &= \chi\left(\widehat{L} = 0 \text{ and } \widehat{b} < 0\right) + \chi\left(\widehat{L} = 1 \text{ and } \widehat{b} > 0\right). \end{aligned}$$

The reader should notice that E_0 commutes with all operators (5.15). Therefore $E_0 \otimes I$ commutes with the multiplicative unitary W . For any positive $c \in A$ we set

$$h(c) = \text{Tr}\left(E_0 \widehat{Q} c \widehat{Q} E_0\right) = \text{Tr}\left(E_0 |b|^{1/2} c |b|^{1/2} E_0\right).$$

Let $c = g(\log a)f(b)$, where $f, g \in C_\infty(\mathbb{R})$. Then $c \in A$. In what follows, $d\mu(z)$ denotes the normalized Haar measure on S^1 . One can verify that the operator $c \widehat{Q} E_0 = c |b|^{\frac{1}{2}} E_0$ is an integral operator:

$$(c |b|^{\frac{1}{2}} E_0 x)(t', z') = \int_{\mathbb{R} \times S^1} K_c(t', z'; t, z) x(t, z) dt d\mu(z)$$

with the kernel

$$K_c(t', z'; t, z) = |t'|^{-1/2} \widetilde{g}(t'/t) f(t) (z'/z)^{\chi(t>0)},$$

where

$$\widetilde{g}(\varrho) = \frac{1}{2\pi\hbar} \int_{\mathbb{R}} g(\tau) \varrho^{i\tau/\hbar} d\tau.$$

for $\varrho > 0$ and $\widetilde{g}(\varrho) = 0$ for $\varrho < 0$. Therefore

$$\begin{aligned} h(c^*c) &= \int_{\mathbb{R} \times S^1 \times \mathbb{R} \times S^1} |K_c(t', z'; t, z)|^2 dt' d\mu(z') dt d\mu(z) \\ &= \int_0^\infty |\widetilde{g}(\varrho)|^2 \frac{d\varrho}{\varrho} \int_{\mathbb{R}} |f(t)|^2 dt \\ &= \frac{1}{\pi\hbar} \int_{\mathbb{R}} |g(\tau)|^2 d\tau \int_{\mathbb{R}} |f(t)|^2 dt < \infty \end{aligned}$$

for $g, f \in L^2(\mathbb{R})$. Let $\epsilon > 0$ and $c_\epsilon = (I + \epsilon \log^2 a)^{-1} (I + \epsilon b^2)^{-1}$. Then $h(c_\epsilon^* c_\epsilon) < \infty$ for any $\epsilon > 0$. Clearly $c_\epsilon \rightarrow I$ in strict topology, when $\epsilon \rightarrow +0$. Therefore the left ideal $\{c \in A : h(c^*c) < \infty\}$ is dense in A . According to the theory developed by Van Daele [10], h is a right Haar weight on our quantum group. See also [20], where the right invariance of h is verified by a straightforward computation.

One can easily construct the reduced dual of our quantum group. By definition (see [2, 16]) this is a quantum group $(\widehat{A}, \widehat{\Delta})$ related to the multiplicative unitary $\widehat{W} = \Sigma W^* \Sigma$ (Σ denotes the flip operator acting on the tensor product of a Hilbert space by itself: $\Sigma(x \otimes y) = y \otimes x$). In particular

$$\widehat{A} = \{(\text{id} \otimes \omega) W^* : \omega \in B(L^2(\mathbb{R} \times S^1))_*\}^{\text{norm closure}}.$$

Let $\widehat{a}, \widehat{b}, \widehat{\beta}$ and \widehat{L} be operators introduced by (5.15). One can show that $\log \widehat{a}, \widehat{b}, \widehat{\beta} |\widehat{b}|$ and \widehat{L} are affiliated with \widehat{A} . Furthermore \widehat{A} is generated by these operators. The action of $\widehat{\Delta}$ is described by the formula:

$$\widehat{\Delta}(c) = \widehat{W}(c \otimes I) \widehat{W}^* = \Sigma W^*(I \otimes c) W \Sigma.$$

In particular

$$\begin{aligned} (5.16) \quad \widehat{\Delta}(\widehat{a}) &= \widehat{a} \otimes \widehat{a}, \\ \widehat{\Delta}(\widehat{b}) &= \left[\widehat{b} \otimes \widehat{a} + I \otimes \widehat{b} \right]_{\widehat{\beta} \otimes \alpha \widehat{w}^{\text{sgn } \widehat{b}} \widehat{\beta}}, \\ \widehat{\Delta}(\widehat{\beta} |\widehat{b}|^p) &= \left[(I \otimes \widehat{w})^{-\text{sgn } \widehat{b}} \otimes I \left(\widehat{\beta} |\widehat{b}|^p \otimes \widehat{a}^p \right) + I \otimes \widehat{\beta} |\widehat{b}|^p \right]_{-\text{sgn}(\widehat{b} \otimes \widehat{b})}, \\ \widehat{\Delta}(\widehat{L}) &= \widehat{L} \otimes I + I \otimes \widehat{L}, \end{aligned}$$

where $\widehat{w} = \alpha^{-2\widehat{L}} = s^{-\widehat{L}}$. To derive the second and third formulae one has to use the second versions of formulae (1.7) and (1.11). The details are left to the reader. The last relation in (5.16) shows that $\widehat{\Delta}(\widehat{w}) = \widehat{w} \otimes \widehat{w}$.

It is easy to verify that operators $\widehat{a}, \widehat{b}, \widehat{\beta}$ and \widehat{w} obey the same commutation relations as a, b, β and w . Using this fact one can show that there exists $\psi \in \text{Mor}(A, \widehat{A})$ such that $\psi(a) = \widehat{a}$, $\psi(b) = \widehat{b}$, $\psi(ib\beta) = i\widehat{b}\widehat{\beta}$ and $\psi(w) = s^{-\widehat{L}}$. Let $\widehat{\Delta}^{\text{opp}}$ be the comultiplication opposite to $\widehat{\Delta}$: $\widehat{\Delta}^{\text{opp}} = \text{flip} \circ \widehat{\Delta}$. Comparing formulae (4.24) with (5.16) we see that

$$(5.17) \quad \widehat{\Delta}^{\text{opp}}(\psi(c)) = (\psi \otimes \psi)\Delta(c)$$

for $c = a, b, \beta |b|^p, w$. Functions of these operators generate A , so (5.17) holds for any $c \in A$.

Let us notice that the operator

$$(\text{id} \otimes \psi)W = G_{\hbar}(\widehat{b} \otimes \widehat{b}, \widehat{\beta} \otimes \widehat{\beta}) e^{\frac{i}{\hbar} \log \widehat{a} \otimes \log \widehat{a}} s^{-\widehat{L} \otimes \widehat{L}}.$$

commutes with Σ . There exists an independent proof of (5.17) based on this observation.

6. 'ax + b'-GROUPS AT ROOTS OF UNITY

In this section we shall assume that $q^2 = e^{-i\hbar}$ is a root of unity. Then $s = \alpha^2 = -e^{\frac{i\pi^2}{\hbar}}$ is a root of unity. Let N be the smallest natural number such that $s^N = 1$. Formula (4.1) shows now that w^N commutes with β . Consequently w^N commutes with all elements of A and the set

$$C_N = \{(w^N - I)c : c \in A\}^{\text{norm closure}}$$

is a two-sided ideal in A . We know that $\Delta(w) = w \otimes w$. Therefore

$$\Delta(w^N - I) = w^N \otimes w^N - I \otimes I = w^N \otimes (w^N - I) + (w^N - I) \otimes I.$$

It shows that $\Delta(C_N) \subset A \otimes C_N + C_N \otimes A$ and the comultiplication Δ goes down to the quotient algebra $A_N = A/C_N$. More precisely there exists $\Delta_N \in \text{Mor}(A_N, A_N \otimes A_N)$ such that $\Delta_N(\pi(c)) = (\pi \otimes \pi)\Delta(c)$. In this formula $\pi \in \text{Mor}(A, A_N)$ denotes the canonical epimorphism from A onto $A_N = A/C_N$.

From now till the end of this section we shall work with the quantum group (A_N, Δ_N) . To simplify notation we shall omit π and write $a, b, \beta |b|$ and w instead of $\pi(a), \pi(b), \pi(\beta |b|)$ and $\pi(w)$. These operators are affiliated with A_N and we have the following commutation relation:

$$(6.1) \quad \begin{aligned} a^* &= a, \quad a > 0, \quad b^* = b, \quad a \overset{\hbar}{\circ} b, \\ \beta^* &= \beta, \quad \beta^2 = I, \quad \beta a \beta = a, \quad \beta b \beta = -b, \\ w^* w &= w w^* = I, \quad w^* a w = a, \quad w^* b w = b, \\ w^* \beta w &= s^{\text{sgn } b} \beta, \quad w^N = I. \end{aligned}$$

The action of Δ_N is described by the formulae identical with (4.24).

It is not difficult to describe A_N as a concrete C*-algebra and find the multiplicative unitary corresponding to (A_N, Δ_N) . To this end one has to repeat the considerations of section 4 replacing S^1 by the cyclic group of N elements:

$$\begin{aligned} \mathbb{Z}_N &= \{s^\ell : \ell = 0, 1, \dots, N-1\} \\ &= \{z \in S^1 : z^N = 1\}. \end{aligned}$$

In particular elements of A_N will be operators acting on $L^2(\mathbb{R} \times \mathbb{Z}_N)$. To define a, b, β and w we shall use the same formulae (4.2) with necessary reinterpretation: now $x \in L^2(\mathbb{R} \times \mathbb{Z}_N)$ and z runs over \mathbb{Z}_N . One can easily verify that a, b, β and w satisfy the relations (6.1). Now the formula (4.5) defines a C*-algebra acting on $L^2(\mathbb{R} \times \mathbb{Z}_N)$. This algebra is isomorphic to A_N .

It is not possible to find a selfadjoint operator L acting on $L^2(\mathbb{R} \times \mathbb{Z}_N)$ such that $\text{Sp } L \subset \mathbb{Z}$ and $w^* L w = L + I$. However there is a replacement for $(I \otimes w)^{L \otimes I}$. Instead of L we shall use an operator u acting on $L^2(\mathbb{R} \times \mathbb{Z}_N)$ according to the formula:

$$(ux)(t, z) = x(t, s^{-1}z).$$

One can easily verify that u is a unitary operator commuting with a, b, β such that $w u w^* = s u$ and $u^N = I$. By the last relation $\text{Sp } u = \mathbb{Z}_N$. Similarly $\text{Sp } w \subset \mathbb{Z}_N$. We shall use the bicharacter describing the selfduality of the group \mathbb{Z}_N :

$$\text{Ch} : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow S^1.$$

By definition $\text{Ch}(s^\ell, s^k) = s^{-k\ell}$ for any $k, \ell \in \mathbb{Z}$. The reader should notice that $\text{Ch}(s^\ell, z) = z^{-\ell}$ and $\text{Ch}(s z', z) = z^{-1} \text{Ch}(z', z)$ for any $\ell \in \mathbb{Z}$ and $z, z' \in \mathbb{Z}_N$. Using the last formula and remembering that

$wuw^* = su$ we obtain $(w \otimes I) \text{Ch}(u \otimes I, I \otimes w)(w^* \otimes I) = \text{Ch}(su \otimes I, I \otimes w) = (I \otimes w^*) \text{Ch}(u \otimes I, I \otimes w)$. Therefore

$$(6.2) \quad \text{Ch}(u \otimes I, I \otimes w)(w \otimes I) \text{Ch}(u \otimes I, I \otimes w)^* = w \otimes w.$$

One should compare this formula with (4.4). It shows that $\text{Ch}(u \otimes I, I \otimes w)$ is the right replacement for $(I \otimes w)^{L \otimes I}$.

For the moment we shall use the Hilbert space K and operators $\widehat{a}, \widehat{b}, \widehat{\beta}$ and \widehat{L} the same as in the section 4. Remembering that $\text{Ch}(s^\ell, z) = z^{-\ell}$ we obtain $\text{Ch}(s^{-\widehat{L}} \otimes I, I \otimes w) = (I \otimes w)^{\widehat{L} \otimes I}$ and formula (4.8) takes the form:

$$(6.3) \quad V = G_{\hbar}(\widehat{b} \otimes b, \widehat{\beta} \otimes \beta)^* e^{\frac{i}{\hbar} \log \widehat{a} \otimes \log a} \text{Ch}(\widehat{w} \otimes I, I \otimes w),$$

where $\widehat{w} = s^{-\widehat{L}}$. Taking into account (4.6) we see that \widehat{w} is a unitary operator with $\text{Sp } \widehat{w} \subset \mathbb{Z}_N$, it commutes with \widehat{a} and \widehat{b} and $\widehat{\beta} \widehat{w} \widehat{\beta} = s^{-\widehat{L} + \text{sgn } \widehat{b}} = s^{\text{sgn } \widehat{b}} \widehat{w}$. We should note that in order to define $V \in M(\mathcal{K}(K) \otimes A_N)$ we need only the operators $\widehat{a}, \widehat{b}, \widehat{\beta}, \widehat{w}$ acting on K . These operators must have the following properties:

$$(6.4) \quad \begin{aligned} &1. \widehat{a}, \widehat{b} \text{ are selfadjoint, } \widehat{a} > 0, \ker \widehat{a} = \ker \widehat{b} = \{0\} \text{ and } \widehat{a} \xrightarrow{\hbar} \widehat{b}, \\ &2. \widehat{\beta} \text{ is a unitary involution, } \widehat{\beta} \text{ commutes with } \widehat{a} \text{ and anticommutes with } \widehat{b}, \\ &3. \widehat{w} \text{ is unitary with } \text{Sp } \widehat{w} \subset \mathbb{Z}_N, \widehat{w} \text{ commutes with } \widehat{a} \text{ and } \widehat{b}, \\ &4. \widehat{\beta} \widehat{w} \widehat{\beta} = s^{\text{sgn } \widehat{b}} \widehat{w}. \end{aligned}$$

In this way operator \widehat{L} disappears from our setup. Let us notice that operators $\widehat{a}, \widehat{b}, \widehat{\beta}, \widehat{w}$ satisfy the same commutation relations as a, b, β, w .

From the beginning of this section we assumed that the deformation parameters s and \hbar are related by the formula $s = \alpha^2 = -e^{i\pi^2/\hbar}$. Repeating (with the necessary modifications indicated above) the considerations of section 4 we obtain the following formulae:

$$(6.5) \quad W = G_{\hbar} \left(e^{i\hbar/2} b^{-1} a \otimes b, \alpha w^{\text{sgn } b} \beta \otimes \beta \right)^* e^{-\frac{i}{\hbar} \log |b| \otimes \log a} \text{Ch}(u \otimes I, I \otimes w),$$

$$(6.6) \quad V_{12} V_{13} = W_{23} V_{12} W_{23}^*,$$

$$(6.7) \quad \Delta_N(c) = W(c \otimes I) W^*$$

for any $c \in A_N$.

Assume that $K = L^2(\mathbb{R} \times \mathbb{Z}_N)$, $\widehat{a} = |b|^{-1}$, $\widehat{\beta} = \alpha w^{\text{sgn } b} \beta$, $\widehat{b} = e^{i\hbar/2} b^{-1} a$, and $\widehat{w} = u$. One can easily verify that these operators obey the properties listed in (6.4). In particular $\widehat{\beta}^2 = I$, $\widehat{\beta}^* = \widehat{\beta}$, $\widehat{\beta} \widehat{w} \widehat{\beta} = w^{\text{sgn } b} u w^{-\text{sgn } b} = s^{\text{sgn } b} u = s^{\text{sgn } b} \widehat{w}$ where in the second step we used the relation $wuw^* = su$. With this choice, operators (6.3) and (6.5) coincide: $V = W$ and (6.6) shows that W is a multiplicative unitary. Using the method described in section 5 one can show that W is modular with $Q = a^{1/2}$ and $\widehat{Q} = |b|^{1/2}$.

This is the multiplicative unitary corresponding to the quantum group (A_N, Δ_N) . It can be used to determine the action of unitary antipode and scaling group, the Haar weight and the reduced dual $(\widehat{A}_N, \widehat{\Delta}_N)$. We know that $\widehat{a}, \widehat{b}, \widehat{\beta}$ and \widehat{w} satisfy the same commutation relations as a, b, β and w . Furthermore formula (6.3) is symmetric: replacing operators without ‘hats’ by corresponding operators with ‘hats’ we obtain an element of $M(\widehat{A} \otimes_{\text{sym}} \widehat{A})$. Using these facts one can show that the quantum group $(\widehat{A}_N, \widehat{\Delta}_N)$ is isomorphic to $(A_N, \Delta_N^{\text{opp}})$.

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