A NEW QUANTUM DEFORMATION OF '$ax+b$' GROUP.

W. PUSZ AND S. L. WORONOWICZ

Abstract. The paper is devoted to locally compact quantum groups that are related to classical '$ax+b$' group. We discuss in detail the quantization of the deformation parameter assumed with no justification in the previous paper. Next we construct (on the C*-level) a new quantum deformation of '$ax+b$' group corresponding to the deformation parameter q^2 being an even root of unit. To this end we add a new reflection operator w commuting with a and b but anticommuting with β.

0. Introduction

In last years a lot of effort was devoted to construct explicit examples of (non-compact) locally compact quantum groups. The present paper inscribes into this line of research. It is devoted to a quantum deformation of the group '$ax+b$' of affine transformations of real line. This deformation was presented first in [21].

We go back to the subject for the following reasons. At first the quantizations of the deformation parameter $\hbar$ crucial for quantum '$ax+b$' was not discussed in detail in the previous paper. Now we give strong arguments that the values of $\hbar$ considered in [21] are the only ones that make the construction possible. The problem is not solved completely and may be subject of further investigation (see the last section of the paper). Secondly one of the important formula in [21] was not proven. We fill this gap. Finally extending a little our setting we construct new quantum deformation of '$ax+b$' group. Our next target is $SL(2,\mathbb{R})$. We are convinced that the groups considered in the present paper will serve as *building blocks* in construction of quantum $SL(2,\mathbb{R})$ group.

Let G be '$ax+b$' group. On the classical level G consists of all transformations of the form

$$\mathbb{R} \ni x \longmapsto ax+b \in \mathbb{R}, \tag{0.1}$$

where a and b are real parameters labeling the elements of the group. We shall assume that $a>0$. Assigning to each element of the group the values of the parameters we define two unbounded continuous real functions on G. To denote the functions we shall use the same letters $a,b \in C(G)$. Then the C*-algebra $C_\infty(G)$ of all continuous functions vanishing at infinity on G is generated by a,b:

$$C_\infty(G) = \left\{ f(a)g(b) : \begin{matrix} f \in C_\infty(]0,\infty[) \\ g \in C_\infty(\mathbb{R}) \end{matrix} \right\}^{\text{uniform closed}}_{\text{linear envelope}} .$$

Functions a and b may be considered as elements affiliated with $C_\infty(G)$. Composing two transformations of the form (0.1) with parameters (a_1,b_1) and (a_2,b_2) one obtains the transformation with parameters $(a_1a_2, a_1b_2+b_1)$. This result leads to the following formulae describing the comultiplication:

$$\begin{aligned} \Delta(a) &= a \otimes a, \\ \Delta(b) &= a \otimes b + b \otimes I. \end{aligned} \tag{0.2}$$

At the moment the elements of G are considered as affine transformations of $\mathbb{R}$. However one may realize them as unitary operators acting on a Hilbert space. To this end, to any transformation of the form (0.1) we assign unitary operator $V_{(a,b)} \in B(L^2(\mathbb{R}))$ introduced by the formula:

$$\left(V_{(a,b)}f\right)(x) = a^{-1/2} f\left(a^{-1}(x-b)\right)$$

for any $f \in L^2(\mathbb{R})$. Then G may be identified with the set of unitary operators:

$$G = \left\{ V_{(a,b)} : a,b \in \mathbb{R}; a>0 \right\}. \tag{0.3}$$

This identification preserves the group structure and the topology. More precisely $V_{(1,0)} = I$ and

$$V_{(a_1,b_1)} V_{(a_2,b_2)} = V_{(a_1a_2, a_1b_2+b_1)} \tag{0.4}$$

for any $a_1, a_2 \in]0,\infty[$ and $b_1, b_2 \in \mathbb{R}$. Moreover a sequence $V_{(a_n,b_n)}$ converges to $V_{(a_\infty,b_\infty)}$ in strong topology if and only if $a_n \to a_\infty > 0$ and $b_n \to b_\infty$. In particular (0.3) with the strong operator topology is a locally compact space. One can also show that (0.3) is a closed subset of $B(L^2(\mathbb{R}))$ (in strong operator topology).

For any Hilbert space H we denote by $\mathcal{K}(H)$ the C*-algebra of all compact operators acting on H. According to the general theory [15] the strongly continuous family of unitaries (0.3) is described by a single unitary $V \in \mathrm{M}(\mathcal{K}(L^2(\mathbb{R})) \otimes C_\infty(G))$. The C*-algebra $C_\infty(G))$ is generated (in the sense of [15]) by V. Formula (0.4) means that

$$(\mathrm{id} \otimes \Delta)V = V_{12}V_{13}.$$

This way we arrive to the notion of (quantum) group of unitary operators. Let H be a Hilbert space. We shall consider pairs (A, V), where A is a C*-algebra and V is a unitary element of the multiplier algebra $\mathrm{M}(\mathcal{K}(H) \otimes A)$. We say that (A, V) is a quantum group of unitary operators if

0. A is generated by $V \in \mathrm{M}(\mathcal{K}(H) \otimes A)$.
1. There exists a morphism $\Delta \in \mathrm{Mor}(A, A \otimes A)$ such that

$$(\mathrm{id} \otimes \Delta)V = V_{13}V_{23}. \tag{0.5}$$

2. There exists an antimultiplicative closed (with respect to the strict topology) linear mapping $\widetilde{\kappa}$ acting on $\mathrm{M}(A)$ such that $(\omega \otimes \mathrm{id})V \in \mathcal{D}(\widetilde{\kappa})$ and

$$\widetilde{\kappa}\left((\omega \otimes \mathrm{id})V\right) = (\omega \otimes \mathrm{id})V^*$$

for any normal functional ω defined on $B(H)$. The antimultiplicativity means that $\mathcal{D}(\widetilde{\kappa})$ is a subalgebra of $\mathrm{M}(A)$ and that $\widetilde{\kappa}(ab) = \widetilde{\kappa}(b)\widetilde{\kappa}(a)$ for any $a, b \in \mathcal{D}(\widetilde{\kappa})$. One also assumes that $A \cap \mathcal{D}(\widetilde{\kappa})$ is norm dense in A.

We are not going to discuss this definition in the present paper. Let us notice only that for finite-dimensional H it essentially coincides with the definition of compact matrix pseudogroup given in [14]. The generalization consists in replacing the set of invertible matrices by the group of unitary operators acting on H. If A is commutative then it is of the form $A = C_\infty(G)$, where G is a locally compact (with respect to the strong operator topology) closed subgroup of the group of unitary operators acting on H. In what follows constructing quantum groups we shall focus on conditions 0 and 1. It is known that condition 2 follows from manageability [18] (or modularity [9]).

Let us go back to the '$ax + b$' group. In the quantum setting functions a and b are replaced by selfadjoint elements $a = a^* > 0$ and $b = b^*$ that no longer commute. Instead they satisfy the relation

$$ab = q^2 ba, \tag{0.6}$$

where the deformation parameter q^2 is a number of modulus 1. Unfortunately in our case elements a and b are represented by unbounded operators and the products ab and ba may not be well defined because of the domain problem. For this reason we replace (0.6) by the so called Zakrzewski relation. It says that for any $\tau \in \mathbb{R}$:

$$a^{i\tau} b a^{-i\tau} = e^{\hbar\tau} b.$$

In this formula $\hbar$ is a real constant such that $q^2 = e^{-i\hbar}$. The reader should notice that for $\tau = -i$ the above relation reduces to (0.6).

The second problem is related to the comultiplication. We would like to keep formulae (0.2). However in general $a \otimes b + b \otimes I$ is not selfadjoint and in the best case we may expect that $\Delta(b)$ is a selfadjoint extension of $a \otimes b + b \otimes I$:

$$a \otimes b + b \otimes I \subset \Delta(b).$$

To choose the extension in a well defined way we have to use an additional operator β independent of a and b: β is a selfadjoint unitary commuting with a and anticommuting with b. It means that the algebra A is no longer generated by a and b.

It is not obvious, how to present quantum '$ax+b$' group as a quantum group of unitary operators (A, V). The crucial point is the formula $V = V(a, b, \dots)$ expressing V in terms of a, b, β and perhaps some other elements related to A. The equation (0.5) takes the form

$$V(a \otimes a, [a \otimes b + b \otimes I], \dots) = V(a \otimes I, b \otimes I, \dots)V(I \otimes a, I \otimes b, \dots),$$

where $[a\otimes b+b\otimes I]$ is a suitable selfadjoint extension of $a\otimes b+b\otimes I$. To find solutions of this equation we spent a lot of time making use of our experience in the area of quantum exponential functions and quantum groups (cf. [17, 16, 12, 20, 19, 21, 8, 10]). As a result we got formulae (3.8) and (4.4) that are starting points in our presentation.

We shortly discuss the content of the paper. Sections 1 and 2 are devoted to mathematical tools used in the paper. In the first one we recall the Zakrzewski commutation relation and related quantum exponential function (with a slightly modified notation). Most of the results presented in that Section come from [19]; the essentially new result is contained in Proposition 1.4. The second Section deals with the notion of a C*-algebra generated by affiliated elements. We prove a number of results used in the main sections of the paper.

Section 3 is devoted to the quantum '$ax+b$' group introduced in [21]. This group exists only for special values of deformation parameter $q^2=e^{-i\hbar}$ with $\hbar=\frac{\pi}{2k+3}$, where $k=0,1,2,\dots$. This fact was not really shown in [21]. The special values of the deformation parameter were chosen to proceed with some computations. It was not clear that (at the expense of some complications) one is not able to construct quantum '$ax+b$' group for larger set of values of the deformation parameter. Now, presenting the '$ax+b$' group as a quantum group of unitary operators we obtain the quantization of the deformation parameter as a precise mathematical statement (cf. Theorem 3.3). More precisely for $q^2=e^{-i\hbar}$ we shall construct a C*-algebra A with distinguished selfadjoint elements a, b and $i\beta b$ affiliated with it (the so called reflection operator β is a unitary involution which is not affiliated with A). These elements satisfy (in a well defined sense) the relations $ab=q^2ba$, $a\beta=\beta a$ and $b\beta=-\beta b$. The algebra A is generated by a unitary element $V\in \mathrm{M}(\mathcal{K}(L^2(\mathbb{R}))\otimes A)$. The pair (A,V) is defined for all $0<\hbar<\pi/2$. However, the existence of Δ satisfying the condition (0.5) selects much smaller subset of admissible $\hbar$'s. We shall prove that Δ exists if and only if $\hbar$ is of the form indicated above. At the end of the section we prove an elegant formula describing the action of the comultiplication on the reflection operator. This formula appeared in the previous paper with no proof (cf. formula (4.16) of [21]).

A new quantum group related to classical '$ax+b$' group is constructed in Section 4. Using the involutive automorphism of the quantum '$ax+b$' group described in [21] we consider the corresponding crossed product. This enlargement of the algebra opens new possibilities. In particular we obtain a new admissible values of the deformation parameter. Now $\hbar=\frac{\pi}{2k+3}$, where $k=0,\frac{1}{2},1,\frac{3}{2},2,\dots$. For integer k we obtain quantum groups closely related to the one considered in [21] (and in Section 3). Half-integer k lead to essentially new examples of locally compact quantum groups.

The next section (Section 5) is devoted the multiplicative unitaries for the quantum group constructed in Section 4. We prove their modularity and find the unitary antipode and scaling group. In particular we show that the objects constructed in Section 4 satisfy all the axioms of Kustermans and Vaes [3] and the ones of Masuda, Nakagami and Woronowicz [5].

In the last section we formulate the problem of quantization of the deformation parameter for '$ax+b$' group in the full generality.

Our approach extensively uses the C*-algebra language and the theory of selfadjoint operators on the Hilbert space. For the basic facts concerning the general C*-algebra theory we refer to [1, 7]. The notation used in the paper follows the one explained in [15, 13]. In particular $\mathrm{M}(A)$ is the multiplier algebra of a C*-algebra A. The affiliation relation in the sense of C*-algebra theory is denoted by "η" and A^η is the set of all affiliated elements ("unbounded multipliers"). It is known that $\mathrm{M}(A)\subset A^\eta$. A morphism from A to a C*-algebra B is by definition any *-homomorphism $\pi: A\longrightarrow \mathrm{M}(B)$ such that $\pi(A)B$ is dense in B. Let us recall that any such π has the unique extension to a unital $*$-homomorphism $\pi:\mathrm{M}(A)\longrightarrow\mathrm{M}(B)$ and to *-preserving map $\pi: A^\eta\longrightarrow B^\eta$ respectively (both denoted by the same symbol). The set of all morphisms from A to B is denoted by $\mathrm{Mor}(A,B)$.

With some abuse of notation, the symbol $\mathrm{Rep}(A)$ will stay for the *set* of all non-degenerate representations of a C*-algebra A. For any $\pi\in\mathrm{Rep}(A)$, we denote by H_π the carrier Hilbert space of π. Then $\pi\in\mathrm{Mor}(A,\mathcal{K}(H_\pi))$.

In the paper we mostly deal with concrete C*-algebras. By definition they are norm closed *-subalgebras of the algebra $B(H)$ of all bounded operators acting on some (separable) Hilbert space H. As a rule, C*-algebras we deal with are separable. Non separable ones will appear only as a multiplier algebras. In particular $B(H) = \mathrm{M}(\mathcal{K}(H))$ where $\mathcal{K}(H)$ is the C*-algebra of all compact operators acting on H. We shall denote by $C^*(H)$ the set of all non-degenerate separable C*-algebras of operators acting on a Hilbert space H. We recall that an algebra $A \subset B(H)$ is non-degenerate if AH is dense in H.

We shall use functional calculus for strongly commuting selfadjoint operators. If T and β are selfadjoint operators acting on a Hilbert space H and T and β strongly commute then

$$T = \int_\Lambda^\oplus r\, dE(r,\varrho), \qquad \beta = \int_\Lambda^\oplus \varrho\, dE(r,\varrho),$$

where $dE(r,\varrho)$ is the common spectral measure supported by the joint spectrum $\Lambda \subset \mathbb{R}^2$ of (T,β). Moreover for any measurable complex valued function on Λ we have

$$f(T,\beta) = \int_\Lambda f(r,\varrho) dE(r,\varrho).$$

In this context the characteristic function χ will appear quite often. By definition for any sentence $\mathcal{R}$, we have

$$\chi(\mathcal{R}) = \begin{cases} 0 & \text{if} \quad \mathcal{R} \text{ is false,} \\ 1 & \text{if} \quad \mathcal{R} \text{ is true.} \end{cases}$$

Typically $\mathcal{R}$ is a formula involving (in)equality sign. For example $\chi(r \le 0)$ is equal 0 for positive r and 1 for $r = 0$ or negative. Consequently $\chi(T \le 0)$ is the spectral projection assigned to the negative part of the spectrum of a selfadjoint operator T. The corresponding spectral subspace will be denoted by $H(T \le 0)$: $H(T \le 0) = \chi(T \le 0)H$. Similarly $\chi(T = \lambda)$ is the orthogonal projection on the eigenspace $H(T = \lambda)$ of T corresponding to the eigenvalue $\lambda \in \mathbb{R}$. We refer to [19] for more detailed explanation of this notation.

Let $\mathrm{Np}(r) = r\chi(r < 0)$. Then for any selfadjoint T,

$$\mathrm{Np}(T) = T\chi(T < 0) \tag{0.7}$$

is a selfadjoint operator acting on H. This is the negative part of the operator T.

1. A special function and selfadjoint extensions.

In this section we recall (in a slightly modified version with a certain loss of generality) the basic definitions and statements of [19]. The only essentially new result is contained in formula (1.10). Later on it will help us to prove the formula announced in [21, formula (4.16)]. We start with a modified version the quantum exponential function introduced in [19]. Let $\hbar \in \mathbb{R}$ and $0 < \hbar < \frac{\pi}{2}$. Instead of function $F_\hbar$ defined on the set $\mathbb{R}_- \times \{-1,1\} \cup \mathbb{R}_+ \times \{0\}$ we shall use function $G_\hbar$ defined on $\mathbb{R} \times \{-1,1\}$. It is related to the function $F_\hbar$ by the formula

$$G_\hbar(r,\varrho) = F_\hbar(r, \varrho\chi(r<0)) \tag{1.1}$$

for any $r \in \mathbb{R}$ and $\varrho = \pm 1$. Taking into account definition [19, formula (1.19)] we obtain

$$G_\hbar(r,\varrho) = \begin{cases} V_\theta(\log r) & \text{for} \quad r > 0 \\ \left[1 + i\varrho|r|^{\frac{\pi}{\hbar}}\right] V_\theta\Big(\log|r| - \pi i\Big) & \text{for} \quad r < 0, \end{cases} \tag{1.2}$$

where $\theta = \frac{2\pi}{\hbar}$ and V_θ is the meromorphic function on $\mathbb{C}$ such that

$$V_\theta(x) = \exp\left\{\frac{1}{2\pi i}\int_0^\infty \log(1+t^{-\theta})\,\frac{dt}{t+e^{-x}}\right\}$$

for all $x \in \mathbb{C}$ such that $|\Im x| < \pi$. In addition $G_\hbar(0,\pm 1) = 1$. Then $G_\hbar(r,\varrho)$ is a continuous function on $\mathbb{R} \times \{-1,1\}$ and

$$\Big(G_\hbar(r,\varrho) = G_\hbar(r,\varrho')\Big) \Longleftrightarrow \Big(\varrho\chi(r<0) = \varrho'\chi(r<0)\Big). \tag{1.3}$$

The asymptotic behavior of $G_\hbar(r,\varrho)$ for large r is described by the formula

$$G_\hbar(r,\varrho) \approx C \exp\left\{\frac{(\log|r|)^2}{2i\hbar}\right\}, \tag{1.4}$$

where C is a phase factor depending only on $\operatorname{sign} r$ and ρ and '$\approx$' means that the difference goes to 0 when $r \to \pm\infty$ (see Statements 9 and 10 of [19, Theorem 1.1]).

It is known that the quantum exponential function assumes values of modulus 1. Therefore if T and β are operators acting on a Hilbert space H, T is selfadjoint and β is unitary selfadjoint commuting with T then $G_\hbar(T,\beta)$ is unitary.

Now we recall the concept of selfadjoint extension of a symmetric operator defined by a reflection operator. Let Q be a symmetric operator acting on a Hilbert space H and ρ be a unitary selfadjoint operator ($\rho^* = \rho$, $\rho^2 = I$) anticommuting with Q. Then we denote by $[Q]_\rho$ the restriction of Q^* to the domain $\{x \in \mathcal{D}(Q^*) : (\rho - I)x \in \mathcal{D}(Q)\}$. It is known (cf. [19, Proposition 5.1]) that $[Q]_\rho$ is a selfadjoint extension of Q. We shall use the following simple

Proposition 1.1. *Let Q, X and ρ be operators acting on a Hilbert space H such that Q is symmetric, X is selfadjoint, ρ is unitary selfadjoint, $\rho Q = -Q\rho$ and $\rho X = -X\rho$. Assume that the restrictions of Q and X to $H(\rho = -1)$ coincide:*

$$Q|_{H(\rho=-1)} = X|_{H(\rho=-1)}. \tag{1.5}$$

Then $X = [Q]_\rho$.

Proof. Let $H_1 = H(\rho = -1)$ and $H_2 = H(\rho = 1)$. Then $H = H_1 \oplus H_2$ and (all) bounded and (some) unbounded operators may be represented by 2×2 matrices. In particular

$$\rho = \begin{pmatrix} -I \ , & 0 \\ 0 \ , & I \end{pmatrix}.$$

Remembering that Q and X anticommute with ρ we obtain:

$$Q = \begin{pmatrix} 0 \ , & Q_- \\ Q_+ \ , & 0 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} 0 \ , & X_- \\ X_+ \ , & 0 \end{pmatrix},$$

where Q_+ and X_+ are operators acting from H_1 to H_2 and Q_- and X_- are operators acting from H_2 to H_1. Clearly $Q_+ \subset Q_-^*$ (Q is symmetric) and $X_- = X_+^*$ (X is selfadjoint). Assumption (1.5) means that $Q_+ = X_+$. Therefore

$$X = \begin{pmatrix} 0 \ , & Q_+^* \\ Q_+ \ , & 0 \end{pmatrix}.$$

On the other hand

$$Q^* = \begin{pmatrix} 0 \ , & Q_+^* \\ Q_-^* \ , & 0 \end{pmatrix}.$$

It shows that $X \subset Q^*$ and $\mathcal{D}(X) = \{x \in \mathcal{D}(Q^*) : (\rho - I)x \in \mathcal{D}(Q)\}$. $\square$

Let $\hbar \in \mathbb{R}$. We shall use the Zakrzewski relation $\overset{\hbar}{\longrightarrow\!\!\circ}$ (cf. [19]). Let R and S be selfadjoint operators acting on a Hilbert space H with the polar decompositions $R = \operatorname{sign} R\,|R|$ and $S = \operatorname{sign} S\,|S|$. For simplicity we shall assume that one of the operators R and S has trivial kernel. If $\ker S = \{0\}$, then $\operatorname{sign} S$ is unitary selfadjoint and

$$\left(R \overset{\hbar}{\longrightarrow\!\!\circ} S\right) \Longleftrightarrow \begin{pmatrix} \operatorname{sign} S \text{ commutes with } R \\ \text{and } |S|^{-i\lambda} R\,|S|^{i\lambda} = e^{\hbar\lambda} R \\ \text{for any } \lambda \in \mathbb{R}. \end{pmatrix}.$$

If $\ker R = \{0\}$, then $\operatorname{sign} R$ is unitary selfadjoint and

$$\left(R \overset{\hbar}{\longrightarrow\!\!\circ} S\right) \Longleftrightarrow \begin{pmatrix} \operatorname{sign} R \text{ commutes with } S \\ \text{and } |R|^{i\lambda} S\,|R|^{-i\lambda} = e^{\hbar\lambda} S \\ \text{for any } \lambda \in \mathbb{R}. \end{pmatrix}.$$

If $\ker R = \ker S = \{0\}$, then the two above conditions are equivalent.

Let R and S be selfadjoint operators with trivial kernels and $R \overset{\hbar}{\longrightarrow\!\!\circ} S$. It is known [19, Example 3.1] that in this case, the operators $e^{i\hbar/2} S^{-1} R$ and $e^{i\hbar/2} S R^{-1}$ are selfadjoint and

$$\operatorname{sign}\left(e^{i\hbar/2} S^{-1} R\right) = \operatorname{sign}\left(e^{i\hbar/2} S R^{-1}\right) = (\operatorname{sign} R)(\operatorname{sign} S).$$

We shall use the following result (cf. [19, Theorem 5.2]):

Proposition 1.2. *Let R, S and τ be operators acting on a Hilbert space H. Assume that R and S are selfadjoint with trivial kernels, $R \overset{\hbar}{\longrightarrow\!\circ}\, S$, and that τ is unitary, selfadjoint anticommuting with R and S. We set $T = e^{i\hbar/2}S^{-1}R$. Then T is a selfadjoint operator with trivial kernel, T commutes with τ, $R+S$ is a closed symmetric operator and the selfadjoint extension*

$$\begin{aligned}[R+S]_\tau &= G_\hbar(T,\tau)^* S G_\hbar(T,\tau)\\ &= G_\hbar(T^{-1},\tau) R G_\hbar(T^{-1},\tau)^*\end{aligned} \tag{1.6}$$

Remark 1.3. If τ' is another unitary, selfadjoint operator anticommuting with R and S and if in addition there exists a unitary selfadjoint operator ρ that commutes with τ, τ' and S and anticommutes with R then

$$\Big(\ [R+S]_\tau \ =\ [R+S]_{\tau'}\ \Big) \Longrightarrow \Big(\ \tau = \tau'\ \Big). \tag{1.7}$$

Indeed if $[R+S]_\tau \ =\ [R+S]_{\tau'}$, then (cf. (1.6))

$$G_\hbar(T,\tau)^* S G_\hbar(T,\tau) \ =\ G_\hbar(T,\tau')^* S G_\hbar(T,\tau').$$

It shows that the unitary operator $U = G_\hbar(T,\tau')G_\hbar(T,\tau)^*$ commutes with S and hence with $|S|$. Clearly

$$G_\hbar(T,\tau)^* = G_\hbar(T,\tau')^* U. \tag{1.8}$$

Moreover $T \overset{\hbar}{\longrightarrow\!\circ}\, S$ due to Zakrzewski relation $R \overset{\hbar}{\longrightarrow\!\circ}\, S$ and ρ anticommutes with T. As we know τ and τ' anticommute with S, hence they commute with $|S|$. We shall use Proposition 2.4 (see the next section). Setting $R_1 = R_2 = T$, $\rho_1 = \tau$, $\rho_2 = \tau'$, $U_1 = I$, $U_2 = U$ and replacing S by $|S|$ we have all the assumptions of that proposition satisfied. Therefore (1.8) implies the equality $\tau\,\mathrm{Np}(T) = \tau'\,\mathrm{Np}(T)$. It means that τ and τ' coincide on $H(T<0)$. Then τ and τ' coincide on $\rho H(T<0)$ for any operator ρ commuting with τ and τ'. If ρ commutes with S and anticommutes with R then it anticommutes with T and $\rho H(T<0) = H(T>0)$. In this case τ and τ' coincide on $H(T<0)\oplus H(T>0) = H$ (this is because $\ker T$ is trivial). Hence $\tau = \tau'$.

We shall prove a result of the same flavor as (1.6):

Proposition 1.4. *Let R and S be strictly positive selfadjoint operators acting on a Hilbert space H such that $R \overset{\hbar}{\longrightarrow\!\circ}\, S$ and τ, ξ and σ be unitary selfadjoint operators commuting with R and S. Assume that τ commutes with ξ and anticommutes with σ. We set: $T = e^{i\hbar/2}S^{-1}R$ and*

$$\rho = \overline{\alpha}\xi\sigma\chi(\tau=1) + \alpha\sigma\xi\chi(\tau=-1), \tag{1.9}$$

where $\alpha = i\,e^{\frac{i\pi^2}{2\hbar}}$. Then T is a positive selfadjoint operator with trivial kernel, ρ is a unitary selfadjoint operator, $\sigma S^{\frac{\pi}{\hbar}} + \rho R^{\frac{\pi}{\hbar}}$ is a closed symmetric operator anticommuting with τ and the selfadjoint extension

$$\left[\sigma S^{\frac{\pi}{\hbar}} + \rho R^{\frac{\pi}{\hbar}}\right]_{-\tau} = G_\hbar(\tau T,\xi)^* \sigma S^{\frac{\pi}{\hbar}} G_\hbar(\tau T,\xi) \tag{1.10}$$

Proof. Inserting S^{-1} instead of R and R instead of S in [19, Example 3.1] we see that T is a positive selfadjoint operator with trivial kernel and

$$T^{ik} = e^{-\frac{i\hbar}{2}k^2} S^{-ik} R^{ik} = e^{\frac{i\hbar}{2}k^2} R^{ik} S^{-ik} \tag{1.11}$$

for any $k \in \mathbb{R}$.

Denote by X the right hand side of (1.10). We know that $G_\hbar(\tau T,\xi)$ is unitary (in what follows we write $G_\hbar(\tau T,\xi)^{-1}$ instead of $G_\hbar(\tau T,\xi)^*$). Operator $S^{\frac{\pi}{\hbar}}$ commutes with σ and τ whereas σ and τ anticommute. Therefore $\sigma S^{\frac{\pi}{\hbar}}$ is a selfadjoint operator anticommuting with τ. So is X.

Remembering that τ anticommutes with $\xi\sigma$ one can easily verify that $\rho^* = \rho$ and $\rho^2 = I$. Furthermore ρ anticommutes with τ. Let $Q = \sigma S^{\frac{\pi}{\hbar}} + \rho R^{\frac{\pi}{\hbar}}$. Clearly Q is a symmetric operator anticommuting with τ. By virtue of Proposition 1.1 it is sufficient to show that

$$Q|_{H(\tau=1)} = X|_{H(\tau=1)}. \tag{1.12}$$

Restricting $G_\hbar(\tau T,\xi)^* \sigma S^{\frac{\pi}{\hbar}} G_\hbar(\tau T,\xi)$ to $H(\tau=1)$ we may replace the second τ by 1 and the first τ by -1 (this is because σ maps $H(\tau=1)$ onto $H(\tau=-1)$):

$$X|_{H(\tau=1)} = G_\hbar(-T,\xi)^{-1}\sigma S^{\frac{\pi}{\hbar}} G_\hbar(T,\xi)|_{H(\tau=1)}$$

and using (1.2) we obtain

$$X|_{H(\tau=1)} = \left[1 + i\xi T^{\frac{\pi}{\hbar}}\right]^{-1} V_\theta \left(\log T - \pi i\right)^{-1} \sigma S^{\frac{\pi}{\hbar}} V_\theta(\log T)|_{H(\tau=1)} \tag{1.13}$$

Now we shall move $\sigma S^{\frac{\pi}{\hbar}}$ to the right end of (1.13). It is known (cf. [19, relation (1.30)]) that the function $V_\theta(x)$ has no poles and no zeroes in strip $\Sigma = \{x \in \mathbb{C} : 0 \leq \Im x \leq \pi\}$. Therefore functions $V_\theta(x)$ and $V_\theta(x)^{-1}$ are continuous on Σ and holomorphic inside Σ. Moreover (cf. [19, the asymptotic formula (1.37)]), $V_\theta(x) \longrightarrow 1$ when $\Re x \longrightarrow -\infty$ whereas $\Im x$ stays bounded and using formula (1.32) of [19] one can easily show that for any $\lambda > 0$, functions $e^{-\lambda x^2} V_\theta(x)$ and $e^{-\lambda x^2} V_\theta(x)^{-1}$ are bounded on Σ. Furthermore T is a strictly positive selfadjoint operator and $T \overset{\hbar}{\multimap} S$. Therefore $T \overset{\pi}{\multimap} S^{\frac{\pi}{\hbar}}$ and using Statement (3) of Theorem 3.1 of [19] we obtain

$$S^{\frac{\pi}{\hbar}}\, V_\theta(\log T) = V_\theta(\log T + i\pi)\, S^{\frac{\pi}{\hbar}}.$$

Inserting this formula into (1.13) and using in the second step formula (1.28) of [19] we get:

$$\begin{aligned} X|_{H(\tau=1)} &= \left[1 + i\xi T^{\frac{\pi}{\hbar}}\right]^{-1} V_\theta\left(\log T - \pi i\right)^{-1} V_\theta(\log T + \pi i)\sigma S^{\frac{\pi}{\hbar}}|_{H(\tau=1)} \\ &= \left[1 + i\xi T^{\frac{\pi}{\hbar}}\right]^{-1} \left[1 + T^{\frac{2\pi}{\hbar}}\right] \sigma S^{\frac{\pi}{\hbar}}|_{H(\tau=1)} \\ &= \left[1 - i\xi T^{\frac{\pi}{\hbar}}\right] S^{\frac{\pi}{\hbar}}\sigma|_{H(\tau=1)}. \end{aligned}$$

On the other hand $\rho|_{H(\tau=1)} = \overline{\alpha}\xi\sigma|_{H(\tau=1)}$ and

$$Q|_{H(\tau=1)} = \left(S^{\frac{\pi}{\hbar}} + \overline{\alpha}\xi R^{\frac{\pi}{\hbar}}\right)\sigma|_{H(\tau=1)}.$$

To end the proof it is sufficient to show that

$$S^{\frac{\pi}{\hbar}} + \overline{\alpha}\xi R^{\frac{\pi}{\hbar}} = \left[1 - i\xi T^{\frac{\pi}{\hbar}}\right] S^{\frac{\pi}{\hbar}}. \tag{1.14}$$

We shall use (1.11). It shows that for any $x, y \in H$ and any $k \in \mathbb{R}$ we have

$$\left(y\middle|S^{ik}\, x\right) - i\, e^{\frac{i\hbar}{2}k^2}\left(y\middle|\xi R^{ik} x\right) = \left(\left(I + i\xi T^{-i\overline{k}}\right) y\middle|S^{ik} x\right)$$

Let $x \in \mathcal{D}\left(S^{\frac{\pi}{\hbar}}\right) \cap \mathcal{D}\left(R^{\frac{\pi}{\hbar}}\right)$. If $y \in \mathcal{D}(T^{\frac{\pi}{\hbar}})$ then both sides of the above formula have continuous holomorphic continuation to the strip $-\frac{\pi}{\hbar} \leq \Im k \leq 0$. Inserting $k = -i\frac{\pi}{\hbar}$ we obtain

$$\left(y\middle|S^{\frac{\pi}{\hbar}}\, x\right) + \overline{\alpha}\left(y\middle|\xi R^{\frac{\pi}{\hbar}} x\right) = \left(\left(I + i\xi T^{\frac{\pi}{\hbar}}\right) y\middle|S^{\frac{\pi}{\hbar}} x\right).$$

This formula holds for any y in the domain of $I + i\xi T^{\frac{\pi}{\hbar}}$. Therefore $S^{\frac{\pi}{\hbar}} x \in \mathcal{D}\left(I - i\xi T^{\frac{\pi}{\hbar}}\right)$ and $S^{\frac{\pi}{\hbar}}\, x + \overline{\alpha}\xi R^{\frac{\pi}{\hbar}} x = \left(I - i\xi T^{\frac{\pi}{\hbar}}\right) S^{\frac{\pi}{\hbar}} x$. This way we showed that

$$S^{\frac{\pi}{\hbar}} + \overline{\alpha}\xi R^{\frac{\pi}{\hbar}} \subset \left(I - i\xi T^{\frac{\pi}{\hbar}}\right) S^{\frac{\pi}{\hbar}}. \tag{1.15}$$

To prove the converse inclusion we use again (1.11). Let $x \in \mathcal{D}\left(S^{\frac{\pi}{\hbar}}\right)$ and $S^{\frac{\pi}{\hbar}} x \in \mathcal{D}\left(T^{\frac{\pi}{\hbar}}\right)$. Then for any $y \in H$ and $k \in \mathbb{R}$:

$$e^{-\frac{i\hbar}{2}k^2}\left(R^{-i\overline{k}} y\middle|S^{\frac{\pi}{\hbar} - ik} x\right) = \left(y\middle|T^{ik} S^{\frac{\pi}{\hbar}} x\right).$$

If $y \in \mathcal{D}(R^{\frac{\pi}{\hbar}})$ then both sides of the above formula have continuous holomorphic continuation to the strip $-\frac{\pi}{\hbar} \leq \Im k \leq 0$. Inserting $k = -i\frac{\pi}{\hbar}$ we obtain

$$i\overline{\alpha}\left(R^{\frac{\pi}{\hbar}} y\middle|x\right) = \left(y\middle|T^{\frac{\pi}{\hbar}} S^{\frac{\pi}{\hbar}} x\right).$$

This formula holds for any $y \in \mathcal{D}(R^{\frac{\pi}{\hbar}})$. Therefore $x \in \mathcal{D}(R^{\frac{\pi}{\hbar}})$. This way we showed the inclusion $\mathcal{D}(T^{\frac{\pi}{\hbar}} S^{\frac{\pi}{\hbar}}) \subset \mathcal{D}(R^{\frac{\pi}{\hbar}})$. Consequently $\mathcal{D}\left(\left(I - i\xi T^{\frac{\pi}{\hbar}}\right) S^{\frac{\pi}{\hbar}}\right) \subset \mathcal{D}\left(S^{\frac{\pi}{\hbar}} + \overline{\alpha}\xi R^{\frac{\pi}{\hbar}}\right)$. Combining this result with (1.15) we get (1.14), (1.12) and finally (1.10).

Operator $\xi T^{\frac{\pi}{\hbar}}$ is selfadjoint. Therefore operator $I - i\xi T^{\frac{\pi}{\hbar}}$ is invertible with the inverse $\left(I - i\xi T^{\frac{\pi}{\hbar}}\right)^{-1} \in B(H)$. Using this fact one can easily show that the composition $\left(I - i\xi T^{\frac{\pi}{\hbar}}\right) S^{\frac{\pi}{\hbar}}\sigma$ is a closed operator. Restricting this operator to $H(\tau = 1)$ we obtain $Q|_{H(\tau=1)}$. Hence $Q|_{H(\tau=1)}$ is closed. Remembering that Q anticommute with τ we conclude that Q is closed. □

We end this section with the reformulation of Theorem 6.1 of [19].

Theorem 1.5. *Let (R, S) be a pair of selfadjoint operators acting on a Hilbert space H such that $\ker R = \ker S = \{0\}$ and $R \overset{\hbar}{—\!\!\circ} S$ and ρ, σ be unitary selfadjoint operators on H. Assume that ρ commutes with R, ρ anticommutes with S, σ commutes with S and σ anticommutes with R. We set:*

$$T = e^{i\hbar/2} S^{-1} R,$$
$$\tau = \alpha\rho\sigma\chi(S < 0) + \overline{\alpha}\sigma\rho\chi(S > 0),$$

where $\alpha = i\, e^{\frac{i\pi^2}{2\hbar}}$. Then

1. *T is selfadjoint, $\operatorname{sign} T = (\operatorname{sign} R)(\operatorname{sign} S)$, $T \overset{\hbar}{—\!\!\circ} R$ and $T \overset{\hbar}{—\!\!\circ} S$*
2. *τ is unitary selfadjoint, τ commutes with T and τ anticommutes with R and S.*
3. *$G_\hbar$ satisfies the following exponential function equality:*

$$\begin{aligned} G_\hbar(R,\rho)G_\hbar(S,\sigma) &= G_\hbar(T,\tau)^* G_\hbar(S,\sigma) G_\hbar(T,\tau) \\ &= G_\hbar\left([R+S]_\tau, \widetilde{\sigma}\right), \end{aligned} \tag{1.16}$$

where $[R+S]_\tau$ is the selfadjoint extension of $R+S$ corresponding to the reflection operator τ and $\widetilde{\sigma} = G_\hbar(T,\tau)^* \sigma G_\hbar(T,\tau)$.

Proof. By direct computation one can easily show that $\tau^2 = I$, $\tau^* = \tau$ and

$$\tau\chi(T<0) = \alpha\rho\chi(R<0)\sigma\chi(S<0) + \overline{\alpha}\sigma\chi(S<0)\rho\chi(R<0).$$

Now, our theorem follows immediately from [19, Theorem 6.1]. □

Remark 1.6. In Theorem 1.5, operator τ may be replaced by $\tau' = \alpha\rho\sigma\chi(R>0) + \overline{\alpha}\sigma\rho\chi(R<0)$. Operator $\widetilde{\sigma}$ is not affected by this change.

Indeed, using the formula $\operatorname{sign} T = \operatorname{sign} R \operatorname{sign} S$, one can verify that $\tau'\chi(T<0) = \tau\chi(T<0)$. It shows that $G_\hbar(T,\tau') = G_\hbar(T,\tau)$.

2. The special functions and affiliation relation.

In this section we shall use the concept of a C*-algebra generated by a set of affiliated elements [15, Definition 4.1, page 501]. Let C, A be C*-algebras and V be an element affiliated with $C \otimes A$. We say that A is generated by an element $V \,\eta\, (C \otimes A)$ if and only if for any $\pi \in \operatorname{Rep}(A)$ and any $B \in C^*(H_\pi)$ we have:

$$\Big((\mathrm{id}\otimes\pi)V \,\eta\, (C\otimes B)\Big) \Longrightarrow \Big(\pi \in \operatorname{Mor}(A,B)\Big) \tag{2.1}$$

In general the above condition is not easy to verify. We shall use the following criterion (cf. [15, Example 10, page 507]):

Proposition 2.1. *Let C, A be C*-algebras and V be a unitary element of $\mathrm{M}(C \otimes A)$. Assume that there exists a faithful representation ϕ of C such that:*

1. *For any ϕ-normal linear functional ω on C we have: $(\omega \otimes \mathrm{id})V \in A$*
2. *The smallest *-subalgebra of A containing $\{(\omega\otimes\mathrm{id})V : \omega$ is ϕ-normal$\}$ is dense in A.*

Then A is generated by $V \in \mathrm{M}(C \otimes A)$.

We recall that a linear functional ω on C is said to be ϕ-normal if there exists a trace-class operator ρ acting on H_ϕ such that $\omega(c) = \operatorname{Tr}(\rho\phi(c))$ for all $c \in C$.

Let Λ be the locally compact space obtained from $\mathbb{R} \times \{-1, 1\}$ by gluing points $(r, -1)$ and $(r, 1)$ for all $r \geq 0$. Then:

$$C_\infty(\Lambda) = \left\{ f \in C_\infty\Big(\mathbb{R}\times\{-1,1\}\Big) : \begin{array}{c} f(r,-1) = f(r,1) \\ \text{for all } r \geq 0 \end{array} \right\}.$$

If R, ρ are operators acting on a Hilbert space H, R is selfadjoint, ρ is unitary selfadjoint and ρ commutes with R then the mapping

$$C_\infty(\Lambda) \ni f \longmapsto \pi(f) = f(R,\rho) \in B(H) \tag{2.2}$$

is a representation of $C_\infty(\Lambda)$ acting on H. Operators R and $\rho\,\mathrm{Np}(R)$ are determined by π. Indeed $R=\pi(f_1)$ and $\rho\,\mathrm{Np}(R)=\pi(f_2)$, where f_1,f_2 are elements of $C_\infty(\Lambda)^\eta=C(\Lambda)$ introduced by the formulae

$$f_1(r,\varrho)=r,\qquad f_2(r,\varrho)=\varrho\,\mathrm{Np}(r) \tag{2.3}$$

for any $r\in\mathbb{R}$ and $\varrho=\pm1$. Using [15, Example 2, page 497] we see that f_1,f_2 generate $C_\infty(\Lambda)$. Therefore for any $\pi\in\mathrm{Rep}\,(C_\infty(\Lambda))$ and any $B\in C^*(H_\pi)$ we have:

$$\Big(\pi(f_1),\pi(f_2)\,\eta\,B\Big)\Longrightarrow\Big(\pi\in\mathrm{Mor}(C_\infty(\Lambda),B)\Big)\Longrightarrow\Big(\pi(f)\,\eta\,B\text{ for any }f\in C(\Lambda)\Big)$$

In particular for π introduced by (2.2) we obtain the following result:

$$\left(\begin{array}{c}R,\rho\,\mathrm{Np}(R)\,\eta\,B\\ f\in C(\Lambda)\end{array}\right)\Longrightarrow\Big(f(R,\rho)\,\eta\,B\Big). \tag{2.4}$$

Our special function $G_\hbar$ is continuous and satisfies the relation $G_\hbar(r,-1)=G_\hbar(r,1)$ for all $r\geq0$. In other words $G_\hbar\in C(\Lambda)$. For any $r\in\mathbb{R}$, $\varrho=\pm1$ and $t>0$ we set:

$$F(t;r,\varrho)=\overline{G_\hbar(r,\varrho)}G_\hbar(tr,\varrho). \tag{2.5}$$

Let $\mathbb{R}_+=\{t\in\mathbb{R}:t>0\}$. Then F is a continuous function on $\mathbb{R}_+\times\Lambda$ with values of modulus 1 and we may treat F as unitary element of $\mathrm{M}\,(C_\infty(\mathbb{R}_+)\otimes C_\infty(\Lambda))$. We shall prove the following

Proposition 2.2. *The* C**-algebra* $C_\infty(\Lambda)$ *is generated by* $F\in\mathrm{M}\,(C_\infty(\mathbb{R}_+)\otimes C_\infty(\Lambda))$.

Proof. We shall use Proposition 2.1 with $C=C_\infty(\mathbb{R}_+)$, $A=C_\infty(\Lambda)$ and $V=F$. Let ϕ be the natural representation of $C_\infty(\mathbb{R}_+)$ acting on $L^2(\mathbb{R}_+)$. For any $g\in C_\infty(\mathbb{R}_+)$, $\phi(g)$ is the multiplication by g. Then ϕ is faithful and a linear functional ω on $C_\infty(\mathbb{R}_+)$ is ϕ-normal if and only if it is of the form

$$\omega(g)=\int_{\mathbb{R}_+}g(t)\varphi(t)\,dt,$$

where $\varphi\in L^1(\mathbb{R}_+)$. Applying $\omega\otimes\mathrm{id}$ to $F\in\mathrm{M}\,(C_\infty(\mathbb{R}_+)\otimes C_\infty(\Lambda))$ we obtain an element of $\mathrm{M}\,(C_\infty(\Lambda))$ i.e. a bounded continuous function on Λ. Clearly for any $r\in\mathbb{R}$ and $\varrho=\pm1$ we have

$$\begin{aligned}(\omega\otimes\mathrm{id})F(r,\varrho)&=\int_{\mathbb{R}_+}F(t;r,\varrho)\varphi(t)\,dt\\&=\overline{G_\hbar(r,\varrho)}\int_{\mathbb{R}_+}G_\hbar(tr,\varrho)\varphi(t)\,dt.\end{aligned} \tag{2.6}$$

Taking into account the asymptotic behavior (1.4) and using the Riemann–Lebesgue lemma one can verify that the integral on right hand side tends to 0 when $r\to\pm\infty$. In other words, $(\omega\otimes\mathrm{id})F\in C_\infty(\Lambda)$.

Using Statement 7 of Theorem 1.1 of [19] one can easily show that

$$\lim_{t\to0^+}\frac{1}{t}\Big[G_\hbar(tr,\varrho)-1\Big]=\frac{r}{2i\sin(\hbar/2)} \tag{2.7}$$

for all $r\in\mathbb{R}$ and $\varrho=\pm1$.

Let $r,r'\in\mathbb{R}$ and $\varrho,\varrho'=\pm1$. Assume for the moment that $(\omega\otimes\mathrm{id})F(r,\varrho)=(\omega\otimes\mathrm{id})F(r',\varrho')$ for all ϕ-normal functionals ω. Then $\overline{G_\hbar(r,\varrho)}G_\hbar(tr,\varrho)=\overline{G_\hbar(r',\varrho')}G_\hbar(tr',\varrho')$ for all $t>0$. Going to the limit when $t\to+0$ we get $\overline{G_\hbar(r,\varrho)}=\overline{G_\hbar(r',\varrho')}$. Comparing this formula with the previous one we see that $G_\hbar(tr,\varrho)=G_\hbar(tr',\varrho')$ for all $t>0$. Formula (2.7) shows now that $r=r'$ and by (1.3) $\varrho\chi(r<0)=\varrho'\chi(r<0)$. This way we have shown that the functions (2.6) separate points of Λ. Now, using the Stone - Weierstrass theorem (applied to the one point compactification of Λ) we conclude that the smallest *-algebra containing all functions (2.6) is dense in $C_\infty(\Lambda)$. ☐

The following Proposition will be very useful in proving many technical details important in the future considerations.

Proposition 2.3. *Let* R, ρ, U, S *be operators acting on a Hilbert space* H *and* $C\in C^*(H)$. *Assume that:*

1. *R is selfadjoint and ρ is unitary selfadjoint commuting with R,*
2. *U is unitary,*

3. S is positive selfadjoint, $\ker S = \{0\}$*, S commutes with ρ and U and $R \overset{\hbar}{\longrightarrow\!\circ} S$,*

4. Operators R, $\rho\,\mathrm{Np}(R)$, U and $\log S$ are affiliated with C.

Then $G_\hbar(R,\rho) \in \mathrm{M}(C)$ and

1. For any $\phi \in \mathrm{Rep}(C)$ and any $B \in C^*(H_\phi)$ we have:

$$\begin{pmatrix} \phi(\log S),\ \phi\left(G_\hbar(R,\rho)^* U\right) \\ \textit{are affiliated with } B \end{pmatrix} \Longrightarrow \begin{pmatrix} \phi(R),\ \phi(\rho\,\mathrm{Np}(R)),\ \phi(U) \\ \textit{are affiliated with } B \end{pmatrix}$$

2. For any $\phi_1, \phi_2 \in \mathrm{Rep}(C)$ such that $H_{\phi_1} = H_{\phi_2}$ we have:

$$\begin{pmatrix} \phi_1(S) = \phi_2(S), \\ \phi_1\left(G_\hbar(R,\rho)^* U\right) = \phi_2\left(G_\hbar(R,\rho)^* U\right) \end{pmatrix} \Longrightarrow \begin{pmatrix} \phi_1(R) = \phi_2(R), \\ \phi_1(\rho\,\mathrm{Np}(R)) = \phi_2(\rho\,\mathrm{Np}(R)), \\ \phi_1(U) = \phi_2(U) \end{pmatrix}$$

Proof. Relation $G_\hbar(R,\rho) \in \mathrm{M}(C)$ follows immediately from (2.4).

Ad 1. Let $\lambda \in \mathbb{R}$. Using the commutation relations satisfied by operators R, ρ, U, S we have:

$$S^{-i\lambda} G_\hbar(R,\rho)^* U S^{i\lambda} = G_\hbar(tR,\rho)^* U,$$

where $t = e^{\hbar\lambda} > 0$. Applying a representation ϕ of C to both sides of the above relation we get

$$\phi(S)^{-i\lambda} \phi\left(G_\hbar(R,\rho)^* U\right) \phi(S)^{i\lambda} = \phi\left(G_\hbar(tR,\rho)^* U\right).$$

If $\phi(\log S)$, $\phi\left(G_\hbar(R,\rho)^* U\right) \eta\, B$, then all factors on the left hand side of the above equation belong to $\mathrm{M}(B)$ and depend continuously on λ (we use strict topology on $\mathrm{M}(B)$). Therefore $\phi\left(G_\hbar(tR,\rho)^* U\right) \in \mathrm{M}(B)$ for any $t \in \mathbb{R}_+$ and the mapping

$$\mathbb{R}_+ \ni t \longmapsto \phi\left(G_\hbar(tR,\rho)^* U\right) \in \mathrm{M}(B)$$

is strictly continuous. Applying the hermitian conjugation and multiplying from the left by $\phi\left(G_\hbar(R,\rho)^* U\right) \in \mathrm{M}(B)$ we see that $\phi\left(G_\hbar(R,\rho)^* G_\hbar(tR,\rho)\right) = \phi\left(F(t;R,\rho)\right) \in \mathrm{M}(B)$ and the mapping

$$\mathbb{R}_+ \ni t \longmapsto \phi\left(F(t;R,\rho)\right) \in \mathrm{M}(B) \tag{2.8}$$

is strictly continuous. In the above relations F is the function introduced by (2.5). According to the general theory [15], strictly continuous bounded mappings from $\mathbb{R}_+$ into $\mathrm{M}(B)$ correspond to elements of $\mathrm{M}(C_\infty(\mathbb{R}_+) \otimes B)$. A moment of reflection shows that the mapping (2.8) corresponds to the element $(\mathrm{id} \otimes \phi\circ\pi)F$, where π is the representation of $C_\infty(\Lambda)$ introduced by (2.2).

This way we have shown that $(\mathrm{id} \otimes \phi\circ\pi)F \in \mathrm{M}(C_\infty(\mathbb{R}_+) \otimes B)$. Using now Proposition 2.2 we conclude that $\phi\circ\pi \in \mathrm{Mor}(C_\infty(\Lambda), B)$. Therefore $\phi\circ\pi$ maps continuous functions on Λ into elements affiliated with B. Applying this rule to functions f_1, f_2 (cf. (2.3)) and $G_\hbar$ we obtain: $\phi(R), \phi(\rho\,\mathrm{Np}(R))\,\eta\, B$ and $\phi(G_\hbar(R,\rho)) \in \mathrm{M}(B)$. Comparing the last relation with the assumed one $\phi\left(G_\hbar(R,\rho)^* U\right) \in \mathrm{M}(B)$ we see that $\phi(U) \in \mathrm{M}(B)$. Statement 1 is shown.

Ad 2. Let $\phi = \phi_1 \oplus \phi_2$. Then $H_\phi = H_{\phi_1} \oplus H_{\phi_2}$ and $\phi(c) = \phi_1(c) \oplus \phi_2(c)$. In our case $H_{\phi_1} = H_{\phi_2}$. We set: $B = \{m \oplus m : m \in \mathcal{K}(H_{\phi_1})\}$. Then $B \in C^*(H_\phi)$. One can easily verify that for any $c\,\eta\, C$ we have:

$$\Big(\phi(c)\,\eta\, B\Big) \Longleftrightarrow \Big(\phi_1(c) = \phi_2(c)\Big).$$

Now Statement 2 follows immediately from Statement 1. □

We shall use slightly different version of Statement 2 of the above proposition.

Proposition 2.4. *Let R_1, ρ_1, U_1, R_2, ρ_2, U_2, S be operators acting on a Hilbert space H. Assume that for each $k = 1,2$ the operators R_k, ρ_k, U_k, S satisfy the assumptions 1-3 of the previous Proposition. Then*

$$\Big(G_\hbar(R_1,\rho_1)^* U_1 = G_\hbar(R_2,\rho_2)^* U_2\Big) \Longrightarrow \begin{pmatrix} R_1 = R_2, \\ \rho_1\,\mathrm{Np}(R_1) = \rho_2\,\mathrm{Np}(R_2), \\ U_1 = U_2. \end{pmatrix} \tag{2.9}$$

Proof. Let $C = \mathcal{K}(H) \oplus \mathcal{K}(H)$ and for any $m_1, m_2 \in \mathcal{K}(H)$ we set $\phi_k(m_1 \oplus m_2) = m_k$ $(k = 1, 2)$. We use Proposition 2.3 with R, ρ, U and S replaced by $R_1 \oplus R_2$, $\rho_1 \oplus \rho_2$, $U_1 \oplus U_2$ and $S \oplus S$, Now (2.9) follows immediately from Statement 2 of Proposition 2.3. □

Proposition 2.5. *Let X and Y be selfadjoint operators acting on Hilbert spaces K and H respectively. Assume that the spectral measure of X is absolutely continuous with respect to the Lebesgue measure. Then for any $A \in C^*(H)$ we have:*

$$\left(\begin{array}{c} e^{iX\otimes Y} \text{ is affiliated} \\ \text{with } \mathcal{K}(K) \otimes A \end{array}\right) \Longrightarrow \Big(Y \text{ is affiliated with } A\Big)$$

Proof. For any normal linear functional ω on $B(K)$ and $t \in \mathbb{R}$ we set

$$f_\omega(t) = \omega\left(e^{itX}\right).$$

Then f_ω is a continuous function on $\mathbb{R}$. Remembering that the spectral measure of X is absolutely continuous with respect to the Lebesgue measure and using the Riemann-Lebesgue lemma one can easily show that $f_\omega(t) \to 0$ when $t \to \pm\infty$. Therefore $f_\omega \in C_\infty(\mathbb{R})$.

Let $t, t' \in \mathbb{R}$, $t \neq t'$. Assume for the moment that $f_\omega(t) = f_\omega(t')$ for all ω. Then $e^{itX} = e^{it'X}$ and $e^{i(t-t')X} = I$. It shows that the spectral measure of X is supported by the set $\frac{2\pi}{t-t'}\mathbb{Z}$, what is in contradiction with the assumption saying that the spectral measure of X is absolutely continuous with respect to the Lebesgue measure. This way we showed that functions f_ω separate points of $\mathbb{R}$. By the Stone − Weierstrass theorem, the smallest *-subalgebra of $C_\infty(\mathbb{R})$ containing all f_ω is dense in $C_\infty(\mathbb{R})$.

By the general theory strongly continuous mappings from $\mathbb{R}$ into the set of unitary operators acting on K correspond to unitary multipliers of $\mathcal{K}(K) \otimes C_\infty(\mathbb{R})$. Let $\mathfrak{X} \in \mathrm{M}(\mathcal{K}(K) \otimes C_\infty(\mathbb{R}))$ be the unitary corresponding to the mapping

$$\mathbb{R} \ni t \longmapsto e^{itX} \in B(K).$$

Then for any normal linear functional ω on $B(K)$ we have

$$(\omega \otimes \mathrm{id})\mathfrak{X} = f_\omega.$$

Using Proposition 2.1 we see that $C_\infty(\mathbb{R})$ is generated by $\mathfrak{X} \in \mathrm{M}(\mathcal{K}(K) \otimes C_\infty(\mathbb{R}))$. For any $f \in C_\infty(\mathbb{R})$ we set:

$$\pi(f) = f(Y).$$

Then π is a representation of $C_\infty(\mathbb{R})$ acting on the Hilbert space $H_\pi = H$. A moment of reflection shows that $(\mathrm{id} \otimes \pi)\mathfrak{X} = e^{iX\otimes Y}$. If $e^{iX\otimes Y}$ is affiliated with $\mathcal{K}(K) \otimes A$ then $\pi \in \mathrm{Mor}(C_\infty(\mathbb{R}), A)$ and π maps continuous functions on $\mathbb{R}$ into elements affiliated with A. Applying this rule to the coordinate function $f(t) = t$ we obtain $Y = \pi(f)\,\eta\, A$. □

3. Constructions related to old quantum '$ax + b$' group.

In this section we recall the main results of [21]. The quantum '$ax+b$' group will be presented as a quantum group of unitary operators. We shall construct a pair (A, V), where A is a C*-algebra and V is a unitary element of $\mathrm{M}(\mathcal{K}(K) \otimes A)$, where K is a Hilbert space endowed with a certain structure and $\mathcal{K}(K)$ denotes the algebra of all compact operators acting on K. (A, V) may be treated as a quantum family of unitary operators acting on K 'labeled by elements' of quantum space related to the C*-algebra A. Our construction will depend on a real parameter $\hbar$. We shall assume that $0 < \hbar < \pi/2$. Negative value of $\hbar$ leads to the C*-algebra anti-isomorphic to that with positive $\hbar$. On the other hand the restriction $\hbar < \pi/2$ is related to the technical assumption used in the theory of the quantum exponential function [19].

The main result of this section is contained in Theorem 3.2. It states that (A, V) is a quantum group if and only if $\hbar = \frac{\pi}{2k+3}$ with $k = 0, 1, 2, \ldots$.

To define A we consider three operators a, b and β acting on the Hilbert space $L^2(\mathbb{R})$. Operator a is strictly positive selfadjoint and such that for any $\tau \in \mathbb{R}$ and any $x \in L^2(\mathbb{R})$ we have

$$\left(a^{i\tau}x\right)(t) = e^{\hbar\tau/2}x(e^{\hbar\tau}t)$$

In other words a is the analytic generator of one-parameter group of unitaries corresponding to the homotheties of $\mathbb{R}$. Operator b is the multiplication operator:

$$(bx)(t) = tx(t).$$

By definition domain $\mathcal{D}(b)$ consists of all $x \in L^2(\mathbb{R})$ such that the right hand side of the above equation is square integrable. Finally, β is the reflection: for any $x \in L^2(\mathbb{R})$ we have:

$$(\beta x)(t) = x(-t)$$

Clearly β is unitary selfadjoint. One can easily verify that $a\beta = \beta a$ and $b\beta = -\beta b$. By the last relation $ib\beta$ is selfadjoint. Moreover

$$a^{i\tau} b a^{-i\tau} = e^{\hbar\tau} b \tag{3.1}$$

for any $\tau \in \mathbb{R}$. This relation means that $a \overset{\hbar}{\multimap} b$.

Theorem 3.1. *Let*

$$A = \left\{ \Big(f_1(b) + \beta f_2(b)\Big)\, g(\log a) : \begin{array}{c} f_1, f_2, g \in C_\infty(\mathbb{R}) \\ f_2(0) = 0 \end{array} \right\}^{\begin{array}{c}\text{norm closed}\\ \text{linear envelope}\end{array}} . \tag{3.2}$$

Then: 1. *A is a nondegenerate C^*-algebra of operators acting on $L^2(\mathbb{R})$,*

2. *$\log a$, b and $ib\beta$ are affiliated with A: $\log a,\ b,\ ib\beta\, \eta\, A$,*

3. *$\log a$, b and $ib\beta$ generate A.*

Proof.

Ad 1. Using the relation $b\beta = -\beta b$ one can easily show that

$$B = \left\{ f_1(b) + \beta f_2(b) : \begin{array}{c} f_1, f_2 \in C_\infty(\mathbb{R}) \\ f_2(0) = 0 \end{array} \right\}^{\begin{array}{c}\text{norm closed}\\ \text{linear envelope}\end{array}} \tag{3.3}$$

is a non-degenerate C*-algebra of operators acting on $L^2(\mathbb{R})$. Let $C_0(\mathbb{R}, B)$ denote the set of all continuous mappings from $\mathbb{R}$ into B with compact support. Then

$$A = \left\{ \int_{\mathbb{R}} f(t) a^{it} dt : f \in C_0(\mathbb{R}, B) \right\}^{\text{norm closure}} . \tag{3.4}$$

To prove this formula it is sufficient to notice that for $f(t) = \big(f_1(b) + \beta f_2(b)\big)\, \varphi(t)$, where $t \in \mathbb{R}$ and $\varphi \in C_0(\mathbb{R})$ we have

$$\int_{\mathbb{R}} f(t) a^{it} dt = \big(f_1(b) + \beta f_2(b)\big)\, g(\log a),$$

where $g(\lambda) = \int_{\mathbb{R}} \varphi(t) e^{i\lambda t} dt$ ($\lambda \in \mathbb{R}$) and by the Riemann-Lebesque Lemma, $g \in C_\infty(\mathbb{R})$. On the other hand (3.1) shows that the unitaries a^{it} ($t \in \mathbb{R}$) implement a one parameter group of automorphisms of B. Using now the standard technique of the theory of crossed products (cf. [7, Section 7.6]) one can easily show that (3.4) is a non-degenerate C*-algebra of operators acting on $L^2(\mathbb{R})$. Statement 1 is proven.

Ad 2. We recall (cf. [6, 15]) that a closed operator T is affiliated with a C*-algebra A if the z-transform $z_T = T(I + T^*T)^{-\frac{1}{2}} \in \mathrm{M}(A)$ and if $(I + T^*T)^{-\frac{1}{2}} A$ is dense in A. Inspecting definition (3.2) one can easily show that $z_{\log a} = (\log a)\left[I + (\log a)^2\right]^{-\frac{1}{2}}$ is a right multiplier of A and that $A\left[I + (\log a)^2\right]^{-\frac{1}{2}}$ is dense in A. Passing to adjoint operators we see that $z^*_{\log a} = z_{\log a}$ is a left multiplier (hence $z_{\log a} \in \mathrm{M}(A)$) and that $\left[I + (\log a)^2\right]^{-\frac{1}{2}} A$ is dense in A. It shows that $\log a$ is affiliated with A.

For $T = b$ and $T = i\beta b$ we have $z_T = b(I + b^2)^{-\frac{1}{2}}$ and $z_T = i\beta b(I + b^2)^{-\frac{1}{2}}$ respectively. In both cases $(I + T^*T)^{-\frac{1}{2}} = (I + b^2)^{-\frac{1}{2}}$. Taking into account definition (3.2) one can easily show that $(I + T^*T)^{-\frac{1}{2}} A$ is dense in A and that z_T is a left multiplier of A. However in both cases z_T is selfadjoint. Therefore z_T is also a right multiplier and $z_T \in \mathrm{M}(A)$. It shows that b and $i\beta b$ are affiliated with A.

Ad 3. We shall use Theorem 3.3 of [15]. By definition (3.2), $(I + b^2)^{-1}\left(I + (\log a)^2\right)^{-1} \in A$. To end the proof it is sufficient to show that $a, b, i\beta b$ separate representations of A. If $c \in A$ is of the form

$$c = \big(f_1(b) + \beta f_2(b)\big)\, g(\log a), \tag{3.5}$$

where $f_1, f_2, g \in C_\infty(\mathbb{R})$, $f_2(0) = 0$ and f_2 is differentiable at point $0 \in \mathbb{R}$, then $f_2(t) = ith(t)$, where $t \in \mathbb{R}$ and $h \in C_\infty(\mathbb{R})$ and

$$\pi(c) = \big(f_1(\pi(b)) + \pi(i\beta b)h(\pi(b))\big)\, g(\pi(\log a)) \tag{3.6}$$

for any representation π of A. One can easily see that elements of the form (3.5) form a dense subset of A. Formula (3.6) shows now that π is determined uniquely by $\pi(\log a)$, $\pi(b)$ and $\pi(i\beta b)$. $\square$

Now we pass to the description of the Hilbert space K (cf. the first paragraph of this Section). The structure of K is determined by a triple of selfadjoint operators $(\widehat{a}, \widehat{b}, \widehat{\beta})$ acting on K and having the following properties:

1. $\widehat{a} > 0$, $\ker \widehat{a} = \ker \widehat{b} = \{0\}$ and $\widehat{a} \overset{\hbar}{-\!\!\circ}\, \widehat{b}$,
2. $\widehat{\beta}$ is a unitary involution, $\widehat{\beta}$ commutes with $\widehat{a}$ and anticommutes with $\widehat{b}$.

One of the possible choices is: $K = L^2(\mathbb{R})$ and $(\widehat{a}, \widehat{b}, \widehat{\beta}) = (a, b, \beta)$. However there is another possibility that is even more interesting:

$$(\widehat{a}, \widehat{b}, \widehat{\beta}) = (|b|^{-1}, e^{i\hbar/2} b^{-1} a, \alpha\beta), \tag{3.7}$$

where $\alpha = \pm 1$. The reader easily verifies that these operators possess required properties.

Zakrzewski relation $\widehat{a} \overset{\hbar}{-\!\!\circ}\, \widehat{b}$ implies that the spectral measures of $\widehat{a}$ and $\widehat{b}$ are absolutely continuous with respect to the Lebesgue measure. Moreover $\mathrm{Sp}(\widehat{a}) = \mathbb{R}_+$ and $\mathrm{Sp}(\widehat{b}) = \mathbb{R}$. The latter follows from the relation $\widehat{\beta}\widehat{b} = -\widehat{b}\widehat{\beta}$.

Let

$$V = G_\hbar(\widehat{b} \otimes b,\, \widehat{\beta} \otimes \beta)^*\, e^{\frac{i}{\hbar} \log \widehat{a} \otimes \log a}. \tag{3.8}$$

This is the basic object considered in this section. We shall prove

Theorem 3.2.

1. *V is a unitary operator and $V \in \mathrm{M}(\mathcal{K}(K) \otimes A)$,*
2. *A is generated by $V \in \mathrm{M}(\mathcal{K}(K) \otimes A)$.*

Proof. Let $R = \widehat{b} \otimes b$, $\rho = \widehat{\beta} \otimes \beta$, $U = e^{\frac{i}{\hbar} \log \widehat{a} \otimes \log a}$, $S = \widehat{a}^{-1} \otimes I$ and $C = \mathcal{K}(K) \otimes A$. Then all the assumptions of Proposition 2.3 are satisfied. Clearly $V = G_\hbar(R, \rho)^* U \in \mathrm{M}(C)$ and Statement 1 is proved.

Let $\pi \in \mathrm{Rep}(A)$ and $B \in C^*(H_\pi)$. Then $\mathrm{id} \otimes \pi$ is a representation of C acting on $K \otimes H_\pi$. The reader should notice that $(\mathrm{id} \otimes \pi)S = \widehat{a}^{-1} \otimes I$ is affiliated with $\mathcal{K}(K) \otimes B$. Assume that $(\mathrm{id} \otimes \pi)V \in \mathrm{M}(\mathcal{K}(K) \otimes B)$. By Statement 1 of Proposition 2.3, operators: $(\mathrm{id} \otimes \pi)R = \widehat{b} \otimes \pi(b)$, $(\mathrm{id} \otimes \pi)(\rho\, \mathrm{Np}(R))$ and $(\mathrm{id} \otimes \pi)U = e^{\frac{i}{\hbar} \log \widehat{a} \otimes \pi(\log a)}$ are affiliated with $\mathcal{K}(K) \otimes B$. Using now Proposition A.1 of [21] we see that $\pi(b)$ is affiliated with B. One can easily verify that $\widehat{\beta} \otimes I$ commutes with ρ and anticommutes with R. Therefore

$$\rho\, \mathrm{Np}(R) - (\widehat{\beta} \otimes I)\rho\, \mathrm{Np}(R)(\widehat{\beta} \otimes I) = \rho(\mathrm{Np}(R) - \mathrm{Np}(-R)) = \rho R$$

and applying $\mathrm{id} \otimes \pi$ to both sides we get

$$\begin{aligned}(\mathrm{id} \otimes \pi)(\rho\, \mathrm{Np}(R)) - (\widehat{\beta} \otimes I)(\mathrm{id} \otimes \pi)(\rho\, \mathrm{Np}(R))(\widehat{\beta} \otimes I) &= (\mathrm{id} \otimes \pi)(\rho R) \\ &= -i\widehat{\beta}\widehat{b} \otimes \pi(i\beta b)\end{aligned}$$

The operators $\widehat{\beta} \otimes I$ and $(\mathrm{id} \otimes \pi)(\rho\, \mathrm{Np}(R))$ appearing on the left hand side are affiliated with $\mathcal{K}(K) \otimes B$. Therefore $i\widehat{\beta}\widehat{b} \otimes \pi(i\beta b)\ \eta\ \mathcal{K}(K) \otimes B$ and using again Proposition A.1 of [21] we see that $\pi(i\beta b)$ is affiliated with B. Moreover, remembering that $e^{\frac{i}{\hbar} \log \widehat{a} \otimes \pi(\log a)}\ \eta\ \mathcal{K}(K) \otimes B$ and using Proposition 2.5 we see that $\pi(\log a)$ is affiliated with B. According to Statement 3 of Theorem 3.1, b, $i\beta b$ and $\log a$ generate A. Therefore $\pi \in \mathrm{Mor}(A, B)$. We showed that $(\mathrm{id} \otimes \pi)V \in \mathrm{M}(\mathcal{K}(K) \otimes B)$ implies $\pi \in \mathrm{Mor}(A, B)$. It means that A is generated by $V \in \mathrm{M}(\mathcal{K}(K) \otimes A)$. $\square$

Now we are able to formulate the main result of this Section:

Theorem 3.3.

$$\left(\begin{array}{c} \textit{There exists } \Delta \in \mathrm{Mor}(A, A \otimes A) \\ \textit{such that} \\ (\mathrm{id} \otimes \Delta)V = V_{12}V_{13} \end{array}\right) \Longleftrightarrow \left(\hbar = \frac{\pi}{2k+3}, \quad k = 0, 1, 2, \ldots\right).$$

Proof. Let

$$
\begin{aligned}
\alpha &= i e^{\frac{i\pi^2}{2\hbar}},\\
T &= I\otimes e^{i\hbar/2}b^{-1}a\otimes b,\\
\tau &= (I\otimes\beta\otimes\beta)\,\Big[\alpha\chi(\widehat{b}\otimes b\otimes I<0)+\overline{\alpha}\chi(\widehat{b}\otimes b\otimes I>0)\Big].
\end{aligned}
\tag{3.9}
$$

and

$$
W' = G_\hbar(T,\tau)^*\, e^{-\frac{i}{\hbar}\left[I\otimes\log|b|\otimes\log a\right]}. \tag{3.10}
$$

Clearly W' is a unitary operator acting on $K\otimes L^2(\mathbb{R})\otimes L^2(\mathbb{R})$. We shall prove that

$$
V_{12}V_{13} = W'V_{12}W'^*. \tag{3.11}
$$

To make our formulae shorter we set

$$
U = e^{\frac{i}{\hbar}\log\widehat{a}\otimes\log a},\qquad Z = e^{-\frac{i}{\hbar}\log|b|\otimes\log a}.
$$

Using the relations $\widehat{a}\overset{\hbar}{\multimap}\widehat{b}$, $\widehat{a}\widehat{\beta}=\widehat{\beta}\widehat{a}$ and $a\overset{\hbar}{\multimap}b$ one can easily verify that

$$
U(\widehat{b}\otimes I)U^* = \widehat{b}\otimes a,\quad U(\widehat{\beta}\otimes I)U^* = \widehat{\beta}\otimes a, \tag{3.12}
$$

$$
Z(a\otimes I)Z^* = a\otimes a. \tag{3.13}
$$

With the above notation $V = G_\hbar(\widehat{b}\otimes b,\,\widehat{\beta}\otimes\beta)^*U$ and

$$
V_{12}V_{13} = G_\hbar(\widehat{b}\otimes b\otimes I,\,\widehat{\beta}\otimes\beta\otimes I)^*\,U_{12}\,G_\hbar(\widehat{b}\otimes I\otimes b,\,\widehat{\beta}\otimes I\otimes\beta)^*\,U_{13}.
$$

Using (3.12) we get

$$
U_{12}\,G_\hbar(\widehat{b}\otimes I\otimes b,\,\widehat{\beta}\otimes I\otimes\beta)^* = G_\hbar(\widehat{b}\otimes a\otimes b,\,\widehat{\beta}\otimes I\otimes\beta)^*\,U_{12}
$$

and

$$
V_{12}V_{13} = \Big[G_\hbar(\widehat{b}\otimes a\otimes b,\,\widehat{\beta}\otimes I\otimes\beta)\,G_\hbar(\widehat{b}\otimes b\otimes I,\,\widehat{\beta}\otimes\beta\otimes I)\Big]^*\,U_{12}U_{13}. \tag{3.14}
$$

Let us consider the first factor in (3.14). We apply Theorem 1.5 with

$$
\begin{aligned}
R &= \widehat{b}\otimes a\otimes b, &\quad \rho &= \widehat{\beta}\otimes I\otimes\beta,\\
S &= \widehat{b}\otimes b\otimes I, &\quad \sigma &= \widehat{\beta}\otimes\beta\otimes I,
\end{aligned}
\tag{3.15}
$$

Then T and τ are given by (3.9) and

$$
G_\hbar(\widehat{b}\otimes a\otimes b,\,\widehat{\beta}\otimes I\otimes\beta)\,G_\hbar(\widehat{b}\otimes b\otimes I,\,\widehat{\beta}\otimes\beta\otimes I) = G_\hbar(T,\tau)^*\,G_\hbar(\widehat{b}\otimes b\otimes I,\,\widehat{\beta}\otimes\beta\otimes I)\,G_\hbar(T,\tau).
$$

Now (3.14) takes the form

$$
V_{12}V_{13} = G_\hbar(T,\tau)^*\,G_\hbar(\widehat{b}\otimes b\otimes I,\,\widehat{\beta}\otimes\beta\otimes I)^*\,G_\hbar(T,\tau)\,U_{12}U_{13}. \tag{3.16}
$$

We shall move $G_\hbar(T,\tau)$ to the end of the right hand side of this formula. Performing simple computations and using (3.13) we obtain:

$$
\begin{aligned}
U_{12}U_{13} &= e^{\frac{i}{\hbar}\log\widehat{a}\otimes\log(a\otimes a)}\\
&= Z_{23}U_{12}Z_{23}^*.
\end{aligned}
$$

It turns out that

$$
\log\widehat{a}\otimes\log(a\otimes a)\text{ commutes with } T, \tag{3.17}
$$

$$
\log\widehat{a}\otimes\log(a\otimes a)\text{ commutes with } \tau. \tag{3.18}
$$

Indeed Zakrzewski relation $a\overset{\hbar}{\multimap}b$ implies $b^{-1}\overset{\hbar}{\multimap}a$. Using the both relations we see that $a\otimes a$ commutes with $e^{i\hbar/2}b^{-1}a\otimes b$. Therefore $\log(a\otimes a)$ commutes with $e^{i\hbar/2}b^{-1}a\otimes b$ and $\log\widehat{a}\otimes\log(a\otimes a)$ commutes with $T = I\otimes e^{i\hbar/2}b^{-1}a\otimes b$. Relation (3.17) is shown.

To prove (3.18) we use Zakrzewski relations $a\overset{\hbar}{\multimap}b$ and $\widehat{a}\overset{\hbar}{\multimap}\widehat{b}$. They show that a commutes with $\operatorname{sign} b$ and $\widehat{a}$ commutes with $\operatorname{sign}\widehat{b}$. Therefore $\log\widehat{a}\otimes\log(a\otimes a)$ commutes with $\operatorname{sign}(\widehat{b}\otimes b\otimes I) = \operatorname{sign}\widehat{b}\otimes\operatorname{sign} b\otimes I$ and (3.18) follows.

Taking into account (3.17) and (3.18) we see that $G_\hbar(T,\tau)$ commutes with $U_{12}U_{13}$. Now relation (3.16) takes the form:

$$
V_{12}V_{13} = G_\hbar(T,\tau)^*\,G_\hbar(\widehat{b}\otimes b\otimes I,\,\widehat{\beta}\otimes\beta\otimes I)^*\,Z_{23}U_{12}Z_{23}^*\,G_\hbar(T,\tau). \tag{3.19}
$$

Finally $b\otimes I$ and $\beta\otimes I$ commute with $\log|b|\otimes\log a$. Therefore $G_\hbar(\widehat{b}\otimes b\otimes I,\, \widehat{\beta}\otimes\beta\otimes I)$ commutes with Z_{23}. Clearly $G_\hbar(\widehat{b}\otimes b\otimes I,\, \widehat{\beta}\otimes\beta\otimes I)^*\, U_{12}=V_{12}$ and $W'=G_\hbar(T,\,\tau)^*Z_{23}$. Now (3.11) follows immediately from (3.19).

By the Zakrzewski relation $\widehat{a}^{i\lambda}\widehat{b}\widehat{a}^{-i\lambda}=e^{\hbar\lambda}\widehat{b}$ for all $\lambda\in\mathbb{R}$. Multiplication by a strictly positive number does not change the sign of an operator. Using this fact one can easily show that τ commutes with $\widehat{a}^{i\lambda}\otimes I\otimes I$. Consequently τ commutes with $\widehat{a}\otimes I\otimes I$. Since $T=I\otimes e^{i\hbar/2}b^{-1}a\otimes b$ and $I\otimes\log|b|\otimes\log a$ obviously commute with $\widehat{a}\otimes I\otimes I$, we conclude that W' commutes with $\widehat{a}\otimes I\otimes I$.

Now we are ready to prove the main statement.

$\Longrightarrow$. Let $\Delta\in\mathrm{Mor}(A,A\otimes A)$ and $(\mathrm{id}\otimes\Delta)V=V_{12}V_{13}$. We go back to the notation used in the proof of Theorem 3.2. In particular $C=\mathcal{K}(K)\otimes A$. For any $c\in C$ we set:

$$\phi_1(c)=(\mathrm{id}\otimes\Delta)(c),$$
$$\phi_2(c)=W'(c\otimes I)W'^*.$$

Then ϕ_1 and ϕ_2 are representations of C acting on the same Hilbert space $K\otimes L^2(\mathbb{R})\otimes L^2(\mathbb{R})$. One can easily verify that $\phi_1(\widehat{a}\otimes I)=\widehat{a}\otimes I\otimes I=\phi_2(\widehat{a}\otimes I)$. Formula (3.11) shows that $\phi_1(V)=\phi_2(V)$. In our notation (cf. the beginning of the proof of Theorem 3.2), $\widehat{a}\otimes I=S$ and $V=G_\hbar(R,\rho)^*U$, where in particular $R=\widehat{b}\otimes b$. Statement 2 of Theorem 2.3 shows now that $\phi_1(R)=\phi_2(R)$. It means that

$$\widehat{b}\otimes\Delta(b)=W'\,(\widehat{b}\otimes b\otimes I)\,W'^*.$$

Taking into account (3.10) and using Proposition 1.2 we get:

$$\begin{aligned}\widehat{b}\otimes\Delta(b)&=G_\hbar(T,\tau)(\widehat{b}\otimes b\otimes I)G_\hbar(T,\tau)^*\\&=\left[\widehat{b}\otimes a\otimes b+\widehat{b}\otimes b\otimes I\right]_\tau.\end{aligned}\tag{3.20}$$

We recall that

$$\tau=(I\otimes\beta\otimes\beta)\,\left[\alpha\chi(\widehat{b}\otimes b\otimes I<0)+\overline{\alpha}\chi(\widehat{b}\otimes b\otimes I>0)\right].$$

Inspecting last two formulae we observe that $\widehat{b}$ is the only operator appearing in the first leg position. We know that $\widehat{b}$ is selfadjoint. Therefore replacing in both sides of (3.20) operator $\widehat{b}$ by a real number λ we obtain a formula that should hold for almost all $\lambda\in\mathrm{Sp}\,\widehat{b}$. For positive λ we get

$$\Delta(b)=\left[a\otimes b+b\otimes I\right]_{\tau_+}\tag{3.21}$$

where

$$\tau_+=(\beta\otimes\beta)\,\left[\alpha\chi(b\otimes I<0)+\overline{\alpha}\chi(b\otimes I>0)\right].\tag{3.22}$$

On the other hand for negative λ we have

$$\Delta(b)=\left[a\otimes b+b\otimes I\right]_{\tau_-}\tag{3.23}$$

where

$$\tau_-=(\beta\otimes\beta)\left[\alpha\chi(b\otimes I>0)+\overline{\alpha}\chi(b\otimes I<0)\right].\tag{3.24}$$

Clearly the two expressions for $\Delta(b)$ must coincide. Let us notice that the operator $I\otimes\beta$ commutes with τ_+, τ_- and $b\otimes I$ and anticommutes with $a\otimes b$. Therefore $\tau_+=\tau_-$ by Remark 1.3. Comparing (3.22) and (3.24) we get $\alpha=\overline{\alpha}$. Remembering that $\alpha=i\,e^{\frac{i\pi^2}{2\hbar}}$ and $0<\hbar<\frac{\pi}{2}$ we conclude that $\hbar=\frac{\pi}{2k+3}$ $(k=0,1,2,\dots)$.

$\Longleftarrow$. Assume that $\hbar=\frac{\pi}{2k+3}$ for some $k=0,1,2,\dots$. Then formula (3.10) essentially simplifies. In this case $\alpha=(-1)^k$, $\tau=(-1)^k(I\otimes\beta\otimes\beta)$ and $W'=W_{23}=I\otimes W$, where

$$W=G_\hbar\left(e^{i\hbar/2}b^{-1}a\otimes b,\,(-1)^k\beta\otimes\beta\right)^*\,e^{-\frac{i}{\hbar}\log|b|\otimes\log a}.\tag{3.25}$$

Formula (3.11) takes the form

$$V_{12}V_{13}=W_{23}V_{12}W_{23}^*.\tag{3.26}$$

For any $c\in A$ we set

$$\Delta(c)=W(c\otimes I)W^*.\tag{3.27}$$

Then Δ is a representation of A acting on $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$. We know that $V \in \mathrm{M}(\mathcal{K}(K) \otimes A)$. Formula (3.26) shows that

$$(\mathrm{id} \otimes \Delta)V = V_{12}V_{13}.$$

Clearly $V_{12}, V_{13} \in \mathrm{M}(\mathcal{K}(K) \otimes A \otimes A)$. Therefore $(\mathrm{id} \otimes \Delta)V = V_{12}V_{13} \in \mathrm{M}(\mathcal{K}(K) \otimes A \otimes A)$. Remembering that A is generated by V we conclude that $\Delta \in \mathrm{Mor}(A, A \otimes A)$. □

Let $\hbar = \frac{\pi}{2k+3}$ $(k = 0, 1, 2, \dots)$. Then formula (3.27) makes it possible to calculate $\Delta(c)$ for any $c \in A$. The same holds for any c affiliated with A. We shall show that

$$\begin{aligned} \Delta(a) &= a \otimes a, \\ \Delta(b) &= \left[a \otimes b + b \otimes I\right]_{(-1)^k\beta\otimes\beta}, \\ \Delta(ib^{2k+3}\beta) &= \left[a^{2k+3} \otimes ib^{2k+3}\beta + ib^{2k+3}\beta \otimes I\right]_{-\operatorname{sign}(b\otimes b)}. \end{aligned} \tag{3.28}$$

Formula for $\Delta(a)$ follows immediately from (3.13); the reader should notice that operators $e^{i\hbar/2}b^{-1}a \otimes b$ and $\beta \otimes \beta$ commute with $a \otimes a$. The formula for $\Delta(b)$ was in fact shown in the proof of Theorem 3.3; in the present case $\tau_+ = \tau_- = (-1)^k\beta \otimes \beta$ and the second formula of (3.28) coincides with (3.21) (and with (3.23) as well).

It remains to prove the third formula. We know that $|b|$ commutes with $ib^{2k+3}\beta$. Taking into account (3.25) we obtain

$$\begin{aligned} \Delta(ib^{2k+3}\beta) &= W(ib^{2k+3}\beta \otimes I)W^* \\ &= G_\hbar\big(e^{i\hbar/2}b^{-1}a\otimes b, (-1)^k\beta\otimes\beta\big)^* \left(ib^{2k+3}\beta\otimes I\right) G_\hbar\big(e^{i\hbar/2}b^{-1}a\otimes b, (-1)^k\beta\otimes\beta\big). \end{aligned} \tag{3.29}$$

To compute the right hand side we use Proposition 1.4 with

$$R = a \otimes |b|\,,\ \ S = |b| \otimes I,\ \ \tau = \operatorname{sign}(b \otimes b),$$
$$\xi = (-1)^k\beta \otimes \beta \ \text{ and } \ \sigma = i(\operatorname{sign} b)\beta \otimes I.$$

Remembering that $\beta^2 = I$ and β anticommutes with b and hence commutes with $|b|$ one can easily check that these operators fulfil all assumption of Proposition 1.4. In this case we have $T = (e^{i\hbar/2}\, |b|^{-1}\, a) \otimes |b|$, $\tau T = e^{i\hbar/2}b^{-1}a \otimes b$ and

$$\xi\sigma = -(-1)^k i(\operatorname{sign} b) \otimes\ \beta = -\sigma\xi.$$

Therefore

$$\rho = -\left[i(\operatorname{sign} b) \otimes \beta\right]\left[\chi(\tau = 1) - \chi(\tau = -1)\right] = -\left[i(\operatorname{sign} b) \otimes \beta\right]\tau = I \otimes i(\operatorname{sign} b)\beta.$$

According to our assumption $\frac{\pi}{\hbar} = 2k + 3$ is an odd positive integer. Therefore

$$\begin{aligned} \sigma S^{\frac{\pi}{\hbar}} &= \Big[i(\operatorname{sign} b)\beta \otimes I\Big]\Big[|b|^{2k+3} \otimes I\Big] &&= ib^{2k+3}\beta \otimes I, \\ \rho R^{\frac{\pi}{\hbar}} &= \Big[I \otimes i(\operatorname{sign} b)\beta\Big]\Big[a^{2k+3} \otimes |b|^{2k+3}\Big] &&= a^{2k+3} \otimes ib^{2k+3}\beta \end{aligned}$$

and formula (1.10) takes the form

$$\begin{aligned} &\Big[ib^{2k+3}\beta \otimes I + a^{2k+3} \otimes ib^{2k+3}\beta\Big]_{-\operatorname{sign}(b\otimes b)} \\ &\quad = G_\hbar\big(e^{i\hbar/2}b^{-1}a\otimes b, (-1)^k\beta\otimes\beta\big)^* \left(ib^{2k+3}\beta\otimes I\right) G_\hbar\big(e^{i\hbar/2}b^{-1}a\otimes b, (-1)^k\beta\otimes\beta\big). \end{aligned} \tag{3.30}$$

Comparing (3.29) with (3.30) we get the last formula of (3.28). This formula appeared without proof in [21].

Remark 3.4. Replacing in the above computations $\sigma = i(\operatorname{sign} b)\beta \otimes I$ by $\sigma = \beta \otimes I$ one can prove that

$$\Delta(|b|^{2k+3}\beta) = \Big[a^{2k+3} \otimes |b|^{2k+3}\beta + |b|^{2k+3}\beta \otimes I\Big]_{-\operatorname{sign}(b\otimes b)}. \tag{3.31}$$

Assume now that $K = L^2(\mathbb{R})$ and that the operators $\widehat{a}, \widehat{b}, \widehat{\beta}$ are given by (3.7). Then operator (3.8) coincides with (3.25): $V = W$. Relation (3.26) takes the form:

$$W_{23}W_{12} = W_{12}W_{13}W_{23}.$$

This is the famous pentagon equation of Baaj and Skandalis [2]. It means that W is a multiplicative unitary. It is known that W is modular [9]. This property enables us to introduce unitary antipode, scaling group and Haar weight (see [9, 18, 21, 11, 22] for details).

4. New quantum deformations of '$ax + b$' group.

In this Section we shall show how to enlarge the set of admissible values of the deformation parameter $\hbar$ beyond the one described in Theorem 3.3. To this end one has to add a new element to the set of generators of the C*-algebra A. This new element denoted by w is a unitary involution commuting with a and b and anticommuting with β.

To define the new C*-algebra A we consider four operators a, b, β and w acting on the Hilbert space $L^2(\mathbb{R}, \mathbb{C}^2)$ introduced in the following way: for any $\tau \in \mathbb{R}$ and any $x \in L^2(\mathbb{R}, \mathbb{C}^2)$ we set:

$$\left(a^{i\tau}x\right)(t) = e^{\hbar\tau/2}x(e^{\hbar\tau}t), \qquad (bx)(t) = tx(t),$$

$$(\beta x)(t) = \begin{pmatrix} 1\ , & 0 \\ 0\ , & -1 \end{pmatrix} x(-t), \quad (wx)(t) = \begin{pmatrix} 0\ , & 1 \\ 1\ , & 0 \end{pmatrix} x(t).$$

As in the previous section a is the analytic generator of the group of unitaries defined by the first formula. b is the selfadjoint operator with domain consisting of all x such that $\|tx(t)\|^2$ is integrable over $\mathbb{R}$. Clearly β and w are unitary selfadjoint. One can easily verify that $a\beta = \beta a$, $b\beta = -\beta b$, $aw = wa$, $bw = wb$, $\beta w = -w\beta$ and $a \overset{\hbar}{\multimap} b$.

Using essentially the same method as in the proof of Theorem 3.1 one can easily show

Theorem 4.1. *Let*

$$A = \left\{ \begin{pmatrix} f_1(b) + \beta f_2(b) \\ +wf_3(b) + w\beta f_4(b) \end{pmatrix} g(\log a) : \begin{array}{c} f_1, f_2, f_3, f_4, g \in C_\infty(\mathbb{R}) \\ f_2(0) = f_4(0) = 0 \end{array} \right\}^{\begin{array}{c}\text{norm closed}\\ \text{linear envelope}\end{array}} . \tag{4.1}$$

Then: 1. *A is a nondegenerate C*-algebra of operators acting on $L^2(\mathbb{R}, \mathbb{C}^2)$,*

2. *$\log a$, b, $ib\beta$ and w are affiliated with A: $\log a$, b, $ib\beta$, $w\,\eta\, A$,*

3. *$\log a$, b, $ib\beta$ and w generate A,*

The reader should notice that the C*-algebra A introduced by (4.1) coincides with the crossed product of the C*-algebra A considered in the previous Section (cf. (3.2)) by the involutive automorphism that leaves a and b invariant and maps β into $-\beta$.

Now we pass to the description of the Hilbert space K. The structure of K is determined by a quadruple of selfadjoint operators $(\widehat{a}, \widehat{b}, \widehat{\beta}, \widehat{w})$ acting on K and having the following properties:

1. $\widehat{a} > 0$, $\ker \widehat{a} = \ker \widehat{b} = \{0\}$ and $\widehat{a} \overset{\hbar}{\multimap} \widehat{b}$,
2. $\widehat{\beta}$ is a unitary involution, $\widehat{\beta}$ commutes with $\widehat{a}$ and anticommutes with $\widehat{b}$.
3. $\widehat{w}$ is a unitary involution, $\widehat{w}$ commutes with $\widehat{a}$ and $\widehat{b}$.
4. $\widehat{w}\widehat{\beta} = \varepsilon\widehat{\beta}\widehat{w}$.

In the last formula ε is a fixed element of the set $\{-1, 1\}$. Our constructions depend very much on the value of ε. What we get for $\varepsilon = 1$ is essentially the crossed product of the quantum group described in the previous Section by the involutive automorphism mentioned above. In particular the set of admissible values of $\hbar$ is the same as before. For $\varepsilon = -1$ we obtain a family of new deformations with $\hbar = \frac{\pi}{2k+4}$ $(k = 0, 1, 2, \dots)$.

For any $z, z' = \pm 1$ we set

$$\mathrm{Ch}(z, z') = \left\{ \begin{array}{ccc} z' & \text{for} & z = -1 \\ 1 & \text{for} & z = 1. \end{array} \right. \tag{4.2}$$

Then Ch is the bicharacter on the multiplicative group $\mathbb{Z}_2 = \{1, -1\}$. One can easily verify that

$$\mathrm{Ch}(z, z') = \frac{1}{2} \sum_{r,s=0}^{1} (-1)^{rs} z^r z'^s. \tag{4.3}$$

Let

$$V = G_\hbar(\widehat{b} \otimes b,\ \widehat{\beta} \otimes \beta)^*\, e^{\frac{i}{\hbar} \log \widehat{a} \otimes \log a}\, \mathrm{Ch}(\widehat{w} \otimes I, I \otimes w). \tag{4.4}$$

This is the basic object considered in this Section. We shall prove

Theorem 4.2.

1. *V is a unitary operator and $V \in \mathrm{M}(\mathcal{K}(K) \otimes A)$,*
2. *A is generated by $V \in \mathrm{M}(\mathcal{K}(K) \otimes A)$.*

Proof. Let $R = \widehat{b} \otimes b$, $\rho = \widehat{\beta} \otimes \beta$, $U = e^{\frac{i}{\hbar}\log \widehat{a} \otimes \log a}\,\mathrm{Ch}(\widehat{w} \otimes I, I \otimes w)$, $S = \widehat{a}^{-1} \otimes I$ and $C = \mathcal{K}(K) \otimes A$. Then all the assumptions of Proposition 2.3 are satisfied. Clearly $V = G_\hbar(R,\rho)^* U \in \mathrm{M}(C)$ and Statement 1 is proved.

Let π be a representation of A and $B \in C^*(H_\pi)$. Then $\mathrm{id} \otimes \pi$ is a representation of C acting on $K \otimes H_\pi$. Assume that $(\mathrm{id} \otimes \pi)V \in \mathrm{M}(\mathcal{K}(K) \otimes B)$. Repeating the reasoning used in the proof of Theorem 3.2 we see that $\pi(b)$ and $\pi(i\beta b)$ are affiliated with B. Furthermore $(\mathrm{id} \otimes \pi)U = e^{\frac{i}{\hbar}\log \widehat{a} \otimes \pi(\log a)}\,\mathrm{Ch}(\widehat{w} \otimes I, I \otimes \pi(w))$ is affiliated with $\mathcal{K}(K) \otimes B$.

We know that $\widehat{a}$ commutes with $\widehat{w}$. Therefore $\widehat{a}$ respects the decomposition of K into direct sum of eigenspaces of $\widehat{w}$. Let $K_\pm = K(\widehat{w} = \pm 1)$. Then

$$\begin{aligned} K &= K_+ \oplus K_-, \\ \widehat{a} &= \widehat{a}_+ \oplus \widehat{a}_-, \\ \widehat{w} &= I \oplus (-I). \end{aligned}$$

With this notation

$$(\mathrm{id} \otimes \pi)U = e^{\frac{i}{\hbar}\log \widehat{a}_+ \otimes \pi(\log a)} \oplus e^{\frac{i}{\hbar}\log \widehat{a}_- \otimes \pi(\log a)}(I \otimes \pi(w)).$$

Remembering that $(\mathrm{id} \otimes \pi)U$ is affiliated with $\mathcal{K}(K) \otimes B$ we see that $e^{\frac{i}{\hbar}\log \widehat{a}_+ \otimes \pi(\log a)}$ is affiliated with $\mathcal{K}(K_+) \otimes B$ and $e^{\frac{i}{\hbar}\log \widehat{a}_- \otimes \pi(\log a)}(I \otimes \pi(w))$ is affiliated with $\mathcal{K}(K_-) \otimes B$. Proposition 2.5 shows now that $\pi(\log a)$ is affiliated with B. Using this fact one can easily show that $\pi(w)$ is also affiliated with B. According to Statement 3 of Theorem 4.1, b, $i\beta b$, $\log a$ and w generate A. Therefore $\pi \in \mathrm{Mor}(A, B)$. We showed that $(\mathrm{id} \otimes \pi)V \in \mathrm{M}(\mathcal{K}(K) \otimes B)$ implies $\pi \in \mathrm{Mor}(A, B)$. It means that A is generated by $V \in \mathrm{M}(\mathcal{K}(K) \otimes A)$. ☐

Now we are able to formulate the main result of this Section:

Theorem 4.3.

$$\left(\begin{array}{c} \textit{There exists } \Delta \in \mathrm{Mor}(A, A \otimes A) \\ \textit{such that} \\ (\mathrm{id} \otimes \Delta)V = V_{12}V_{13} \end{array}\right) \Longleftrightarrow \left(\hbar = \frac{\pi}{\ell + 3}, \ \textit{where} \ \begin{array}{c} \ell \in \mathbb{Z}, \quad \ell \geq 0 \\ \textit{and } (-1)^\ell = \varepsilon \end{array}\right).$$

Proof. We essentially repeat the proof of Theorem 3.3. Since in the great part calculations are very similar, we sketch the main steps only and point out necessary modifications. Together with ε we shall use another parameter $s = \frac{1-\varepsilon}{2}$. Then $\mathrm{Ch}(\varepsilon, w) = w^s$.

Let γ be an operator acting on $L^2(\mathbb{R}, \mathbb{C}^2)$ defined by the formula

$$(\gamma x)(t) = \begin{pmatrix} 1\ , & 0 \\ 0\ , & -1 \end{pmatrix} x(t) \tag{4.5}$$

Then γ is a unitary selfadjoint operator, γ commutes with a, b, β and anticommutes with w.

Let

$$\begin{aligned} \alpha &= ie^{\frac{i\pi^2}{2\hbar}}, \\ T &= I \otimes e^{i\hbar/2} b^{-1} a \otimes b, \\ \tau &= (I \otimes w^s \beta \otimes \beta)\left[\alpha\chi(\widehat{b} \otimes b \otimes I < 0) + \varepsilon\overline{\alpha}\chi(\widehat{b} \otimes b \otimes I > 0)\right]. \end{aligned} \tag{4.6}$$

and

$$W' = G_\hbar(T, \tau)^*\, e^{-\frac{i}{\hbar} I \otimes \log|b| \otimes \log a}\,\mathrm{Ch}(I \otimes \gamma \otimes I, I \otimes I \otimes w). \tag{4.7}$$

Clearly W' is a unitary operator acting on $K \otimes L^2(\mathbb{R}, \mathbb{C}^2) \otimes L^2(\mathbb{R}, \mathbb{C}^2)$. We shall prove that

$$V_{12}V_{13} = W' V_{12} W'^*. \tag{4.8}$$

In order to make our formulae shorter we set

$$\begin{aligned} U &= e^{\frac{i}{\hbar}\log \widehat{a} \otimes \log a}\,\mathrm{Ch}(\widehat{w} \otimes I, I \otimes w), \\ Z &= e^{-\frac{i}{\hbar}\log|b| \otimes \log a}\,\mathrm{Ch}(\gamma \otimes I, I \otimes w). \end{aligned}$$

Using the commutation relations, one can easily verify that

$$U(\widehat{b}\otimes I)U^* = \widehat{b}\otimes a, \qquad U(\widehat{\beta}\otimes I)U^* = \widehat{\beta}\otimes w^s, \tag{4.9}$$

$$Z(a\otimes I)Z^* = a\otimes a, \qquad Z(w\otimes I)Z^* = w\otimes w. \tag{4.10}$$

With the above notation $V = G_\hbar(\widehat{b}\otimes b,\, \widehat{\beta}\otimes\beta)^* U$ and

$$V_{12}V_{13} = G_\hbar(\widehat{b}\otimes b\otimes I,\, \widehat{\beta}\otimes\beta\otimes I)^*\, U_{12}\, G_\hbar(\widehat{b}\otimes I\otimes b,\, \widehat{\beta}\otimes I\otimes\beta)^*\, U_{13}. \tag{4.11}$$

Taking into account (4.9) we get

$$U_{12}\, G_\hbar(\widehat{b}\otimes I\otimes b,\, \widehat{\beta}\otimes I\otimes\beta)^* = G_\hbar(\widehat{b}\otimes a\otimes b,\, \widehat{\beta}\otimes w^s\otimes\beta)^*\, U_{12} \tag{4.12}$$

and

$$V_{12}V_{13} = \Big[G_\hbar(\widehat{b}\otimes a\otimes b,\, \widehat{\beta}\otimes w^s\otimes\beta)\, G_\hbar(\widehat{b}\otimes b\otimes I,\, \widehat{\beta}\otimes\beta\otimes I)\Big]^*\, U_{12}U_{13}. \tag{4.13}$$

Let us consider the first factor in (3.14). We apply Theorem 1.5 with

$$\begin{aligned} R &= \widehat{b}\otimes a\otimes b, \quad \rho = \widehat{\beta}\otimes w^s\otimes\beta,\\ S &= \widehat{b}\otimes b\otimes I, \quad \sigma = \widehat{\beta}\otimes\beta\otimes I, \end{aligned} \tag{4.14}$$

Then T and τ are given by (4.6) and

$$G_\hbar(\widehat{b}\otimes a\otimes b,\, \widehat{\beta}\otimes w^s\otimes\beta)\, G_\hbar(\widehat{b}\otimes b\otimes I,\, \widehat{\beta}\otimes\beta\otimes I) = G_\hbar(T,\tau)^*\, G_\hbar(\widehat{b}\otimes b\otimes I,\, \widehat{\beta}\otimes\beta\otimes I)\, G_\hbar(T,\tau).$$

Now (4.13) takes the form

$$V_{12}V_{13} = G_\hbar(T,\tau)^*\, G_\hbar(\widehat{b}\otimes b\otimes I,\, \widehat{\beta}\otimes\beta\otimes I)^*\, G_\hbar(T,\tau)\, U_{12}U_{13}. \tag{4.15}$$

Performing simple computations and using (4.10) we obtain

$$\begin{aligned} U_{12}U_{13} &= e^{\frac{i}{\hbar}\log\widehat{a}\otimes\log(a\otimes a)}\, \mathrm{Ch}(\widehat{w}\otimes I\otimes I, I\otimes w\otimes w)\\ &= Z_{23}U_{12}Z_{23}^*. \end{aligned}$$

Repeating the arguments used in the proof of Theorem 3.3 we see that $G_\hbar(T,\tau)$ commutes with $e^{\frac{i}{\hbar}\log\widehat{a}\otimes\log(a\otimes a)}$. Furthermore T and τ commute with $\widehat{w}\otimes I\otimes I$ and $I\otimes w\otimes w$. Therefore $G_\hbar(T,\tau)$ commutes with $\mathrm{Ch}(\widehat{w}\otimes I\otimes I, I\otimes w\otimes w)$ and in (4.15) we may move $G_\hbar(T,\tau)$ to the most right position:

$$V_{12}V_{13} = G_\hbar(T,\tau)^*\, G_\hbar(\widehat{b}\otimes b\otimes I,\, \widehat{\beta}\otimes\beta\otimes I)^*\, Z_{23}U_{12}Z_{23}^*\, G_\hbar(T,\tau) \tag{4.16}$$

Finally one easily verifies that $b\otimes I$ and $\beta\otimes I$ commute with $\log|b|\otimes\log a$, $\gamma\otimes I$ and $I\otimes w$. Therefore $G_\hbar(\widehat{b}\otimes b\otimes I,\, \widehat{\beta}\otimes\beta\otimes I)$ commutes with Z_{23}. Clearly $G_\hbar(\widehat{b}\otimes b\otimes I,\, \widehat{\beta}\otimes\beta\otimes I)^*\, U_{12} = V_{12}$ and $W' = G_\hbar(T,\, \tau)^* Z_{23}$. Now (4.8) follows immediately from (4.16).

Also in the present case W' commutes with $\widehat{a}\otimes I\otimes I$. The same proof applies.

Now we are ready to prove the main statement.

$\Longrightarrow$. Let $\Delta\in\mathrm{Mor}(A, A\otimes A)$ and $(\mathrm{id}\otimes\Delta)V = V_{12}V_{13}$. Repeating the reasoning used in the proof of Theorem 3.3 we easily arrive to the formula

$$\Delta(b) = \big[a\otimes b + b\otimes I\big]_{\tau_+} = \big[a\otimes b + b\otimes I\big]_{\tau_-} \tag{4.17}$$

where

$$\begin{aligned} \tau_+ &= (w^s\beta\otimes\beta)\,\big[\alpha\chi(b\otimes I<0) + \varepsilon\overline{\alpha}\chi(b\otimes I>0)\big]\,,\\ \tau_- &= (w^s\beta\otimes\beta)\,\big[\alpha\chi(b\otimes I>0) + \varepsilon\overline{\alpha}\chi(b\otimes I<0)\big]\,. \end{aligned} \tag{4.18}$$

Clearly the two expressions for $\Delta(b)$ must coincide. Let us notice that the operator $I\otimes\beta$ commutes with τ_+, τ_- and $b\otimes I$ and anticommutes with $a\otimes b$. Therefore $\tau_+ = \tau_-$ by Remark 1.3. Using (4.18) we get $\alpha = \varepsilon\overline{\alpha}$. Remembering that $\alpha = i\,e^{\frac{i\pi^2}{2\hbar}}$ and $0<\hbar<\frac{\pi}{2}$ we conclude that $\hbar = \frac{\pi}{\ell+3}$, where ℓ is a non-negative integer such that $(-1)^\ell = \varepsilon$.

$\Longleftarrow$. Assume that $\hbar = \frac{\pi}{\ell+3}$ for some $\ell = 0,1,2,\dots$ such that $(-1)^\ell=\varepsilon$. Then formula (4.7) essentially simplifies. In this case $\alpha = i^\ell$, $w^s = w^\ell$ (this is because $s\equiv\ell \mod 2$), $\tau = \alpha I\otimes w^s\beta\otimes\beta = I\otimes(iw)^\ell\beta\otimes\beta$ and $W' = W_{23} = I\otimes W$, where

$$W = G_\hbar\left(e^{i\hbar/2}b^{-1}a\otimes b,\, (iw)^\ell\beta\otimes\beta\right)^*\, e^{-\frac{i}{\hbar}\log|b|\otimes\log a}\,\mathrm{Ch}(\gamma\otimes I, I\otimes w). \tag{4.19}$$

Formula (4.8) takes the form

$$V_{12}V_{13} = W_{23}V_{12}W_{23}^{*}. \tag{4.20}$$

For any $c \in A$ we set

$$\Delta(c) = W(c \otimes I)W^{*}. \tag{4.21}$$

Then Δ is a representation of A acting on $L^2(\mathbb{R}, \mathbb{C}^2) \otimes L^2(\mathbb{R}, \mathbb{C}^2)$. We know that $V \in \mathrm{M}(\mathcal{K}(K) \otimes A)$. Formula (4.20) shows that

$$(\mathrm{id} \otimes \Delta)V = V_{12}V_{13}.$$

Clearly V_{12}, $V_{13} \in \mathrm{M}(\mathcal{K}(K) \otimes A \otimes A)$. Therefore $(\mathrm{id} \otimes \Delta)V = V_{12}V_{13} \in \mathrm{M}(\mathcal{K}(K) \otimes A \otimes A)$. Remembering that A is generated by V we conclude that $\Delta \in \mathrm{Mor}(A, A \otimes A)$. ☐

Let $\varepsilon = (-1)^\ell$ and $\hbar = \frac{\pi}{\ell+3}$ $(\ell = 0, 1, 2, \dots)$ as in section 3. Formula (4.21) enables us to calculate $\Delta(c)$ for any $c \in A$. The same holds for any c affiliated with A. We shall show that

$$\begin{aligned} \Delta(a) &= a \otimes a, \\ \Delta(b) &= \left[a \otimes b + b \otimes I\right]_{(iw)^\ell \beta \otimes \beta}, \\ \Delta\left((ib)^{\ell+3}\beta\right) &= \left[w^\ell a^{\ell+3} \otimes (ib)^{\ell+3}\beta + (ib)^{\ell+3}\beta \otimes I\right]_{-\operatorname{sign}(b\otimes b)}, \\ \Delta(w) &= w \otimes w. \end{aligned} \tag{4.22}$$

Formulae for $\Delta(a)$ and $\Delta(w)$ follow immediately from (4.10); the reader should notice that operators $e^{i\hbar/2}b^{-1}a \otimes b$ and $(iw)^\ell \beta \otimes \beta$ commute with $a \otimes a$ and $w \otimes w$. The formula for $\Delta(b)$ coincides with (4.17).

It remains to prove the third formula. The reader should notice that $(ib)^{\ell+3}\beta$ is a selfadjoint operator. We know that $|b|$ and γ commutes with $(ib)^{\ell+3}\beta$. Taking into account (4.19) we obtain

$$\begin{aligned} \Delta((ib)^{\ell+3}\beta) &= W\left((ib)^{\ell+3}\beta \otimes I\right)W^{*} \\ &= G_\hbar\left(e^{i\hbar/2}b^{-1}a\otimes b, (iw)^\ell\beta\otimes\beta\right)^{*}\left((ib)^{\ell+3}\beta\otimes I\right) G_\hbar\left(e^{i\hbar/2}b^{-1}a\otimes b, (iw)^\ell\beta\otimes\beta\right). \end{aligned} \tag{4.23}$$

To compute the right hand side we use Proposition 1.4 with

$$R = a \otimes |b|\,,\ S = |b| \otimes I,\ \tau = \operatorname{sign}(b \otimes b),$$

$$\xi = (iw)^\ell \beta \otimes \beta \ \text{ and } \ \sigma = (i \operatorname{sign} b)^{\ell+3}\beta \otimes I.$$

One can easily check that these operators fulfil all assumption of Proposition 1.4. In this case we have $T = (e^{i\hbar/2}\,|b|^{-1}\,a) \otimes |b|$ and $\tau T = e^{i\hbar/2}b^{-1}a \otimes b$. Operators ξ and σ anticommute. Indeed $\beta(i \operatorname{sign} b)^{\ell+3} = (-i \operatorname{sign} b)^{\ell+3}\beta$ and $(iw)^\ell\beta = \beta(-iw)^\ell$. Therefore $\xi\sigma = (-1)^{2\ell+3}\sigma\xi = -\sigma\xi$. Inserting in (1.9) $\alpha = i^\ell$ we obtain

$$\rho = (-i)^\ell \xi\sigma\left[\chi(\tau = 1) - (-1)^\ell \chi(\tau = -1)\right] = (-i)^\ell \xi\sigma\tau^{\ell+1} = \mu \otimes \nu,$$

where $\mu = (-i)^\ell (iw)^\ell \beta (i \operatorname{sign} b)^{\ell+3}\beta(\operatorname{sign} b)^{\ell+1} = (-i)^{\ell+3}w^\ell$ and $\nu = \beta(\operatorname{sign} b)^{\ell+1} = (-\operatorname{sign} b)^{\ell+1}\beta$. Finally $\rho = (-i)^{\ell+3}w^\ell \otimes (-\operatorname{sign} b)^{\ell+1}\beta = w^\ell \otimes (i \operatorname{sign} b)^{\ell+3}\beta$.

According to our assumption $\frac{\pi}{\hbar} = \ell + 3$. Therefore

$$\begin{aligned} \sigma S^{\frac{\pi}{\hbar}} &= \left[(i \operatorname{sign} b)^{\ell+3}\beta \otimes I\right]\left[|b|^{\ell+3} \otimes I\right] = (ib)^{\ell+3}\beta \otimes I, \\ \rho R^{\frac{\pi}{\hbar}} &= \left[w^\ell \otimes (i \operatorname{sign} b)^{\ell+3}\beta\right]\left[a^{\ell+3} \otimes |b|^{\ell+3}\right] = w^\ell a^{\ell+3} \otimes (ib)^{\ell+3}\beta \end{aligned}$$

and formula (1.10) takes the form

$$\begin{aligned} &\left[(ib)^{\ell+3}\beta \otimes I + w^\ell a^{\ell+3} \otimes (ib)^{\ell+3}\beta\right]_{-\operatorname{sign}(b\otimes b)} \\ &= G_\hbar\left(e^{i\hbar/2}b^{-1}a\otimes b, (iw)^\ell\beta\otimes\beta\right)^{*}\left((ib)^{\ell+3}\beta\otimes I\right) G_\hbar\left(e^{i\hbar/2}b^{-1}a\otimes b, (iw)^\ell\beta\otimes\beta\right). \end{aligned} \tag{4.24}$$

Comparing (4.23) with (4.24) we get the third formula of (4.22).

Remark 4.4. Replacing in the above computations $\sigma = (i \operatorname{sign} b)^{\ell+3}\beta \otimes I$ by $\sigma = \beta \otimes I$ one can prove that

$$\Delta(|b|^{\ell+3}\,\beta) = \left[w^\ell a^{\ell+3} \otimes |b|^{\ell+3}\,\beta + |b|^{\ell+3}\,\beta \otimes I\right]_{-\operatorname{sign}(b\otimes b)}.$$

For odd ℓ this formula is equivalent to the third formula of (4.22). For even $\ell = 2k$ the formula coincides with (3.31).

5. Modularity and all that

Now we shall investigate the unitary W introduced by (4.19). We shall prove that W is a modular multiplicative unitary. Throughout this Section $\varepsilon=(-1)^\ell$ and $\hbar=\frac{\pi}{\ell+3}$ where $\ell=0,1,2,\dots$.

Let $\overline{K}$ be the Hilbert space complex conjugate to K. The structure of $\overline{K}$ is established by an antiunitary mapping $K\ni x\longleftrightarrow\overline{x}\in\overline{K}$. For any closed operator c acting on K, we denote by $c^\top$ the transpose of c. By definition $c^\top$ is an operator acting on $\overline{K}$ with domain $\mathcal{D}(c^\top)=\{\overline{x}:x\in\mathcal{D}(c^*)\}$ such that

$$c^\top\overline{x}=\overline{c^*x}$$

for any $x\in\mathcal{D}(c^*)$. In what follows $Q=a^{\frac12}$.

Proposition 5.1. *Let V be the unitary operator introduced by* (4.4) *and*

$$\widetilde{V}=G_\hbar\Big(-\widehat{b}^\top\otimes e^{i\hbar/2}ba^{-1},-\widehat{\beta}^\top\otimes(iw)^\ell\beta\Big)\,e^{\frac{i}{\hbar}\log\widehat{a}^\top\otimes\log a}\,\mathrm{Ch}(\widehat{w}^\top\otimes I,I\otimes w).\tag{5.1}$$

Then $\widetilde{V}$ is unitary and for any $x,z\in K$, $y\in D(Q^{-1})$, $u\in D(Q)$ we have:

$$(x\otimes u|V|z\otimes y)=\Big(\overline{z}\otimes Qu\Big|\widetilde{V}\Big|\overline{x}\otimes Q^{-1}y\Big).\tag{5.2}$$

Proof. We shall follow the proof of Proposition 2.3 of [21]. The reader should notice that in large part that proof is independent of the particular value of $\hbar$. To make our formulae shorter we set:

$$\begin{aligned}
U'&=e^{\frac{i}{\hbar}\log\widehat{a}\otimes\log a}, & \widetilde{U}'&=e^{\frac{i}{\hbar}\log\widehat{a}^\top\otimes\log a},\\
U&=U'\,\mathrm{Ch}(\widehat{w}\otimes I,I\otimes w), & \widetilde{U}&=\widetilde{U}'\,\mathrm{Ch}(\widehat{w}^\top\otimes I,I\otimes w),\\
B&=\left|\widehat{b}\otimes b\right|, & \widetilde{B}&=\left|\widehat{b}^\top\otimes e^{i\hbar/2}ba^{-1}\right|.
\end{aligned}\tag{5.3}$$

We know that $\operatorname{sign}b$ and Q commute. Therefore we may assume that u and y are eigenvectors of $\operatorname{sign}b$. Similarly we may assume that x and z are common eigenvectors of $\operatorname{sign}\widehat{b}$. Proceeding in the same way as in [21] we reduce (5.2) to the following three equations (cf. [21, formula (2.23) and next two in sequence]):

$$(x\otimes u|V_\theta(\log B)^*U|z\otimes y)=\Big(\overline{z}\otimes Qu\Big|V_\theta(\log\widetilde{B}-\pi i)\widetilde{U}\Big|\overline{x}\otimes Q^{-1}y\Big),\tag{5.4}$$

$$(x\otimes u|V_\theta(\log B-\pi i)^*U|z\otimes y)=\Big(\overline{z}\otimes Qu\Big|V_\theta(\log\widetilde{B})\widetilde{U}\Big|\overline{x}\otimes Q^{-1}y\Big),\tag{5.5}$$

$$\begin{aligned}
&\Big(x\otimes u\Big|\Big[i\Big(\widehat{\beta}\otimes\beta\Big)B^{\frac{\pi}{\hbar}}V_\theta(\log B-\pi i)\Big]^*U\Big|z\otimes y\Big)\\
&\qquad=\Big(\overline{z}\otimes Qu\Big|-i\Big(\widehat{\beta}^\top\otimes(iw)^\ell\beta\Big)\widetilde{B}^{\frac{\pi}{\hbar}}V_\theta(\log\widetilde{B}-\pi i)\widetilde{U}\Big|\overline{x}\otimes Q^{-1}y\Big).
\end{aligned}\tag{5.6}$$

In these formulae $\theta=\frac{2\pi}{\hbar}=2(\ell+3)$, The left hand side of the last formula

$$\text{LHS of (5.6)}=-i\Big(\widehat{\beta}x\otimes\beta u\Big|B^{\frac{\pi}{\hbar}}V_\theta(\log B-\pi i)^*U\Big|z\otimes y\Big)$$

To compute the right hand side we use anticommutativity of w and β: $(iw)^\ell\beta=(-i)^\ell\beta w^\ell=\overline{\alpha}\beta w^\ell$. We also know that β commutes with Q. Therefore

$$\text{RHS of (5.6)}=-i\overline{\alpha}\Big(\overline{z}\otimes Q\beta u\Big|\Big(\widehat{\beta}^\top\otimes w^\ell\Big)\widetilde{B}^{\frac{\pi}{\hbar}}V_\theta(\log\widetilde{B}-\pi i)\widetilde{U}\Big|\overline{x}\otimes Q^{-1}y\Big).$$

We shall move $\widehat{\beta}^\top\otimes w^\ell$ to the most right position. Clearly this operator commutes with $\widetilde{B}$ and $\widetilde{U}'=e^{\frac{i}{\hbar}\log\widehat{a}^\top\otimes\log a}$. Furthermore

$$\Big(\widehat{\beta}^\top\otimes w^\ell\Big)\,\mathrm{Ch}(\widehat{w}^\top\otimes I,I\otimes w)=\mathrm{Ch}\left(\widehat{w}^\top\otimes I,I\otimes w\right)\Big(\widehat{\beta}^\top\otimes I\Big)$$

Indeed for $w=1$ this formula is obvious, for $w=-1$ it reduces to $(-1)^\ell\widehat{\beta}^\top\widehat{w}^\top=\widehat{w}^\top\widehat{\beta}^\top$. The latter is equivalent to assumed relation $\widehat{\beta}\widehat{w}=\varepsilon\widehat{w}\widehat{\beta}=(-1)^\ell\widehat{w}\widehat{\beta}$. Now the right hand side of (5.6) takes the form:

$$\text{RHS of (5.6)}=-i\overline{\alpha}\Big(\overline{z}\otimes Q\beta u\Big|\widetilde{B}^{\frac{\pi}{\hbar}}V_\theta(\log\widetilde{B}-\pi i)\widetilde{U}\Big|\overline{\widehat{\beta}x}\otimes Q^{-1}y\Big).$$

Replacing $\widehat{\beta}x$ and βu by x and u respectively we see that (5.6) is equivalent to the equation

$$(x \otimes u \middle| \alpha B^{\frac{\pi}{\hbar}} V_\theta(\log B - \pi i)^* U \middle| z \otimes y) = \left(\overline{z} \otimes Qu \middle| \widetilde{B}^{\frac{\pi}{\hbar}} V_\theta(\log \widetilde{B} - \pi i)\widetilde{U} \middle| \overline{x} \otimes Q^{-1}y\right) \tag{5.7}$$

Let us notice that our crucial formulae (5.4), (5.5) and (5.7) fit the same pattern:

$$(x \otimes u | f_i(B)U | z \otimes y) = \left(\overline{z} \otimes Qu \middle| g_i(\widetilde{B})\widetilde{U} \middle| \overline{x} \otimes Q^{-1}y\right), \tag{5.8}$$

where f_i and g_i $(i = 1, 2, 3)$ are functions on positive reals:

$$\begin{aligned} f_1(t) &= \overline{V_\theta(\log t)}, & g_1(t) &= V_\theta(\log t - \pi i), \\ f_2(t) &= \overline{V_\theta(\log t - \pi i)}, & g_2(t) &= V_\theta(\log t), \\ f_3(t) &= \alpha t^{\frac{\pi}{\hbar}} \overline{V_\theta(\log t - \pi i)}, & g_3(t) &= t^{\frac{\pi}{\hbar}} V_\theta(\log t - \pi i). \end{aligned}$$

for all $t > 0$.

Replacing operators U and $\widetilde{U}$ by U' and $\widetilde{U}'$ (cf. (5.3)) we obtain a simplified version of (5.8):

$$(x \otimes u | f_k(B)U' | z \otimes y) = \left(\overline{z} \otimes Qu \middle| g_k(\widetilde{B})\widetilde{U}' \middle| \overline{x} \otimes Q^{-1}y\right). \tag{5.9}$$

It is known that the last equality holds in all three cases $k = 1, 2, 3$ (cf. [21, proof of Proposition 2.3]). We shall show that (5.8) follows from (5.9). Inserting in (5.9), $w^r z$ and $w^s y$ instead of z and y we obtain:

$$(x \otimes u | f_k(B)U' (w^r \otimes w^s) | z \otimes y) = \left(\overline{z} \otimes Qu \middle| g_k(\widetilde{B})\widetilde{U}' \left(\widehat{w}^{\top r} \otimes w^s\right) \middle| \overline{x} \otimes Q^{-1}y\right).$$

Multiplying both sides by $(-1)^{rs}$, summing over $r, s = 0, 1$ and using (4.3) we obtain (5.8). The proof is complete. □

We recall the basic definitions [2, 18, 9]. Let H be a Hilbert space and W be a unitary operator acting on $H \otimes H$. We say that W is multiplicative unitary if it satisfies the pentagonal equation

$$W_{23}W_{12} = W_{12}W_{13}W_{23}.$$

A multiplicative unitary W is said to be modular if there exist strictly positive selfadjoint operators $\widehat{Q}$ and Q acting on H and a unitary operator $\widetilde{W}$ acting on $\overline{H} \otimes H$ such that $\widehat{Q} \otimes Q$ commutes with W and

$$(x \otimes u | W | z \otimes y) = \left(\overline{z} \otimes Qu \middle| \widetilde{W} \middle| \overline{x} \otimes Q^{-1}y\right) \tag{5.10}$$

for any $x, z \in H$, $u \in \mathcal{D}(Q)$ and $y \in \mathcal{D}(Q^{-1})$. In this definition $\overline{H}$ is the complex conjugate Hilbert space related to H by an antiunitary mapping $H \ni x \longleftrightarrow \overline{x} \in \overline{H}$. The main result of this Section is contained in the following

Theorem 5.2. *The operator W introduced by* (4.19) *is a modular multiplicative unitary acting on* $L^2(\mathbb{R}, \mathbb{C}^2) \otimes L^2(\mathbb{R}, \mathbb{C}^2)$.

Proof. Assume that $K = L^2(\mathbb{R}, \mathbb{C}^2)$. One can easily verify that operators

$$\begin{aligned} \widehat{a} &= |b|^{-1}, & \widehat{\beta} &= (iw)^\ell \beta, \\ \widehat{b} &= e^{i\hbar/2} b^{-1} a, & \widehat{w} &= \gamma \end{aligned} \tag{5.11}$$

satisfy the requirements listed in the first part of Section 4. In particular $\widehat{\beta}\widehat{w} = \varepsilon \widehat{w}\widehat{\beta}$ with $\varepsilon = (-1)^\ell$. With this choice, right hand side of (4.4) coincides with that of (4.19): $V = W$ and relation (4.20) takes the form:

$$W_{23}W_{12} = W_{12}W_{13}W_{23}.$$

Hence W is a multiplicative unitary operator.

Let $Q = a^{1/2}$ and $\widehat{Q} = |b|^{1/2}$. Inserting in (5.1) operators (5.11) we obtain a unitary operator $\widetilde{W}$ satisfying formula (5.10). To end the proof we have to show that W commutes with $\widehat{Q} \otimes Q$.

We know that a commutes with β and w. One can easily check that $|b|$ commutes with $(iw)^\ell\beta$ and γ. Therefore $\widehat{Q} \otimes Q = |b|^{1/2} \otimes a^{1/2}$ commutes with $(iw)^\ell\beta \otimes \beta$, $\gamma \otimes I$ and $I \otimes w$. Clearly it commutes with $\log|b| \otimes \log a$. Moreover due to the Zakrzewski relation $a \overset{\hbar}{\multimap} b$, $\widehat{Q} \otimes Q$ commutes with $e^{i\hbar/2}b^{-1} \otimes b$. Inspecting formula (4.19) we see that $\widehat{Q} \otimes Q$ commutes with W. □

Now we can use the full power of the theory of multiplicative unitaries [2, 18, 9]. Denoting by $B(L^2(\mathbb{R},\mathbb{C}^2))_*$ the set of all normal functionals on $B(L^2(\mathbb{R},\mathbb{C}^2))$ we have:

$$A = \left\{(\omega\otimes\mathrm{id})W : \omega\in B(L^2(\mathbb{R},\mathbb{C}^2))_*\right\}^{\text{norm closure}}.$$

Indeed according to Theorem 1.5 of [18], the set on the right hand side is a C*-algebra generated by W and above equality follows immediately from Theorem 4.2 (in the present setting $V=W$).

Formula (4.20) shows that (4.4) is an adapted operator in the sense of [18, Definition 1.3]. Comparing (5.1) with Statement 5 of Theorem 1.6 of [18] one can easily find the unitary antipode R of our quantum group. It acts on a, b, β, w as follows:

$$\begin{aligned} a^R &= a^{-1}, & \beta^R &= -(iw)^\ell\beta,\\ b^R &= -e^{i\hbar/2}ba^{-1}, & w^R &= w. \end{aligned}$$

The action of the scaling group is described by the formulae:

$$\begin{aligned} \tau_t(a) &= a, & \tau_t(\beta) &= \beta,\\ \tau_t(b) &= e^{\hbar t}b, & \tau_t(w) &= w. \end{aligned}$$

In the following, Tr denotes the trace of operators acting on $L^2(\mathbb{R},\mathbb{C}^2)$ and tr denotes the trace of 2×2 matrices. For any positive $c\in A$ we set

$$h(c) = \mathrm{Tr}\left(\widehat{Q}c\widehat{Q}\right) = \mathrm{Tr}\left(|b|^{1/2}\,c\,|b|^{1/2}\right).$$

Let $c = g(\log a)\left[f_1(b)+\beta f_2(b)+wf_3(b)+w\beta f_4(b)\right]$, where f_1, f_2, f_3, f_4, $g\in C_\infty(\mathbb{R})$ and $f_2(0)=f_4(0)=0$. For any $\varrho>0$ we set:

$$\widetilde{g}(\varrho) = \frac{1}{2\pi\hbar}\int_{\mathbb{R}} g(\tau)\varrho^{i\tau/\hbar}d\tau.$$

One can verify that the operator $c\widehat{Q} = c\,|b|^{\frac12}$ is an integral operator:

$$(c\,|b|^{\frac12}\,x)(t') = \int_{\mathbb{R}} K_c(t',t)\,x(t)dt$$

with the kernel

$$K_c(t',t) = |t'|^{-1/2}\,\widetilde{g}(t'/t)\begin{pmatrix} f_1(t) & , & f_3(t)\\ f_3(t) & , & f_1(t)\end{pmatrix}$$

for $t't>0$ and

$$K_c(t',t) = |t'|^{-1/2}\,\widetilde{g}(-t'/t)\begin{pmatrix} f_2(t) & , & -f_4(t)\\ f_4(t) & , & -f_2(t)\end{pmatrix}$$

for $t't<0$. Therefore

$$\begin{aligned} h(c^*c) &= \int_{\mathbb{R}\times\mathbb{R}} \mathrm{tr}\left[K_c(t',t)^*K_c(t',t)\right]\,dt'dt\\ &= \int_0^\infty |\widetilde{g}(\varrho)|^2\,\frac{d\varrho}{\varrho}\int_{\mathbb{R}} 2\left(|f_1(t)|^2+|f_3(t)|^2+|f_2(t)|^2+|f_4(t)|^2\right)dt\\ &= \frac{1}{\pi\hbar}\int_{\mathbb{R}} |g(\tau)|^2\,d\tau\int_{\mathbb{R}}\left(|f_1(t)|^2+|f_3(t)|^2+|f_2(t)|^2+|f_4(t)|^2\right)dt. \end{aligned}$$

This formula shows that $\{c\in A : h(c^*c)<\infty\}$ is dense in A. According to the theory elaborated by Van Daele [11], h is a right Haar weight on our quantum group. See also [22], where the right invariance of h is verified by a straightforward computation.

One can easily construct the regular dual of our quantum group. By definition this is a quantum group related to the multiplicative unitary $\widehat{W} = \Sigma W^*\Sigma$ (see [2, 18]). For odd ℓ, the dual is antiisomorphic to the original group. More precisely there is a C*-algebra isomorphism $\psi : A \longrightarrow \widehat{A}$ such that $\widehat{\Delta}(\psi(c)) = \Sigma(\psi\otimes\psi)\Delta(c)\Sigma$ for any $c\in A$. For even ℓ the situation is even simpler. The details are left to the reader.

6. An open problem

Let $\hbar$ be a real number. We say that $\hbar$ is admissible if there exists a locally compact quantum group (A, Δ) in the sense of Kustermans and Vaes [3] (see also [5]), with selfadjoint elements a and $b \neq 0$ affiliated to A such that $a \overset{\hbar}{\multimap} b$ and $\Delta(b) \supset a \otimes b + b \otimes I$.

Replacing A by the opposite algebra we see that $-\hbar$ is admissible for any admissible $\hbar$. Results of [21] and of the present paper show that values $\hbar = \frac{\pi}{\ell+3}$ $(\ell = 0, 1, 2, 3, \dots)$ are admissible. Taking into account the third formulae of (3.28) and (4.22) we see that $\hbar = (\ell + 3)\pi$ are also admissible for the same values of ℓ. It would be interesting to determine the set of all admissible $\hbar$. As far as we know this is an open problem. We believe that this set is closed, nowhere dense in $\mathbb{R}$ and that $\hbar/\pi$ is rational for any admissible $\hbar$. The latter would mean that '$ax + b$' exists only *at roots of unity.*

7. Acknowledgment

The authors acknowledge the financial support of the Polish Committee for Scientific Research (KBN, grant 2 P03A 040 22) and of the Foundation for Polish Science.

References

[1] W. Arveson: An invitation to C^*-Algebra, Springer-Verlag New York, Heidelberg, Berlin, 1976
[2] S. Baaj et G. Skandalis, *Unitaries multiplicatifs et dualité pour les produits croisé de C^*-algèbres,* Ann. Sci. Ec. Norm. Sup., 4^e série, **26** (1993), 425-488.
[3] J. Kustermans and S. Vaes: Locally compact quantum groups. *Annales Scientifiques de l'Ecole Normale Suprieure.* 33 (6) (2000), 837–934.
[4] T. Masuda and Y. Nakagami, *A von Neumann algebra framework for the duality of the quantum groups,* Publ. RIMS Kyoto Univ., **30** (1994), 799-850.
[5] T. Masuda, Y. Nakagami and S.L. Woronowicz: *A C^*-algebraic framework for the quantum groups,* International Journal of Mathematics Vol **14**, No 9 (2003), 903-1001
[6] K. Napiórkowski and S.L. Woronowicz: *Operator theory in the C^*-algebra framework.* Reports on Mathematical Physics, **31**, No. 3 (1992) 353–371.
[7] Gert K. Pedersen: C*-algebras and their Automorphism Groups, Academic Press, London, New York, San Francisco 1979.
[8] M. Rowicka: *Exponential Equations Related to the Quantum '$ax + b$' Group,* Communication in Mathematical Physics, **244**, (2004) 419–453.
[9] P.M. Sołtan and S.L. Woronowicz, *A remark on manageable multiplicative unitaries,* Letters on Math. Phys., **57** (2001), 239 - 252.
[10] P.M. Sołtan, *New deformations of the group of affine transformations of the plane,* Doctor dissertation (in Polish), University of Warsaw (2003).
[11] A. Van Daele: The Haar measure on some locally compact quantum groups, Preprint of the K.U. Leuven (2000), arXiv:math.OA/0109004v1 (2001).
[12] A. Van Daele and S.L. Woronowicz: *Duality for the quantum $E(2)$ group.* Pacific Journal of Mathematics **173** (1996) 375 - 385
[13] S.L. Woronowicz, *Unbounded elements affiliated with C^*-algebras and non-compact quantum groups,* Commun. Math. Phys. **136** (1991), 399–432.
[14] S.L. Woronowicz: Compact matrix pseudogroup. Commun. math. Physics 111 (1987), 613–665
[15] S.L. Woronowicz: C*-algebras generated by unbounded elements. Reviews on Mathematical Physics, Vol. **7**, No. 3, (1995) 481 - 521
[16] S.L. Woronowicz: *Quantum $E(2)$ group and its Pontryagin dual.* Letters in Mathematical Physics **23** (1991) 251–263
[17] S.L. Woronowicz: *Operator equalities related to quantum E(2) group.* Communications in mathematical Physics **144** (1992) 417–428
[18] S.L. Woronowicz, *From multiplicative unitaries to quantum groups,* International Journal of Mathematics, **7** (1996), 127-149.
[19] S.L. Woronowicz, *Quantum exponential function,* Reviews in Mathematical Physics, Vol. **12**, No. 6 (2000), 873 – 920.
[20] S.L. Woronowicz, *Quantum '$az + b$' group on complex plane,* International Journal of Mathematics, **12** (2001), 461-503.
[21] S.L. Woronowicz and S. Zakrzewski, *Quantum '$ax + b$' group,* Reviews in Mathematical Physics, **14**, Nos 7 & 8 (2002), 797 - 828.
[22] S.L. Woronowicz *Haar weight on some quantum groups.* Group 24: Physical and mathematical aspects of symmetries, Proceedings of the 24th International Colloquium on Group Theoretical Methods in Physics Paris, 15 - 20 July 2002, Institute of Physics, Conference Series Number **173**, 763 – 772.

(W. PUSZ) Department of Mathematical Methods in Physics, Faculty of Physics, University of Warsaw, Hoża 74, 00-682 Warszawa, Poland.
E-mail address, W. PUSZ: `wieslaw.pusz@fuw.edu.pl`

(S. L. WORONOWICZ) Department of Mathematical Methods in Physics, Faculty of Physics, University of Warsaw, Hoża 74, 00-682 Warszawa, Poland.
E-mail address, S. L. WORONOWICZ: `stanislaw.woronowicz@fuw.edu.pl`