QUANTUM '$az + b$' GROUP ON COMPLEX PLANE.

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ABSTRACT. '$az + b$' is the group of affine transformations of complex plane $\mathbb{C}$. The coefficients $a, b \in \mathbb{C}$. In quantum version a, b are normal operators such that $ab = q^2 ba$, where q is the deformation parameter. We shall assume that q is a root of unity. More precisely $q = e^{\frac{2\pi i}{N}}$, where N is an even natural number. To construct the group we write an explicit formula for the Kac Takesaki operator W. It is shown that W is a manageable multiplicative unitary in the sense of [3, 18]. Then using the general theory we construct a C*-algebra A and a comultiplication $\Delta \in \mathrm{Mor}(A, A \otimes A)$. A should be interpreted as the algebra of all continuous functions vanishing at infinity on quantum '$az+b$'- group. The group structure of is encoded by Δ. The existence of coinverse also follows from the general theory [18]. In the appendix, we briefly discuss the case of real q.

0. INTRODUCTION.

The group '$az + b$' considere in this paper is the group of affine transformations of the complex plane $\mathbb{C}$. The group will be denoted by G. The *-algebra $\mathcal{A}$ of polynomial functions on G is generated by three normal commuting elements a, a^{-1}, b subject to the one relation: $a^{-1}a = I$. The comultiplication Δ encoding the group structure is the *-algebra homomorphism from $\mathcal{A}$ into $\mathcal{A} \otimes \mathcal{A}$ such that

$$\begin{aligned} \Delta(a) &= a \otimes a, \\ \Delta(b) &= a \otimes b + b \otimes I. \end{aligned} \tag{0.1}$$

One can easily verify that $(\mathcal{A}, \Delta)$ is a Hopf *-algebra. In particular counit e and coinverse κ are given by the formulae:

$$\begin{aligned} e(a) &= 1, & \kappa(a) &= a^{-1}, \\ e(b) &= 0, & \kappa(b) &= -a^{-1}b. \end{aligned} \tag{0.2}$$

Now we perform quantum deformation of G. The quantum '$az + b$'-group on the level of Hopf *-algebra is an object with no problems. The deformation parameter q is a complex number of modulus 1.

Let $\mathcal{A}_o$ be the *-algebra generated by three elements a, a^{-1}, b subject to the following relations:

$$\begin{aligned} a^{-1}a &= aa^{-1} = 1, \\ aa^* &= a^*a, \ bb^* = b^*b, \\ ab &= q^2 ba, \ ab^* = b^*a. \end{aligned} \tag{0.3}$$

The comultiplication $\Delta : \mathcal{A}_o \to \mathcal{A}_o \otimes \mathcal{A}_o$ is the *-algebra homomorphism acting on generators in the way described in (0.1).

In what follows[1], q is a root of unity of the form

$$q = e^{\frac{2\pi i}{N}}. \tag{0.4}$$

In this paper N is an even natural number. The case when N is odd seems to be much more complicated. To justify certain computations (the one in Appendix B) we shall assume that $N \geq 6$. However it is not difficult to show that all the results remain in force for $N = 4$. The case $N = 2$ should be treated separately.

[1]except the Appendix A, where the case of real q is considered

By (0.3), $a^{\frac{N}{2}}$ and $b^{\frac{N}{2}}$ commute with all elements of $\mathcal{A}_o$. So do $\left(a^{\frac{N}{2}}\right)^*$ and $\left(b^{\frac{N}{2}}\right)^*$. Using the q-deformed binomial coefficient, one can easily verify that:

$$\Delta\left(a^{\frac{N}{2}}\right) = a^{\frac{N}{2}} \otimes a^{\frac{N}{2}},$$

$$\Delta\left(b^{\frac{N}{2}}\right) = a^{\frac{N}{2}} \otimes b^{\frac{N}{2}} + b^{\frac{N}{2}} \otimes I.$$

If $a^{\frac{N}{2}}$ and $b^{\frac{N}{2}}$ are selfadjoint, so are $\Delta\left(a^{\frac{N}{2}}\right)$ and $\Delta\left(b^{\frac{N}{2}}\right)$. It means that the commutation relations (0.3) and comultiplication (0.1) are compatible with the relations:

$$\left(a^{\frac{N}{2}}\right)^* = a^{\frac{N}{2}}, \qquad \left(b^{\frac{N}{2}}\right)^* = b^{\frac{N}{2}}. \tag{0.5}$$

In other words the *-algebra $\mathcal{A}$ generated by three elements a, a^{-1}, b subject to the relations (0.3) and (0.5) is equipped with the comultiplication Δ such that the formulae (0.1) hold. One can easily verify that the object $(\mathcal{A}, \Delta)$ introduced above is a Hopf *-algebra. The counit e and coinverse κ are given by the same formulae (0.2) as in the classical case. Moreover the matrix

$$w = \left(\begin{array}{cc} a & , & b \\ 0 & , & I \end{array}\right)$$

is a corepresentation of $(\mathcal{A}, \Delta)$.

By definition the Hopf *-algebra $(\mathcal{A}, \Delta)$ is the algebra of polynomials on the quantum '$az + b$' - group. Therefore w is a two dimensional representation of this group.

On the Hilbert space level, the generators a, a^{-1} and b should be treated as unbounded[2] operators acting on a Hilbert space. Since for unbounded operators the algebraic operations are often ill defined, one has to give a more precise meaning to the formulae (0.3). Moreover the relations (0.5) give rise to a spectral condition localizing the spectra of a and b. This condition is necessary to obtain the comultiplication on the C*-level (cf [16] where the similar condition was used to construct the quantum $E(2)$ - group). This is why we included relations (0.5) in the definition of $\mathcal{A}$.

Let q be the number introduced by (0.4). The reader should notice that $q^{\mathbb{Z}}$ is a finite set: $q^{\mathbb{Z}} = \left\{1, q, q^2, \ldots q^{N-1}\right\}$. To formulate the spectral condition in the convenient way, we shall use the following notation:

$$\mathbb{R}_+ = \left\{r \in \mathbb{R} : r > 0\right\},$$

$$\Gamma = \left\{z \in \mathbb{C} - \{0\} : \text{Phase } z \in q^{\mathbb{Z}}\right\} = \bigcup_{k=0}^{N-1} q^k \mathbb{R}_+, \tag{0.6}$$

$$\overline{\Gamma} = \Gamma \cup \{0\} = \left\{z \in \mathbb{C} : z^{\frac{N}{2}} \in \mathbb{R}\right\}.$$

Clearly Γ is a multiplicative subgroup of $\{z \in \mathbb{C} : z \neq 0\}$ and $\overline{\Gamma}$ is the closure of Γ in $\mathbb{C}$.

Now we are able to formulate our commutation relations: let H be a Hilbert space and a and b be closed operators acting on H. We say that (a, b) is a G-pair if

$$\left.\begin{array}{c} a, b \text{ are normal operators,} \\ \text{Sp } a, \text{Sp } b \subset \overline{\Gamma}, \ker a = \{0\}, \\ (\text{Phase } a)b(\text{Phase } a)^* = qb \\ \text{and for any } t \in \mathbb{R} : \\ |a|^{-it}b|a|^{it} = e^{\frac{2\pi}{N}t}b. \end{array}\right\} \tag{0.7}$$

The set of all pair G-pairs acting on a Hilbert space H will be denoted by G_H. Performing the analytic continuation up to the point $t = i$ we obtain: $|a|b = qb|a|$. Now the connection of (0.3) and (0.5) with (0.7) is clear. In particular relations (0.5) correspond to the spectral condition $\text{Sp } a, \text{Sp } b \subset \overline{\Gamma}$.

We use the terminology introduced in [17]. By the procedure described in [17, Sec. 7], relations (0.7) gives rise to a C*-algebra A. This C*-algebra is generated by three unbounded elements a, a^{-1}, b affiliated with it and

$$\pi \longleftrightarrow \left(\pi(a), \pi(b)\right) \tag{0.8}$$

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[2] one can easily check that relations (0.3) cannot be satisfied by bounded operators a and $b \neq 0$

defines continuous (in both directions) one to one correspondence between the set $\text{Rep}(A, H)$ of all representation of A acting on a Hilbert space H and the set G_H of all G-pairs acting on H.

We shall give an explicit construction of the algebra A. We shall also prove that there exists a comultiplication $\Delta \in \text{Mor}(A, A \otimes A)$ such that

$$
\begin{aligned}
\Delta(a) &= a \otimes a, \\
\Delta(b) &= a \otimes b \dotplus b \otimes I.
\end{aligned}
\tag{0.9}
$$

In this formula '$\dotplus$' denotes the sum of operators (defined on the intersection of their domains) followed by the closure.

We shall briefly describe the content of the paper. Sections 1 and 2 are devoted to the basic framework used in our theory. In Section 1 we introduce and investigate special function F_N playing the fundamental role in our computations. This function is defined on $\overline{\Gamma}$ and its values $F_N(\gamma)$ are complex number of modulus 1. We find a functional equation characterizing this function. Finally we present a formula expressing the Fourier transform of F_N by a holomorphic continuation of F_N. The proof of this formula is given in Appendix B. In Section 2 we consider pairs of normal operators (R, S) satisfying certain commutation relations, more restrictive than (0.7). In particular R and S have spectra contained in $\overline{\Gamma}$. We prove that the closure of the sum $R + S$ is normal and have the spectrum contained in the same set. Moreover the function F_N satisfies the following exponential equation:

$$
F_N(R \dotplus S) = F_N(R)F_N(S). \tag{0.10}
$$

The multiplicative unitary for the quantum '$az+b$'-group is constructed in Section 3. We start with a G-pair (a, b). The simplest operator satisfying the pentagonal equation of Baaj and Skandalis [3] is the one given by:

$$
W = F_N\left(ab^{-1} \otimes b\right) \chi\left(b^{-1} \otimes I, I \otimes a\right). \tag{0.11}
$$

Unfortunately this operator is not manageable in the sense of [18]. The operator Q appearing in the definition of manageability must be of the form $Q = r|a|$, where r is a strictly positive operator commuting with a and b. The point is that the operator W introduced by (0.11) does not commute with $Q \otimes Q = r|a| \otimes r|a|$ as it is required in the definition of manageability. To overcome this problem we modify slightly the formula (0.11) replacing it by (3.2). In Section 3 we show that operator (3.2) is a manageable multiplicative unitary. The proof of pentagonal equation is based on (0.10); to verify manageability we use the formula for the Fourier transform of F_N presented in Section 1.

In Section 4 we describe in details the crossed product algebra $A_{cp} = C_\infty(\overline{\Gamma}) \times_\sigma \Gamma$. We prove that this is the C*-algebra corresponding to the relations (0.7). In Section 6 A_{cp} is identified with the C*-algebra A of *continuous functions vanishing at infinity on* G. The latter algebra is computed according to the theory of multiplicative unitaries [3, 18]. By definition A is the norm closure of $\{(\text{id} \otimes \omega)W : \omega \in B(H)\}$. To prove the equality $A = A_{cp}$ we use some elementary facts concerning special functions, affiliation relation and generators of C*-algebras. These facts are collected in Section 5.

We would like to point out the further development of the subject. A. Van Daele [12] has found left and right invariant Haar weights on quantum '$az + b$'-group constructed in this paper. He has shown that the quantum '$az + b$'-group is a locally compact quantum group in the sense of Kustermans and Vaes [6]. It turned out that the Haar weights are scaled by the scaling group (6.14) in a non-trivial way. This is the first example of this phenomena. It was foreseen by the theory of Kustermans and Vaes, however some of the experts believed that in the proper theory the Haar weights should be invariant with respect to the scaling group.

Next topic that seems to be interesting is the double group construction [9, 21]. This is the dual version of quantum double of Drinfeld [5], that suits better to the C^*-language than the original Drinfeld construction. Applying the double group construction to the quantum '$az + b$' we should obtain a quantum deformation of $GL(2, \mathbb{C})$-group. This is the work in progress [10]. The first step in this direction is done in Section 7, where the dual of the quantum '$az + b$' is investigated.

Let us discuss the other possible values of the deformation parameter q. We believe that after not very essential changes the theory will work for all primitive roots of unity of even order. The

case of odd N seems to be much more difficult. The most likely it will require the technique of reflection operators used in the theory of quantum '$ax + b$'-group[3] [22].

In general the parameter q need not to be of modulus 1. The case of real q is considered in Appendix A. The special function and operator framework for this case has been created in [15, 16]. Then the group Γ consists of system of concentric circles. For general possible q, Γ is composed of a finite number of logarithmic spirals. The work is in progress [11].

In this paper we intensively use unbounded operators acting on a Hilbert space and C^*-algebras generated by unbounded elements. We refer to [1, 17, 20] for the basic concepts and results.

Let R, S be closed operators acting on a Hilbert space H. $R + S$ and $R \circ S$ will denote the sum and the composition of operators R and S. By definition $D(R + S) = D(R) \cap D(S)$ and $D(R \circ S) = \{x \in D(S) : Sx \in D(R)\}$. If $R + S$ is densely defined and closeable then its closure will be denoted by $R \dotplus S$. Similarly if $R \circ S$ is densely defined and closeable then its closure will be denoted by RS.

Let R be a closed operator acting on a Hilbert space. We recall that R is called normal if $RR^* = R^*R$. It means that $D(R) = D(R^*)$ and $\|R^*x\| = \|Rx\|$ for all $x \in D(R)$. Normal operators have no normal extensions. Indeed, if $R \subset S$ and R, S are normal then $S^* \subset R^*$, $D(S) = D(S^*) \subset D(R^*) = D(R) \subset D(S)$ and $S = R$.

We shall use functional calculus of normal operators: Any normal operator R is of the form $R = \int_\Lambda \lambda dE_R(\lambda)$, where $\Lambda = \operatorname{Sp} R$ is the spectrum of R and dE_R is the spectral measure related to R. Then for any function F on Λ we set

$$F(R) = \int_\Lambda F(\lambda)dE_R(\lambda).$$

One may also consider functions of two strongly commuting normal operators. In particular for any normal operators a, b and any function χ on $\Lambda = \operatorname{Sp} a \times \operatorname{Sp} b$ we set

$$\chi(b^{-1} \otimes I, I \otimes a) = \int_\Lambda \chi(\lambda^{-1}, \lambda')dE_b(\lambda) \otimes dE_a(\lambda').$$

1. SPECIAL FUNCTIONS.

Let Γ be the multiplicative subgroup of $\{z \in \mathbb{C} : z \neq 0\}$ introduced by (0.6). For any $\gamma, \gamma' \in \Gamma$ we set

$$\chi(\gamma, \gamma') = q^{kk'} e^{\frac{N}{2\pi i}(\log r)(\log r')},$$

where k, k' are integers and r, r' are strictly positive real numbers such that $\gamma = q^k r$ and $\gamma' = q^{k'} r'$. Then χ is a bicharacter on Γ: for any $\gamma, \gamma', \gamma'' \in \Gamma$ we have:

$$\begin{array}{rcl}
|\chi(\gamma, \gamma')| & = & 1, \\
\chi(\gamma\gamma', \gamma'') & = & \chi(\gamma, \gamma'')\chi(\gamma', \gamma''), \\
\chi(\gamma, \gamma'\gamma'') & = & \chi(\gamma, \gamma')\chi(\gamma, \gamma''), \\
\chi(\gamma, \gamma') & = & \chi(\gamma', \gamma).
\end{array} \tag{1.1}$$

Moreover the family of characters $\{\chi(\cdot, \gamma) : \gamma \in \Gamma\}$ separates points of Γ: if $\chi(\gamma', \gamma) = \chi(\gamma'', \gamma)$ for all $\gamma \in \Gamma$, then $\gamma' = \gamma''$. One can easily verify that

$$\chi(\gamma, \gamma') = \frac{\alpha(\gamma\gamma')}{\alpha(\gamma)\alpha(\gamma')}, \tag{1.2}$$

where

$$\alpha(q^k r) = e^{\frac{\pi i}{N}k^2} e^{\frac{N}{4\pi i}(\log r)^2} \tag{1.3}$$

for any $k \in \mathbb{Z}$ and $r > 0$. The reader should notice that the right hand side of the last equation remains unchanged, when k is replaced by $k + N$ (we recall that N is even).

Any character on Γ is determined by its values at $q \in \Gamma$ and on $\mathbb{R}_+ \subset \Gamma$. By simple computations:

$$\chi(\gamma, q) = \operatorname{Phase} \gamma,$$
$$\chi(\gamma, r) = |\gamma|^{\frac{N}{2\pi i} \log r} \tag{1.4}$$

for any $r \in \mathbb{R}_+$ and $\gamma \in \Gamma$.

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[3]This is the group of affine transformations of real line.

The following function will play an important role:

$$
F_N(q^k r) = \begin{cases} \prod_{s=1}^{\frac{k}{2}} \left(\frac{1+q^{2s}r}{1+q^{-2s}r} \right) \frac{f_0(qr)}{1+r} & \text{for } k \text{ - even} \\[3em] \prod_{s=0}^{\frac{k-1}{2}} \left(\frac{1+q^{2s+1}r}{1+q^{-2s-1}r} \right) f_0(r) & \text{for } k \text{ - odd} \end{cases} \tag{1.5}
$$

where

$$
f_0(z) = \exp\left\{ \frac{1}{\pi i} \int_0^\infty \log(1+a^{-\frac{N}{2}}) \frac{da}{a+z^{-1}} \right\} \tag{1.6}
$$

for any $z \in \mathbb{C}$ such that $z \neq 0$ and Phase $z \neq -1$. Let us notice that for any integer s',

$$
\prod_{s=1}^{\frac{N}{2}} \frac{1+q^{2s+s'}r}{1+q^{-2s-s'}r} = 1.
$$

This formula shows that the right hand side of (1.5) depends only on $k \pmod N$ and r. It means that (1.5) defines a function on Γ. Setting $F_N(0) = 0$ we extend it to $\overline{\Gamma}$. One can easily verify that $F_N \in C\left(\overline{\Gamma}\right)$.

Proposition 1.1. *For any $\gamma \in \Gamma$ we have:*

$$
|F_N(\gamma)| = 1 \tag{1.7}
$$

$$
F_N\left(q^{-2}\gamma\right) F_N\left(\gamma^{-1}\right) = e^{\frac{\pi}{6i}\left(\frac{2}{N}+\frac{N}{2}\right)} \alpha\left(q^{-1}\gamma\right), \tag{1.8}
$$

$$
\left(1+q^{-1}\overline{\gamma}\right) F_N(q\gamma) = (1+q\gamma) F_N\left(q^{-1}\gamma\right). \tag{1.9}
$$

Moreover $F_N(0) = 1$

Proof. The function f_0 introduced by (1.6) is related to the function V_θ investigated in [19] in the following way:

$$
f_0(z) = V_{\frac{N}{2}}(\log z)^2.
$$

Taking into account the formulae (1.31), (1.32) and (1.34) of [19] we have:

$$
f_0(q^2 z) = (1+qz)^2 \, f_0(z), \tag{1.10}
$$

$$
\overline{f_0(\overline{z})} f_0(z) = 1, \tag{1.11}
$$

$$
f_0(z) f_0\left(z^{-1}\right) = C \, e^{\frac{N}{4\pi i}(\log z)^2}, \tag{1.12}
$$

where $C = e^{\left(\frac{N}{2}+\frac{2}{N}\right)\frac{\pi}{6i}}$.

Remembering that in the formula (1.5), $r \in \mathbb{R}$ and $|q| = 1$ we obtain:

$$
\left|F_N(q^k r)\right| = \begin{cases} \left| \dfrac{f_0(qr)}{1+r} \right| & \text{for } k \text{ - even} \\[2em] |f_0(r)| & \text{for } k \text{ - odd} \end{cases}
$$

Inserting in (1.10) and (1.11) $z = q^{-1}r$, where $r > 0$ we get:

$$
f_0(qr) = (1+r)^2 f_0\left(q^{-1}r\right), \qquad \overline{f_0(qr)} f_0\left(q^{-1}r\right) = 1. \tag{1.13}
$$

Therefore $f_0(qr)\overline{f_0(qr)} = (1+r)^2$ and $|f_0(qr)| = 1+r$. On the other hand it follows directly from (1.6) that $|f_0(r)| = 1$. Using these relations one can easily verify equality (1.7).

The proof of (1.8) is purely computational. Let $\gamma = q^k r$. Then $q^{-2}\gamma = q^{k-2}r$ and $\gamma^{-1} = q^{N-k}r^{-1}$. The reader should notice that (1.8) remains unchanged, when $(k-2, r)$ is replaced by $(N-k, r^{-1})$. Therefore we may assume that $k - 2 \leq N - k$. For even k we have:

$$
F_N\left(q^{-2}\gamma\right) F_N\left(\gamma^{-1}\right) = \frac{f_0(qr)f_0(qr^{-1})}{(1+r)(1+r^{-1})} \prod_{s=1}^{\frac{k}{2}-1}\left(\frac{1+q^{2s}r}{1+q^{-2s}r}\right) \prod_{s=1}^{\frac{N-k}{2}}\left(\frac{1+q^{2s}r^{-1}}{1+q^{-2s}r^{-1}}\right)
$$

$$
= \frac{rf_0(qr)f_0(qr^{-1})}{(1+r)^2} \prod_{s=1}^{\frac{k}{2}-1}\left(\frac{1+q^{2s}r}{1+q^{-2s}r}\right) \prod_{s=1}^{\frac{N-k}{2}} q^{4s}\left(\frac{1+q^{-2s}r}{1+q^{2s}r}\right).
$$

By virtue of (1.13) and (1.12) we have:

$$\frac{rf_0(qr)f_0(qr^{-1})}{(1+r)^2} = rf_0\left(q^{-1}r\right)f_0\left(qr^{-1}\right) = Cre^{\frac{N}{4\pi i}\left(\log r - \frac{2\pi i}{N}\right)^2}$$

$$= Cre^{\frac{N}{4\pi i}(\log r)^2 - \log r + \frac{\pi i}{N}} = Ce^{\frac{\pi i}{N}}e^{\frac{N}{4\pi i}(\log r)^2}.$$

By elementary computations:

$$e^{\frac{\pi i}{N}}\prod_{s=1}^{\frac{N-k}{2}}q^{4s} = q^{\frac{1}{2}(k-1)^2}.$$

Finally

$$\prod_{s=1}^{\frac{k}{2}-1}\left(\frac{1+q^{2s}r}{1+q^{-2s}r}\right)\prod_{s=1}^{\frac{N-k}{2}}\left(\frac{1+q^{-2s}r}{1+q^{2s}r}\right) = \prod_{s=\frac{k}{2}}^{\frac{N-k}{2}}\left(\frac{1+q^{-2s}r}{1+q^{2s}r}\right) = 1$$

Inserting these results into the main computations we get:

$$F_N\left(q^{-2}\gamma\right)F_N\left(\gamma^{-1}\right) = C\,q^{\frac{1}{2}(k-1)^2}\,e^{\frac{N}{4\pi i}(\log r)^2} = C\alpha\left(q^{-1}\gamma\right)$$

and (1.8) follows. For odd k the computations are simpler:

$$F_N\left(q^{-2}\gamma\right)F_N\left(\gamma^{-1}\right) = f_0(r)f_0(r^{-1})\prod_{s=0}^{\frac{k-1}{2}-1}\left(\frac{1+q^{2s+1}r}{1+q^{-2s-1}r}\right)\prod_{s=0}^{\frac{N-k-1}{2}}\left(\frac{1+q^{2s+1}r^{-1}}{1+q^{-2s-1}r^{-1}}\right)$$

$$= C\,e^{\frac{N}{4\pi i}(\log r)^2}\prod_{s=0}^{\frac{k-1}{2}-1}\left(\frac{1+q^{2s+1}r}{1+q^{-2s-1}r}\right)\prod_{s=0}^{\frac{N-k-1}{2}}q^{4s+2}\left(\frac{1+q^{-2s-1}r}{1+q^{2s+1}r}\right)$$

$$= C\,e^{\frac{N}{4\pi i}(\log r)^2}\,q^{\frac{1}{2}(N-k+1)^2}\prod_{s=\frac{k-1}{2}}^{\frac{N-k-1}{2}}\left(\frac{1+q^{-2s-1}r}{1+q^{2s+1}r}\right)$$

$$= C\,q^{\frac{1}{2}(k-1)^2}\,e^{\frac{N}{4\pi i}(\log r)^2} = C\alpha\left(q^{-1}\gamma\right).$$

The formula (1.8) is proved.

It follows immediately from (1.5) that

$$\frac{F_N(q^k r)}{F_N(q^{k-2}r)} = \frac{1+q^k r}{1+q^{-k}r}.$$

Replacing in this formula $q^{k-1}r$ by γ we obtain (1.9).

<div style="text-align: right">□</div>

Remark: Formula (1.8) shows that for $\gamma \to \infty$ the function F_N has the following asymptotic behavior:

$$F_N(\gamma) \approx e^{\frac{\pi}{6i}\left(\frac{2}{N}+\frac{N}{2}\right)}\alpha\left(q\gamma\right). \tag{1.14}$$

The following Theorem reveals the analytic properties of F_N.

Theorem 1.2. *Let $\Omega = \left\{z \in \mathbb{C} : z \neq 0 \text{ and } 0 < \arg z < \frac{2\pi}{N}\right\}$ and $\overline{\Omega}$ be the closure of Ω in $\mathbb{C}$. Then there exists a continuous function $\widetilde{F}$ defined on $\overline{\Omega} \times q^{\mathbb{Z}}$ such that:*

1. *For any fixed $k \in \mathbb{Z}$ the function $z \longrightarrow \widetilde{F}(z, q^k)$ is holomorphic on Ω.*

2. *For any nonnegative real r and any $k \in \mathbb{Z}$ we have:*

$$\widetilde{F}(r, q^k) = F_N(q^k r), \tag{1.15}$$

$$\widetilde{F}(qr, q^k) = \left(1+q^{-k-1}r\right)F_N(q^{k+1}r) = \left(1+q^{k+1}r\right)F_N(q^{k-1}r). \tag{1.16}$$

3. *The value $\widetilde{F}(z, q^k) \neq 0$ for any $z \in \Omega$ and $k \in \mathbb{Z}$.*

4. *Asymptotic behavior: for any $k \in \mathbb{Z}$:*

$$|\widetilde{F}(z, q^k)| = |z|^{\frac{N}{2\pi}\arg z}\,\Theta_k(z), \tag{1.17}$$

where $\displaystyle\lim_{z \to \infty}\Theta_k(z) = 1$.

Proof. For any $z \in \overline{\Omega}$ we set:

$$\widetilde{F}(z, q^k) = \begin{cases} \displaystyle\prod_{s=1}^{\frac{k}{2}} \left(\frac{1 + q^{2s}z}{1 + q^{-2s}z}\right) \frac{f_0(qz)}{1 + z} & \text{for } k \text{ - even} \\[4mm] \displaystyle\prod_{s=0}^{\frac{k-1}{2}} \left(\frac{1 + q^{2s+1}z}{1 + q^{-2s-1}z}\right) f_0(z) & \text{for } k \text{ - odd} \end{cases} \tag{1.18}$$

Then $\widetilde{F}$ is a continuous function on $\overline{\Omega} \times q^{\mathbb{Z}}$, for fixed $k \in \mathbb{Z}$, $\widetilde{F}(\cdot, q^k)$ is holomorphic on Ω and (1.15) holds. We shall prove (1.16). One has to notice that the second equality in this formula follows from (1.9). Let $k \in \mathbb{Z}$. If k is even, then using (1.10) and (1.9) we have:

$$\widetilde{F}(qr, q^k) = \prod_{s=1}^{\frac{k}{2}} \left(\frac{1 + q^{2s+1}r}{1 + q^{-2s+1}r}\right) \frac{f_0(q^2 r)}{1 + qr} = \prod_{s=1}^{\frac{k}{2}} \left(\frac{1 + q^{2s+1}r}{1 + q^{-2s+1}r}\right) (1 + qr) f_0(r)$$

$$= \left(1 + q^{-k-1}r\right) \prod_{s=0}^{\frac{k}{2}} \left(\frac{1 + q^{2s+1}r}{1 + q^{-2s-1}r}\right) f_0(r) = \left(1 + q^{-k-1}r\right) F_N(q^{k+1}r).$$

If k is odd, then:

$$\widetilde{F}(qr, q^k) = \prod_{s=0}^{\frac{k-1}{2}} \left(\frac{1 + q^{2s+2}r}{1 + q^{-2s}r}\right) f_0(qr) = \left(1 + q^{k+1}r\right) \prod_{s=1}^{\frac{k-1}{2}} \left(\frac{1 + q^{2s}r}{1 + q^{-2s}r}\right) \frac{f_0(qr)}{1 + r}$$

$$= \left(1 + q^{k+1}r\right) F_N(q^{k-1}r).$$

Formula (1.16) is proven.

To prove Statement 3 we notice that the function (1.6) has no zero in the region $0 \leq \arg z \leq \frac{4\pi}{N}$. Moreover the factors $1 + q^{s'}z \neq 0$ for any $z \in \Omega$. Inspecting the formula (1.18) we see that the function $\widetilde{F}$ has no zero in $\Omega \times \overline{\Gamma}$.

To prove Statement 4 we rewrite formula (1.8) in the following way:

$$F_N\left(q^{-2}\gamma\right) = e^{\frac{\pi}{6i}\left(\frac{2}{N} + \frac{N}{2}\right)} \alpha\left(q^{-1}\gamma\right) \overline{F_N\left(\gamma^{-1}\right)}.$$

Inserting $\gamma = q^{k+2}r$ (where $k \in \mathbb{Z}$ and $r \in \mathbb{R}_+$) and using (1.15) we obtain:

$$\widetilde{F}\left(r, q^k\right) = e^{\frac{\pi}{6i}\left(\frac{2}{N} + \frac{N}{2}\right)} \alpha\left(q^{k+1}r\right) \overline{\widetilde{F}\left(r^{-1}, q^{-(k+2)}\right)}.$$

This relation holds for $r \in \mathbb{R}_+$. Performing the holomorphic continuation into the region Ω we obtain

$$\widetilde{F}\left(z, q^k\right) = e^{\frac{\pi}{6i}\left(\frac{2}{N} + \frac{N}{2}\right)} e^{\frac{\pi i}{N}(k+1)^2} e^{\frac{N}{4\pi i}(\log z)^2} \overline{\widetilde{F}\left(z^{-1}, q^{-(k+2)}\right)}.$$

One can easily check that $\left|e^{\frac{N}{4\pi i}(\log z)^2}\right| = |z|^{\frac{N}{2\pi}\arg z}$. This way we obtain (1.17) with

$$\Theta_k(z) = \left|\widetilde{F}\left(\overline{z^{-1}}, q^{-(k+2)}\right)\right|.$$

When $z \to \infty$ then $\overline{z^{-1}} \to 0$ and $\Theta_k(z) \to \left|\widetilde{F}(0, q^{-(k+2)})\right| = 1$.

$\square$

The function $f_0(\cdot)$ introduced by (1.6) is holomorphic on $\{z \in \mathbb{C} : \text{Phase } z \neq -1\}$. i.e. on complex plane with the cut along negative real axis. By (1.18) the functions $\widetilde{F}(\cdot, q^k)$ are meromorphic on the same region. Clearly the formula (1.17) remains valid for $-\pi < \arg z < \pi$. We shall use this fact in Appendix B.

The Haar measure on Γ will be denoted by $d\gamma$:

$$\int_\Gamma f(\gamma)\, d\gamma = \sum_{k=0}^{N-1} \int_0^\infty f\left(q^k r\right) \frac{dr}{r}. \tag{1.19}$$

In Section 3 we shall use the following Fourier transform formula:

$$C_F F_N(\gamma') = \int_\Gamma \frac{(\text{Phase } \gamma)^{\frac{N}{2}} \gamma}{(1 - \overline{\gamma}) F_N(-\gamma)} \chi(\gamma, \gamma') d\gamma$$

$$\text{(1.20)}$$

$$= \sum_{k=0}^{N-1} \int_{\mathbb{R}_+} \frac{(-q)^k \chi(q^k r, \gamma')}{(1 - q^{-k} r) F_N(-q^k r)} \, dr,$$

where $C_F = -2\pi i F_N(-1)^{-1}$. This formula needs an explanation. For $k = 0$, the integrand has a pole at the point $r = 1$. In this case the integral over $\mathbb{R}_+$ should be replaced by the integral over the path ℓ going along the real axis from 0 to ∞ rounding the pole of the integrand from below. The integral should be understood in the sense of the distribution theory; before comparing the numerical values of the both sides one has to multiply them by a test function of γ' and integrate with respect to γ' (on the right hand side the integral with respect to γ should be taken after that with respect to γ'). The proof of (1.20) is given in Appendix B

2. OPERATOR EQUALITIES.

In this Section we shall investigate a pair of operators R, S satisfying certain special commutation relations. The operators of that kind will play an essential role in our theory of quantum '$az + b$' group.

For any Hilbert space H we denote by $\mathcal{D}_H$ the set of all pairs (R, S) of closed operators acting on H such that:

$$\left.\begin{array}{c} R \text{ and } S \text{ are normal,} \\[4pt] \text{Sp } R \text{ and Sp } S \text{ are contained in } \overline{\Gamma}, \\[4pt] \ker R = \ker S = \{0\}, \\[4pt] \chi(S, \gamma)\chi(R, \gamma') = \chi(\gamma, \gamma')\chi(R, \gamma')\chi(S, \gamma) \\[4pt] \text{for all } \gamma, \gamma' \in \Gamma. \end{array}\right\} \qquad \text{(2.1)}$$

The fourth (last) condition in (2.1) will be called Weyl pentagonal relation.

At first we give an example of such a pair of operators. Let $H = L^2(\Gamma, d\gamma)$, where $d\gamma$ is the Haar measure. By R we denote the multiplication by variable:

$$(Rx)(\gamma') = \gamma' x(\gamma')$$

for any $x \in H$ such that the right hand side is square integrable. Then R is a normal operator, $\text{Sp } R \subset \overline{\Gamma}$ and $\ker R = \{0\}$. For any $\gamma \in \Gamma$ we denote by U_γ the shift operator:

$$(U_\gamma x)(\gamma') = x(\gamma\gamma')$$

for any $x \in H$. Then $\Gamma \ni \gamma \longrightarrow U_\gamma \in B(H)$ is a unitary representation of the abelian group Γ. By virtue of the SNAG theorem ([4, Chapter 6, §2, Theorem 1]), there exists a spectral measure $dE(\gamma)$ on Γ such that

$$U_\gamma = \int_\Gamma \chi(\gamma', \gamma) \, dE(\gamma')$$

for all $\gamma \in \Gamma$. Let

$$S = \int_\Gamma \gamma' \, dE(\gamma').$$

Then S is a normal operator, $\text{Sp } S \subset \overline{\Gamma}$ and $\ker S = \{0\}$. Moreover $U_\gamma = \chi(S, \gamma)$ for any $\gamma \in \Gamma$. Therefore

$$(\chi(S, \gamma)x)(\gamma'') = x(\gamma\gamma''),$$

$$(\chi(R, \gamma')x)(\gamma'') = \chi(\gamma', \gamma'')x(\gamma'')$$

for any $x \in H$ and $\gamma, \gamma' \in \Gamma$. The second equation follows immediately from the definition of R. Now one can easily verify that the operators R and S satisfy the Weyl pentagonal relation:

$$\begin{aligned} \left(\chi(S, \gamma)\chi(R, \gamma')x\right)(\gamma'') &= \left(\chi(R, \gamma')x\right)(\gamma\gamma'') = \chi(\gamma\gamma'', \gamma')x(\gamma\gamma'') \\ &= \chi(\gamma, \gamma')\chi(\gamma'', \gamma')x(\gamma\gamma'') = \chi(\gamma, \gamma')\chi(\gamma'', \gamma')\left(\chi(S, \gamma)x\right)(\gamma'') \\ &= \chi(\gamma, \gamma')\left(\chi(R, \gamma')\chi(S, \gamma)x\right)(\gamma''). \end{aligned}$$

It shows that $(R, S) \in \mathcal{D}_{L^2(\Gamma, d\gamma)}$. This pair will be called the Schrödinger pair. The name comes from the clear analogy with the Schrödinger representation of canonical commutation relations in quantum mechanics. By the famous uniqueness theorem of Mackey, Stone and von Neumann [8] any pair of operators (R, S) satisfying (2.1) is a direct sum of irreducible pairs, each of which is unitarily equivalent to the Schrödinger pair.

Let H be a Hilbert space and $(R, S) \in \mathcal{D}_H$. Rewriting the Weyl pentagonal relation in the form: $\chi(S, \gamma)\chi(R, \gamma')\chi(S, \gamma)^* = \chi(\gamma R, \gamma')$ and remembering that the set of characters $\{\chi(\cdot, \gamma') : \gamma' \in \Gamma\}$ separates points of Γ we conclude that

$$\chi(S, \gamma)R\chi(S, \gamma)^* = \gamma R. \tag{2.2}$$

for any $\gamma \in \Gamma$. Inserting $\gamma = q, e^{-\frac{2\pi}{N}t}$ and using (1.4) we obtain:

$$(\text{Phase } S)R = qR(\text{Phase } S),$$
$$|S|^{it}R = e^{-\frac{2\pi}{N}t}R|S|^{it}, \tag{2.3}$$

where $t \in \mathbb{R}$. Writing R in the form of polar decomposition we obtain

$$\begin{aligned}
(\text{Phase } S)|R| &= |R|(\text{Phase } S), \\
|S|(\text{Phase } R) &= (\text{Phase } R)|S|, \\
(\text{Phase } S)(\text{Phase } R) &= q(\text{Phase } R)(\text{Phase } S), \\
|S|^{it}|R|^{it'} &= e^{-\frac{2\pi i}{N}tt'}|R|^{it'}|S|^{it},
\end{aligned} \tag{2.4}$$

where in the fourth formula $t, t' \in \mathbb{R}$. Clearly each of the Statements (2.2), (2.3) and (2.4) is equivalent to the Weyl pentagonal relation. Setting $t = -i$ in the second equations of (2.3) we obtain formally: $|S| R = q R |S|$. Combining this with the first equations we obtain: $SR = q^2 RS$ and $SR^* = R^*S$. We shall get these relations in a more rigorous way:

Proposition 2.1. *Let H be a Hilbert space and $(R, S) \in \mathcal{D}_H$. Then the compositions $S \circ R$, $R \circ S$, $S \circ R^*$, $R^* \circ S$ and $R^* \circ S^*$ are densely defined closeable operators and denoting by SR, RS, SR^*, R^*S and R^*S^* their closures we have:*

$$SR = q^2 RS, \quad SR^* = R^*S, \quad (SR)^* = R^*S^*$$

Moreover SR is a normal operator, $\ker SR = \{0\}$, $\text{Sp } SR \subset \overline{\Gamma}$ and

$$\chi(q^{-1}SR, \gamma) = \alpha(\gamma)\chi(R, \gamma)\chi(S, \gamma). \tag{2.5}$$

for any $\gamma \in \Gamma$.

Proof. The last relation of (2.4) means that operators $|R|$ and $|S|$ satisfy the Zakrzewski relation $|R| \multimap |S|$ of [19] with $\hbar = \frac{2\pi}{N}$. Therefore (see the example following Theorem 3.4 of [19]) operator $e^{\frac{\pi i}{N}}|R||S|$ is selfadjoint and strictly positive and

$$\left(e^{\frac{\pi i}{N}}|R||S|\right)^{it} = e^{-\frac{\pi i}{N}t^2}|R|^{it}|S|^{it}$$

for any $t \in \mathbb{R}$. Selfadjointness of $e^{\frac{\pi i}{N}}|R||S|$ imply that $e^{-\frac{\pi i}{N}}|S||R| = e^{\frac{\pi i}{N}}|R||S|$ and $|S||R| = q|R||S|$. Multiplying both sides of this relation by $(\text{Phase } R)(\text{Phase } S)$ and using (2.4) we obtain $SR = q^2 RS$. Similarly multiplying both sides of $|S||R| = q|R||S|$ by $(\text{Phase } R)^*(\text{Phase } S)$ we obtain $SR^* = R^*S$. The formula

$$SR = e^{\frac{\pi i}{N}}(\text{Phase } S)(\text{Phase } R) \, e^{\frac{\pi i}{N}}|R||S|$$

shows that Phase $SR = e^{\frac{\pi i}{N}}(\text{Phase } S)(\text{Phase } R)$ and $|SR| = e^{\frac{\pi i}{N}}|R||S|$. By (2.4) these two operators commute. Therefore SR is normal. The spectral condition imposed on R and S means that $(\text{Phase } R)^N = (\text{Phase } S)^N = I$. Taking into account the third relation of (2.4) we compute:

$$(\text{Phase } SR)^N = -\left((\text{Phase } S)(\text{Phase } R)\right)^N = -q^{\frac{N(N+1)}{2}}I.$$

Remembering that N is even we have: $\frac{N(N+1)}{2} \equiv \frac{N}{2} \pmod{N}$, $q^{\frac{N(N+1)}{2}} = q^{\frac{N}{2}} = -1$ and $(\text{Phase } SR)^N = I$. It shows that SR satisfies the spectral condition $\text{Sp } SR \subset \overline{\Gamma}$.

Let $\gamma = q^k r$ ($r \in \mathbb{R}_+$, $k \in \mathbb{Z}$). We compute:

$$\chi(q^{-1}SR, \gamma) = \chi\left(q^{-1}e^{\frac{\pi i}{N}}(\text{Phase } S)(\text{Phase } R), \gamma\right) \chi\left(e^{\frac{\pi i}{N}}|R|\,|S|, \gamma\right)$$

$$= \left(q^{-1}e^{\frac{\pi i}{N}}(\text{Phase } S)(\text{Phase } R)\right)^k \left(e^{\frac{\pi i}{N}}|R|\,|S|\right)^{\frac{N}{2\pi i}\log r}$$

$$= q^{-k}e^{\frac{k\pi i}{N}}q^{\frac{k(k+1)}{2}}(\text{Phase } R)^k(\text{Phase } S)^k e^{-\frac{\pi i}{N}(\frac{N}{2\pi}\log r)^2}|R|^{\frac{N}{2\pi i}\log r}|S|^{\frac{N}{2\pi i}\log r}$$

$$= e^{\frac{\pi i}{N}k^2}e^{\frac{N}{4\pi i}(\log r)^2}(\text{Phase } R)^k|R|^{\frac{N}{2\pi i}\log r}(\text{Phase } S)^k|S|^{\frac{N}{2\pi i}\log r}$$

$$= \alpha(\gamma)\chi(R, \gamma)\chi(S, \gamma).$$

$\square$

Let H be a Hilbert space and $(R, S) \in \mathcal{D}_H$. We shall prove that the operator $q^{-1}SR = qRS$ is unitarily equivalent to S. Taking into account (2.2) we obtain $\chi(S, \gamma)\alpha(R)\chi(S, \gamma)^* = \alpha(\gamma R)$. According to (1.2): $\alpha(\gamma R) = \alpha(R)\alpha(\gamma)\chi(R, \gamma)$. Therefore, using (2.5) we obtain:

$$\alpha(R)^*\chi(S, \gamma)\alpha(R) = \alpha(\gamma)\chi(R, \gamma)\chi(S, \gamma) = \chi(q^{-1}SR, \gamma).$$

It shows that

$$\alpha(R)^*S\alpha(R) = q^{-1}SR. \tag{2.6}$$

Proposition 2.2. *Let H be a Hilbert space and $(R, S) \in \mathcal{D}_H$. Then (R^*, S^*), (S^{-1}, R), (S, R^{-1}) and (R, SR) belong to $\mathcal{D}_H$.*

Proof. Clearly the spectra of R^{-1}, R^*, S^{-1}, S^* are contained in $\overline{\Gamma}$. Let $\gamma \in \Gamma$. Remembering that $\chi(\cdot, \cdot)$ is a bicharacter, we obtain:

$$\chi(R, \gamma)^* = \chi(R^{-1}, \gamma),$$
$$\chi(R, \overline{\gamma}) = \chi(R^*, \gamma).$$

The same equalities hold for operator S. Using the above equalities and the Weyl pentagonal relation for the pair (R, S) one can easily verify that the pairs (R^*, S^*), (S^{-1}, R), (S, R^{-1}) satisfy the Weyl pentagonal relation. Therefore $(R^*, S^*), (S^{-1}, R), (S, R^{-1}) \in \mathcal{D}_H$. If $(R, S) \in \mathcal{D}_H$ then $(R, qS) \in \mathcal{D}_H$. Using (2.6) and the obvious relation $\alpha(R)^*R\alpha(R) = R$ we see that the pair (R, SR) is unitarily equivalent to (R, qS). Therefore $(R, SR) \in \mathcal{D}_H$.

$\square$

To prove our main theorems concerning pairs of operators satisfying (2.1) we shall use a regularization procedure. It is based on the following proposition:

Proposition 2.3. *Let H be a Hilbert space and $(R, S) \in \mathcal{D}_H$. Then there exists a one parameter group of unitaries $\{\mathcal{R}_t\}_{t \in \mathbb{R}_+}$ acting on H such that for any $t \in \mathbb{R}_+$:*

$$\mathcal{R}_t \text{ commutes with } \text{Phase } R \text{ and } \text{Phase } S,$$

$$\mathcal{R}_t^*|S|\mathcal{R}_t = |S|^t, \qquad \mathcal{R}_t|R|\mathcal{R}_t^* = |R|^t.$$

Proof. We may assume that $H = L^2(\Gamma)$ and that $(R, S) \in \mathcal{D}_H$ is the Schrödinger pair. For any $x \in L^2(\Gamma)$ we set:

$$(\mathcal{R}_t x)(q^k r) = t^{\frac{1}{2}}x\left(q^k r^t\right).$$

By the obvious formula: $\frac{dr^t}{r^t} = t\frac{dr}{r}$, $\mathcal{R}_t$ are unitary operators acting on $L^2(\Gamma)$. The simple computations:

$$(\mathcal{R}_t R\mathcal{R}_t^* x)(q^k r) = t^{\frac{1}{2}}(R\mathcal{R}_t^* x)(q^k r^t) = t^{\frac{1}{2}}q^k r^t(\mathcal{R}_t^* x)(q^k r^t) = q^k|r|^t x(q^k r)$$

shows that $\mathcal{R}_t R\mathcal{R}_t^* = \text{Phase } R\, |R|^t$. Therefore $\mathcal{R}_t(\text{Phase } R)\mathcal{R}_t^* = \text{Phase } R$ and $\mathcal{R}_t|R|\mathcal{R}_t^* = |R|^t$. Similarly

$$(\mathcal{R}_t^*\chi(S, \gamma)\mathcal{R}_t x)(q^k r) = t^{-\frac{1}{2}}(\chi(S, \gamma)\mathcal{R}_t x)(q^k r^{t^{-1}}) = t^{-\frac{1}{2}}(\mathcal{R}_t x)(\gamma q^k r^{t^{-1}})$$

$$= x(\text{Phase }\gamma|\gamma|^t q^k r) = \left(\chi\left(S, \text{Phase }\gamma|\gamma|^t\right)x\right)(q^k r)$$

shows that $\mathcal{R}_t^*\chi(S, \gamma)\mathcal{R}_t = \chi(S, \text{Phase }\gamma|\gamma|^t)$. Setting $\gamma = q$ and using (1.4) we see that $\mathcal{R}_t^*\text{Phase } S\mathcal{R}_t = \text{Phase } S$. Setting $\gamma = e^{-\frac{2\pi}{N}\tau}$ (where $\tau \in \mathbb{R}$) and using (1.4) we see that $\mathcal{R}_t^*|S|^{i\tau}\mathcal{R}_t = |S|^{i\tau t}$. Therefore $\mathcal{R}_t^*|S|\mathcal{R}_t = |S|^t$.

$\square$

Now we are able to formulate our main theorems.

Theorem 2.4. *Let H be a Hilbert space and $(R, S) \in \mathcal{D}_H$. Then $S + S{\circ}R$ is densely defined closeable operator and its closure*

$$S \dotplus SR = F_N(R)^* S F_N(R). \tag{2.7}$$

In particular $S \dotplus SR$ is normal and $\mathrm{Sp}\,(S \dotplus SR) \subset \overline{\Gamma}$.

Proof. Let us recall some notation used in [19]: For any $0 < \hbar \leq \frac{2\pi}{N}$ we set:

$$\Omega_\hbar = \{ z \in \mathbb{C} : z \neq 0 \text{ and } 0 < \arg z < \hbar \},$$

$$\overline{\Omega}_\hbar = \text{ the closure of } \Omega_\hbar \text{ in } \mathbb{C},$$

$$\mathcal{H}_\hbar = \left\{ f \in C(\overline{\Omega}_\hbar) : \begin{array}{c} f \text{ is holomorphic on } \Omega_\hbar \text{ and} \\ \text{for any } \lambda > 0 \text{ the function} \\ e^{-\lambda \log(z)^2} f(z) \text{ is bounded on } \Omega_\hbar \end{array} \right\}.$$

For $\hbar = \frac{2\pi}{N}$, $\overline{\Omega}_\hbar$ coincides with $\overline{\Omega}$ introduced in Theorem 1.2. Using the asymptotic formula (1.17) one can easily show that for any $k \in \mathbb{Z}$, the function $\widetilde{F}(\cdot, q^k)$ belongs to $\mathcal{H}_{\frac{2\pi}{N}}$. For $k = \frac{N}{2} - 1$, this function has zero at $z = q \in \overline{\Omega}$ (cf (1.16)). Therefore for this k, the inverse $\widetilde{F}(\cdot, q^k)^{-1}$ does not belong to $\mathcal{H}_{\frac{2\pi}{N}}$. On the other hand replacing $\overline{\Omega}$ by $\overline{\Omega}_\hbar$, where $\hbar < \frac{2\pi}{N}$ we excise the zero point of $\widetilde{F}(\cdot, q^k)$. Using Statements 3 and 4 of Theorem 1.2 one can easily show that for any $k \in \mathbb{Z}$, the inverse $\widetilde{F}(\cdot, q^k)^{-1}$ belongs to $\mathcal{H}_\hbar$ for all $\hbar < \frac{2\pi}{N}$.

Let $0 < t < 1$. The last relation of (2.4) shows that operators R and S satisfy the Zakrzewski relation $|R| {\multimap} |S|^t$ with $\hbar = \frac{2\pi}{N} t$. Let $k \in \mathbb{Z}$. Using Statement 4 of Theorem 3.3 of [19] we obtain:

$$\widetilde{F}(e^{\frac{2\pi i t}{N}} |R|, q^k) |S|^t = |S|^t \widetilde{F}(|R|, q^k).$$

For $t = 1$, only Statement 3 of this Theorem may be used:

$$\widetilde{F}(q|R|, q^k)|S| \subset |S| \widetilde{F}(|R|, q^k).$$

We know that $\mathrm{Phase}\, R$ commutes with $|R|$ and S. Therefore the eigenspaces of $\mathrm{Phase}\, R$ are invariant under functions of $|R|$ and S. Restricting the above relations to these eigenspaces, setting q^k to be the corresponding eigenvalue and taking the direct sum we obtain:

$$\widetilde{F}(e^{\frac{2\pi i t}{N}} |R|, \mathrm{Phase}\, R)|S|^t = |S|^t \widetilde{F}(|R|, \mathrm{Phase}\, R),$$

$$\widetilde{F}(q|R|, \mathrm{Phase}\, R)|S| \subset |S| \widetilde{F}(|R|, \mathrm{Phase}\, R).$$

Using now relation (1.15) we get:

$$\widetilde{F}(e^{\frac{2\pi i t}{N}} |R|, \mathrm{Phase}\, R)|S|^t = |S|^t F_N(R), \tag{2.8}$$

$$\widetilde{F}(q|R|, \mathrm{Phase}\, R)|S| \subset |S| F_N(R).$$

By (1.16), the second relation leads to

$$(I + qR)|S| \subset F_N(q^{-1}R)^* |S| F_N(R).$$

Multiplying (from the left) the both sides by $\mathrm{Phase}\, S$ and using (2.4) we obtain:

$$(I + q^2 R)S \subset F_N(R)^* S F_N(R). \tag{2.9}$$

We have to show the inverse inclusion. To this end we shall use the operators $\mathcal{R}_t$ introduced in Proposition 2.3.

Let $x \in D(F_N(R)^* S F_N(R))$. Then $F_N(R)x \in D(S)$ and $F_N(R)x \in D(|S|^t)$ for any $0 < t < 1$. By (2.8), $x \in D(\widetilde{F}(e^{\frac{2\pi i t}{N}} |R|, \mathrm{Phase}\, R)|S|^t)$. Using now Proposition 2.3 we obtain:

$$\mathcal{R}_t x \in D\left(\widetilde{F}\left(e^{\frac{2\pi i t}{N}} |R|^{t^{-1}}, \mathrm{Phase}\, R \right) |S| \right). \tag{2.10}$$

By Theorem 1.2 (Statements 3 and 4), the function $\mathbb{R}_+ \ni r \longrightarrow \left| \widetilde{F}(e^{\frac{2\pi i t}{N}} r^{t^{-1}}, q^k) \right| \in \mathbb{R}_+$ is separated from 0 and behaves like r, when $r \to \infty$. Therefore (2.10) implies that $\mathcal{R}_t x \in D(S)$ and $|S| \mathcal{R}_t x \in D(R)$. Remembering that $D(R)$ is $\mathrm{Phase}\, S$ - invariant we obtain: $S \mathcal{R}_t x \in D(R)$. It shows that

$$\mathcal{R}_t x \in D(S + q^2 R{\circ}S).$$

This relation holds for $t < 1$. When $t \to 1$, then $\mathcal{R}_t x \to x$. Moreover the formula

$$F_N(R)^* S F_N(R) \mathcal{R}_t x = \mathcal{R}_t F_N \left((\text{Phase } R)|R|^{t^{-1}} \right)^* |S|^t F_N \left((\text{Phase } R)|R|^{t^{-1}} \right) x$$

shows that in this case

$$F_N(R)^* S F_N(R) \mathcal{R}_t x \longrightarrow F_N(R)^* S F_N(R) x$$

This way we showed that $D(S + q^2 R \circ S)$ is a core for $F_N(R)^* S F_N(R)$. Taking into account (2.9) and remembering that $q^2 RS = SR$ and we obtain (2.7).

$\square$

Corollary 2.5. *Let H be a Hilbert space and $(R, S) \in \mathcal{D}_H$. Then*

$$R \dotplus S = F_N(S^{-1}R)^* S F_N(S^{-1}R), \tag{2.11}$$

$$R \dotplus S = F_N(SR^{-1}) R F_N(SR^{-1})^*. \tag{2.12}$$

In particular $R \dotplus S$ is normal and $\mathrm{Sp}\,(R \dotplus S) \subset \overline{\Gamma}$.

Proof. Using repeatedly Proposition 2.2 we get: $(S^{-1}, R), (S^{-1}, RS^{-1}), (RS^{-1}, S) \in \mathcal{D}_H$. Therefore $(S^{-1}R, S) = (q^{-2}RS^{-1}, S) \in \mathcal{D}_H$. Replacing (R, S) by $(S^{-1}R, S)$ in (2.7) we obtain (2.11).

Replacing (R, S) by $(qS^{-1}R, S)$ in (2.6) we obtain:

$$\alpha(qS^{-1}R)^* S \alpha(qS^{-1}R) = R.$$

Combining this formula with (2.11) we get:

$$R \dotplus S = F_N(S^{-1}R)^* \alpha(qS^{-1}R) R \alpha(qS^{-1}R)^* F_N(S^{-1}R).$$

Inserting in (1.8) $\gamma = q^2 S^{-1}R$ and remembering that $q^{-2}R^{-1}S = SR^{-1}$ we obtain

$$F_N \left(S^{-1}R \right)^* \alpha \left(qS^{-1}R \right) = e^{-\frac{\pi}{6i}\left(\frac{2}{N} + \frac{N}{2} \right)} F_N \left(SR^{-1} \right).$$

Combining the two last relations we get (2.12)

$\square$

Theorem 2.6. *Let H be a Hilbert space and $(R, S) \in \mathcal{D}_H$. Then*

$$F_N(R \dotplus S) = F_N(R) F_N(S). \tag{2.13}$$

Proof. We shall use the method developed in [15, 19]. Let $T = SR^{-1}$. Using Proposition 2.2 one can easily show that $(R, T), (S, T) \in \mathcal{D}_H$. Clearly $S^{-1}R = q^{-2}T^{-1}$. Therefore T commutes with $F_N(S^{-1}R)$ and by (2.11), $(R \dotplus S, T)$ is unitarily equivalent to (S, T). It shows that $(R \dotplus S, T) \in \mathcal{D}_H$. By virtue of Theorem 2.4:

$$F_N(R \dotplus S)^* T F_N(R \dotplus S) = T \dotplus T(R \dotplus S). \tag{2.14}$$

By the same argument

$$F_N(R)^* T F_N(R) = T \dotplus TR$$

and

$$F_N(S)^* F_N(R)^* T F_N(R) F_N(S) = F_N(S)^* T F_N(S) \dotplus F_N(S)^* TR F_N(S).$$

The second term equals to TR (because $TR = S$ commutes with $F_N(S)$). To compute the first one we use again Theorem 2.4:

$$F_N(S)^* F_N(R)^* T F_N(R) F_N(S) = (T \dotplus TS) \dotplus TR. \tag{2.15}$$

We shall prove later that the right hand sides of (2.14) and (2.15) coincide (cf (2.17)). Therefore

$$F_N(R \dotplus S)^* T F_N(R \dotplus S) = F_N(S)^* F_N(R)^* T F_N(R) F_N(S).$$

It shows that $F_N(R \dotplus S) F_N(S)^* F_N(R)^*$ commutes with T. Therefore it commutes with $|T|^{it}$ for any $t \in \mathbb{R}$:

$$F_N(R \dotplus S) F_N(S)^* F_N(R)^* = |T|^{it} F_N(R \dotplus S) F_N(S)^* F_N(R)^* |T|^{-it}. \tag{2.16}$$

Using the relations (2.3) (with (R, S) replaced by (R, T)) we obtain $|T|^{it} R |T|^{-it} = e^{-\frac{2\pi t}{N}} R$. The same equation holds for S and $R \dotplus S$. Therefore the right hand side of (2.16) equals to $F_N(e^{-\frac{2\pi t}{N}}(R \dotplus S)) F_N(e^{-\frac{2\pi t}{N}} S)^* F_N(e^{-\frac{2\pi t}{N}} R)^*$ and tends strongly to I, when $t \to \infty$. Formula (2.16) shows now that $F_N(R \dotplus S) F_N(S)^* F_N(R)^* = I$ and (2.13) follows.

To end the proof we have to show that

$$T \dotplus T(R \dotplus S) = (T \dotplus TS) \dotplus TR. \tag{2.17}$$

To this end it is sufficient to show that

$$D(T) \cap D(TR) \cap D(TS)$$
$$\text{is core for } T \dot{+} T(R \dot{+} S). \tag{2.18}$$

Indeed the both sides of (2.17) coincides on $D(T) \cap D(TS) \cap D(TR)$. Therefore (2.18) implies that $T \dot{+} T(R \dot{+} S) \subset (T \dot{+} TS) \dot{+} TR$. However normal operators have no proper normal extensions and (2.17) follows. Before (2.18) we shall prove that

$$D(T^2) \cap D(T(R \dot{+} S))$$
$$\text{is core for } T \dot{+} T(R \dot{+} S). \tag{2.19}$$

Let $\tau > 0$. For any $z \in \overline{\Omega}$ we set:

$$f_\tau(z) = \begin{cases} e^{-\tau(\log z)^2} & \text{for } z \neq 0, \\ 0 & \text{for } z = 0, \end{cases}$$

Then f_τ is a continuous bounded function on $\overline{\Omega}$, holomorphic on Ω. When $\tau \to \infty$ then f_τ converges almost uniformly to 1. Remembering that $(R \dot{+} S, T) \in \mathcal{D}_H$ and using again Theorem 3.3 of [19] we obtain:

$$(R \dot{+} S) f_\tau(|T|) \subset f_\tau(q|T|)(R \dot{+} S).$$

By definition $T \dot{+} T(R \dot{+} S)$ is the closure of $T + T(R \dot{+} S)$. Therefore the intersection $D(T) \cap D(T(S \dot{+} R))$ is a core for $T \dot{+} T(R \dot{+} S)$. Let $x \in D(T) \cap D(T(S \dot{+} R))$. Then by the above inclusion $f_\tau(|T|)x \in D(T(S \dot{+} R))$. The function f_τ is rapidly decreasing: $|z|^2 f_\tau(z)$ is bounded, when $z \to \infty$. Therefore $f_\tau(|T|)x \in D(T^2)$ and $f_\tau(|T|)x \in D(T^2 + T(S \dot{+} R))$. Using the almost uniform convergence mentioned above one can easily show that $f_\tau(|T|)x \to x$ and $\left(T + T(R \dot{+} S)\right) f_\tau(|T|)x \to Tx + T(R \dot{+} S)x$ when $t \to 0$. Relation (2.19) is shown.

Using Proposition 2.2 one can easily show that $(S^{-1}R, TS) \in \mathcal{D}_H$. Replacing (R, S) by $(S^{-1}R, TS)$ in (2.7) we get:

$$TR \dot{+} TS = F_N(S^{-1}R)^* TS F_N(S^{-1}R). \tag{2.20}$$

Remembering that T commutes with $F_N(S^{-1}R) = F_N(q^{-2}T^{-1})$ and comparing with (2.11) we obtain

$$T(R \dot{+} S) = TR \dot{+} TS.$$

Applying Proposition 2.3 to the pair $(S^{-1}R, TS)$ we find a the group of unitaries $(\mathcal{R}_t)_{t \in \mathbb{R}_+}$ such that $\mathcal{R}_t^* |TS| \mathcal{R}_t = |TS|^t$ and $\mathcal{R}_t |S^{-1}R| \mathcal{R}_t^* = |S^{-1}R|^t$. Let $t < 1$ and $x \in D(TR \dot{+} TS)$. By the last part of the proof of Theorem 2.4, $\mathcal{R}_t x \in D(TR) \cap D(TS)$ and for $t \to 1$, $\mathcal{R}_t x$ converges to x in the graph topology of $TR + TS$.

Remembering that $S^{-1}R = q^{-2}T^{-1}$ we obtain $\mathcal{R}_t |T| \mathcal{R}_t^* = |T|^t$ and $\mathcal{R}_t^* |T| \mathcal{R}_t = |T|^{t^{-1}}$. Let $\frac{1}{2} \leq t < 1$ and $x \in D(T^2)$. Then $t^{-1} \leq 2$, $x \in D(|T|^{t^{-1}})$ and $\mathcal{R}_t x \in D(T)$. Moreover for $t \to 1$, $\mathcal{R}_t x$ converges to x in the graph topology of T. Indeed $|T| \mathcal{R}_t x = \mathcal{R}_t |T|^{t^{-1}} x$ converges to x in norm, when $t \to 1$.

Summarizing the last two paragraphs we see that for any $x \in D(T^2) \cap D(T(R \dot{+} S))$ we have: $\mathcal{R}_t x \in D(T) \cap D(TR) \cap D(TS)$ and for $t \to 1$, $\mathcal{R}_t x$ converges to x in the graph topology of $T + TR + TS$. Now (2.18) follows from (2.19).

$\square$

Let H be a Hilbert space and $(R, S) \in \mathcal{D}_H$. Combining (2.13) with (2.11) and (2.12) we obtain the following pentagonal equations:

$$F_N(R)F_N(S) = F_N(S^{-1}R)^* F_N(S) F_N(S^{-1}R), \tag{2.21}$$

$$F_N(R)F_N(S) = F_N(SR^{-1}) F_N(R) F_N(SR^{-1})^*. \tag{2.22}$$

3. The Kac Takesaki operator.

We start with a short remark concerning the connection between commutation relations (0.7) and (2.1). Let a, b be closed operators acting on a Hilbert space H. Comparing (0.7) with (2.3) we obtain:

$$(b, a) \in \mathcal{D}_H \quad \Longleftrightarrow \quad \left\{ \begin{array}{c} \ker b = \{0\} \text{ and} \\ (a, b) \in G_H \end{array} \right\}. \tag{3.1}$$

In general case, for any $(a, b) \in G_H$, the Hilbert space: $H = \ker b \oplus (\ker b)^\perp$ and we have the corresponding decomposition $(a, b) = (a_o, 0) \oplus (a', b')$, where $(b', a') \in \mathcal{D}_{(\ker b)^\perp}$.

The main result of this Section is contained in the following

Theorem 3.1. *Let H be a Hilbert space, $(a, b) \in G_H$ and s, r be strictly positive selfadjoint operators acting on H. Assume that $\ker b = \{0\}$, r and s strongly commute with a and b and $r^{-it} s r^{it} = e^{\frac{2\pi}{N} t} s$ for all $t \in \mathbb{R}$. Then the operator*

$$W = F_N \left(ab^{-1} \otimes b \right) \chi \left(sb^{-1} \otimes I, I \otimes a \right) \tag{3.2}$$

is a manageable multiplicative unitary. Moreover

$$W(a \otimes I) W^* = \quad a \otimes a, \tag{3.3}$$

$$W(b \otimes I) W^* = a \otimes b \dotplus b \otimes I. \tag{3.4}$$

Proof. According to (3.1), $(b, a) \in \mathcal{D}_H$. Using Proposition 2.2 we obtain: $(a, b^{-1}) \in \mathcal{D}_H$. Remembering that s is strictly positive and strongly commutes with a and b we get: $(a, sb^{-1}) \in \mathcal{D}_H$. Formula (2.2) shows now that $\chi(sb^{-1}, \gamma) a \chi(sb^{-1}, \gamma)^* = \gamma a$. for any $\gamma \in \Gamma$. Using this relation one can easily verify that

$$\chi(sb^{-1} \otimes I, I \otimes a)(a \otimes I)\chi(sb^{-1} \otimes I, I \otimes a)^* = a \otimes a. \tag{3.5}$$

Relations $(b, a), (a, b^{-1}) \in \mathcal{D}_H$ imply that $a \otimes a$ strongly commutes with $ab^{-1} \otimes b$. Therefore $a \otimes a$ commutes with $F_N(ab^{-1} \otimes b)$ and (3.3) follows immediately from (3.5). To prove (3.4) we notice that $\chi(sb^{-1} \otimes I, I \otimes a)$ commutes with $b \otimes I$. Therefore the right hand side of (3.4) coincides with $F_N(ab^{-1} \otimes b)(b \otimes I)F_N(ab^{-1} \otimes b)^*$. The relation $(b, a) \in \mathcal{D}_H$ imply that the pair $(R, S) = (b \otimes I, a \otimes b) \in \mathcal{D}_{H \otimes H}$. For this pair $SR^{-1} = ab^{-1} \otimes b$. Now (3.4) follows immediately from (2.12). This way we proved (3.3) and (3.4).

The pentagonal equation for (3.2) will follow from

Proposition 3.2. *Let H and K be Hilbert spaces, $(a, b) \in G_H$, $(\hat{a}, \hat{b}) \in G_K$ and s be a strictly positive selfadjoint operators acting on H. Assume that $\ker b = \{0\}$ and s strongly commute with a and b. Then the operators (3.2) and*

$$V = F_N \left(\hat{b} \otimes b \right) \chi \left(\hat{a} \otimes I, I \otimes a \right) \tag{3.6}$$

satisfy the pentagonal equation:

$$W_{23} V_{12} = V_{12} V_{13} W_{23}. \tag{3.7}$$

Proof. Using formulae (3.3) and (3.4) we obtain:

$$W_{23} V_{12} W_{23}^* = F_N \left(\hat{b} \otimes (a \otimes b \dotplus b \otimes I) \right) \chi(\hat{a} \otimes I \otimes I, I \otimes a \otimes a). \tag{3.8}$$

Remembering that $\chi(\cdot, \cdot)$ is a bicharacter on Γ we rewrite the second factor on the right hand side of (3.8) in the following way:

$$\chi(\hat{a} \otimes I \otimes I, I \otimes a \otimes a) = \chi(\hat{a} \otimes I \otimes I, I \otimes a \otimes I)\chi(\hat{a} \otimes I \otimes I, I \otimes I \otimes a). \tag{3.9}$$

If $\hat{b} = 0$ then $V = \chi(\hat{a} \otimes I, I \otimes a)$, right hand side of (3.9) equals to $V_{12} V_{13}$, the first factor on the right hand side of (3.8) disappears ($F_N(0) = I$), (3.8) reduces to $W_{23} V_{12} W_{23}^* = V_{12} V_{13}$ and (3.7) follows.

Therefore we may assume that $\ker \hat{b} = \{0\}$. In this case $(\hat{b}, \hat{a}) \in \mathcal{D}_K$ and by (2.2), $\gamma \hat{b} \chi(\hat{a}, \gamma) = \chi(\hat{a}, \gamma) \hat{b}$ for any $\gamma \in \Gamma$. Using this relation one can easily verify that $(\hat{b} \otimes a)\chi(\hat{a} \otimes I, I \otimes a) = \chi(\hat{a} \otimes I, I \otimes a)(\hat{b} \otimes I)$ and

$$(\hat{b} \otimes a \otimes b)\chi(\hat{a} \otimes I \otimes I, I \otimes a \otimes I) = \chi(\hat{a} \otimes I \otimes I, I \otimes a \otimes I)(\hat{b} \otimes I \otimes b) \tag{3.10}$$

We already know that $(b \otimes I, a \otimes b) \in \mathcal{D}_{H \otimes H}$. Hence the pair $(R, S) = (\hat{b} \otimes b \otimes I, \hat{b} \otimes a \otimes b)$ belongs to $\mathcal{D}_{K \otimes H \otimes H}$ and using (2.13) we obtain:

$$F_N \left(\hat{b} \otimes (a \otimes b \dotplus b \otimes I) \right) = F_N \left(\hat{b} \otimes b \otimes I \right) F_N \left(\hat{b} \otimes a \otimes b \right). \tag{3.11}$$

Inserting (3.9) and (3.11) into (3.8) and using (3.10) we obtain:

$$W_{23} V_{12} W_{23}^* = \begin{cases} F_N \left(\hat{b} \otimes b \otimes I \right) \chi(\hat{a} \otimes I \otimes I, I \otimes a \otimes I) \\[2mm] F_N \left(\hat{b} \otimes I \otimes b \right) \chi(\hat{a} \otimes I \otimes I, I \otimes I \otimes a). \end{cases}$$

Clearly the right hand side of the above equality equals to $V_{12} V_{13}$ and (3.7) follows.

$\square$

We continue the proof of Theorem 3.1. Let H be a Hilbert space, $(a, b) \in G_H$ and s be a strictly positive selfadjoint operator acting on H. Assume that $\ker b = \{0\}$ and s strongly commutes with a and b. Then one can easily verify that $(sb^{-1}, ab^{-1}) \in G_H$. For $K = H$, $\hat{a} = sb^{-1}$ and $\hat{b} = ab^{-1}$, the operator (3.6) coincides with (3.2) and formula (3.7) takes the form:

$$W_{23} W_{12} = W_{12} W_{13} W_{23}. \tag{3.12}$$

It shows that the operator W introduced by (3.2) is a multiplicative unitary.

For any Hilbert space K we denote by $\overline{K}$ the complex conjugate Hilbert space. Then we have canonical antiunitary bijection:

$$K \ni x \longrightarrow \overline{x} \in \overline{K}. \tag{3.13}$$

If m is a closed operator acting on K, then its transpose $m^\top$ is introduced by the formula

$$m^\top \overline{x} = \overline{m^* x}$$

for any $x \in D(m^*)$. Clearly $m^\top$ is a closed operator acting on $\overline{K}$ with the domain $D(m^\top) = \{\overline{x} : x \in D(m^*)\}$. If $x \in D(m^*)$ and $z \in D(m)$, then

$$\left(\overline{z} \big| m^\top \big| \overline{x} \right) = (x | m | z). \tag{3.14}$$

Indeed: $\left(\overline{z} | m^\top \overline{x} \right) = \left(\overline{z} | \overline{m^* x} \right) = (m^* x | z) = (x | mz)$.

One can easily verify that the transposition commutes with the adjoint operation: $(m^*)^\top = (m^\top)^*$, so $m^\top$ is normal for normal m. Moreover the transposition inverses the order of multiplication: $(ab)^\top = b^\top a^\top$. Therefore for invertible b, $(a, b) \in G_H$ implies $(b^\top, a^\top) \in G_{\overline{H}}$. If $\hat{a}$ is a normal operator on K and f is a bounded measurable function on $\operatorname{Sp} \hat{a}$ then $\operatorname{Sp} \hat{a} = \operatorname{Sp} \hat{a}^\top$ and $f(\hat{a})^\top = f(\hat{a}^\top)$.

Let $\hat{a}$ and a be normal operators acting on K and H respectively. Then $\hat{a} \otimes I$ and $I \otimes a$ are strongly commuting normal operators acting on $K \otimes H$. Their joint spectrum coincides with $\operatorname{Sp} \hat{a} \times \operatorname{Sp} a$. We have the following 'partial transposition' formula

$$\left(\overline{z} \otimes u \big| f(\hat{a}^\top \otimes I, I \otimes a) \big| \overline{x} \otimes y \right) = (x \otimes u | f(\hat{a} \otimes I, I \otimes a) | z \otimes y). \tag{3.15}$$

In this formula $x, z \in K$, $u, y \in H$ and $f(\cdot, \cdot)$ is a bounded measurable function on $\operatorname{Sp} \hat{a} \times \operatorname{Sp} a$. By linearity and continuity it is sufficient to prove this formula for functions of the form $f = f_1 \otimes f_2$, where f_1 and f_2 are functions of one variable. In this case the formula follows immediately from (3.14).

To prove the manageability of the multiplicative unitary (3.2) we shall use the following

Proposition 3.3. *Let H and K be Hilbert spaces, $(a, b) \in G_H$ and $(\hat{a}, \hat{b}) \in G_K$ and let V be the unitary operator introduced by (3.6). Moreover let Q be a strictly positive selfadjoint operator acting on H such that Q strongly commutes with a and $Q^{-it} b Q^{it} = e^{\frac{2\pi t}{N}} b$ for all $t \in \mathbb{R}$. We set:*

$$\widetilde{V} = F_N \left(-\hat{b}^\top \otimes q a^{-1} b \right)^* \chi \left(\hat{a}^\top \otimes I, I \otimes a \right). \tag{3.16}$$

Then $\widetilde{V}$ is unitary and for any $x, z \in K$, $y \in D(Q^{-1})$, $u \in D(Q)$ we have:

$$(x \otimes u | V | z \otimes y) = \left(\overline{z} \otimes Qu \big| \widetilde{V} \big| \overline{x} \otimes Q^{-1} y \right). \tag{3.17}$$

Remark: Formula (3.7) shows that (3.6) is an adapted operator in the sense of [18, Definition 1.3]. Comparing (3.16) with Statement 5 of Theorem 1.6 of [18] one can easily find the unitary antipode R of our quantum group. It acts on a, b as follows:

$$a^R = a^{-1},$$
$$b^R = -qa^{-1}b. \tag{3.18}$$

Proof. To make our formulae shorter we set:

$$U = \chi\left(\hat{a} \otimes I, I \otimes a\right), \quad \widetilde{U} = \chi\left(\hat{a}^\top \otimes I, I \otimes a\right),$$
$$B = \hat{b} \otimes b, \qquad \widetilde{B} = -\hat{b}^\top \otimes qa^{-1}b. \tag{3.19}$$

Then equation (3.17) reduces to

$$(x \otimes u | F_N(B)U | z \otimes y) = \left(\overline{z} \otimes Qu \Big| F_N(\widetilde{B})^* \widetilde{U} \Big| \overline{x} \otimes Q^{-1}y\right) \tag{3.20}$$

Let $x, z \in K$ and $y, u \in H$. Inserting in (3.15), $f = \chi$ we obtain:

$$\left(\overline{z} \otimes u \Big| \widetilde{U} \Big| \overline{x} \otimes y\right) = (x \otimes u | U | z \otimes y). \tag{3.21}$$

If either $\hat{b} = 0$ or $b = 0$, then $V = U$, $\widetilde{V} = \widetilde{U}$ and (3.20) follows immediately from (3.21) (the reader should remember that Q commutes with a). Therefore we may assume that $\ker \hat{b} = \{0\}$ and $\ker b = \{0\}$. To prove formulae (3.20) we shall use the following

Proposition 3.4. *Let a, b, Q be operators acting on a Hilbert space H and $\hat{a}, \hat{b}$ be operators acting on a Hilbert space K. Assume that: $(a, b) \in G_H$, $\ker b = \{0\}$, Q is strictly positive, Q strongly commutes with a and $Q^{-it}bQ^{it} = e^{\frac{2\pi t}{N}}b$ for all $t \in \mathbb{R}$. Assume also that $(\hat{a}, \hat{b}) \in G_K$ and $\ker \hat{b} = \{0\}$. Moreover, let $x, z \in K$, $y \in D(Q^{-1})$, $u \in D(Q)$ and for any $\gamma \in \Gamma$,*

$$\varphi(\gamma) = \quad (x \otimes u | \chi(B, \gamma)U | z \otimes y),$$
$$\psi(\gamma) = \left(\overline{z} \otimes Qu \Big| \chi(\widetilde{B}, \gamma)\widetilde{U} \Big| \overline{x} \otimes Q^{-1}y\right), \tag{3.22}$$

where B, U, $\widetilde{B}$ and $\widetilde{U}$ are operators introduced by (3.19). Then

$$\psi(\gamma) = |\gamma| \, (\text{Phase } \gamma)^{\frac{N}{2}} \, \overline{\alpha(\gamma)} \, \varphi(\gamma) \tag{3.23}$$

for any $k \in \mathbb{R}$.

Proof. Let $\gamma \in \Gamma$ and $t \in \mathbb{R}$. Using the formula $Q^{-it}bQ^{it} = e^{\frac{2\pi t}{N}}b$ we obtain

$$\left(I \otimes Q^{-it}\right) \widetilde{B} \left(I \otimes Q^{it}\right) = e^{\frac{2\pi t}{N}} \widetilde{B},$$

$$\begin{aligned}
\left(I \otimes Q^{-it}\right) \chi(\widetilde{B}, \gamma) \left(I \otimes Q^{it}\right) &= \chi\left(e^{\frac{2\pi t}{N}} \widetilde{B}, \gamma\right) \\
&= \chi\left(e^{\frac{2\pi t}{N}}, \gamma\right) \chi\left(\widetilde{B}, \gamma\right) \\
&= |\gamma|^{-it} \chi\left(\widetilde{B}, \gamma\right).
\end{aligned}$$

Remembering that Q strongly commutes with a, one can easily show that $I \otimes Q^{it}$ commutes with $\widetilde{U}$. Therefore, for any $x, z \in K$ and $y, u \in H$ we have:

$$\left(\overline{z} \otimes Q^{it}u \Big| \chi(\widetilde{B}, \gamma)\widetilde{U} \Big| \overline{x} \otimes Q^{it}y\right) = |\gamma|^{-it} \left(\overline{z} \otimes u \Big| \chi(\widetilde{B}, \gamma)\widetilde{U} \Big| \overline{x} \otimes y\right).$$

If $y \in D(Q^{-1})$, $u \in D(Q)$ then performing the holomorphic continuation up to the point $t = i$ and using definition (3.22) of ψ we obtain:

$$\psi(\gamma) = |\gamma| \left(\overline{z} \otimes u \Big| \chi(\widetilde{B}, \gamma)\widetilde{U} \Big| \overline{x} \otimes y\right).$$

Remembering that χ is a bicharacter and using the definition (3.19) of $\widetilde{B}$ we obtain:

$$\begin{aligned}
\chi(\widetilde{B}, \gamma) &= \chi(-1, \gamma) \, \chi(\hat{b}, \gamma)^\top \otimes \chi\left(qa^{-1}b, \gamma\right) \\
&= (\text{Phase } \gamma)^{\frac{N}{2}} \, \chi(\hat{b}, \gamma)^\top \otimes \chi\left(qa^{-1}b, \gamma\right).
\end{aligned}$$

Therefore, using in the third step (3.21) we obtain:

$$
\begin{aligned}
\psi(\gamma) &= |\gamma|\,(\text{Phase}\,\gamma)^{\frac{N}{2}}\left(\overline{z}\otimes u\,\Big|\,\left[\chi(\hat{b},\gamma)^{\top}\otimes\chi\left(qa^{-1}b,\gamma\right)\right]\widetilde{U}\,\Big|\,\overline{x}\otimes y\right)\\[2mm]
&= |\gamma|\,(\text{Phase}\,\gamma)^{\frac{N}{2}}\left(\overline{\chi(\hat{b},\gamma)z}\otimes\chi\left(qa^{-1}b,\gamma\right)^{*}u\,\Big|\,\widetilde{U}\,\Big|\,\overline{x}\otimes y\right)\\[2mm]
&= |\gamma|\,(\text{Phase}\,\gamma)^{\frac{N}{2}}\left(x\otimes\chi\left(qa^{-1}b,\gamma\right)^{*}u\,\Big|\,U\,\Big|\,\chi(\hat{b},\gamma)z\otimes y\right)\\[2mm]
&= |\gamma|\,(\text{Phase}\,\gamma)^{\frac{N}{2}}\left(x\otimes u\,\Big|\,\left[I\otimes\chi\left(qa^{-1}b,\gamma\right)\right]U\left[\chi(\hat{b},\gamma)\otimes I\right]\,\Big|\,z\otimes y\right)\\[2mm]
&= |\gamma|\,(\text{Phase}\,\gamma)^{\frac{N}{2}}\left(x\otimes u\,\Big|\,\chi\left(I\otimes qa^{-1}b,\gamma\right)U\chi(\hat{b}\otimes I,\gamma)\,\Big|\,z\otimes y\right)\\[2mm]
&= |\gamma|\,(\text{Phase}\,\gamma)^{\frac{N}{2}}\left(x\otimes u\,\Big|\,\chi\left(I\otimes qa^{-1}b,\gamma\right)\chi(\hat{b}\otimes a,\gamma)U\,\Big|\,z\otimes y\right),
\end{aligned}
$$

where in the last step we used (3.10). Now, to end the proof of (3.23) it is sufficient to show that

$$
\alpha(\gamma)\chi\left(I\otimes qa^{-1}b,\gamma\right)\chi(\hat{b}\otimes a,\gamma)=\chi(\hat{b}\otimes b,\gamma)
$$

To this end it is sufficient to use Proposition 2.1 applied to the operators $R = I\otimes qa^{-1}b$ and $S = \hat{b}\otimes a$. One can easily verify that $(R,S)\in\mathcal{D}_H$.

$\square$

In the following we shall use the distribution theory on Γ. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}$. We say that a function φ on Γ is a function of Schwartz class if for any $k = 0, 1, \ldots, N-1$, the function $t \to \varphi\left(q^k e^t\right)$ belongs to $\mathcal{S}(\mathbb{R})$. The space of all Schwartz class functions on Γ equipped with the obvious Frechet topology will be denoted by $\mathcal{S}(\Gamma)$. By definition tempered distributions on Γ are elements of the adjoint space $\mathcal{S}'(\Gamma)$. We shall use standard operations on $\mathcal{S}(\Gamma)$ and $\mathcal{S}'(\Gamma)$. In particular for any $\varphi \in \mathcal{S}(\Gamma)$, its Fourier transform is given by

$$
(\mathcal{F}\varphi)(\gamma)=\int_{\Gamma}\varphi(\gamma')\chi(\gamma',\gamma)d\gamma'.
$$

The same formula works for tempered distributions.

We have to investigate the regularity properties of functions φ and ψ introduced by (3.22). If

$$
x\in D(\hat{b}^{\pm 1}),\qquad u\in D(b^{\pm 1}Q^{\pm 3}),\qquad y\in D(Q^{\pm 3})\tag{3.24}
$$

for all possible combinations of signs, then the functions φ and ψ belong to the Schwartz space $\mathcal{S}(\Gamma)$. Indeed using the relation $Q^{-it}bQ^{it}=e^{\frac{2\pi t}{N}}b$ one can easily show that $\left(I\otimes Q^{-it}\right)\chi(B,\gamma)\left(I\otimes Q^{it}\right)=\chi\left(e^{\frac{2\pi t}{N}}B,\gamma\right)=|\gamma|^{-it}\chi(B,\gamma)$ and

$$
\left(x\otimes Q^{it}u\,\big|\,\chi(B,\gamma)U\,\big|\,z\otimes Q^{it}y\right)=|\gamma|^{-it}\left(x\otimes u\,|\,\chi(B,\gamma)U\,|\,z\otimes y\right)=|\gamma|^{-it}\varphi(\gamma).
$$

Performing the holomorphic continuation up to the point $t = \pm 3i$ we obtain:

$$
\left(x\otimes Q^{\pm 3}u\,\big|\,\chi(B,\gamma)U\,\big|\,z\otimes Q^{\mp 3}y\right)=|\gamma|^{\pm 3}\varphi(\gamma).
$$

For $\gamma = q^k e^t$ we have:

$$
e^{\pm 3t}\varphi(q^k e^t)=\left(x\otimes Q^{\pm 3}u\,\Big|\,(\text{Phase}\,B)^k|B|^{\frac{N}{2\pi i}t}U\,\Big|\,z\otimes Q^{\mp 3}y\right).
$$

By (3.24), $x\otimes Q^{\pm 3}u\in D\left(B^{\pm 1}\right)$. Therefore the functions $e^{\pm 3t}\varphi(q^k e^t)$ admit holomorphic continuation to functions bounded on the strip $\left\{t\in\mathbb{C}:-\frac{2\pi}{N}<\Im t<\frac{2\pi}{N}\right\}$. It implies that the functions $t\mapsto\varphi(q^k e^t)$ (where $k = 0, 1, \ldots, N-1$) belong to $\mathcal{S}(\mathbb{R})$ and $\varphi\in\mathcal{S}(\Gamma)$. Moreover using (3.23) we see that the functions $e^{\pm t}\psi(q^k e^t)$ admit holomorphic continuation to functions bounded on the same strip. It shows that $\psi\in\mathcal{S}(\Gamma)$.

Let f and g be measurable bounded functions on Γ. Then these functions may be considered as a tempered distributions on $\mathbb{R}$. We denote by $\hat{f}$ and $\hat{g}$ the inverse Fourier transform of these

distributions. Then

$$f(\gamma) = \int_\Gamma \hat{f}(\gamma')\,\chi(\gamma',\gamma)d\gamma'$$

$$g(\gamma) = \int_\Gamma \hat{g}(\gamma')\,\chi(\gamma',\gamma)d\gamma' \tag{3.25}$$

for almost all $\gamma \in \Gamma$.

Proposition 3.5. *Let f, g be bounded measurable functions on $\mathbb{R}_+$ and $\hat{f}$ and $\hat{g}$ be tempered distributions related to f and g via formulae (3.25). Assume that*

$$\hat{f}(\gamma) = |\gamma|\,(\text{Phase}\,\gamma)^{\frac{N}{2}}\,\overline{\alpha(\gamma)}\,\hat{g}(\gamma). \tag{3.26}$$

Then, using the notation and assumptions of Proposition 3.4 we have:

$$(x \otimes u | f(B)U | z \otimes y) = \left(\overline{z} \otimes Qu \Big| g(\widetilde{B})\widetilde{U} \Big| \overline{x} \otimes Q^{-1}y\right). \tag{3.27}$$

Proof. Assume for the moment that vectors x, y, z, u satisfy conditions (3.24). Then the functions (3.22) belong to $\mathcal{S}(\mathbb{R})$. Comparing (3.25) with (3.22) we obtain:

$$(x \otimes u | f(B)U | z \otimes y) \qquad = \int_\mathbb{R} \hat{f}(k)\varphi(k)dk,$$

$$\left(\overline{z} \otimes Qu \Big| g\left(\widetilde{B}\right)\widetilde{U} \Big| \overline{x} \otimes Q^{-1}y\right) = \int_\mathbb{R} \hat{g}(k)\psi(k)dk,$$

Using now (3.26) and (3.23) we see that the right hand sides of the above formulae coincide and (3.27) follows. To end the proof we notice that the conditions (3.24) select sufficiently large sets of vectors: $D(\hat{b}) \cap D(\hat{b}^{-1})$ is dense in H, $D(bQ^3) \cap D(bQ^{-3}) \cap D(b^{-1}Q^3) \cap D(b^{-1}Q^{-3})$ is a core for Q and $D(Q^3) \cap D(Q^{-3})$ is a core for Q^{-1}.

$$\square$$

We continue the proof of Proposition 3.3. For any $\gamma \in \Gamma$ we set

$$f(\gamma) = F_N(\gamma), \qquad g(\gamma) = \overline{F_N(\gamma)} \tag{3.28}$$

Let $\hat{f}$ and $\hat{g}$ be the tempered distributions related to the above functions via formulae (3.25). We already have know that (3.17) resolves itself into (3.20). By virtue of Proposition 3.5, to prove this relation it is sufficient to verify formula (3.26). Comparing (1.20) with (3.25) we obtain:

$$\hat{f}(\gamma) = \frac{(\text{Phase}\,\gamma)^{\frac{N}{2}}\,\gamma}{C_F\,(1-\overline{\gamma})\,F_N(-\gamma)}.$$

Let us notice that $g(\gamma) = \overline{f(\gamma)}$. Therefore $\hat{g}(\gamma) = \overline{\hat{f}(\gamma^{-1})}$ and

$$\hat{g}(\gamma) = \frac{(\text{Phase}\,\gamma)^{\frac{N}{2}}\,\overline{\gamma}^{-1}F_N\left(-\gamma^{-1}\right)}{\overline{C_F}\,(1-\gamma^{-1})}.$$

So we have:

$$\frac{\hat{g}(\gamma)}{\hat{f}(\gamma)} = \frac{C_F(1-\overline{\gamma})}{\overline{C_F}(\gamma-1)\overline{\gamma}}\,F_N\left(-\gamma^{-1}\right) F_N\left(-\gamma\right).$$

Inserting in (1.9), $q^{-1}\gamma$ instead of γ we get:

$$F_N\left(q^{-2}\gamma\right) = \frac{1+\overline{\gamma}}{1+\gamma}F_N(\gamma).$$

Combining this formula with (1.8) we obtain:

$$F_N\left(\gamma^{-1}\right) F_N(\gamma) = e^{\frac{\pi}{6i}\left(\frac{2}{N}+\frac{N}{2}\right)}\alpha\left(q^{-1}\gamma\right)\frac{1+\gamma}{1+\overline{\gamma}}. \tag{3.29}$$

Therefore

$$\frac{\hat{g}(\gamma)}{\hat{f}(\gamma)} = \frac{C_F(1-\overline{\gamma})}{\overline{C_F}(\gamma-1)\overline{\gamma}}\,e^{\frac{\pi}{6i}\left(\frac{2}{N}+\frac{N}{2}\right)}\alpha\left(-q^{-1}\gamma\right)\frac{1-\gamma}{1-\overline{\gamma}}$$

$$= -\frac{C_F}{\overline{C_F}\overline{\gamma}}\,e^{\frac{\pi}{6i}\left(\frac{2}{N}+\frac{N}{2}\right)}\alpha\left(-q^{-1}\gamma\right).$$

To compute the last factor we use (1.2) and (1.4):

$$\alpha\left(-q^{-1}\gamma\right) = \alpha\left(-q^{-1}\right)\alpha\left(\gamma\right)\chi\left(-q^{-1},\gamma\right)$$

$$= \alpha\left(-q^{-1}\right)\left(\text{Phase}\,\gamma\right)^{\frac{N}{2}-1}\alpha(\gamma).$$

Therefore

$$\frac{\hat{g}(\gamma)}{\hat{f}(\gamma)} = -\frac{C_F}{\overline{C_F}}\, e^{\frac{\pi}{6i}\left(\frac{2}{N}+\frac{N}{2}\right)}\alpha\left(-q^{-1}\right)|\gamma|^{-1}\left(\text{Phase}\,\gamma\right)^{\frac{N}{2}}\alpha(\gamma). \tag{3.30}$$

According to (1.20), $C_F = -2\pi i F_N(-1)^{-1}$. Therefore $-C_F/\overline{C_F} = F_N(-1)^{-2}$. Setting in (3.29), $\gamma \in \mathbb{R}$, $\gamma \to -1$ we see that

$$F_N(-1)^2 = e^{\frac{\pi}{6i}\left(\frac{2}{N}+\frac{N}{2}\right)}\alpha\left(-q^{-1}\right).$$

Using these data, one can easily verify that the numerical factor

$$-\frac{C_F}{\overline{C_F}}\, e^{\frac{\pi}{6i}\left(\frac{2}{N}+\frac{N}{2}\right)}\alpha\left(-q^{-1}\right) = 1$$

and (3.30) is equivalent to (3.26). This ends the proof of formula (3.17) and of Proposition 3.3. $\qquad\square$

Now we are able to prove Theorem 3.1. Let a, b, r, s be operators acting on a Hilbert space H, satisfying the assumptions of Theorem 3.1. Setting $K = H$, $\hat{a} = sb^{-1}$ and $\hat{b} = ab^{-1}$ and $Q = r|a|$ we satisfy all the assumptions of Propositions 3.2 and 3.3. Introducing the above data into (3.6) and (3.16) we obtain unitary operators $W \in B(H \otimes H)$ and $\widetilde{W} \in B(\overline{H} \otimes H)$. Clearly W is given by (3.2). Proposition 3.2 shows that the operator W satisfies the pentagonal equation (3.7). In the present setting formula (3.17) shows that

$$(x \otimes u|W|z \otimes y) = \left(\overline{z} \otimes Qu\middle|\widetilde{W}\middle|\overline{x} \otimes Q^{-1}y\right).$$

for any $x, z \in H$, $y \in D(Q^{-1})$, $u \in D(Q)$.

To finish the proof of manageability of W we have to show that W commutes with $Q \otimes Q$. To this end we recall that $|a|^{-it}b|a|^{it} = e^{\frac{2\pi}{N}t}b$ and $r^{-it}sr^{it} = e^{\frac{2\pi}{N}t}s$. Moreover r, s strongly commute with a, b. Using these informations one can easily show that $Q^{it} \otimes Q^{it}$ commutes with $ab^{-1} \otimes b$, $sb^{-1} \otimes I$ and $I \otimes a$. So does it with W. This is the end of the proof of Theorem 3.1.

4. CROSSED PRODUCT ALGEBRA.

In this Section we construct the C^*-algebra related to the commutation relations (0.7). We shall freely use the notion of affiliated elements to C^*-algebras and the concept of C^*-algebra generated by a finite sequence of unbounded elements affiliated with it [17]. The affiliation relation will be denoted by η. Consequently A^η will denotes the set of all elements affiliated to a C^*-algebra A: $a \in A^\eta$ means $a\,\eta\,A$.

Let $B = C_\infty\left(\overline{\Gamma}\right)$ be the C^*-algebra of all continuous functions vanishing at infinity on $\overline{\Gamma}$. For any $\gamma \in \overline{\Gamma}$ we set:

$$b(\gamma) = \gamma. \tag{4.1}$$

Then $b \in C\left(\overline{\Gamma}\right)$. According to [17, formula (2.6)], b is an element affiliated with B. Clearly b is normal (any element affiliated with a commutative C*-algebra is normal) and $\text{Sp}\,b \subset \overline{\Gamma}$.

Let $\gamma \in \Gamma$ and $f \in B$. For any $\tau \in \overline{\Gamma}$ we set:

$$(\sigma_\gamma f)(\tau) = f\left(\gamma\tau\right).$$

Then $\sigma_\gamma f \in B$, $\sigma_\gamma \in \text{Aut}\left(B\right)$ and $(\sigma_\gamma)_{\gamma\in\Gamma}$ is a pointwise continuous one parameter group of automorphisms of B labeled by Γ. In other words, $\left(B, (\sigma_\gamma)_{\gamma\in\Gamma}\right)$ is a C^*-dynamical system. Let

$$A_{\text{cp}} = B \times_\sigma \Gamma \tag{4.2}$$

be the corresponding C^*-crossed product algebra [7]. The canonical embedding $B \hookrightarrow M(A_{\text{cp}})$ is a morphism from B into A_{cp}. Therefore the elements affiliated with B are affiliated with A_{cp}. In particular $b\,\eta\,A_{\text{cp}}$. By the definition of crossed product, $M(A_{\text{cp}})$ contains a strictly continuous one parameter group of unitaries $(U_\gamma)_{\gamma\in\Gamma}$ implementing the action σ of Γ on B:

$$U_\gamma f U_\gamma^* = \sigma_\gamma f$$

for any $f \in B$ and $\gamma \in \Gamma$. Using SNAG Theorem ([4, Chapter 6, §2, Theorem 1]) and Theorem 5.2 proven in the next Section one can easily show that any strictly continuous one parameter group of unitaries $(U_\gamma)_{\gamma \in \Gamma}$ contained in $M(A_{\mathrm{cp}})$ is of the form: $U_\gamma = \chi(a, \gamma)$, where a is a normal element affiliated with A_{cp} such that $\mathrm{Sp}\, a \subset \overline{\Gamma}$. Moreover a is invertible and $a^{-1}\, \eta\, A_{\mathrm{cp}}$.

One can easily verify that $\sigma_\gamma b = \gamma b$. Therefore $U_\gamma b = \gamma b U_\gamma$ for any $\gamma \in \Gamma$. It means that (a, b) is a G-pair.

By the construction, the set

$$\left\{ fg(a) : f \in B,\ g \in C_\infty(\Gamma) \right\}^{\text{linear envelope}} \tag{4.3}$$

is a dense subset of the C^*-crossed product $A_{\mathrm{cp}} = B \times_\sigma \Gamma$.

Proposition 4.1. *The C^*-algebra A_{cp} is generated (in the sense explained in [17]) by the three affiliated elements $a, a^{-1}, b\ \eta\ A_{\mathrm{cp}}$.*

Proof. We shall use Theorem 3.3 of [17]. We know that any element of B is a function of b. Therefore b separates representations of B. Remembering that (4.3) is dense in A_{cp} we see that elements a and b separate representations of A_{cp}. This way we verified Assumption 1 of Theorem 3.3 of [17].

Let $r_1 = (I + b^*b)^{-1}$, $r_2 = (I + a^*a)^{-1}$ and $r_2 = \left(I + \left(a^{-1}\right)^* a^{-1}\right)^{-1}$. To end the proof it is sufficient to notice, that

$$r_1 r_2 r_3 = fg(a),$$

where $f = (I + b^*b)^{-1}$ and $g(\gamma) = \left(|\gamma| + |\gamma|^{-1}\right)^2$. Clearly $f \in B$ and $g \in C_\infty(\Gamma)$. Therefore $r_1 r_2 r_3$ belongs to (4.3) and consequently $r_1 r_2 r_3 \in A_{\mathrm{cp}}$. It shows that assumption 2 of Theorem 3.3 of [17] holds. Now this Theorem says that A_{cp} is generated by a, a^{-1} and b. $\square$

Let H be a Hilbert space and π be a non-degenerate representation of A_{cp} acting on H: $\pi \in \mathrm{Rep}\,(A_{\mathrm{cp}}, H)$. According to the general theory, π admits a natural extension to the set of affiliated elements A_{cp}^η. Clearly $(\pi(a), \pi(b))$ is a G-pair. It turns out that any G-pair is of this form.

Proposition 4.2. *Let H be a Hilbert space and $(a_o, b_o) \in G_H$. Then there exists unique representation $\pi \in \mathrm{Rep}\,(A_{\mathrm{cp}}, H)$ such that $a_o = \pi(a)$ and $b_o = \pi(b)$. If A is a non-degenerate C^*-algebra of operators on H and $a_o, a_o^{-1}, b_o\, \eta\, A$, then $\pi \in \mathrm{Mor}(A_{\mathrm{cp}}, A)$.*

Proof. For any $f \in B$ we set:

$$\pi_o(f) = f(b_o). \tag{4.4}$$

Elementary computations show that the π_o is a non-degenerate representation of B. The action of π_o on elements affiliated with B is described by the same formula (4.4). In particular for element (4.1) we have: $\pi_o(b) = b_o$

Now we shall use the relation $(a_o, b_o) \in G_H$. It means that $\chi(a_o, \gamma) b_o \chi(a_o, \gamma)^* = \gamma b_o$. Therefore

$$\chi(a_o, \gamma) \pi_o(f) \chi(a_o, \gamma)^* = \chi(a_o, \gamma) f(b_o) \chi(a_o, \gamma)^*$$
$$= f(\gamma b_o) = (\sigma_\gamma f)(b_o) = \pi_o(\sigma_\gamma f). \tag{4.5}$$

It shows that the pair $\left(\pi_o, (\chi(a_o, \gamma))_{\gamma \in \Gamma}\right)$ is a covariant representation of the C^*-dynamical system $\left(B, (\sigma_\gamma)_{\gamma \in \gamma}\right)$. Let π be the corresponding representation of the crossed product algebra A_{cp}. Then $\pi\left(\chi(a, \gamma)\right) = (\chi(a_o, \gamma))$ and $\pi(a) = a_o$. Moreover π restricted to $B \subset M(A_{\mathrm{cp}})$ coincides with π_o. In particular $\pi(b) = b_o$.

This way we constructed representation $\pi \in \mathrm{Rep}\,(A_{\mathrm{cp}}, H)$ having desired properties. The uniqueness of π and the last Statement of the Proposition follows immediately from Proposition 4.1 (cf Definition 3.1 and Theorem 6.2 of [17]). $\square$

5. Special functions and the affiliation relation.

In this Section we freely use many notions of the theory of non-unital C^*-algebras. In particular for any C^*-algebra A, $M(A)$ is the multiplier algebra. The natural topology on $M(A)$ is the topology of strict convergence. The set of all elements affiliated with A will be denoted by A^η. We write $R\,\eta\,A$ instead of $R \in A^\eta$. These notions are described in [17]. This Section is devoted to the proof of the following two theorems.

Theorem 5.1. *Let R be a normal operator acting on a Hilbert space H and A be a non-degenerate C^*-subalgebra of $B(H)$. Assume that $\operatorname{Sp} R \subset \overline{\Gamma}$. Then the following two Statements are equivalent:*

1. For any $\gamma \in \overline{\Gamma}$ we have $F_N(\gamma R) \in M(A)$. Moreover the mapping

$$\overline{\Gamma} \ni \gamma \longmapsto F_N(\gamma R) \in M(A), \tag{5.1}$$

is continuous provided $M(A)$ is considered with the strict topology.

2. Operator R is affiliated with A.

Proof. We shall consider the commutative C^*-algebra $B = C_\infty(\overline{\Gamma})$. Elements of B^η are continuous functions f on $\overline{\Gamma}$. If in addition f is bounded, then $f \in M(B)$. Let $\gamma \in \overline{\Gamma}$. For any $\gamma' \in \overline{\Gamma}$ we set:

$$f_1(\gamma') = \gamma',$$

$$F^\gamma(\gamma') = F_N(\gamma\gamma').$$

Then $f_1\,\eta\,B$. Using [17, Example 2, page 497] we see that f_1 generates B. The continuity of F_N and Statement 1 of Theorem 1.2 easily imply that F^γ is a unitary element of $M(B)$. Remembering, that any continuous function on a compact set is uniformly continuous one can show that F^γ converges almost uniformly to F^{γ_o}, when $\gamma \to \gamma_o$. For bounded sets, the strict topology on $M(C_\infty(\overline{\Gamma}))$ coincides with that of almost uniform convergence. Therefore the mapping

$$\overline{\Gamma} \ni \gamma \longmapsto F^\gamma \in M(B) \tag{5.2}$$

is continuous and for any function $\varphi \in L^1(\Gamma)$ we may consider integral

$$F^\varphi = \int_\Gamma F^\gamma\, \varphi(\gamma) d\gamma \in M(B).$$

Taking into account the asymptotic behavior (1.14) one can show, that $F^\varphi(\gamma') \to 0$, when $\gamma' \to \infty$. It means that $F^\varphi \in B$.

Let $\gamma_1, \gamma_2 \in \overline{\Gamma}$. Assume that $F^\gamma(\gamma_1) = F^\gamma(\gamma_2)$ for and all $\gamma \in \overline{\Gamma}$. Then $F_N(q\gamma_1) = F_N(q\gamma_2)$ and $F_N(r\gamma_1) = F_N(r\gamma_2)$ for all $r \in \mathbb{R}_+$. By holomorphic continuation: $\widetilde{F}(z|\gamma_1|, \operatorname{Phase}\gamma_1) = \widetilde{F}(z|\gamma_2|, \operatorname{Phase}\gamma_2)$ for all $z \in \overline{\Omega}$. Setting $z = q$ and using (1.16) we obtain $(1 + q^{-1}\overline{\gamma}_1)F_N(q\gamma_1) = (1 + q^{-1}\overline{\gamma}_2)F_N(q\gamma_2)$. Therefore $\gamma_1 = \gamma_2$. It shows that the family of functions: $\{F^\gamma : \gamma \in \Gamma\}$ separates points of $\overline{\Gamma}$. The same is valid for the family $\{F^\varphi : \varphi \in L^1(\Gamma)\}$. Now, using the Stone - Weierstrass theorem (applied to the one point compactification of $\overline{\Gamma}$) we conclude that

$$\begin{array}{l}\text{The }^*\text{-subalgebra of } B \text{ generated by} \\ \left\{F^\varphi : \varphi \in L^1(\Gamma)\right\} \text{ is dense in } B\end{array} \tag{5.3}$$

Now, let R be the operator satisfying the assumptions of the Theorem. Then the mapping

$$\pi : B \ni f \longrightarrow f(R) \in B(H)$$

is a non-degenerate representation of B. Clearly $\pi(f_1) = R$. If $R\,\eta\,A$, then $\pi \in \operatorname{Mor}(B, A)$. In this case

$$F_N(\gamma R) = \pi(F^\gamma) \in M(A)$$

and the continuity of (5.1) follows immediately from that of (5.2). We showed that Statement 1 follows from Statement 2. Conversely assume that Statement 1 holds. Then $\pi(F^\gamma) \in M(A)$ for all $\gamma \in \overline{\Gamma}$ and the mapping

$$\overline{\Gamma} \ni \gamma \to \pi(F^\gamma) \in M(A) \tag{5.4}$$

is continuous. Integrating over γ we obtain $\pi(F^\varphi) \in M(A)$ for any $\varphi \in L^1(\Gamma)$. By (5.3), $\pi(B) \subset M(A)$. We shall use once more the continuity of (5.4). It implies that F^φ tends strictly to $F^0 = I$ when φ is nonnegative, $\int_\Gamma \varphi(\gamma)d\gamma = 1$ and the support of φ shrinks to $0 \in \overline{\Gamma}$. Therefore $\pi(B)$ contains an approximate unit for A and $\pi \in \operatorname{Mor}(B, A)$. Applying π to $f_1\,\eta\,B$ we obtain $R\,\eta\,A$.

$\square$

In the same way we shall prove

Theorem 5.2. *Let R be a normal operator acting on a Hilbert space H and A be a non-degenerate C^*-subalgebra of $B(H)$. Assume that $\ker R = \{0\}$ and $\operatorname{Sp} R \subset \overline{\Gamma}$. Then the following two Statements are equivalent:*

1. For any $\gamma \in \Gamma$ we have $\chi(R, \gamma) \in M(A)$. Moreover the mapping

$$\Gamma \ni \gamma \longmapsto \chi(R, \gamma) \in M(A), \tag{5.5}$$

is continuous provided $M(A)$ is considered with the strict topology.

2. Operators R and R^{-1} are affiliated with A.

Proof. We shall consider the commutative C^*-algebra $B = C_\infty(\Gamma)$. Elements of B^η are continuous functions f on Γ. If in addition f is bounded, then $f \in M(B)$. Let $\gamma \in \Gamma$. For any $\gamma' \in \Gamma$ we set:

$$f_1(\gamma') = \gamma', \qquad f_2(\gamma') = \gamma'^{-1}$$

$$G^\gamma(\gamma') = \chi(\gamma', \gamma).$$

Then $f_1, f_2 \, \eta \, B$. Using [17, Example 2, page 497] we see that f_1, f_2 generate B. The continuity of χ easily imply that G^γ is a unitary element of $M(B)$. Remembering, that any continuous function on a compact set is uniformly continuous one can show that G^γ converges almost uniformly to G^{γ_o}, when $\gamma \to \gamma_o$. For bounded sets, the strict topology on $M(C_\infty(\Gamma))$ coincides with that of almost uniform convergence. Therefore the mapping

$$\Gamma \ni \gamma \longmapsto G^\gamma \in M(B) \tag{5.6}$$

is continuous and for any function $\varphi \in L^1(\Gamma)$ we may consider integral

$$G^\varphi = \int_\Gamma G^\gamma \, \varphi(\gamma) d\gamma \in M(B).$$

Clearly G^φ is a Fourier transform of φ. It is well known that Fourier transforms of L^1-functions form a dense subset of $C_\infty(\Gamma)$:

$$\begin{array}{c} \left\{ G^\varphi : \varphi \in L^1(\Gamma) \right\} \text{ is} \\ \text{a dense subset of } B \end{array} \tag{5.7}$$

Now, let R be the operator satisfying the assumptions of the Theorem. Then the mapping

$$\pi : B \ni f \longrightarrow f(R) \in B(H)$$

is a non-degenerate representation of B. Clearly $\pi(f_1) = R$ and $\pi(f_2) = R^{-1}$. If $R \, \eta \, A$ and $R^{-1} \, \eta \, A$, then $\pi \in \operatorname{Mor}(B, A)$. In this case

$$\chi(R, \gamma) = \pi(G^\gamma) \in M(A)$$

and the continuity of (5.5) follows immediately from that of (5.6). We showed that Statement 1 follows from Statement 2. Conversely assume that Statement 1 holds. Then $\pi(G^\gamma) \in M(A)$ for all $\gamma \in \Gamma$ and the mapping

$$\Gamma \ni \gamma \to \pi(G^\gamma) \in M(A) \tag{5.8}$$

is continuous. Integrating over γ we obtain $\pi(G^\varphi) \in M(A)$ for any $\varphi \in L^1(\Gamma)$. By (5.7), $\pi(B) \subset M(A)$. We shall use once more the continuity of (5.8). It implies that G^φ tends strictly to $G^1 = I$ when φ is nonnegative , $\int_\Gamma \varphi(\gamma) d\gamma = 1$ and the support of φ shrinks to $1 \in \Gamma$. Therefore $\pi(B)$ contains an approximate unit for A and $\pi \in \operatorname{Mor}(B, A)$. Applying π to $f_1, f_2 \, \eta \, B$ we obtain $R, R^{-1} \, \eta \, A$.

$$\square$$

We shall also use

Proposition 5.3. *Let R_1, R_2 be strongly commuting normal operator acting on Hilbert space H and A be a non-degenerate C^*-subalgebra of $B(H)$. Assume that $\operatorname{Sp} R_i \subset \overline{\Gamma}$, $\ker R_i = \{0\}$ and $R_i, R_i^{-1} \, \eta \, A$ for $i = 1, 2$. Then $f(R_1, R_2) \in M(A_1 \otimes A_2)$ for any $f \in C_{\mathrm{bounded}}(\Gamma \times \Gamma)$.*

Proof. We shall consider the commutative C^*-algebra $B = C_\infty(\Gamma \times \Gamma)$. Elements of B^η are continuous functions f on $\Gamma \times \Gamma$. If in addition f is bounded, then $f \in M(B)$: $C_{\mathrm{bounded}}(\Gamma \times \Gamma) = M(B)$. For any $\gamma, \gamma' \in \Gamma$ we set:

$$f_1(\gamma, \gamma') = \gamma, \qquad f_2(\gamma, \gamma') = \gamma^{-1},$$

$$f_3(\gamma, \gamma') = \gamma', \qquad f_4(\gamma, \gamma') = \gamma'^{-1}.$$

Then $f_1, f_2, f_3, f_4 \, \eta \, B$. Using [17, Example 2, page 497] we see that f_1, f_2, f_3, f_4 generate the C*-algebra B.

Now, let R_1, R_2 be the operators satisfying the assumptions of the Proposition. Then the mapping

$$\pi : B \ni f \longrightarrow f(R_1, R_2) \in B(H)$$

is a non-degenerate representation of B. Clearly $\pi(f_1) = R_1$, $\pi(f_2) = R_1^{-1}$, $\pi(f_3) = R_2$, $\pi(f_4) = R_2^{-1}$. We assumed that $R_1, R_1^{-1}, R_2, R_1^{-1} \, \eta \, A$. Therefore $\pi \in \mathrm{Mor}(B, A)$ and

$$f(R_1, R_2) = \pi(f) \in M(A)$$

for any $f \in M(B)$.

$\square$

6. FROM MULTIPLICATIVE UNITARY TO QUANTUM GROUP.

Let G be the quantum space corresponding to the C^*-algebra A_{cp}. In other words, elements of A_{cp} are interpreted as continuous functions vanishing at infinity on G. In this Section we endow G with a group structure introducing a comultiplication $\Delta \in \mathrm{Mor}(A_{\mathrm{cp}}, A_{\mathrm{cp}} \otimes A_{\mathrm{cp}})$. It will be shown that the quantum group G coincides with the '$az + b$'-group introduced in Section 0.

Any C^*-algebra may be embedded in a non-degenerate way into $B(H)$, where H is a Hilbert space. Then affiliated elements become closed operators acting on H. Let

$$\jmath : A_{\mathrm{cp}} \hookrightarrow B(H) \tag{6.1}$$

be a non-degenerate embedding. Then $\jmath \in \mathrm{Rep}\,(A_{\mathrm{cp}}, H)$ and $\jmath(a)$ and $\jmath(b)$ are normal operators acting on H. To simplify the notation we will drop the embedding symbol '$\jmath$' writing a, b instead of $\jmath(a), \jmath(b)$. Clearly $(a, b) \in G_H$. With this notation $A_{\mathrm{cp}} \subset B(H)$.

One can check that the subspace $\ker b$ is A_{cp}-invariant. Replacing if necessary H by $(\ker b)^\perp$ we may assume that $\ker b = \{0\}$. We may also assume that the commutant $A'_{\mathrm{cp}} = \{a' \in B(H) : a'c = ca' \text{ for any } c \in A_{\mathrm{cp}}\}$ contains a W^*-algebra isomorphic to $B(K)$, where K is an infinite-dimensional Hilbert space. If this is not the case, then we replace (6.1) by $\jmath' : A_{\mathrm{cp}} \hookrightarrow B(K \otimes H)$ introduced by the formula $\jmath'(c) = I_{B(K)} \otimes \jmath(c)$ for any $c \in A_{\mathrm{cp}}$. Since the commutant A'_{cp} is large enough, there exist strictly positive selfadjoint operators r, s acting on H such that r, s strongly commute with a, b and $r^{-it}sr^{it} = e^{\frac{2\pi t}{N}}s$ for all $t \in \mathbb{R}$.

Let $\hat{b} = ab^{-1}$ and $\hat{a} = sb^{-1}$. Then the operator (3.2) equals to

$$W = F_N \left(\hat{b} \otimes b \right) \chi \left(\hat{a} \otimes I, I \otimes a \right) \tag{6.2}$$

This operator acts on $H \otimes H$. By Theorem 3.1 W is a manageable multiplicative unitary. Corresponding operators Q and $\widetilde{W}$ are given by: $Q = r|a|$ and

$$\widetilde{W} = F_N \left(-\hat{b}^\top \otimes qa^{-1}b \right)^* \chi \left(\hat{a}^\top \otimes I, I \otimes a \right). \tag{6.3}$$

We shall use the theory developed in [3, 18]. Let $B(H)_*$ be the set of all normal linear functionals defined on $B(H)$ and

$$A = \left\{ (\omega \otimes \mathrm{id})W : \omega \in B(H)_* \right\}^{\mathrm{norm\ closure}}. \tag{6.4}$$

According to the general theory, A is a C^*-algebra and $W \in M(CB(H) \otimes A)$, where $CB(H)$ the C^*-algebra of all compact operators acting on H. The algebra A is interpreted as the algebra of all 'continuous functions vanishing at infinity on the quantum group'. We shall prove that this algebra coincides with the crossed product algebra A_{cp}:

$$A = A_{\mathrm{cp}}. \tag{6.5}$$

Any closed operator acting on H is affiliated with $CB(H)$. In particular $\hat{a}, \hat{a}^{-1}, \hat{b} \in CB(H)^\eta$. Remembering that $a, a^{-1}, b \in A_{\mathrm{cp}}^\eta$ we see that $\hat{a} \otimes I$, $\hat{a}^{-1} \otimes I$, $I \otimes a$, $I \otimes a^{-1}$ and $\hat{b} \otimes b$ are affiliated with $CB(H) \otimes A_{\mathrm{cp}}$. By virtue of Proposition 5.3 and Theorem 5.1 we have:

$$\chi(\hat{a} \otimes I, I \otimes a) \in M \left(CB(H) \otimes A_{\mathrm{cp}} \right),$$

$$F_N(\hat{b} \otimes b) \quad \in M \left(CB(H) \otimes A_{\mathrm{cp}} \right).$$

Consequently $W \in M \left(CB(H) \otimes A_{\mathrm{cp}} \right)$. Now using (6.4) we obtain $A \subset M(A_{\mathrm{cp}})$ and $AA_{\mathrm{cp}} \subset A_{\mathrm{cp}}$.

W is a unitary element of the multiplier algebra. Therefore $W\left(CB(H)\otimes A_{\mathrm{cp}}\right)=CB(H)\otimes A_{\mathrm{cp}}$ and the set

$$\left\{W(m\otimes c):m\in CB(H),c\in A_{\mathrm{cp}}\right\} \tag{6.6}$$

is linearly dense in $CB(H)\otimes A_{\mathrm{cp}}$. For any $\omega\in B(H)_*$, $m\in CB(H)$ and $c\in A_{\mathrm{cp}}$ we have: $(\omega\otimes\mathrm{id})\left(W(m\otimes c)\right)=\left((m\omega\otimes\mathrm{id})W\right)c\in AA_{\mathrm{cp}}$. Applying $\omega\otimes\mathrm{id}$ to all elements of (6.6) we see that

$$AA_{\mathrm{cp}}\text{ is a linearly dense subset of }A_{\mathrm{cp}}. \tag{6.7}$$

We shall prove that

$$a,\ a^{-1},b\ \eta\ A. \tag{6.8}$$

For all $\gamma\in\overline{\Gamma}$ we set

$$V(\gamma)=F_N(\gamma\hat{b}\otimes b)\chi(\hat{a}\otimes I,I\otimes a). \tag{6.9}$$

Then $V(\gamma)\in B(H\otimes H)=M\left(CB(H)\otimes CB(H)\right)$. In what follows we endow multiplier algebras with the strict topology. Using Theorem 5.1 one can easily show that $\left(V(\gamma)\right)_{\gamma\in\overline{\Gamma}}$ is a continuous family of elements of $M\left(CB(H)\otimes CB(H)\right)$. Tensoring by $I\in M(A)$ and using the leg numbering notation $V_{12}(\gamma)=V(\gamma)\otimes I$ we obtain a continuous family $\left(V_{12}(\gamma)\right)_{\gamma\in\overline{\Gamma}}$ of elements of $M\left(CB(H)\otimes CB(H)\otimes A\right)$.

By Proposition 3.2, operators (6.9) satisfy the pentagonal equation (3.7). Therefore

$$V_{13}(\gamma)=V_{12}(\gamma)^*W_{23}V_{12}(\gamma)W_{23}^*. \tag{6.10}$$

Using this formula and remembering that $W\in M(CB(H)\otimes A)$ we see that $\left(V_{13}(\gamma)\right)_{\gamma\in\overline{\Gamma}}$ is a continuous family of elements of $M\left(CB(H)\otimes CB(H)\otimes A\right)$. It implies that $\left(V(\gamma)\right)_{\gamma\in\overline{\Gamma}}$ is a continuous family of elements of $M\left(CB(H)\otimes A\right)$.

Therefore $F_N(\gamma\hat{b}\otimes b)=V(\gamma)V(0)^*\in M(CB(H)\otimes A)$ depends continuously on $\gamma\in\overline{\Gamma}$. Now, Theorem 5.1 shows that $\hat{b}\otimes b$ is affiliated with $CB(H)\otimes A$. Taking into account Proposition A.1 of [22] we get $b\ \eta\ A$.

Let $\gamma\in\Gamma$. Inserting in Proposition 3.2: $K=\mathbb{C}$, $\hat{b}=0$ and $\hat{a}=\gamma$ we see that the operator

$$V(\gamma)=\chi(\gamma,a)=\chi(a,\gamma) \tag{6.11}$$

satisfy the pentagonal equation (3.7). In the present case equation (6.10) takes the form

$$I\otimes\chi(a,\gamma)=\left(\chi(a,\gamma)^*\otimes I\right)W\left(\chi(a,\gamma)\otimes I\right)W^*. \tag{6.12}$$

It shows that $\left(I\otimes\chi(a,\gamma)\right)_{\gamma\in\Gamma}$ is a continuous one parameter group of unitary elements of $M\left(CB(H)\otimes A\right)$. Consequently $\left(\chi(a,\gamma)\right)_{\gamma\in\Gamma}$ is a continuous one parameter group of unitary elements of $M(A)$. Now Theorem 5.2 shows that a and a^{-1} are affiliated with A. This way (6.8) is shown.

Now we combine Proposition 4.1 with (6.8). By virtue of Definition 3.1 of [17], the embedding (6.1) belongs to $\mathrm{Mor}(A_{\mathrm{cp}},A)$. It means that $A_{\mathrm{cp}}A$ is a linearly dense subset of A. Comparing this result with (6.7) we obtain (6.5). This way we revealed the structure of the algebra of 'continuous functions vanishing at infinity on G.

According to the general theory [3, 18] the comultiplication Δ is introduced by the formula:

$$\Delta(c)=W(c\otimes I)W^*.$$

It is known that $\Delta(c)\in M(A\otimes A)$ for any $c\in A$ and that $\Delta\in\mathrm{Mor}(A,A\otimes A)$. Clearly the action of Δ on elements affiliated with A is described by the same formula. By the pentagonal equation we have

$$(\mathrm{id}\otimes\Delta)W=W_{12}W_{13}.$$

Using this formula one can easily show that Δ is coassociative.

We have computed the action of Δ on generators a, b of A. Formula (3.3) and (3.4) show that

$$\Delta(a)=\quad a\otimes a,$$

$$\Delta(b)=a\otimes b\dotplus b\otimes I.$$

Now we shall discuss the coinverse map κ. According to [18] we have polar decomposition

$$\kappa(c) = \left(\tau_{i/2}(c)\right)^R, \tag{6.13}$$

where $\tau_{i/2}$ is the analytic generator of the scaling group and the map $A \ni c \mapsto c^R$ is the unitary antipode. The action of the scaling group is described by the formula: $\tau_t(c) = Q^{2it}cQ^{-2it}$. Remembering that Q commutes with a and that $Q^{-it}cQ^{it} = e^{\frac{2\pi t}{N}}b$ we obtain:

$$\tau_t(a) = a, \qquad \tau_t(b) = e^{\frac{-4\pi t}{N}}b. \tag{6.14}$$

Consequently: $\tau_{i/2}(a) = a$ and $\tau_{i/2}(b) = e^{\frac{-2\pi i}{N}}b = q^{-1}b$. The unitary antipode is given by (3.18). Now, formula (6.13) shows that:

$$\kappa(a) = a^{-1}, \qquad \kappa(b) = -a^{-1}b.$$

This result coincides with the formula (0.2) derived within the Hopf algebra framework.

7. THE DUAL OF 'az + b' QUANTUM GROUP.

The theory of multiplicative unitaries provide a simple method of constructing of group duals. Let W be the multiplicative unitary introduced by (6.2). The algebra of 'continuous functions vanishing at infinity' on the dual of the group is introduced by the formula:

$$\hat{A} = \left\{(\mathrm{id} \otimes \omega)W^* : \omega \in B(H)_*\right\}^{\text{norm closure}}$$

The dual group structure is given by the comultiplication

$$\widehat{\Delta}(c) = W^*(I \otimes c)W$$

for any $c \in \hat{A}$. Following Baaj and Skandalis we denote by $\Sigma : H \otimes H \longrightarrow H \otimes H$ the flip operator: $\Sigma(x \otimes y) = y \otimes x$ for any $x, y \in H$. The corresponding flip acting on operators will be denoted by σ:

$$\sigma(c \otimes c') = \Sigma(c \otimes c')\Sigma = c' \otimes c$$

for any $c, c' \in B(H)$. The following theorem reduces the description of the dual of 'az + b' group to the original group.

Theorem 7.1. *Operators $\hat{a}$, $\hat{b}$ are affiliated with $\hat{A}$. There exists a C^*-isomorphism $\psi : A \longrightarrow \hat{A}$ such that $\psi(a) = \hat{a}$ and $\psi(b) = \hat{b}$. This isomorphism reverses order of the group operation:*

$$\widehat{\Delta}(\psi(c)) = \sigma(\psi \otimes \psi)\Delta(c) \tag{7.1}$$

for any $c \in A$.

Proof One can easily verify that $(b, a) \in \mathcal{D}_H$ and $(\hat{b}, \hat{a}) \in \mathcal{D}_H$. Each of the two pairs is of infinite multiplicity. Therefore they are unitarily equivalent. Let $Z \in B(H)$ be a unitary operator such that $\hat{a} = Z^*aZ$, $\hat{b} = Z^*bZ$ and ψ be the automorphism of $B(H)$ implemented by Z:

$$\psi(c) = Z^*cZ$$

Then $\psi(a) = \hat{a}$, $\psi(b) = \hat{b}$ and taking into account definition (6.2) we see that operator $(\mathrm{id} \otimes \psi)W$ is invariant with respect to the flip:

$$\sigma(\mathrm{id} \otimes \psi)W = (\mathrm{id} \otimes \psi)W.$$

Therefore, for any $\omega \in B(H)_*$ we have:

$$\psi\left((\omega \otimes \mathrm{id})W\right) = (\omega \otimes \mathrm{id})(\mathrm{id} \otimes \psi)W$$
$$= (\mathrm{id} \otimes \omega)(\mathrm{id} \otimes \psi)W = (\mathrm{id} \otimes \omega_Z)W, \tag{7.2}$$

where $\omega_Z = \omega \circ \psi \in B(H)_*$. Formula (7.2) shows that $\psi(A) = \hat{A}$. We shall verify (7.1). Let $c = (\omega \otimes \mathrm{id})W \in A$. Then by the above formula $\psi(c) = (\mathrm{id} \otimes \omega_Z)W$ and

$$\widehat{\Delta}(\psi(c)) = (\mathrm{id} \otimes \mathrm{id} \otimes \omega_Z)(\widehat{\Delta} \otimes \mathrm{id})W = (\mathrm{id} \otimes \mathrm{id} \otimes \omega_Z)W_{23}W_{13}.$$

Therefore

$$\sigma\widehat{\Delta}(\psi(c)) = (\mathrm{id} \otimes \mathrm{id} \otimes \omega_Z)W_{13}W_{23}. \tag{7.3}$$

On the other hand

$$(\psi \otimes \psi)\Delta(c) = (\psi \otimes \psi)\Delta\left((\omega \otimes \mathrm{id})W\right) = (\omega \otimes \psi \otimes \psi)W_{12}W_{13}$$
$$= (\omega \otimes \mathrm{id} \otimes \mathrm{id})\left[(\mathrm{id} \otimes \psi)W\right]_{12}\left[(\mathrm{id} \otimes \psi)W\right]_{13}.$$

Remembering that $(\mathrm{id} \otimes \psi)W$ is flip-invariant and that ψ is multiplicative we obtain:

$$(\psi \otimes \psi)\Delta(c) = (\mathrm{id} \otimes \mathrm{id} \otimes \omega)\left[(\mathrm{id} \otimes \psi)W\right]_{13}\left[(\mathrm{id} \otimes \psi)W\right]_{23}$$

$$= (\mathrm{id} \otimes \mathrm{id} \otimes \omega)(\mathrm{id} \otimes \mathrm{id} \otimes \psi)W_{13}W_{23} = (\mathrm{id} \otimes \mathrm{id} \otimes \omega_Z)W_{13}W_{23}.$$

Comparing this formula with (7.3) we obtain (7.1)

$$\square$$

Appendices

A. '$az + b$'-GROUP FOR REAL q.

In this Appendix we briefly describe the quantum deformation of '$az + b$'-group corresponding to the real value of the deformation parameter q. We shall assume that $0 < q < 1$. Since most of the formulae have the same form as in the case of root of unity, we indicate only the changes. The most important one consists in replacing (0.6) by

$$\Gamma = \left\{z \in \mathbb{C} - \{0\} : |z| \in q^{\mathbb{Z}}\right\}. \tag{A.1}$$

In [15, 16], the group (A.1) is denoted by $\mathbb{C}^q$. Any element $\gamma \in \Gamma$ is of the form $\gamma = q^{i\varphi + k}$ where $\varphi \in \mathbb{R}$ and $k \in \mathbb{Z}$. The bicharacter χ and the Frenel function α are introduced by:

$$\chi\left(q^{i\varphi+k}, q^{i\psi+\ell}\right) = q^{i(\varphi\ell + k\psi)},$$

$$\alpha\left(q^{i\varphi+k}\right) = q^{i\varphi k}.$$

The reader easily verifies that the formula (1.2) remains valid. In the definition (0.7) of G-pair we replace $e^{\frac{2\pi}{N}t}$ by q^{-it}. We also keep the definition (2.1) of $\mathcal{D}_H$ formally unchanged. However its meaning is now different: relations (2.4) take the form:

$$(\text{Phase } S)|R| = q|R|(\text{Phase } S), \qquad |S|(\text{Phase } R) = q(\text{Phase } R)|S|,$$

$$(\text{Phase } S)(\text{Phase } R) = (\text{Phase } R)(\text{Phase } S), \quad |S| \text{ and } |R| \text{ strongly commute.} \tag{A.2}$$

The properties of pairs of operators $(R, S) \in \mathcal{D}_H$ are investigated in details in [15]. In particular it is shown that the closure of the sum $R + S$ is again a normal operator with the spectrum in $\overline{\Gamma}$. Moreover:

$$R \dotplus S = F_q(R^{-1}S)RF_q(R^{-1}S)^*$$

$$= F_q(RS^{-1})^*SF_q(RS^{-1}), \tag{A.3}$$

$$F_q(R \dotplus S) = F_q(R)F_q(S).$$

In these formulae, F_q is the special function on Γ introduced by the formula:

$$F_q(\gamma) = \prod_{k=0}^{\infty} \frac{1 + q^{2k}\overline{\gamma}}{1 + q^{2k}\gamma}.$$

Clearly $|f(\gamma)| = 1$ for any $\gamma \in \Gamma$, so the operators of the form $F_q(R)$, $F_q(S)$, $F_q(R \dotplus S)$ are unitary.

Using the above formulae one can show the following

Theorem A.1. *Let H be a Hilbert space, $(a, b) \in G_H$ and s be unitary and r be strictly positive selfadjoint operators acting on H. Assume that $\ker b = \{0\}$, r and s strongly commute with a and b, $\mathrm{Sp}\, r \subset q^{\mathbb{Z}} \cup \{0\}$ and $r^{-it}sr^{it} = q^{-it}s$ for all $t \in \mathbb{R}$. Then the operator*

$$W = F_q\left(b^{-1}a \otimes b\right)\chi\left(sb^{-1} \otimes I, I \otimes a\right) \tag{A.4}$$

is a manageable multiplicative unitary. Moreover

$$W(a \otimes I)W^* = \quad a \otimes a, \tag{A.5}$$

$$W(b \otimes I)W^* = a \otimes b \dotplus b \otimes I. \tag{A.6}$$

Proof. It is very easy to show that $(b, a) \in \mathcal{D}_H$, $(b \otimes I, a \otimes b) \in \mathcal{D}_{H \otimes H}$ and $(b^{-1}a \otimes b \otimes I, b^{-1}a \otimes a \otimes b)$ $\in \mathcal{D}_{H \otimes H \otimes H}$. Then using the formulae (A.3) one can show that (A.5) and (A.6) hold and that the operator (A.4) satisfies the pentagonal equation. This part of the proof essentially repeats the corresponding part of the proof of Theorem 3.1 and Proposition 3.2.

To prove the manageability we shall use the Fourier transform of F_q with respect to the angle variable:
$$F_q(\gamma) = \sum_{k \in \mathbb{Z}} F_k(|\gamma|)(\text{Phase } \gamma)^k.$$

The numbers $F_k(q^n)$ satisfy a number of equalities discovered in [2]. In particular $F_k(q^n)$ are real and
$$F_k(q^n) = (-q)^k F_{-k}(q^{n-k}). \tag{A.7}$$

Let $Q = r|a|$ and
$$\widetilde{W} = F_q \left(\left(b^{-1} a \right)^\top \otimes q a^{-1} b \right)^* \chi \left(\left(s b^{-1} \right)^\top \otimes I, I \otimes a \right).$$

Then Q is a strictly positive selfadjoint operator acting on H and $\widetilde{W}$ is a unitary operator acting on $\overline{H} \otimes H$. One can easily verify that $Q \otimes Q$ commutes with W. To prove manageability of (A.4) it is sufficient to show that
$$(x \otimes u | W | z \otimes y) = \left(\overline{z} \otimes Qu \big| \widetilde{W} \big| \overline{x} \otimes Q^{-1} y \right) \tag{A.8}$$

for any $x, z \in H$, $y \in D(Q^{-1})$, $u \in D(Q)$. In what follows the operators $b^{-1}a$ and sb^{-1} will be denoted by $\hat{b}$ and $\hat{a}$ respectively. One can easily show that $(\hat{b}, \hat{a}) \in \mathcal{D}_H$.

The operators $|a|, |b|$ and Q are strictly positive, selfadjoint and strongly commuting. Their spectrum is discrete and consists of integer powers of q (plus the accumulation point 0 with no eigenspace assigned to it). The same statement is true for operators $\hat{a}$ and $\hat{b}$. Therefore proving (A.8) we may assume that x, z are eigenvectors of $|\hat{a}|$ and $|\hat{b}|$ and u, y are eigenvectors of $|a|, |b|$ and Q:
$$\begin{aligned} |\hat{a}|x &= q^{n_x} x, & |\hat{b}|x &= q^{m_x} x, \\ |\hat{a}|z &= q^{n_z} z, & |\hat{b}|z &= q^{m_z} z, \\ |a|u &= q^{n_u} u, & |b|u &= q^{m_u} u, & Qu &= q^{\ell_u} u, \\ |a|y &= q^{n_y} y, & |b|y &= q^{m_y} y, & Qy &= q^{\ell_y} y, \end{aligned} \tag{A.9}$$
where $n_x, n_z, n_u, n_y, m_x, m_z, m_u, m_y, \ell_u, \ell_y \in \mathbb{Z}$. Then
$$F_q(\hat{b} \otimes b)^*(x \otimes u) = F_q(q^{m_x+m_u} \text{Phase } \hat{b} \otimes \text{Phase } b)^*(x \otimes u)$$

$$= \sum_{k \in \mathbb{Z}} \overline{F_k(q^{m_x+m_u})}(\text{Phase } \hat{b})^{-k} x \otimes (\text{Phase } b)^{-k} u,$$

$$\chi(\hat{a} \otimes I, I \otimes a)(z \otimes y) = \chi(q^{n_z} \text{Phase } \hat{a} \otimes I, I \otimes q^{n_y} \text{Phase } a)(z \otimes y)$$

$$= (\text{Phase } \hat{a})^{n_y} z \otimes (\text{Phase } a)^{n_z} y$$
and the left hand side of (A.8) equals:
$$\text{LHS} = \sum_{k \in \mathbb{Z}} F_k(q^{m_x+m_u}) \left\{ \begin{array}{l} \left(x \big| (\text{Phase } \hat{b})^k (\text{Phase } \hat{a})^{n_y} \big| z \right) \\ \times \left(u \big| (\text{Phase } b)^k (\text{Phase } a)^{n_z} \big| y \right). \end{array} \right.$$

Remembering that $(b, a) \in \mathcal{D}_H$ and using relations (A.2) one can easily show that $(\text{Phase } b)^k$ $(\text{Phase } a)^{n_z} y$ is an eigenvector of $|a|, |b|, Q$ corresponding to the eigenvalue $q^{k+n_y}, q^{-n_x+m_y}, q^{k+\ell_y}$ respectively. Similarly starting from the relation $(\hat{b}, \hat{a}) \in \mathcal{D}_H$ one can show that $(\text{Phase } \hat{b})^k$ $(\text{Phase } \hat{a})^{n_y} z$ is an eigenvector of $|\hat{a}|, |\hat{b}|$ corresponding to the eigenvalue $q^{k+n_z}, q^{-n_y+m_z}$ respectively. The eigenvectors corresponding to different eigenvalues are orthogonal. Therefore
$$\text{LHS} = F_{\ell_u - \ell_y}(q^{m_x+m_u}) \left\{ \begin{array}{l} \left(x \big| (\text{Phase } \hat{b})^{\ell_u - \ell_y} (\text{Phase } \hat{a})^{n_y} \big| z \right) \\ \times \left(u \big| (\text{Phase } b)^{\ell_u - \ell_y} (\text{Phase } a)^{n_z} \big| y \right) \end{array} \right.$$

and LHS$\neq 0$ implies that:
$$m_y - n_z = m_u, \ m_z - n_y = m_x \text{ and } n_x - n_z = n_u - n_y = \ell_u - \ell_y. \tag{A.10}$$

To compute the right hand side of (A.8) we notice that $qa^{-1}b = (\text{Phase}\,a)^*(\text{Phase}\,b)|a|^{-1}|b|$. Therefore:

$$F_q(-\hat{b}^\top \otimes qa^{-1}b)(\overline{z} \otimes Qu) = F_q(-q^{m_z-n_u+m_u}\text{Phase}\,\hat{b}^\top \otimes (\text{Phase}\,a)^*\text{Phase}\,b)(\overline{z} \otimes q^{\ell_u}u)$$

$$= \sum_{k\in\mathbb{Z}}q^{\ell_u}F_k(q^{m_z-n_u+m_u})(-\text{Phase}\,\hat{b}^\top)^k\overline{z} \otimes (\text{Phase}\,a)^{-k}(\text{Phase}\,b)^k u$$

$$= \sum_{k\in\mathbb{Z}}q^{\ell_u}F_k(q^{m_z-n_u+m_u})\overline{(-\text{Phase}\,\hat{b})^{-k}}z \otimes (\text{Phase}\,a)^{-k}(\text{Phase}\,b)^k u$$

$$\chi(\hat{a}^\top \otimes I, I \otimes a)(\overline{x} \otimes Q^{-1}y) = \chi(q^{n_x}\text{Phase}\,\hat{a}^\top \otimes I, I \otimes q^{n_y}\text{Phase}\,a)(\overline{x} \otimes q^{-\ell_y}y)$$

$$= q^{-\ell_y}(\text{Phase}\,\hat{a}^\top)^{n_y}\overline{x} \otimes (\text{Phase}\,a)^{n_x}y$$

$$= q^{-\ell_y}\overline{(\text{Phase}\,\hat{a})^{-n_y}x} \otimes (\text{Phase}\,a)^{n_x}y,$$

and the right hand side of (A.8) equals:

$$\text{RHS} = \sum_{k\in\mathbb{Z}}q^{\ell_u-\ell_y}(-1)^k F_k(q^{m_z-n_u+m_u})\begin{cases} \left(x\middle|(\text{Phase}\,\hat{b})^{-k}(\text{Phase}\,\hat{a})^{n_y}\middle|z\right) \\[2mm] \times\left(u\middle|(\text{Phase}\,b)^{-k}(\text{Phase}\,a)^{n_z+k}\middle|y\right)\end{cases}$$

The same analysis as for LHS shows that

$$\text{RHS} = (-q)^{\ell_u-\ell_y}F_{\ell_y-\ell_u}(q^{m_z-n_u+m_u})\begin{cases} \left(x\middle|(\text{Phase}\,\hat{b})^{\ell_u-\ell_y}(\text{Phase}\,\hat{a})^{n_y}\middle|z\right) \\[2mm] \times\left(u\middle|(\text{Phase}\,b)^{\ell_u-\ell_y}(\text{Phase}\,a)^{n_z+\ell_y-\ell_u}\middle|y\right)\end{cases}$$

and RHS$\neq 0$ implies (A.10). Now (A.8) follows immediately from (A.7).

$\square$

The passage from multiplicative unitary (A.4) to the quantum group goes in the same way as for the case of root of unity. All the text of Section 4 remains in power for real q. In Section 5 one has to replace $F_N(\gamma)$ by $F_q(q^2\gamma)$ (with this substitution formulae (2.11), (2.12) and (3.2) convert into the first two formulae of (A.3) and (A.4) respectively. We would like also to refer to the paper [15], which contains some of the results of Section 5 for real q (compare Theorem 5.1 with Proposition 5.2 of [15]). Also Section 6 works (with suitable changes) for real q.

B. Fourier transform of F_N.

This Section is devoted to the proof of the Fourier transform formula (1.20). Let $\Phi(\gamma')$ be the right hand side of (1.20):

$$\Phi(\gamma') = \sum_{k=0}^{N-1}\int_\ell \frac{(-q\text{Phase}\,\gamma')^k|\gamma'|^{\frac{N}{2\pi i}\log r}}{(1-q^{-k}r)\,F_N(-q^k r)}\,dr, \tag{B.1}$$

where ℓ is the path in the complex plane connecting 0 with ∞ going along $\mathbb{R}_+$ but rounding the point 1 from below (for $k=0$ the integrand in the above formula has a pole at point $r=1$). The integrals in (B.1) are not absolutely convergent. Indeed all factors in the integrands except $(1-q^{-k}r)^{-1}$ are of constant modulus; whereas the integral $\int_0^\infty \frac{dr}{1-q^{-k}r}$ is logarithmically divergent at ∞. In fact the integrals in (B.1) are very like Frenel integral. We shall prove that they exist as improper integrals: $\int_\ell = \lim_{R\to\infty}\int_{\ell^R}$, where ℓ^R is the part of ℓ ending at point R. To this end we use the holomorphic continuation (1.15):

$$\Phi(\gamma') = \sum_{k=0}^{N-1}\int_\ell \frac{(-q\text{Phase}\,\gamma')^k|\gamma'|^{\frac{N}{2\pi i}\log r}}{(1-q^{-k}r)\,\widetilde{F}(r,-q^k)}\,dr \tag{B.2}$$

Let ℓ_1, ℓ_2 and ℓ_3 be the paths described as follows. Path ℓ_1 starts at 0 and goes to ∞ along $e^{\frac{\pi i}{N}}\mathbb{R}_+$. Path ℓ_2 starts at 0, goes along ℓ up to the point 2, next along the circle of radius 2 up to the point $2e^{\frac{3\pi i}{N}}$ and then to ∞ along $e^{\frac{3\pi i}{N}}\mathbb{R}_+$. Path ℓ_3 coincides with ℓ_2 except it rounds the point $r=1$ from above. We advice the reader to draw the corresponding picture.

Using the asymptotic behaviour (1.17) one can easily show that the integral of the integrand in (B.2) over the arc of circle of radius R connecting ℓ with ℓ_1 and ℓ_2 goes to 0 when $R \to \infty$. Therefore we can deform the integration path replacing ℓ by ℓ_1 and ℓ_2. For ℓ_1, this procedure works for all k except $k = 0$. For $k = 0$ we have to take into account the pole of the integrand at point $r = 1$. Its contribution equals to $2\pi i$ times the residuum of the integrand at the pole. The latter equals to $-\widetilde{F}(1, -1)^{-1} = -F_N(-1)^{-1}$. Therefore we have:

$$\Phi(\gamma') = -2\pi i\, F_N(-1)^{-1} + \sum_{k=0}^{N-1} \int_{\ell_1} \frac{(-q\mathrm{Phase}\,\gamma')^k |\gamma'|^{\frac{N}{2\pi i} \log r}}{(1 - q^{-k}r)\, \widetilde{F}(r, -q^k)}\, dr, \tag{B.3}$$

$$\Phi(\gamma') = \sum_{k=0}^{N-1} \int_{\ell_2} \frac{(-q\mathrm{Phase}\,\gamma')^k |\gamma'|^{\frac{N}{2\pi i} \log r}}{(1 - q^{-k}r)\, \widetilde{F}(r, -q^k)}\, dr. \tag{B.4}$$

If $r \in \ell_1$, then $\arg r = \frac{\pi}{N}$ and using (1.17) we see that the modulus of the integrands in (B.10) equal

$$\left| \frac{(-q\mathrm{Phase}\,\gamma')^k |\gamma'|^{\frac{N}{2\pi i} \log r}}{(1 - q^{-k}r)\, \widetilde{F}(r, -q^k)} \right| = \frac{|\gamma'|^{\frac{1}{2}}}{|1 - q^{-k}r|\,|\widetilde{F}(r, -q^k)|} = \frac{|\gamma'|^{\frac{1}{2}}}{|1 - q^{-k}r|\,|r|^{\frac{1}{2}}|\Theta_k(r)|}.$$

Remembering that $|\Theta_k(r)| \to 1$, when $r \to \infty$ we see that the integrals in (B.3) are absolutely converging and that

$$\lim_{\gamma' \to 0} \Phi(\gamma') = -2\pi i\, F_N(-1)^{-1}. \tag{B.5}$$

Performing the holomorphic continuation in (1.16) we obtain:

$$\widetilde{F}(qr, q^k) = \left(1 + q^{-k-1}r\right) \widetilde{F}(r, q^{k+1}). \tag{B.6}$$

We shall prove

Proposition B.1. *Let Ω and $\overline{\Omega}$ be the subsets of $\mathbb{C}$ introduced in Theorem 1.2. Then there exists a continuous function $\widetilde{\Phi}$ defined on $\overline{\Omega} \times q^{\mathbb{Z}}$ such that:*

1. *For any fixed $k \in \mathbb{Z}$ the function $z \longrightarrow \widetilde{\Phi}(z, q^k)$ is holomorphic on Ω.*

2. *For any nonnegative real r and any $k \in \mathbb{Z}$ we have:*

$$\widetilde{\Phi}(r, q^k) = \Phi(q^k r), \tag{B.7}$$

$$\widetilde{\Phi}(qr, q^k) = \left(1 + q^{-k-1}r\right) \widetilde{\Phi}(r, q^{k+1}). \tag{B.8}$$

3. *For any $k \in \mathbb{Z}$ and $z \in \overline{\Omega}$:*

$$|\widetilde{\Phi}(z, q^k)| \leq C_1 + C_2 |z|^{\frac{3}{2}}, \tag{B.9}$$

where C_1 and C_2 are positive constants.

Proof. Let

$$\widetilde{\Phi}(z, q^{k'}) = \sum_{k=0}^{N-1} \int_{\ell_2} \frac{\left(-q^{1+k'}\right)^k z^{\frac{N}{2\pi i} \log r}}{(1 - q^{-k}r)\, \widetilde{F}(r, -q^k)}\, dr. \tag{B.10}$$

To analyze this expression we replace the path ℓ_2 by ℓ_3 taking into account the contribution of the pole at the point $r = 1$:

$$\widetilde{\Phi}(z, q^{k'}) = 2\pi \widetilde{F}(1, -1)^{-1} + \sum_{k=0}^{N-1} \int_{\ell_3} \frac{\left(-q^{1+k'}\right)^k z^{\frac{N}{2\pi i} \log r}}{(1 - q^{-k}r)\, \widetilde{F}(r, -q^k)}\, dr. \tag{B.11}$$

The modulus of the integrands in (B.11) equals

$$I_k(r) = \left| \frac{\left(-q^{1+k'}\right)^k z^{\frac{N}{2\pi i} \log r}}{(1 - q^{-k}r)\, \widetilde{F}(r, -q^k)} \right| = \frac{|z|^{\frac{N \arg r}{2\pi}} |r|^{\frac{N \arg z}{2\pi}}}{|1 - q^{-k}r|\,|\widetilde{F}(r, -q^k)|}$$

If $r \in \ell_3$ and $z \in \overline{\Omega}$, then $0 \leq \arg r \leq \frac{3\pi}{N}$, and $0 \leq \arg z \leq \frac{2\pi}{N}$ and

$$I_k(r) \leq \frac{\max\left(1, |z|^{\frac{3}{2}}\right) \max\left(1, |r|\right)}{|1 - q^{-k}r|\,|\widetilde{F}(r, -q^k)|}$$

If moreover $|r| > 2$, then $\arg r = \frac{3\pi}{N}$ and using (1.17) we see that

$$I_k(r) = \frac{|z|^{\frac{3}{2}} |r|^{\frac{N \arg z}{2\pi}}}{|1 - q^{-k}r|\, |\widetilde{F}(r, -q^k)|} = \frac{|z|^{\frac{3}{2}} |r|^{\frac{N \arg z}{2\pi} - \frac{3}{2}}}{|1 - q^{-k}r|\, |\Theta_k(r)|}$$

Using these estimates one can easily show that the integrals in (B.11) are absolutely convergent for $z \in \overline{\Omega}$, that the functions $\widetilde{\Phi}(\cdot, q^{k'})$ are holomorphic on $\overline{\Omega}$ and that the estimate (B.9) holds.

Comparing (B.10) with (B.4) we see that (B.7) holds.

Using (B.6) we rewrite (B.10) in the following form:

$$\widetilde{\Phi}(z, q^{k'}) = \sum_{k=0}^{N-1} \int_{\ell_2} \frac{\left(-q^{1+k'}\right)^k z^{\frac{N}{2\pi i} \log r}}{\widetilde{F}(qr, -q^{k-1})}\, dr. \tag{B.12}$$

Now we replace qr by r (so the new r runs over $q\ell_2$) and next $k - 1$ by k:

$$\widetilde{\Phi}(z, q^{k'}) = \sum_{k=0}^{N-1} \int_{q\ell_2} \frac{\left(-q^{1+k'}\right)^k z^{-1+\frac{N}{2\pi i} \log r}}{\widetilde{F}(r, -q^{k-1})}\, \frac{dr}{q}$$

$$= -q^{k'} z^{-1} \sum_{k=0}^{N-1} \int_{q\ell_2} \frac{\left(-q^{1+k'}\right)^k z^{\frac{N}{2\pi i} \log r}}{\widetilde{F}(r, -q^k)}\, dr.$$

One can easily check that the integrand has no singularities between $q\ell_2$ and ℓ_2. Therefore we may replace $q\ell_2$ by ℓ_2:

$$q^{-k'} z \widetilde{\Phi}(z, q^{k'}) = -\sum_{k=0}^{N-1} \int_{\ell_2} \frac{\left(-q^{1+k'}\right)^k z^{\frac{N}{2\pi i} \log r}}{\widetilde{F}(r, -q^k)}\, dr.$$

Combining this formula with (B.10) we obtain:

$$\left(1 + q^{-k'} z\right) \widetilde{\Phi}(z, q^{k'}) = \sum_{k=0}^{N-1} \int_{\ell_2} \frac{\left(-q^{1+k'}\right)^k q^{-k} r\, z^{\frac{N}{2\pi i} \log r}}{(1 - q^{-k}r)\, \widetilde{F}(r, -q^k)}\, dr.$$

On the other hand, replacing in (B.10), z and k' by qz and $k' - 1$ we get:

$$\widetilde{\Phi}(qz, q^{k'-1}) = \sum_{k=0}^{N-1} \int_{\ell_2} \frac{\left(-q^{k'}\right)^k r\, z^{\frac{N}{2\pi i} \log r}}{(1 - q^{-k}r)\, \widetilde{F}(r, -q^k)}\, dr.$$

Therefore $\widetilde{\Phi}(qz, q^{k'-1}) = \left(1 + q^{-k'} z\right) \widetilde{\Phi}(z, q^{k'})$ and (B.8) follows.

$$\square$$

We continue the proof of (1.20). Let $r \in \mathbb{R}_+$. Comparing (B.8) with (B.6) we obtain:

$$\frac{\widetilde{\Phi}(qr, q^k)}{\widetilde{F}(qr, q^k)} = \frac{\widetilde{\Phi}(r, q^{k+1})}{\widetilde{F}(r, q^{k+1})}. \tag{B.13}$$

For any $z \in \mathbb{C}$ of the form $z = q^k r$, where $r \in \overline{\Omega}$ and $k \in \mathbb{Z}$ we set:

$$\varphi(z) = \frac{\widetilde{\Phi}(r, q^k)}{\widetilde{F}(r, q^k)}. \tag{B.14}$$

The reader should notice that due to (B.13), the right hand side of (B.14) does not depend on the choice of the decomposition $z = q^k r$, which is not unique for $\arg z \in q^{\mathbb{Z}}$. Clearly φ is a holomorphic function on $\mathbb{C}$ (the singularity at the point $z = 0$ is removable). Combining (B.9) with (1.17) we see that $|\varphi(z)|$ grows no faster than $|z|^{\frac{3}{2}}$ when $z \to \infty$. Using this fact one can easily show that the Taylor-Maclaurin expansion of φ reduces to two terms. In other words, φ is a first-order polynomial: $\varphi(z) = \varphi(0) + \varphi'(0)z$. However for Phase $z = \frac{2\pi}{N}(1 - \epsilon)$ (where ϵ is a small positive number), $\varphi(z) = \frac{\widetilde{\Phi}(z,1)}{\widetilde{F}(z,1)}$ grows no faster than $|z|^{\frac{1}{2}+\epsilon}$ (cf (B.9) and (1.17)). This behaviour shows

that $\varphi'(0) = 0$ and $\varphi(z) = \varphi(0)$. This way we showed that $\widetilde{\Phi}(r, q^k) = \varphi(0)\widetilde{F}(r, q^k)$. For $r \in \mathbb{R}_+$ this formula means that

$$\Phi(q^k r) = \varphi(0)F_N(q^k r).$$

Setting $r \to 0$ and using (B.5) and remembering that $F_N(0) = 1$ we obtain

$$\varphi(0) = -2\pi i F_N(-1)^{-1}.$$

The formula (1.20) is proved.

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