Quantum transport

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Dictionary

$$\vec{D} = \varepsilon \vec{E}$$

ε_0 **vacuum permittivity, permittivity of free space** (przenikalność elektryczna próżni)
ε_r relative permittivity (względna przenikalność elektryczna)
$\varepsilon = \varepsilon_0 \varepsilon_r$ permittivity (przenikalność elektryczna)

$$\vec{B} = \mu \vec{H}$$

μ_0 **vacuum permeability, permeability of free space** (przenikalność magnetyczna)

$$\mu_0 = 4\pi \cdot 10^{-7} \text{H/m}$$

μ_r relative permeability (względna przenikalność magnetyczna)
$\mu = \mu_0 \mu_r$ permeability (przenikalność magnetyczna)

magnetic susceptibility $\chi_m = \mu_r - 1$

electric field $\vec{E}$ and the magnetic field $\vec{B}$
displacement field $\vec{D}$ and the magnetizing field $\vec{H}$

Scalar and vector fields

Maxwell's equations in matter

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{j}_{sw} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \vec{D} = \rho_{sw}$$

$$\nabla \vec{B} = 0$$

Material equations (linear)

$$\vec{B} = \mu_0 \vec{H} + \vec{M} = \mu_0(1 + \chi_m)\vec{H} = \mu \vec{H} = \mu_r \mu_0 \vec{H}$$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0(1 + \chi_e)\vec{E} = \varepsilon \vec{E} = \varepsilon_0 \varepsilon_r \vec{E}$$

$$\vec{j}_{sw} = \hat{\sigma} \vec{E}$$

$$v^2 = \frac{1}{\mu_0 \varepsilon_0} \frac{1}{\mu_r \varepsilon_r} = \frac{c^2}{\mu_r \varepsilon_r} = \frac{c^2}{n^2}$$

The equations written in the form of a scalar φ and vector A potentials:

$$\boxed{\vec{B} = \nabla \times \overrightarrow{A}}$$

Then $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \overrightarrow{A}) \Rightarrow \nabla \times \vec{E} + \frac{\partial}{\partial t}(\nabla \times \overrightarrow{A}) = 0 \Rightarrow \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$

If the rotation of the gradient is zero, then: $-\nabla \varphi = \vec{E} + \frac{\partial \vec{A}}{\partial t}$ thus $\boxed{\vec{E} = -\nabla \varphi - \frac{\partial \vec{A}}{\partial t}}$

Scalar and vector fields

Maxwell's equations in matter

$$\boxed{\vec{B} = \nabla \times \overrightarrow{A}} \qquad \boxed{\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}}$$

Example: $\varphi = -\vec{E}\vec{r} + C_\varphi$ $\qquad \vec{A} = -\vec{E}t + C_A$

Not only constants C_φ and C_A we can add for the scalar and vector potentials:

$$\varphi \to \varphi - \frac{d\chi}{dt} \qquad \vec{A} \to \vec{A} + \nabla\chi$$ eg.: $\chi = \pm\vec{E}\vec{r}t$

We call it **the gauge**

Landau gauge: field $\vec{B} = (0,0,B_z) \Rightarrow B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$ $\qquad A_y = B_z x$ lub $A_x = -B_z y$

(unfortunately distinguishes direction)

Coulomb gauge: $\nabla\vec{A} = 0$ field $\vec{B} = (0,0,B_z) \Rightarrow \vec{A} = \frac{1}{2}B_z(-y, x, 0) = \frac{1}{2}\vec{B} \times \vec{r}$

(unfortunately complicates calculations)

Lorentz gauge: $\nabla\vec{A} + \frac{\partial\varphi}{\partial t} = 0$

Scalar and vector fields

Schrodinger equation in the $\vec{E}$ and $\vec{B}$ fields:

$$\boxed{\vec{B} = \nabla \times \overrightarrow{A}}$$

$$\boxed{\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}}$$

$$\left\{\frac{1}{2m}\left[\hat{p} - q\,\vec{A}(\vec{r},t)\right]^2 + q\varphi(\vec{r},t) + U(\vec{r},t)\right\}\psi(\vec{r},t) = i\hbar\frac{d}{dt}\psi(\vec{r},t)$$

The sum: kinetic momentum

Canonical momentum $\hat{p}$

Continuity equation

$$J(\vec{r},t) = \frac{\hbar\, q}{2\, i\, m}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) - \frac{q}{m}|\Psi|^2\vec{A}(\vec{r},t)$$

or else:

$$J(\vec{r},t) = \frac{q}{2}\left[\Psi^*\left(\frac{\hat{p} - q\,\vec{A}(\vec{r},t)}{m}\Psi\right) + \left(\frac{\hat{p} - q\,\vec{A}(\vec{r},t)}{m}\Psi\right)^*\Psi\right]$$

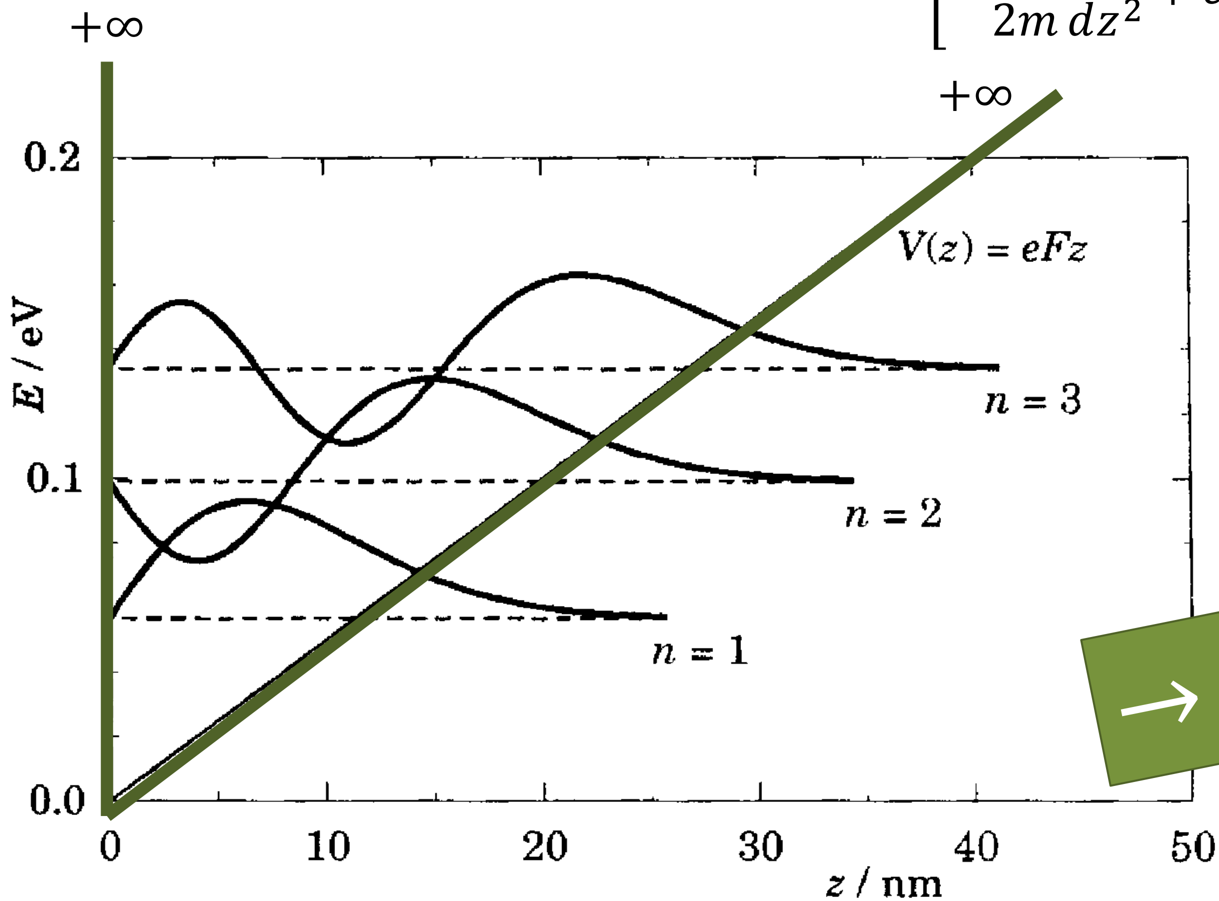
On exercises

Scalar and vector fields

Example: constant electric field, TRIANGULAR well:

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2} + eFz\right]\psi(z) = \varepsilon\psi(z)$$

$$\varepsilon_n = c_n\left[\frac{(eFz)^2}{2m}\right]^{1/3}$$



→ Lecture 3

FIGURE 4.6. Triangular potential well $V(z) = eFz$, showing the energy levels and wave functions. The scales are for electrons in GaAs and a field of 5 MV m^{-1}.

Triangular well

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2} + eFz\right]\psi(z) = \varepsilon\psi(z)$$

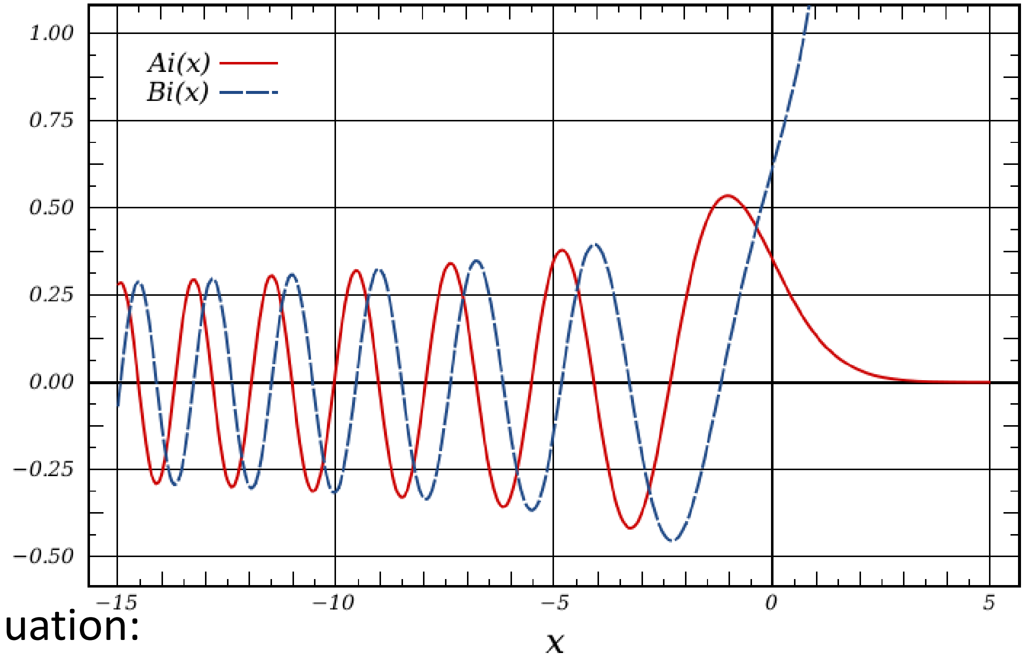
Transformation:

$$\frac{d^2}{dz^2}\psi(z) = \frac{2m}{\hbar^2}(eFz - \varepsilon)\psi(z)$$

Substituting: $z_0 = \left(\frac{\hbar^2}{2meF}\right)^{1/3}$

$$\varepsilon_0 = eFz_0 = \left[\frac{(eF\hbar)^2}{2m}\right]^{1/3}, \bar{z} = \frac{z}{z_0}, \ \bar{\varepsilon} = \frac{\varepsilon}{\varepsilon_0}$$

$$\frac{d^2}{dz^2}\psi(z) = \frac{2m}{\hbar^2}(eFz - \varepsilon)\psi(z)$$



The equation reduces to Stokes or Airy equation:

$$\frac{d^2}{dz^2}f(z) = zf(z)$$

Its two independent solutions the Airy functions $Ai(z)$ and $Bi(z)$. The solutions of the equation are the zeros of a function $Ai(z)$ (after some rearrangements).

Scalar and vector fields

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}+q\varphi(\vec{r},t)\right]\psi(z)=\varepsilon\psi(z)$$ selecting the gauging $q\varphi(\vec{r},t)=eFz$

The stationary solutions as an Airy function:

$$\psi(z,\varepsilon)=Ai\left(\frac{1}{z_0}\left(z-\frac{\varepsilon}{eF}\right)\right)=Ai\left(\frac{eFz-\varepsilon}{\varepsilon_0}\right)$$

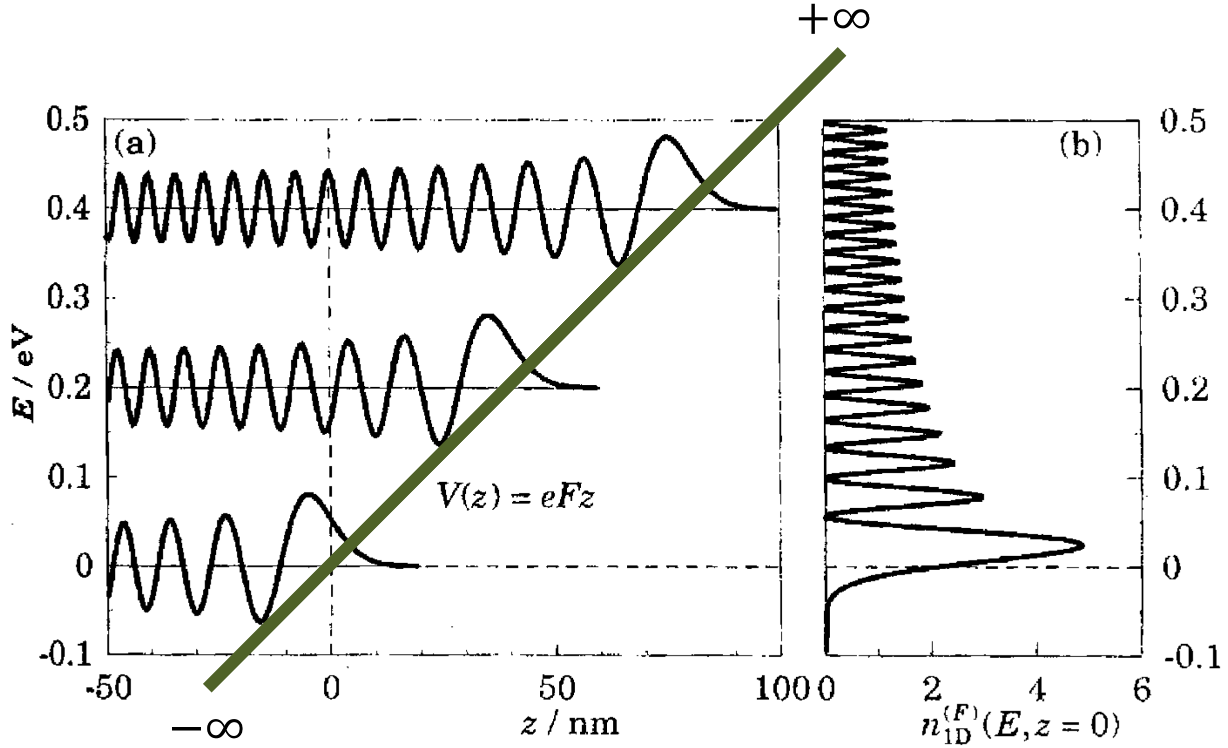


FIGURE 6.1. (a) Potential energy eFz, three wave functions, and energies for electrons in GaAs in a uniform electric field of $5\,\mathrm{MV\,m^{-1}}$. (b) Local density of states at $z = 0$, showing how the features correspond to the wave functions.

$$\varepsilon_0=eFz_0=\left[\frac{(eF\hbar)^2}{2m}\right]^{1/3}$$

$$z_0=\left(\frac{\hbar^2}{2meF}\right)^{1/3}$$

Scalar and vector fields

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}+q\varphi(\vec{r},t)\right]\psi(z)=\varepsilon\psi(z)$$ selecting the gauging $q\varphi(\vec{r},t)=eFz$

The stationary solutions as an Airy function:

And where is the motion of the electron in the field???

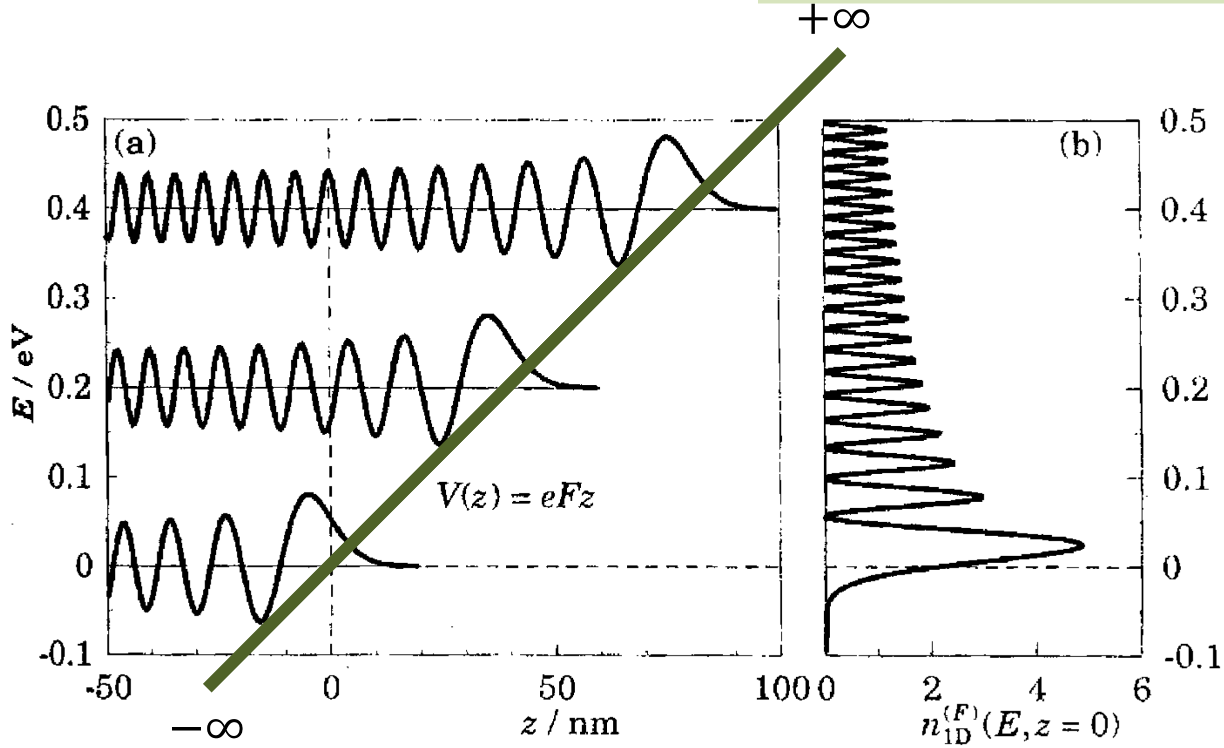


FIGURE 6.1. (a) Potential energy eFz, three wave functions, and energies for electrons in GaAs in a uniform electric field of 5 MV m^{-1}. (b) Local density of states at $z = 0$, showing how the features correspond to the wave functions.

The wave function:

- solution: standing waves-like!
- the function tunnels in the barier $z > \frac{\varepsilon}{eF}$,
- for the increasing potential it decays more quickly,
- oscillates for $z < \frac{\varepsilon}{eF}$ faster if $z - \frac{\varepsilon}{eF} \to (-\infty)$, it means it increases the kinetic energy
- Adding a constant to the potential CHANGES wave functions!

Local density of states

The density of states (in general) can be defined as:

$$N(E) = \sum_n \delta(E - \varepsilon_n)$$

After integration

$$\int_{E_1}^{E_2} N(E)\, dE = \int_{E_1}^{E_2} \sum_n \delta(E - \varepsilon_n)\, dE = \sum_n \int_{E_1}^{E_2} \delta(E - \varepsilon_n) dE$$

For instance:

$$N^{1D}(E) = \sum_k \delta\big(E - \varepsilon(k)\big) = \int \frac{1}{E'(k)}\, \delta(k - k')\, 2\, dk = \frac{1}{\pi}\sqrt{\frac{2m}{E}}$$

$$N^{2D}(E) = \sum_k \delta\big(E - \varepsilon(k)\big) = \int \frac{1}{E'(k)}\, \delta(k - k')\, 2\pi k\, dk = \frac{m}{\pi\hbar^2}$$

$$N^{3D}(E) = \sum_k \delta\big(E - \varepsilon(k)\big) = \int \frac{1}{E'(k)}\, \delta(k - k')\, 4\pi k^2\, dk = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{E}$$

On exercises

Local density of states

The density of states (in general) can be defined as:

$$N(E) = \sum_n \delta(E - \varepsilon_n)$$

Local denisty of states:

$$N(E) = \sum_n |\phi_k(\vec{r})|^2 \delta(E - \varepsilon_n)$$

For instance: $$N^{1D}(E,z) \sim \int_{-\infty}^{\infty} Ai^2\left(\frac{eFz-\varepsilon}{\varepsilon_0}\right)\delta(E-\varepsilon)\,d\varepsilon \sim \frac{2}{\hbar}\sqrt{\frac{2m}{\varepsilon_0}}\,Ai^2\left(\frac{eFz-\varepsilon}{\varepsilon_0}\right)$$

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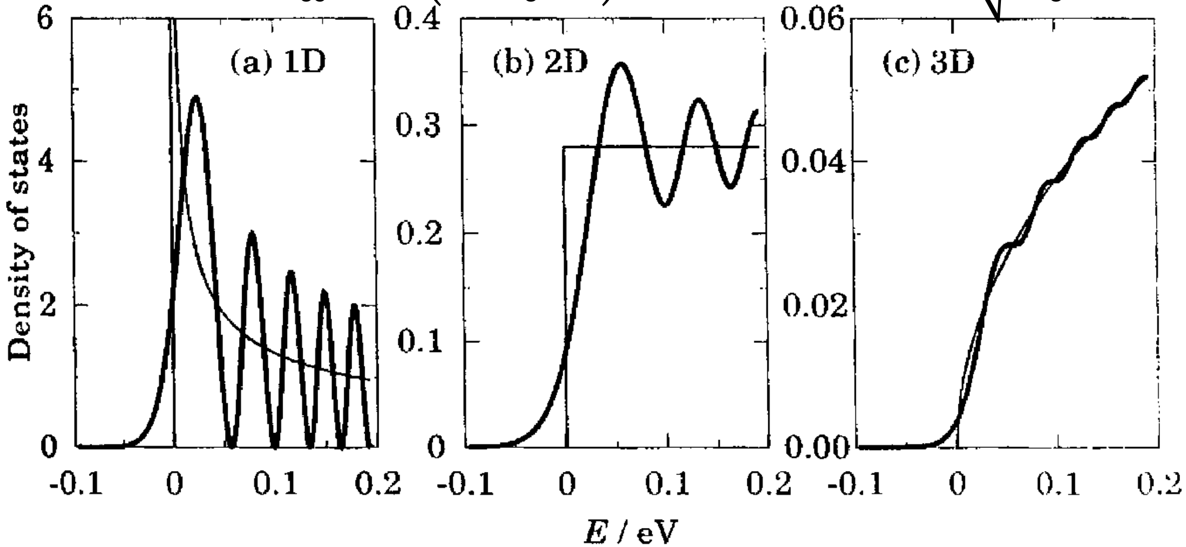


FIGURE 6.2. Local density of states $n^{(F)}(E,z)$ for electrons in GaAs in an electric field of 5 MV m^{-1} as a function of local kinetic energy, $\varepsilon = E - eFz$. The thin curves are the results for free electrons. The units of $n(E,z)$ are eV^{-1} nm^{-d} in d dimensions.

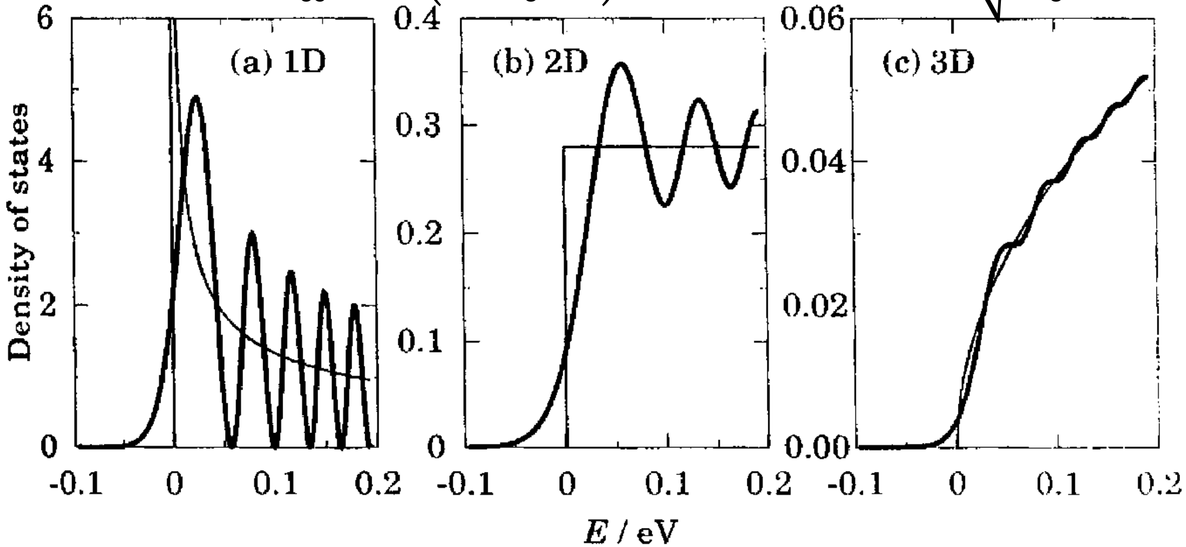
Local density of states

The density of states (in general) can be defined as:

$$N^{3D}(E,z) \sim \frac{m}{\pi\hbar^3}\sqrt{2m\varepsilon_0}\int_{-\infty}^{E} Ai^2\left(\frac{eFz-\varepsilon}{\varepsilon_0}\right)d\varepsilon = \frac{m}{\pi\hbar^3}\sqrt{2m\varepsilon_0}\left[[Ai'(s)]^2 - s[Ai(s)]^2\right]$$

$$s = \frac{eFz-E}{\varepsilon_0}$$

For instance: $$N^{1D}(E,z) \sim \int_{-\infty}^{\infty} Ai^2\left(\frac{eFz-\varepsilon}{\varepsilon_0}\right)\delta(E-\varepsilon)\,d\varepsilon \sim \frac{2}{\hbar}\sqrt{\frac{2m}{\varepsilon_0}}\,Ai^2\left(\frac{eFz-\varepsilon}{\varepsilon_0}\right)$$



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Franz-Keldysh effect - in the electric field optical transitions occur at lower energies - the energy gap is „blurred", the wavefunctions are „leaking" into the band gap:

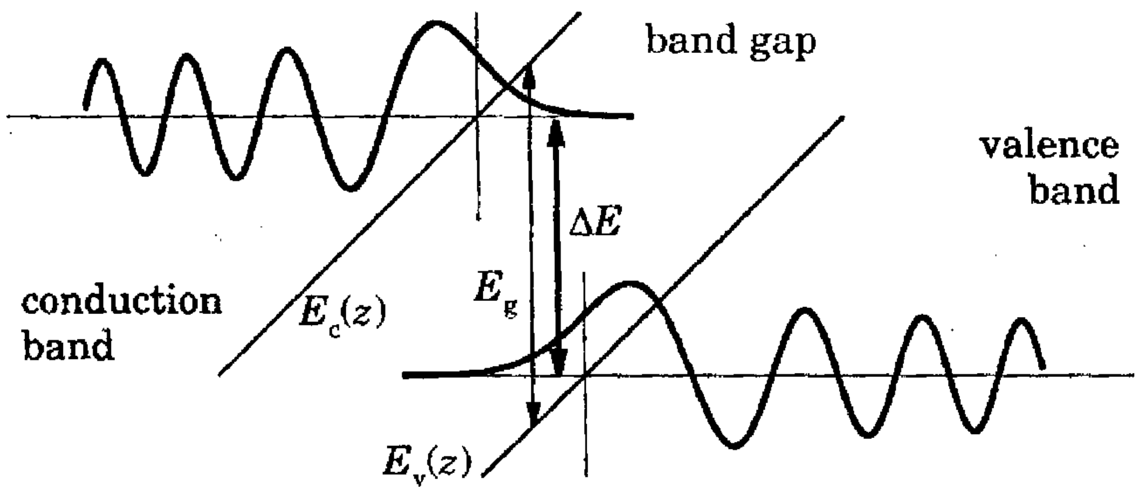

FIGURE 6.3. The Franz–Keldysh effect on interband absorption. The states shown in the valence and conduction bands are separated by $\Delta E < E_g$ but overlap because of the tail that tunnels into the band gap.

Scalar and vector fields

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}+q\varphi(\vec{r},t)\right]\psi(z)=\varepsilon\psi(z)$$ We choose a gauge $q\varphi(\vec{r},t)=eFz$

$$\left\{\frac{1}{2m}\left[\hat{p}-q\,\vec{A}(\vec{r},t)\right]^2+q\varphi(\vec{r},t)+U(\vec{r},t)\right\}\psi(\vec{r},t)=i\hbar\frac{d}{dt}\psi(\vec{r},t)$$

We choose a gauge $\varphi(\vec{r},t)=0$, but then $\vec{A}(\vec{r},t)=-\vec{E}t$

$$\left\{\frac{1}{2m}\left[\hat{p}-e\,\vec{E}t\right]^2\right\}\psi(\vec{r},t)=i\hbar\frac{d}{dt}\psi(\vec{r},t)$$

No stationary states

Sign $-$, because $q=-e$

The potential is not position-depended – solution of $\exp\left(i\,\vec{k}\vec{r}\right)$

$$\psi_{\vec{k}}(\vec{r},t)=\exp\left(i\,\vec{k}\vec{r}\right)\mathrm{T}\left(\vec{k},t\right)$$

$$\frac{1}{2m}\left[\hat{p}-e\,\vec{E}t\right]^2\psi_{\vec{k}}(\vec{r},t)=\frac{1}{2m}\left[\hbar\vec{k}-e\,\vec{E}t\right]^2\psi_{\vec{k}}(\vec{r},t)=\frac{\hbar^2}{2m}\left[\vec{k}-\frac{e}{\hbar}\vec{E}t\right]^2\psi_{\vec{k}}(\vec{r},t)=i\hbar\frac{d}{dt}\psi_{\vec{k}}(\vec{r},t)$$

$$\frac{\hbar^2}{2m}\left[\vec{k}-\frac{e}{\hbar}\vec{E}t\right]^2\exp\left(i\,\vec{k}\vec{r}\right)\mathrm{T}\left(\vec{k},t\right)=i\hbar\frac{d}{dt}\exp\left(i\,\vec{k}\vec{r}\right)\mathrm{T}\left(\vec{k},t\right)$$

Scalar and vector fields

$$\psi_{\vec{k}}(\vec{r},t) = \exp\left(i\,\vec{k}\vec{r}\right)\mathrm{T}\left(\vec{k},t\right) = \exp\left(i\,\vec{k}\vec{r}\right)\exp\left(-\frac{i}{\hbar}\int^{t}\frac{\hbar^2}{2m}\left[\vec{k}-\frac{e}{\hbar}\vec{E}t'\right]^2 dt'\right) =$$

$$= \exp\left(i\left[\vec{k}\vec{r}-\frac{1}{\hbar}\int^{t}\frac{\hbar^2}{2m}\left[\vec{k}-\frac{e}{\hbar}\vec{E}t'\right]^2 dt'\right]\right)$$

The particle accelerates with time with a momentum $\hbar\vec{k}-e\vec{E}t'$, corresponding to a constant force $-e\vec{E}$. The momentum of the particle increases. On the other hand, we would expect that this change in momentum can be observed in the change of spatial part of the wavefunction $\exp\left(i\,\vec{k}\vec{r}\right)$ (changes the wavelength, or changes the wave vector $\vec{k}$) – which does not occure.

It is also difficult to define the density of states.

The current density is OK - constant in space and increases with time (constant acceleration)

$$J(\vec{r},t) = \frac{q}{2}\left[\Psi^*\left(\frac{\hat{p}-q\,\vec{A}(\vec{r},t)}{m}\Psi\right)+\left(\frac{\hat{p}-q\,\vec{A}(\vec{r},t)}{m}\Psi\right)^*\Psi\right] = -\frac{e}{m}\left(\hbar\vec{k}-e\vec{E}t\right)$$

$$\frac{\hbar^2}{2m}\left[\vec{k}-\frac{e}{\hbar}\vec{E}t\right]^2\exp\left(i\,\vec{k}\vec{r}\right)\mathrm{T}\left(\vec{k},t\right) = i\hbar\frac{d}{dt}\exp\left(i\,\vec{k}\vec{r}\right)\mathrm{T}\left(\vec{k},t\right)$$

Local density of states

The density of states (in general) can be defined as:

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For instance:

$$N^{1D}(E) = \sum_k \delta\big(E - \varepsilon(k)\big) = \int \frac{1}{E'(k)}\, \delta(k - k')\, 2\, dk = \frac{1}{\pi}\sqrt{\frac{2m}{E}}$$

$$N^{2D}(E) = \sum_k \delta\big(E - \varepsilon(k)\big) = \int \frac{1}{E'(k)}\, \delta(k - k')\, 2\pi k\, dk = \frac{m}{\pi\hbar^2}$$

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On exercises

Homogenous magnetic field

The Landau gauge solution

$$\left\{\frac{1}{2m}\left[\hat{p} - q\,\vec{A}(\vec{r},t)\right]^2 + q\varphi(\vec{r},t) + U(\vec{r},t)\right\}\psi(\vec{r},t) = i\hbar\frac{d}{dt}\psi(\vec{r},t)$$

Homogenous magnetic field

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Landau gauge: magnetic field $\vec{B} = (0,0,B_z) \Rightarrow B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$ (unfortunately distinguishes direction)

$\vec{A} = [0, B_z x, 0]$ czyli $A_y = B_z x \stackrel{\text{def}}{=} Bx$ $\qquad q = -e$

We assume that in a plane xy there is no other potential

$$\left\{\frac{1}{2m}\left[-\hbar^2\frac{\partial^2}{\partial x^2} + \left(-i\hbar\frac{\partial}{\partial y} + eBx\right)^2 - \hbar^2\frac{\partial^2}{\partial z^2}\right] + U(z)\right\}\psi(\vec{r}) = E\psi(\vec{r})$$

Which gives: $$\left[-\frac{\hbar^2}{2m}\nabla^2 - \frac{ie\hbar}{m}Bx\frac{\partial}{\partial y} + \frac{(eBx)^2}{2m} + U(z)\right]\psi(\vec{r}) = E\psi(\vec{r})$$

The evidence of the Lorentz force

Parabolic potential!

Homogenous magnetic field

Time-reversal invariance, T-symmetry (*symetria względem odwrócenia czasu*): if the solution of the Schrodinger equation is the function $\Psi(t)$, then $\Psi^*(-t)$ must be also the solution – only for a real Hamiltonian. For the magnetic field, we have to reverse also the direction of the magnetic field: $\Psi\left(t,\vec{B}\right) \rightarrow \Psi^*\left(-t,-\vec{B}\right)$; we reverse the sign of kinetic momentum $\left[\hat{p} - q\,\vec{A}(\vec{r},t)\right]$.

Imaginary value - lack of time reversal symmetry

The potential in z-direction is independent of other variables (factorization) → thus we consider 2D + z problem

Which gives: $$\left[-\frac{\hbar^2}{2m}\nabla^2 - \frac{ie\hbar}{m}Bx\frac{\partial}{\partial y} + \frac{(eBx)^2}{2m} + U(z)\right]\psi(\vec{r}) = E\psi(\vec{r})$$

The evidence of the Lorentz force

Parabolic potential!

Vector potential does not depend on y, we can assume the function of the form:

$$\psi(\vec{r}) = w(z)u(x)\exp\left(ik_y y\right)$$

Homogenous magnetic field

$$\left[-\frac{\hbar^2}{2m}\nabla^2 - \frac{ie\hbar}{m}Bx\frac{\partial}{\partial y} + \frac{(eBx)^2}{2m} + U(z)\right]\psi(\vec{r}) = E\psi(\vec{r})$$

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$$\psi(\vec{r}) = w(z)u(x)\exp\left(ik_y y\right)$$

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\,\omega_c^2\left(x + \frac{\hbar k_y}{eB}\right)^2\right]u(x) = \varepsilon u(x) \qquad \omega_c = \left|\frac{eB}{m}\right| \qquad R_c = \frac{v}{\omega_c} = \frac{\sqrt{2mE}}{|eB|}$$

Cyclotron frequency

Cyclotron radius (*gyroradius*)

k_y wave vector. What interesting in ε THERE IS NO k_y.

The parabolic potential of the form of $x_k = -\hbar k_y/eB$

Homogenous magnetic field

$$\left[-\frac{\hbar^2}{2m}\nabla^2 - \frac{ie\hbar}{m}Bx\frac{\partial}{\partial y} + \frac{(eBx)^2}{2m} + U(z)\right]\psi(\vec{r}) = E\psi(\vec{r})$$

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$$\psi(\vec{r}) = w(z)u(x)\exp\left(ik_y y\right)$$

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Magnetic length: $l_B = \sqrt{\frac{\hbar}{m\omega_c}} = \sqrt{\frac{\hbar}{|eB|}}$ does not depend on mass m, but ONLY on magnetic field B!

The typical value for $B = 1.0$ T is $l_B = 26$ nm.

Solutions $\varepsilon_{nk} = \left(n - \frac{1}{2}\right)\hbar\omega_c$ (does not depend on k_y).

$$\phi_{nk}(x,y) \propto H_{n-1}\left(\frac{x - x_k}{l_B}\right)\exp\left[-\frac{(x - x_k)^2}{2l_B^2}\right]\exp\left(ik_y y\right)$$

$n = 1, 2, 3$ ... they are subsequent **Landau levels**.

Homogenous magnetic field

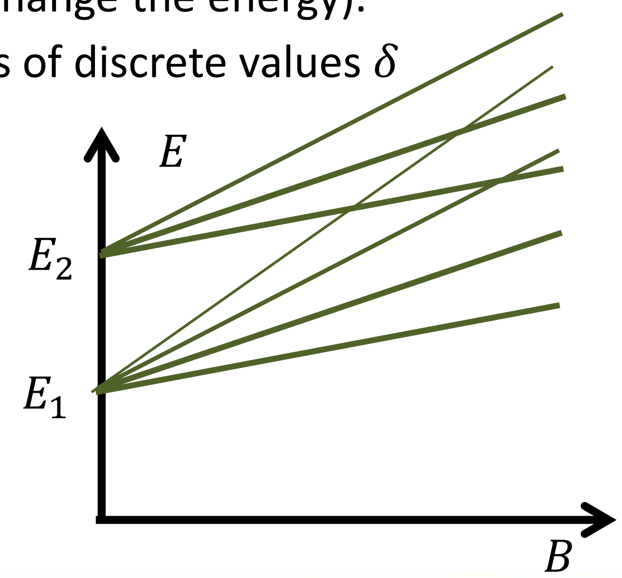
The 2D case:

Solutions $\varepsilon_{nk} = \left(n - \frac{1}{2}\right)\hbar\omega_c + E_n$ (does not depend on k_y; E_n- is any 2D energy).

$$\phi_{nk}(x,y) \propto H_{n-1}\left(\frac{x - x_k}{l_B}\right)\exp\left[-\frac{(x - x_k)^2}{2l_B^2}\right]\exp(ik_y y) \qquad n = 1, 2, 3 \ldots$$

Wave functions are the functions of the oscillator (along x, of the order of $l_B/\sqrt{2}$) and travelling waves (along y) – weird, right? Why?

The energy does not depend on k vector – states of different k have the same energy, so they are degenerated (therefore any combination of them does not change the energy).
The density of states is reduced from the constant $\frac{m}{\pi\hbar^2}$ to a series of discrete values δ given by the equation of ε_{nk} - they are called **Landau levels**.



Full energy (including binding potential in z direction):

$$E = E_z + \varepsilon_{nk} = E_z + \left(n - \frac{1}{2}\right)\hbar\omega_c$$

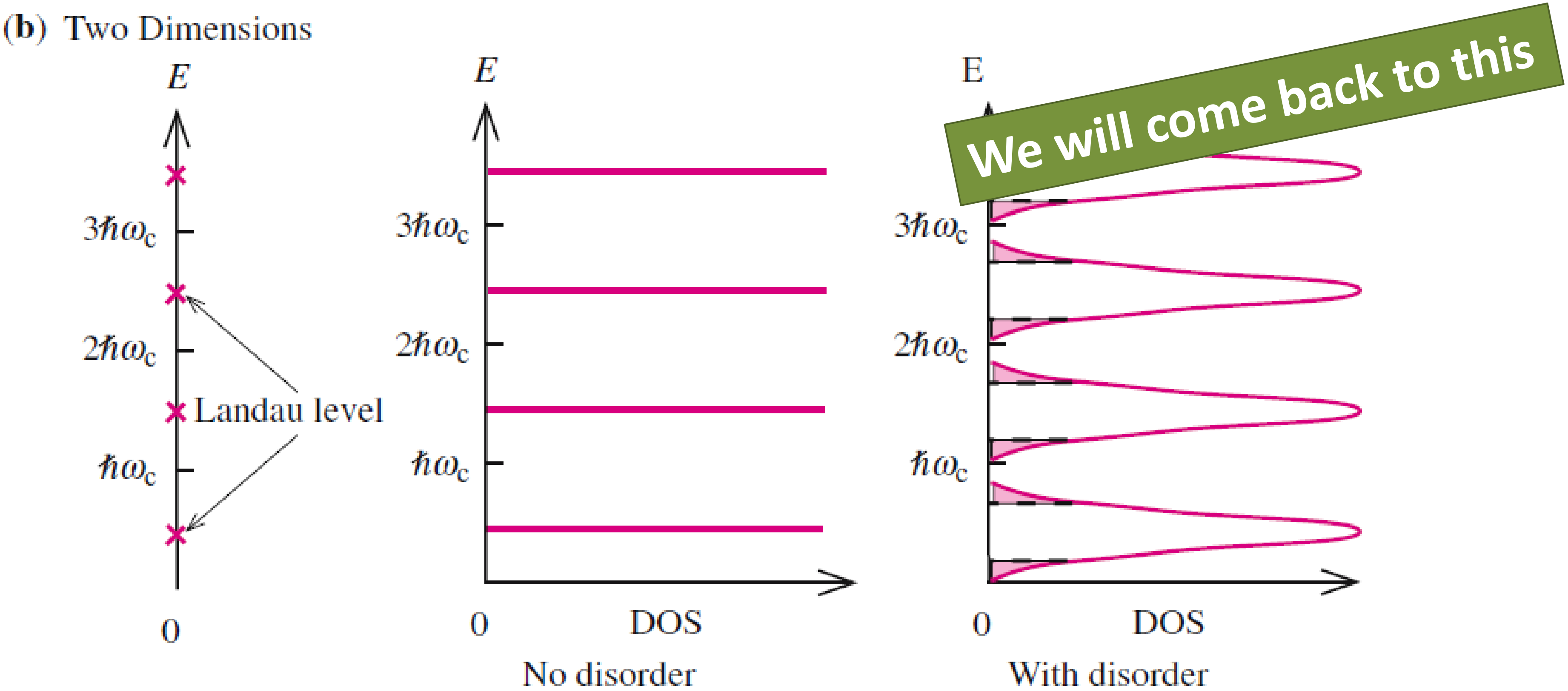
$$n = 1, 2, 3 \ldots$$

Homogenous magnetic field

The 2D case:

Solutions $\varepsilon_{nk} = \left(n - \frac{1}{2}\right)\hbar\omega_c + E_n$ (does not depend on k_y; E_n- is any 2D energy).

$$\phi_{nk}(x,y) \propto H_{n-1}\left(\frac{x-x_k}{l_B}\right)\exp\left[-\frac{(x-x_k)^2}{2l_B^2}\right]\exp(ik_y y) \qquad n = 1,2,3\,...$$



Homogenous magnetic field

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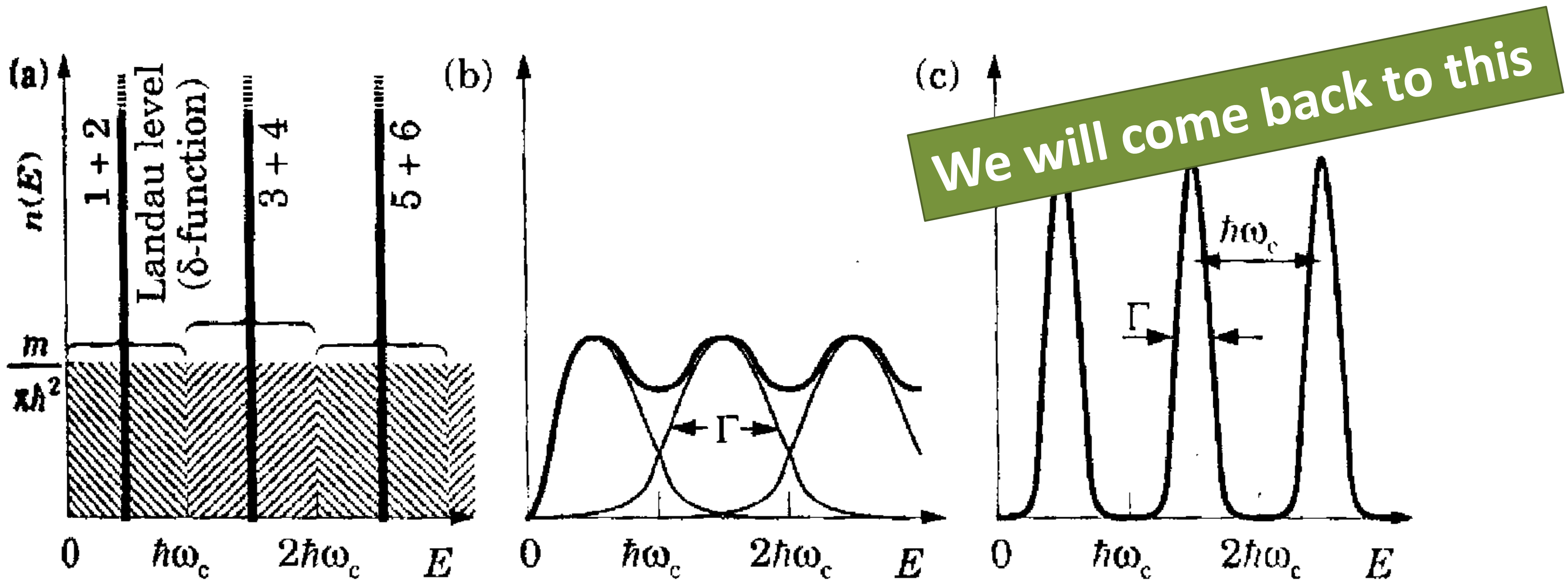


FIGURE 6.7. Density of states in a magnetic field, neglecting spin splitting. (a) The states in each range $\hbar\omega_c$ are squeezed into a δ-function Landau level. (b) Landau levels have a non-zero width Γ in a more realistic picture and overlap if $\hbar\omega_c < \Gamma$. (c) The levels become distinct when $\hbar\omega_c > \Gamma$.

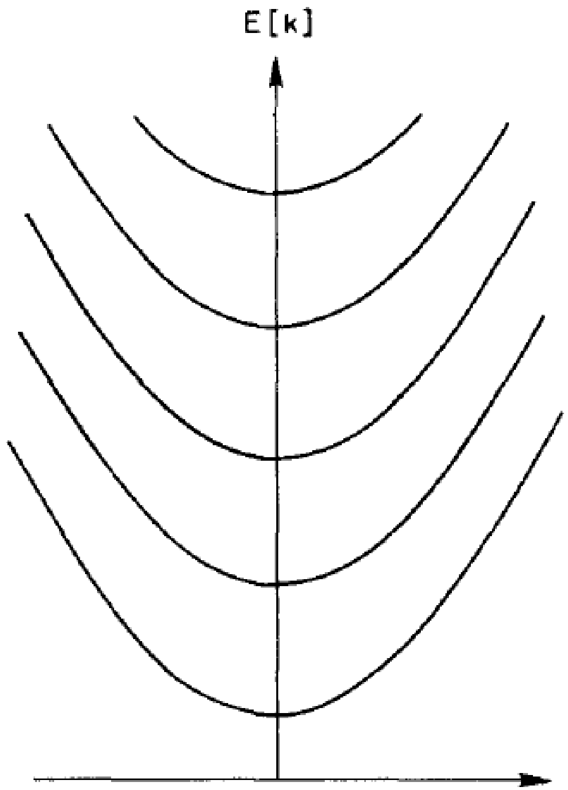
Homogenous magnetic field

The 3D case (no $U(z)$ potential)

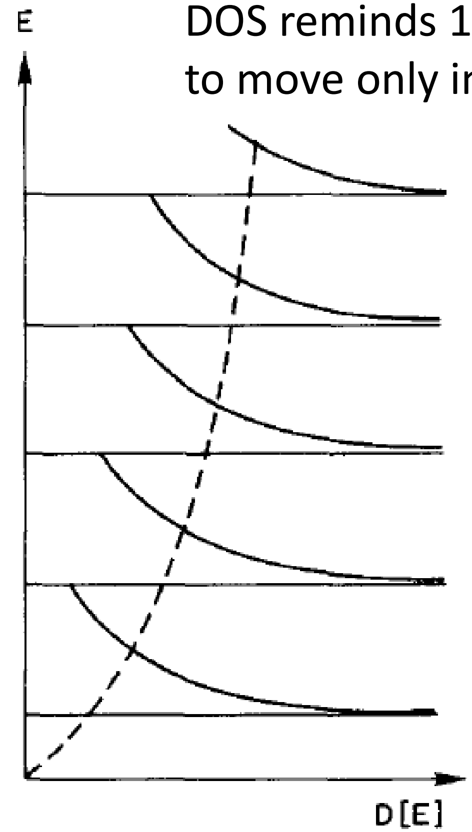
Solution:

$$\varepsilon_{nk} = \left(n - \frac{1}{2}\right)\hbar\omega_c + \frac{\hbar^2 k_z^2}{2m^*}$$

$n = 1, 2, 3 \ldots$ are subsequent Landau levels.



k_z



DOS reminds 1D because it is possible to move only in the direction z

http://www2.warwick.ac.uk/fac/sci/physics/current/postgraduate/regs/mpags/ex5/mag/

Homogenous magnetic field

The solution in the symmetric gauge:

$$\left\{\frac{1}{2m}\left[\hat{p} - q\,\vec{A}(\vec{r},t)\right]^2 + q\varphi(\vec{r},t) + U(\vec{r},t)\right\}\psi(\vec{r},t) = i\hbar\frac{d}{dt}\psi(\vec{r},t)$$

The symmetric gauge: field $\vec{B} = (0,0,B_z) \Rightarrow A_\theta = \frac{1}{2}Br, A_r = 0, A_z = 0$

$$\left\{-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right] - \frac{i\hbar eB}{m}\frac{\partial}{\partial\theta} + \frac{e^2B^2r^2}{8m} + U(z)\right\}\psi(r,\theta,z) = E\psi(r,\theta,z)$$

This time a rotation angle θ is the invariant, which can be associated with angular momentum and the function in the form of $\exp(il\theta)$

$$\varepsilon_{nl} = \left(n + \frac{1}{2}l + \frac{1}{2}|l| - \frac{1}{2}\right)\hbar\omega_c \qquad n = 1,2,3\ldots \qquad l = 0,\pm1,\pm2,\pm3\ldots$$

$$\phi_{nk}(r,\theta) \propto \exp(il\theta)\exp\left[-\frac{r^2}{4l_B^2}\right] r^{|l|} L_{n-1}^{(|l|)}\left(\frac{r^2}{2l_B^2}\right)$$

The symmetrical potential also has its drawbacks - where is the origin of ALL cyclotron orbits?
What are the solutions with negative sign?

Associate Laguerre polynomial

Homogenous magnetic field

In a magnetic field, we cannot forget about spin!

Electron spin: $\mu_B = \frac{e\hbar}{2m_0}$ (Bohr magneton = magnitude of the magnetic moment of the electron on the orbit of the total angular momentum $1\hbar$)

$$H' = \mu_B \vec{B} \underline{g} \hat{S}$$

In general g-factor may be the tensor

In the case of free electron $g = 2{,}0023$ …, but in the solid state it may have very different values (eg. $g = -0.44$ in GaAs and $g = +0.4$ in $Al_{0.3}Ga_{0.7}As$).

Homogenous magnetic field

We return to the Landau gauge:

Solutions $\varepsilon_{nk} = \left(n - \frac{1}{2}\right)\hbar\omega_c + E_n$ (does not depend on k_y; E_n- is any 2D energy).

$$\phi_{nk}(x,y) \propto H_{n-1}\left(\frac{x - x_k}{l_B}\right)\exp\left[-\frac{(x - x_k)^2}{2l_B^2}\right]\exp(ik_y y) \qquad n = 1, 2, 3 \ldots$$

Question: for a given n (i.e. Landau level) how many different states $\phi_{nk}(x,y)$ of the same energy there are – i.e. what is the degeneration of the Landau levels?

Let's calculate how many different functions of quantum numbers k_y (only k_y counts, because in Landau gauge x_k depends only on k_y) – similar considerations can be worked out in an arbitrary gauge.

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What is the number of states per one level? The sample $\mathrm{S} = L_x \times L_y$, in the Landau gauge for y coordnate we have plane wave condition $k = (2\pi/L_y)n_y$ (where n_y is an integer number).

How many states of different n_y there are?

For x coordinate the wavefunction is centered in $x_k = -\frac{\hbar k}{eB} = -(2\pi\hbar n_y/eBL_y)$.

If n_y is too large then x_k can be outside the sample – no harmonic force, no harmonic solution.

Landau levels

The solution of the Schrödinger equation in a magnetic field gives a discrete spectrum.

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For x coordinate the wavefunction is centered in $x_k = -\frac{\hbar k}{eB} = -(2\pi\hbar n_y/eBL_y)$.
The condition for x_k to be in the sample (rather than outside):

$$-L_x < \frac{2\pi\hbar n_y}{eBL_y} < 0 \qquad \text{czyli} \qquad 0 < n_y < \frac{eB}{h} L_x L_y = n_B S \qquad \text{(the absolute value)}$$

There is no factor 2 associated with the degeneracy of the spin (because spin in the magnetic field is not degenerated)

The dimension of $n_B = \frac{eB}{h}$ is "amount" per unit area

$$n_B = \frac{eB}{h}$$

The degeneration of Landau levels – is the number of allowed states for each of the Landau level per unit area – **it increases with increasing field B**

Landau levels

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flux $\Phi_0 = \frac{h}{e} = 4.135667516 \times 10^{-15}$ Wb [Wb]=[T m^2]

The magnetic flux quantum (pol. *flukson*) (In a superconductor $h/2e$, so this is not a „quantum")

$\Phi = BS$ the total magnetic flux in the sample $S = L_x \times L_y$

$$0 < n_y \Phi_0 < \Phi$$

The amount of allowed states is related to the amount of magnetic flux quanta passing through the sample!

Local density of states

The density of states (in general) can be defined as:

$$N(E) = \sum_n \delta(E - \varepsilon_n)$$

After integration

$$\int_{E_1}^{E_2} N(E)\, dE = \int_{E_1}^{E_2} \sum_n \delta(E - \varepsilon_n)\, dE = \sum_n \int_{E_1}^{E_2} \delta(E - \varepsilon_n) dE$$

For instance:

$$N^{1D}(E) = \sum_k \delta\big(E - \varepsilon(k)\big) = \int \frac{1}{E'(k)}\, \delta(k - k')\, 2\, dk = \frac{1}{\pi}\sqrt{\frac{2m}{E}}$$

$$N^{2D}(E) = \sum_k \delta\big(E - \varepsilon(k)\big) = \int \frac{1}{E'(k)}\, \delta(k - k')\, 2\pi k\, dk = \frac{m}{\pi \hbar^2}$$

$$N^{3D}(E) = \sum_k \delta\big(E - \varepsilon(k)\big) = \int \frac{1}{E'(k)}\, \delta(k - k')\, 4\pi k^2\, dk = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{E}$$

On exercises

Landau levels

$$N^{2D}(E) = \frac{m}{\pi\hbar^2}$$

Broadening of levels due to the scattering $\Gamma = \hbar/\tau_i$

τ_i this is single-particle (or quantum) lifetime – this is NOT the same time, which we discussed with Drude model (transport lifetime)

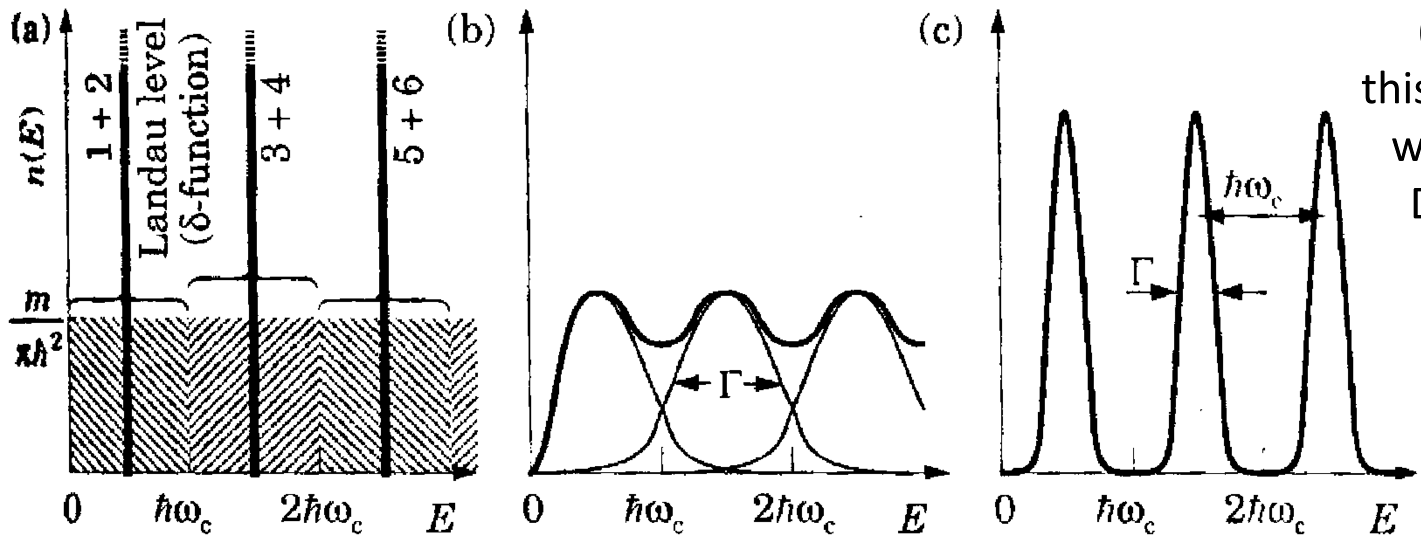

FIGURE 6.7. Density of states in a magnetic field, neglecting spin splitting. (a) The states in each range $\hbar\omega_c$ are squeezed into a δ-function Landau level. (b) Landau levels have a non-zero width Γ in a more realistic picture and overlap if $\hbar\omega_c < \Gamma$. (c) The levels become distinct when $\hbar\omega_c > \Gamma$.

Counting 2 spins: $2n_B = \frac{2eB}{h} = \frac{2m\omega_c}{2\pi\hbar} = \frac{m}{\pi\hbar^2}\ \hbar\omega_c$

Each of the states on the Landau level occupies an area $\frac{h}{eB} = 2\pi l_B^2$

$$l_B = \sqrt{\frac{\hbar}{m\omega_c}} = \sqrt{\frac{\hbar}{|eB|}}$$

Landau levels

$$n_B = \frac{eB}{h}$$

The degeneration of Landau levels – is the number of allowed states for each of the Landau level per unit area – **it increases with increasing field** B

The carrier concentration in 2D: n_{2D} – on how many Landau levels these carriers can be hold?

Filling factor ν (*współczynnik wypełnienia*) – usually this is not an integer

$$\nu = \frac{n_{2D}}{n_B} = \frac{hn_{2D}}{eB} = \frac{\Phi_0 n_{2D}}{B} = 2\pi l_B^2 n_{2D}$$

(taking into account the spin degeneracy)

Increasing the magnetic field we are successively filling the Landau levels. You can completely fill n-th level ($\nu = n$) and then $B_n = hn_{2D}/en$, until we reach $n = 1$, i.e. all electrons are at the same Landau level (ie. the *quantum limit*).

For $\nu < 1$ nteresting things happens (which'll be right back!)

Landau levels

$n_B = \frac{eB}{h}$ The degeneration of Landau levels – is the number of allowed states for each of the Landau level per unit area – **it increases with increasing field** B

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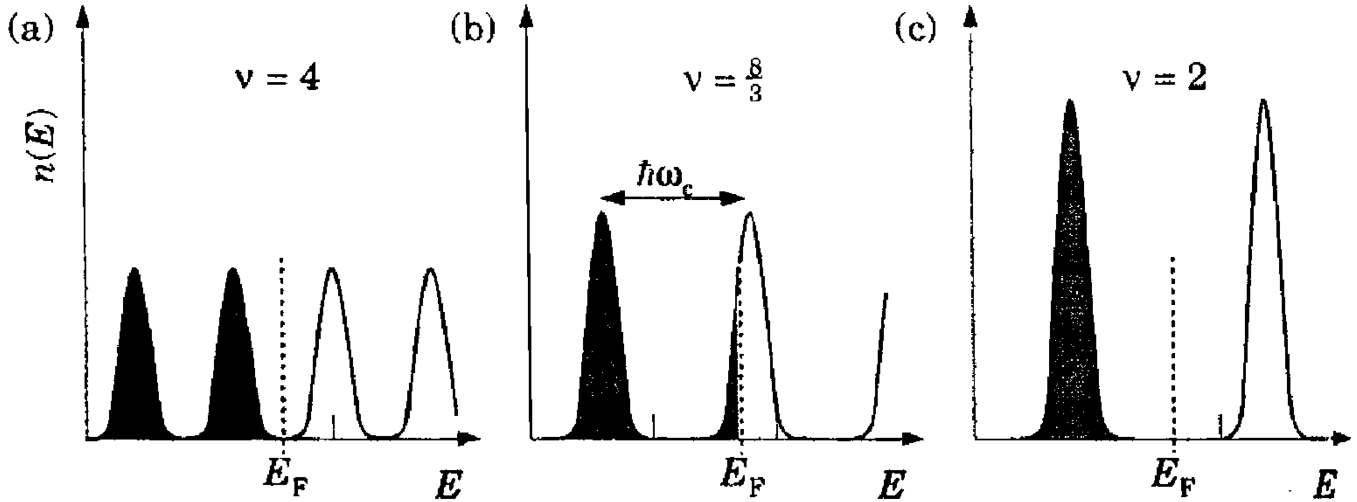
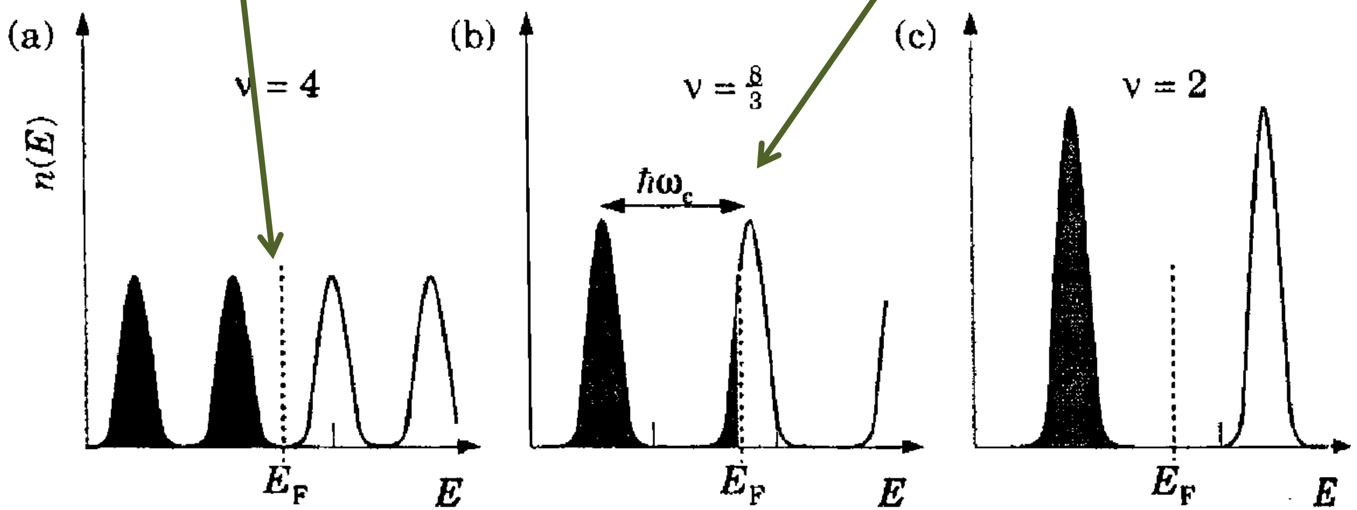


FIGURE 6.8. Occupation of Landau levels in a magnetic field neglecting the spin splitting, showing how the Fermi level moves to maintain a constant density of electrons. The fields are in the ratio 2 : 3 : 4 and give $\nu = 4, \frac{8}{3}$, and 2.

Landau levels

The Fermi level lies **between** Landau levels - there is no DOS, change of E_F does not change DOS –incompressible states (*stany nieściśliwe*)

The Fermi level lies **inside** the Landau level – large DOS, change of E_F strongly affects the DOS – *compressible states* (*stany* ściśliwe)



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Landau levels

The Fermi level in the magnetic field:

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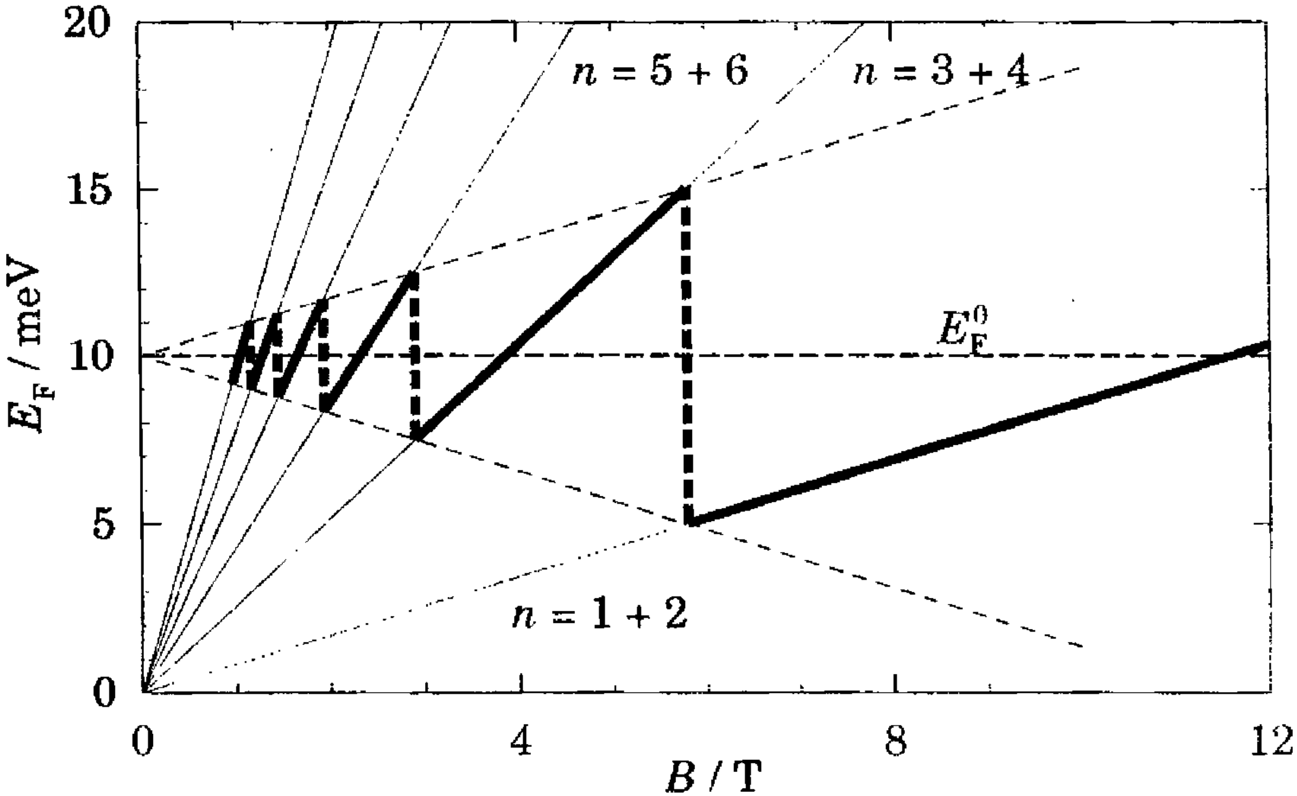

FIGURE 6.9. Variation of the Fermi level as a function of magnetic field for a two-dimensional electron gas in GaAs with $E_F^0 = 10$ meV before the field was applied. Spin splitting is neglected. The fan of thin lines shows the Landau levels, while the discontinuous thick line is E_F.

Landau levels

The Fermi level in the magnetic field:

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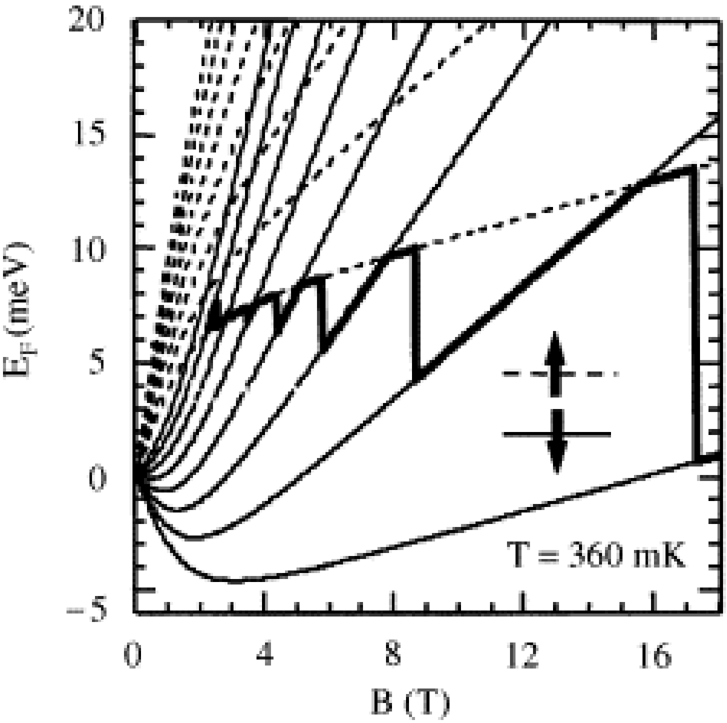

Fig. 16. Landau level fan diagram for the magnetic 2DEG sample described in Fig. 15. Solid (dashed) lines correspond to spin-down (spin-up) states. The dark solid line shows the variation of the Fermi energy with magnetic field. Parameters used in this calculation are: E_F=7 meV at B=0, and T=360 mK. The spin-splitting parameters used are obtained by fitting the magneto-optical data in Fig. 3: T_0=2.1 K and a saturation conduction band spin splitting of 12.9 meV.

Spin dynamics and quantum transport in magnetic semiconductor quantum structures
D.D Awschalom, N. Samarth, Journal of Magnetism and Magnetic Materials **200** (1999) 130-147