Semiconductor heterostructures – quantum wells

"FACE IT — ANECDOTAL EVIDENCE JUST DOESN'T CARRY ANY WEIGHT IN MATH."

anecdotal evidence = niepotwierdzony dowód

Faculty of Physics UW

Jacek.Szczytko@fuw.edu.pl

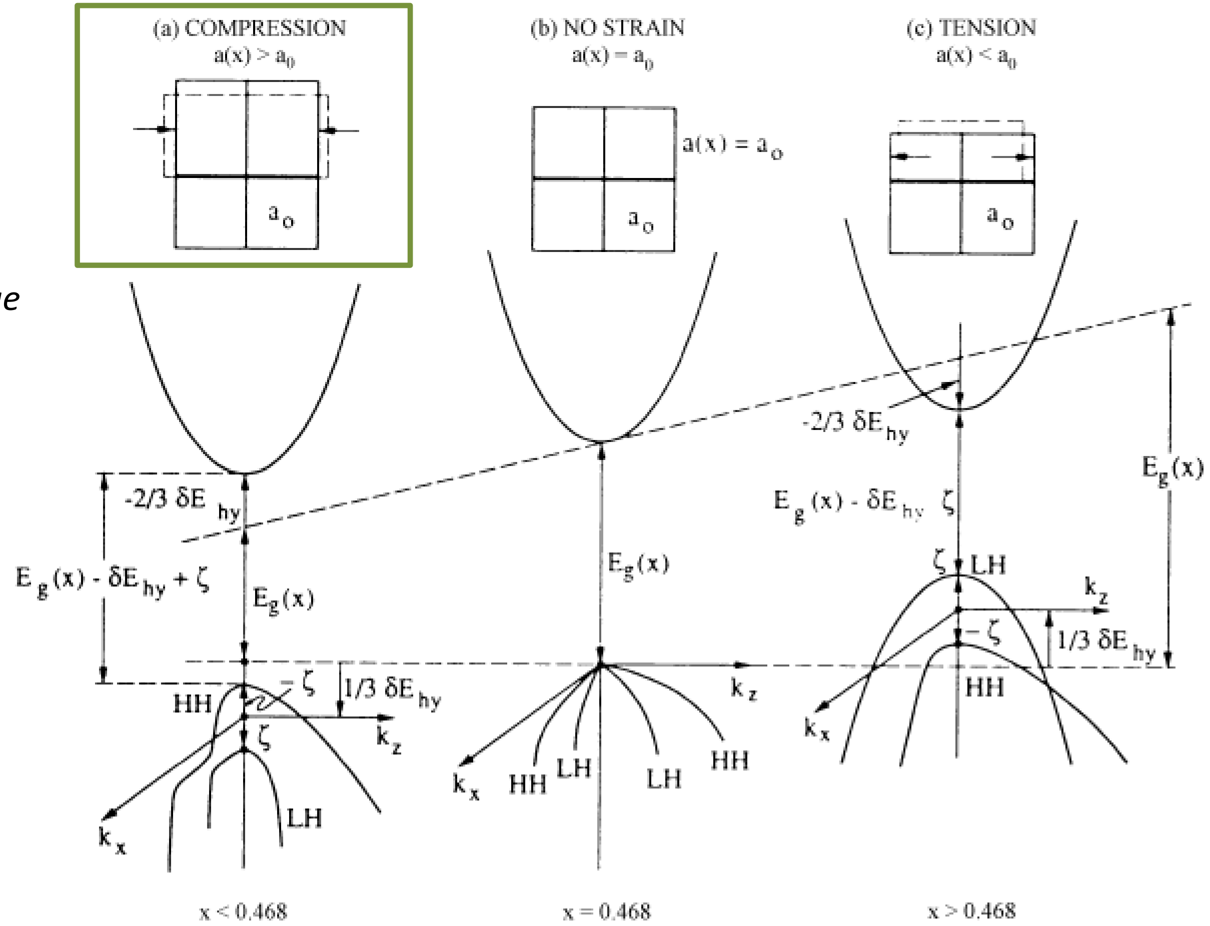
inżynieria nanostruktur

Bandgap engineering

The presence of the well changes the symmetry of the crystal (eg. quantum wells in the direction of [001] corresponds to an uniaxial pressure applied perpendicular to the layer). You have to solve the $\boldsymbol{kp}$ perturbation theory (Chemla 1983):

Effects of biaxial strain: decrease of the degeneracy of the valence band and change of the effective masses in the $Ga_xIn_{1-x}As$ / $Ga_xIn_{1-x}As_y yP_{1-y}$ *material system.*

S.L. Chuang, *Phys. Rev. B* 43, p. 9649 (1991). 9, 10



Triangular well

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2} + eFz\right]\psi(z) = \varepsilon\psi(z)$$

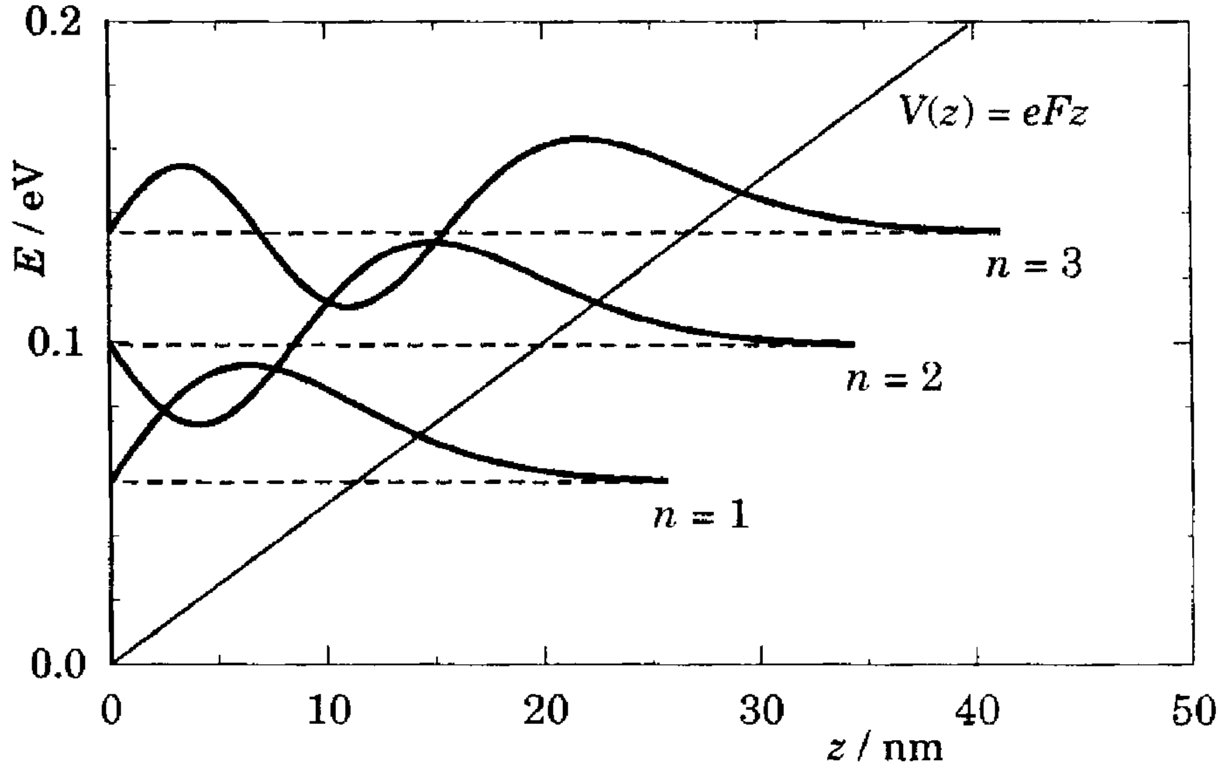


FIGURE 4.6. Triangular potential well $V(z) = eFz$, showing the energy levels and wave functions. The scales are for electrons in GaAs and a field of 5 MV m^{-1}.

Triangular well

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2} + eFz\right]\psi(z) = \varepsilon\psi(z)$$

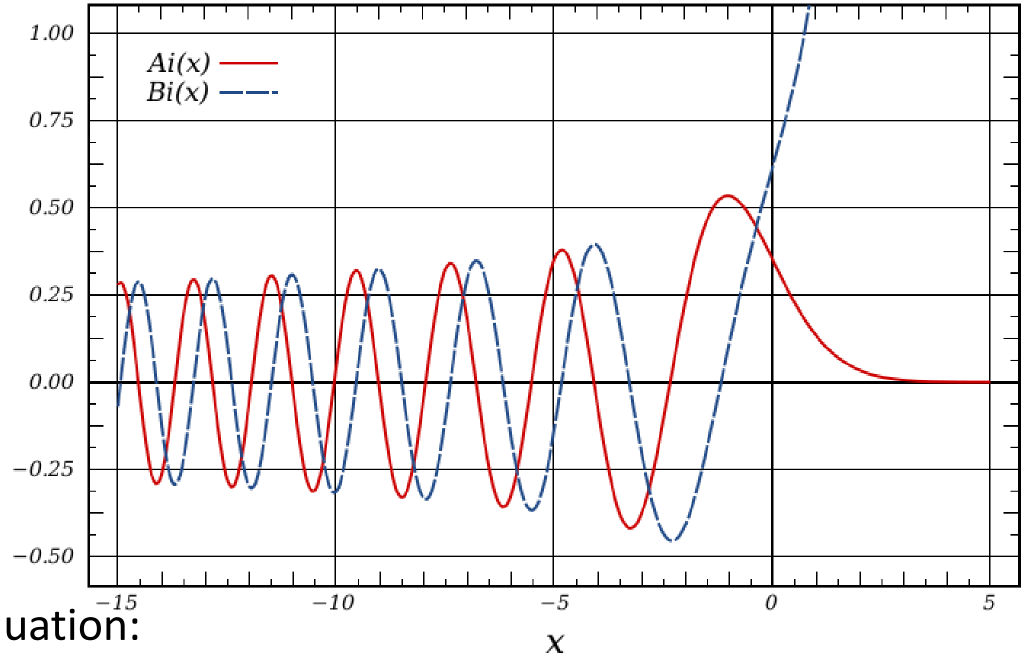
Transformation:

$$\frac{d^2}{dz^2}\psi(z) = \frac{2m}{\hbar^2}(eFz - \varepsilon)\psi(z)$$

Substituting: $z_0 = \left(\frac{\hbar^2}{2meF}\right)^{1/3}$

$$\varepsilon_0 = eFz_0 = \left[\frac{(eF\hbar)^2}{2m}\right]^{1/3},\ \bar{z} = \frac{z}{z_0},\ \bar{\varepsilon} = \frac{\varepsilon}{\varepsilon_0}$$

$$\frac{d^2}{dz^2}\psi(z) = \frac{2m}{\hbar^2}(eFz - \varepsilon)\psi(z)$$



The equation reduces to Stokes or Airy equation:

$$\frac{d^2}{dz^2}f(z) = zf(z)$$

Its two independent solutions the Airy functions $Ai(z)$ and $Bi(z)$. The solutions of the equation are the zeros of a function $Ai(z)$ (after some rearrangements).

Triangular well

$V(z) = eFz$

$$\varepsilon_n = c_n \left[\frac{(eFz)^2}{2m} \right]^{1/3}$$

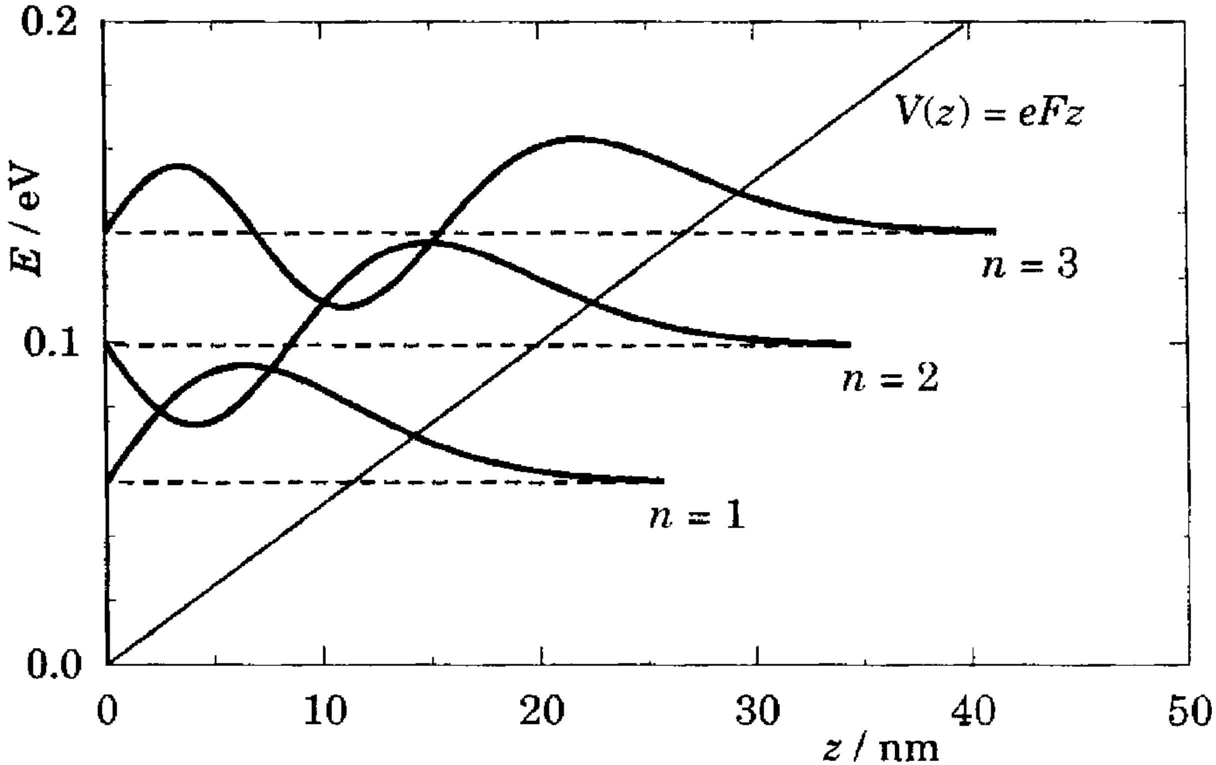


FIGURE 4.6. Triangular potential well $V(z) = eFz$, showing the energy levels and wave functions. The scales are for electrons in GaAs and a field of 5 MV m^{-1}.

WKB approximation

WKB approximation (Wentzel – Krammers – Brillouin) – for slowly changing potential

It is also known as the **LG** or **Liouville–Green** method or **JWKB** and **WKBJ**, where the "J" stands for Jeffreys or *phase integral method* **or** *semi-classical approximation*.

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right]\psi(x)=E\psi(x)$$

What is slowly varing potential? It is for sure $V(x)=V_0=const$. The solution for this potential is a plane wave $\psi(x)=e^{ikx}$ - phase of the wavefunction $k(x)=k=const$ is constant in the whole space $k^2=\frac{2m}{\hbar}[E-V_0]$

Let's define $k^2(x)=\frac{2m}{\hbar}[E-V(x)]$ - we want the phase $k(x)$ to be slowly varying in space, i.e.

$$\left|\frac{dk}{dx}\right|\ll k^2$$

(such condition).
We are looking for the solution $\psi(x)=e^{i\chi(x)}$ where $\chi(x)$ is the phase of the wavefunction.

WKB approximation

WKB approximation (Wentzel – Krammers – Brillouin) – for slowly changing potential

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right]\psi(x)=E\psi(x)$$

Let's define $k^2(x)=\frac{2m}{\hbar}[E-V(x)]$ - slowly varing in real space $k(x)$.

We are looking for the solutuion $\psi(x)=e^{i\chi(x)}$ where $\chi(x)$ its the phase of the wavefunction.

Inserting into Schrodinger equation:

$[\chi'(x)]^2-i\chi''(x)=\frac{2m}{\hbar}[E-V(x)]\equiv k^2(x)$ - this is rigorous.

Zero-order WKB approximation assumes $[\chi'(x)]^2 \gg |\chi''(x)|$ czyli $\chi''(x)\approx 0$

$[\chi'(x)]^2=k^2(x)$ czyli

$$\chi(x)=\pm\int^{x}k(x')dx'$$

Usually we expand more

$$[\chi'(x)]^2=k^2(x)+i\chi''(x)=k^2(x)+i[\chi'(x)]'\approx k^2(x)\pm i[k(x)]'$$

Thus:

$$\chi'(x)\approx\pm k(x)\sqrt{1+\frac{ik'(x)}{k^2(x)}}\approx\pm k(x)+\frac{ik'(x)}{2k(x)}$$

WKB approximation

WKB approximation (Wentzel – Krammers – Brillouin) – for slowly changing potential

Typically, the WKB method continues into

$$[\chi'(x)]^2 = k^2(x) + i\chi''(x) = k^2(x) + i[\chi'(x)]' \approx k^2(x) \pm i[k(x)]'$$

Thus:

$$\chi'(x) \approx \pm k(x)\sqrt{1+\frac{ik'(x)}{k^2(x)}} \approx \pm k(x) + \frac{ik'(x)}{2k(x)}$$

therefore:

$$\chi(x) = \pm\int^x k(x')dx' + \frac{i}{2}\ln k(x)$$

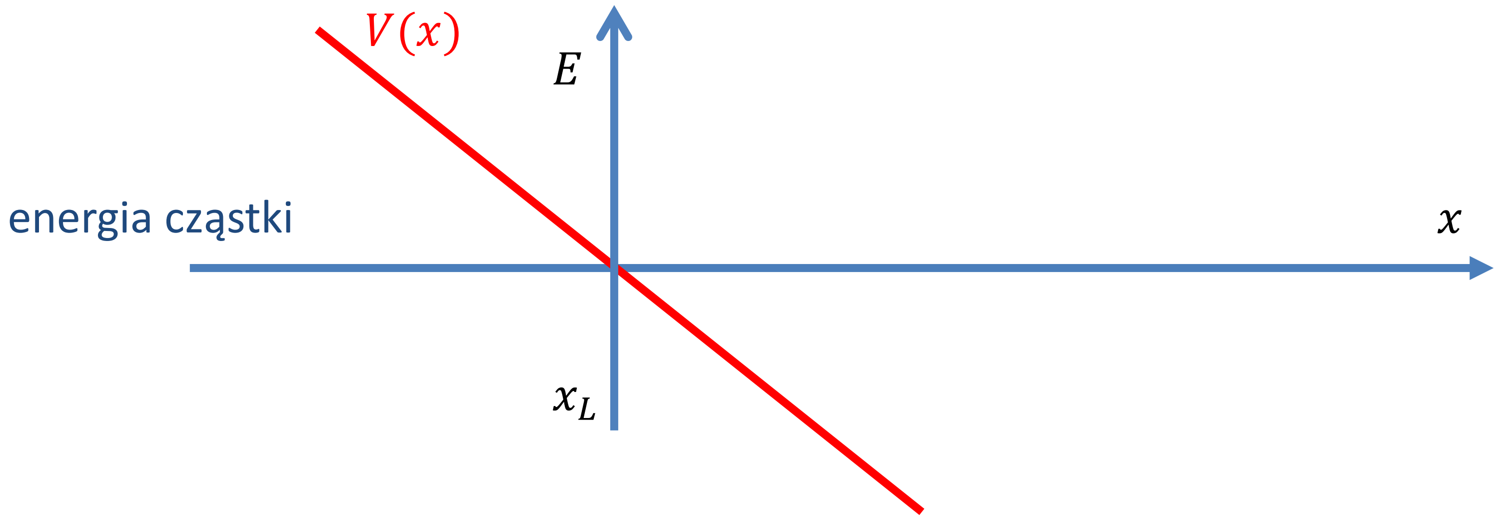
We get:

$$\psi(x) \approx \frac{1}{\sqrt{k(x)}}\exp\left[\pm i\int^x k(x')dx'\right]$$

The term $1/\sqrt{k(x)}$ - the density of probability of fast-moving particles is small for large k - OK!

WKB approximation

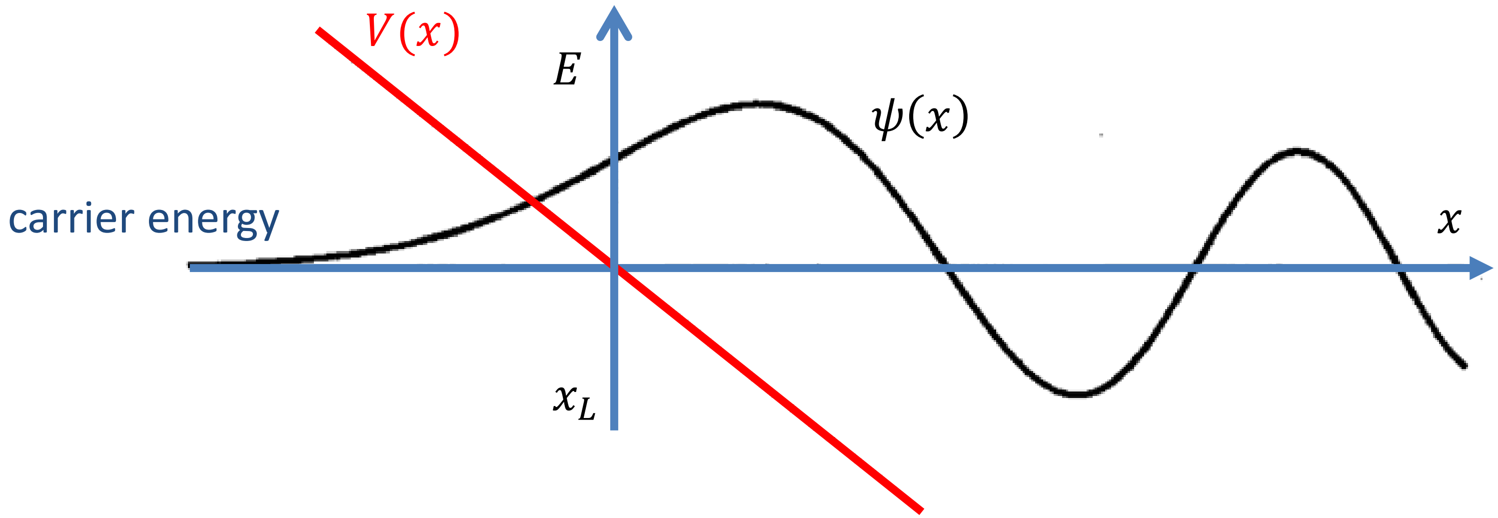
WKB approximation (Wentzel – Krammers – Brillouin) – for slowly varying potential



In the case of turning points (ie. the edges of the barriers well) potential changes rapidly compared with the wavelength $k(x)$ - WKB approximation is not valid (a rigorous approach avoids it by moving into the complex plane). Another way is to note that the potential is linear for a small region around the turning point Δx_L - and the Airy functions are the exact solutions. Then the solution must match on both regions near x_L.
The problem sounds complicated but fortunately the results are simple (we got additional phase).

WKB approximation

WKB approximation (Wentzel – Krammers – Brillouin) – for slowly varying potential



$$\psi(x) \sim \frac{2}{\sqrt{k(x)}} \cos\left[\int_{x_L}^{x} k(x')dx' - \frac{\pi}{4}\right], \qquad x \gg x_L$$

$$\psi(x) \sim \frac{1}{\sqrt{\kappa(x)}} \exp\left[-\int_{x_L}^{x} \kappa(x')dx'\right], \qquad x \ll x_L$$

WKB approximation

WKB approximation (Wentzel – Krammers – Brillouin) – for slowly varying potential

$$\psi(x) \sim \frac{2}{\sqrt{k(x)}} \cos\left[\int_{x_L}^{x} k(x')dx' - \frac{\pi}{4}\right], \qquad x \gg x_L$$

Examples:

1. „Hard" (infinitely steep) wall – the wave function goes to zero at the boundaries, so an exact number of half-wavelengths must fit between them

$$\int_{x_L}^{x_R} k(x')dx' = n\pi$$

For : $k(x) = const$ we have $k_n = \frac{n\pi}{L}$

2. „Soft wall" – an allowed state must obey the matching conditions ($x \gg x_L$, above) and at x_L it has additional phase $\left(-\frac{\pi}{4}\right)$ and similarly at x_R - next $\left(-\frac{\pi}{4}\right)$. Altogether:

$$\int_{x_L}^{x_R} k(x')dx' = \left(n - \frac{1}{2}\right)\pi$$

WKB approximation

WKB approximation (Wentzel – Krammers – Brillouin) – for slowly varying potential

$$\psi(x) \sim \frac{2}{\sqrt{k(x)}} \cos\left[\int_{x_L}^{x} k(x')dx' - \frac{\pi}{4}\right], \qquad x \gg x_L$$

3. Triangular well of „hard” and „soft” well:

$$\int_{x_L}^{x_R} k(x')dx' = \left(n - \frac{1}{4}\right)\pi$$

Eg. Triangular well: $V(x) = eFx$ then $k_n(x) = \frac{1}{\hbar}\sqrt{2m\big(E_n - V(x)\big)} = \frac{1}{\hbar}\sqrt{2m(E_n - eFx)}$

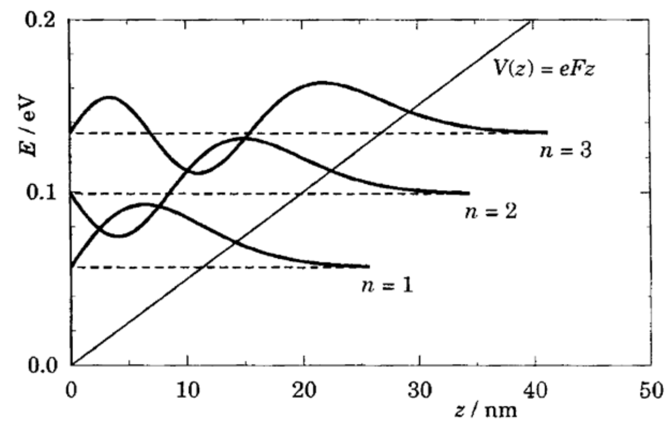
$$s = \frac{x}{x_R}$$

$$\int_{x_L}^{x_R} k_n(x')dx' = \int_0^{E_n/eF} \frac{1}{\hbar}\sqrt{2m(E_n - eFx')}dx' = \left[\frac{2mE_n}{\hbar^2}\right]^{1/2} \frac{E_n}{eF}\int_0^1 \sqrt{1-s}\, ds =$$

$$\left[\frac{2mE_n}{\hbar^2}\right]^{1/2} \frac{E_n}{eF}\int_0^1 \sqrt{1-s}\, ds = \frac{2}{3}\left[\frac{2mE_n}{\hbar^2}\right]^{1/2} \frac{E_n}{eF}$$

$$\int \sqrt{1-x}\; dx = -\frac{2}{3}(1-x)^{3/2} + \text{constant}$$

$$E_n = \left[\frac{3}{2}\pi\left(n - \frac{1}{4}\right)\right]^{2/3} \left[\frac{(eF\hbar)^2}{2m}\right]^{1/3}$$

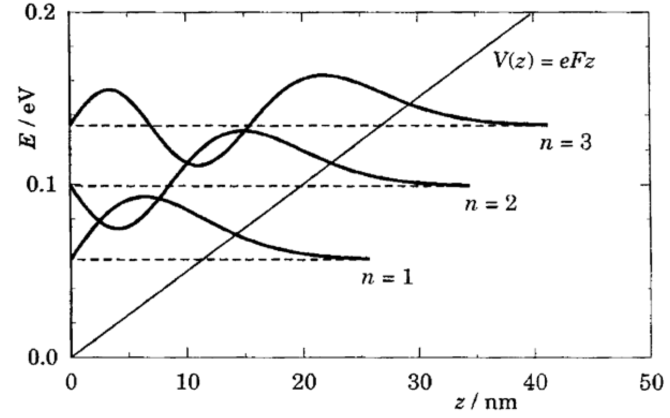


Triangular well

WKB approximation (Wentzel – Krammers – Brillouin) – for slowly varying potential

TABLE 7.1 A comparison of various approximate methods for energy levels in a triangular potential, in units of $\varepsilon_0 = [(eF\hbar)^2/(2m)]^{1/3}$, and the exact results from the Airy function.

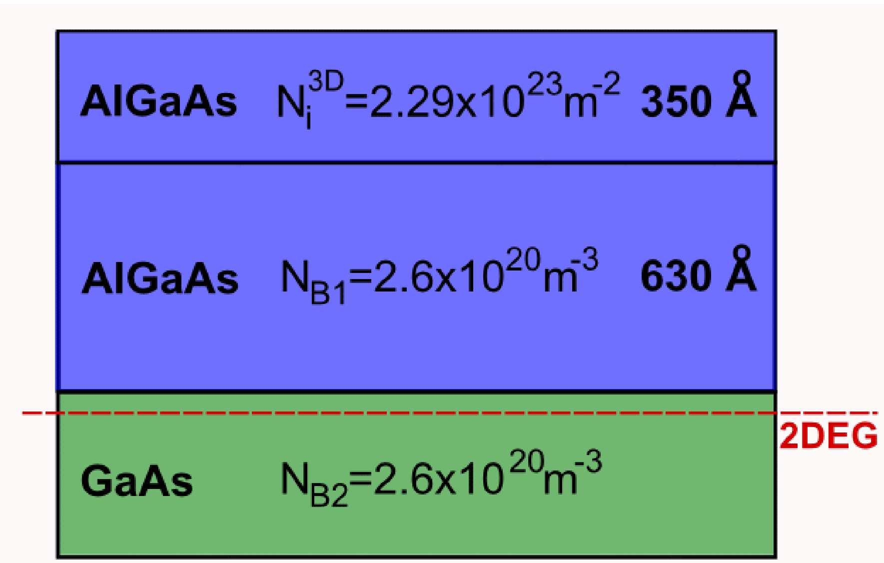
n	Airy function (exact)	WKB	Variational (Fang–Howard)	Variational (Gaussian)
1	2.3381	2.3203	2.4764	2.3448
2	4.0879	4.0818		
3	5.5206	5.5172		
⋮	⋮	⋮		
10	12.8288	12.8281		

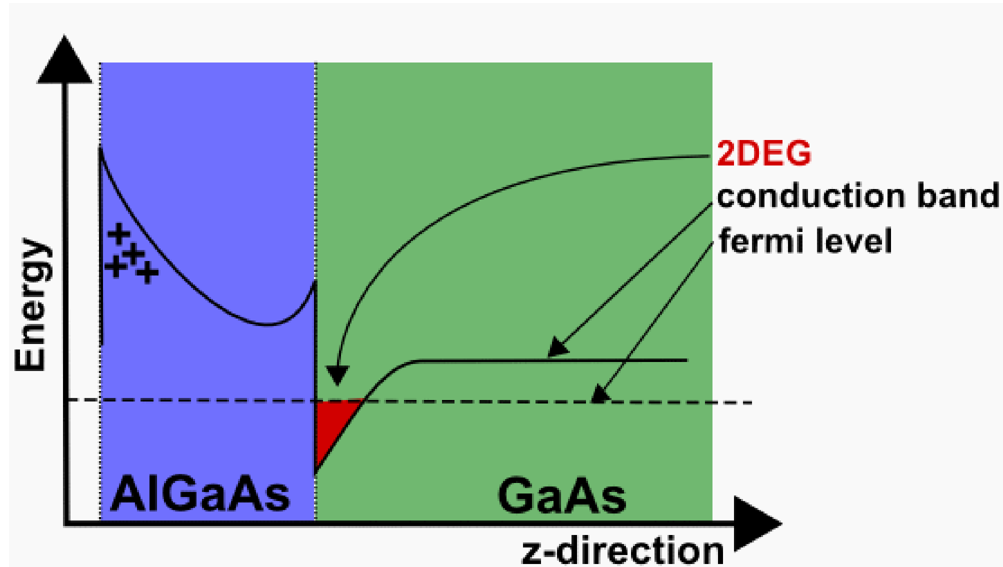


$$E_n = \left[\frac{3}{2}\pi\left(n - \frac{1}{4}\right)\right]^{2/3} \left[\frac{(eF\hbar)^2}{2m}\right]^{1/3}$$

Triangular well

WKB approximation (Wentzel – Krammers – Brillouin) – for slowly varying potential





http://www.phys.unsw.edu.au/QED/research/2D_scattering.htm

$$E_n = \left[\frac{3}{2}\pi\left(n-\frac{1}{4}\right)\right]^{2/3}\left[\frac{(eF\hbar)^2}{2m}\right]^{1/3}$$

WKB approximation

$$V(x) = V_b\left[1 - \left(\frac{x}{d}\right)^2\right]$$

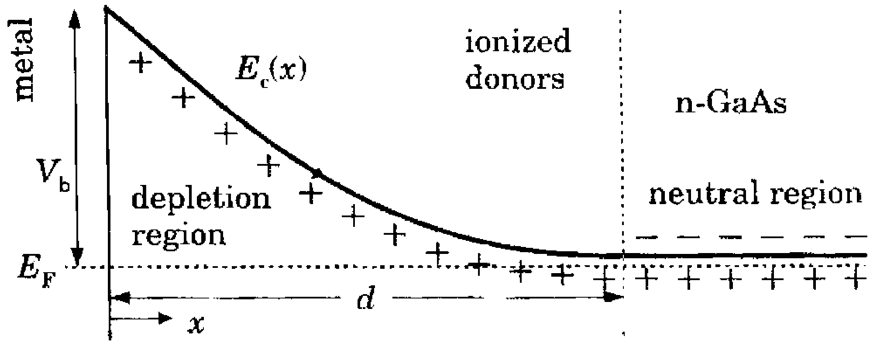


FIGURE 7.7. Schottky barrier in the conduction band $E_c(x)$ between a metal and n-GaAs. The potential is parabolic with height V_b and thickness d.

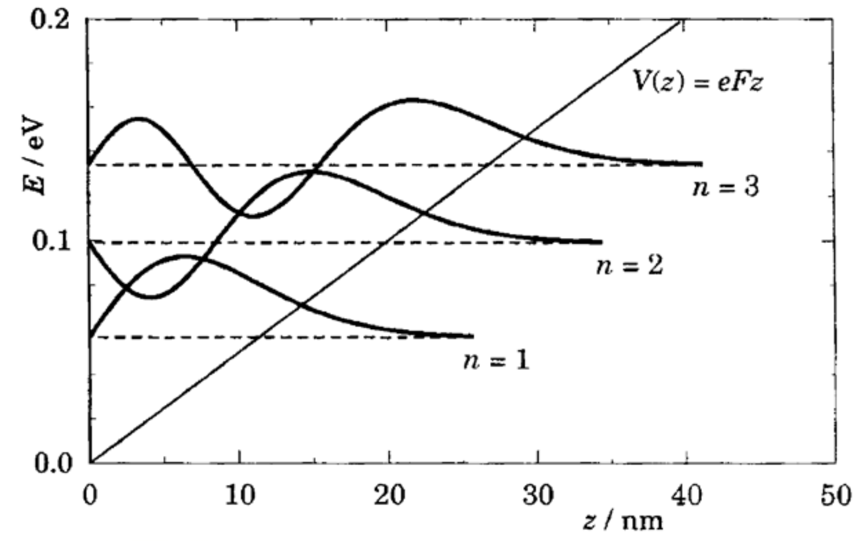
$$k_n(x) = \frac{1}{\hbar}\sqrt{2mV(x)} = \frac{1}{\hbar}\sqrt{2mV_b\left[1 - \left(\frac{x}{d}\right)^2\right]}$$

We will return discussing the transport

Low dimensional structures

Praca domowa:

Prosiłbym o narysowanie funkcji Airy w przestrzeni nie w jednostkach niemianowanych, ale metrycznych - czyli rysunek w nanometrach. Proszę przyjąć wartość pola elektrycznego dla dwóch przypadków: taką, żeby zmiana o 20 nm zwiększała potencjał o 250 meV, oraz zmiana o 20 nm zwiększała potencjał o 100 meV. Proszę podać wartości pola elektrycznego (w jednostkach SI). Obliczenia przeprowadzić dla pasma przewodnictwa i pasma walencyjnego GaAs (masy efektywne znajdziecie sami).



131

Quantum harmonic oscillator

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}+V(z)\right]\psi(z)=E\psi(z) \qquad V(z)=\frac{1}{2}Kz^2=\frac{1}{2}m\omega_0^2 z^2 \qquad E_n=\left(n-\frac{1}{2}\right)\hbar\omega_0$$

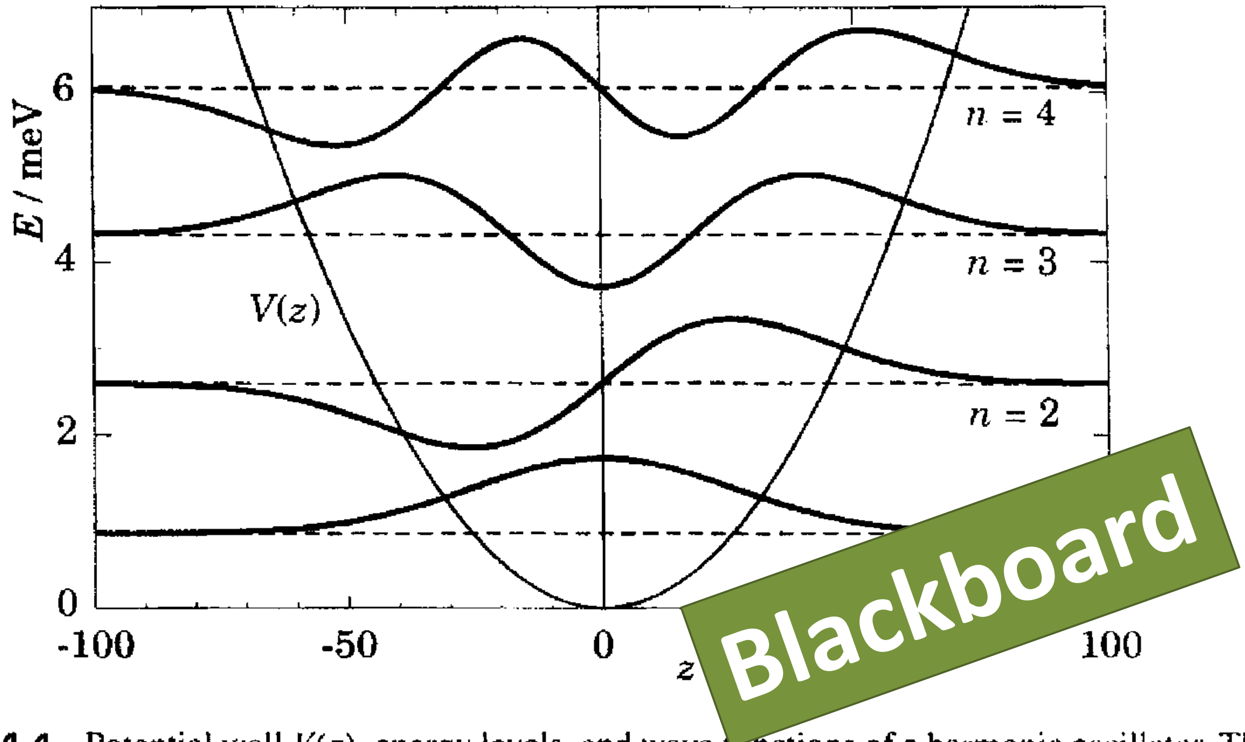


FIGURE 4.4. Potential well $V(z)$, energy levels, and wave functions of a harmonic oscillator. The potential is generated by a magnetic field of 1 T acting on electrons in GaAs.

Quantum harmonic oscillator

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}+\frac{1}{2}m\omega_0^2z^2\right]\psi(z)=E\psi(z) \qquad \varepsilon=\frac{E}{\hbar\omega_0} \qquad \xi=\sqrt{\frac{m\omega_0}{\hbar}}z$$

Quantum harmonic oscillator

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}+\frac{1}{2}m\omega_0^2 z^2\right]\psi(z)=E\psi(z) \qquad \varepsilon=\frac{E}{\hbar\omega_0} \qquad \xi=\sqrt{\frac{m\omega_0}{\hbar}}z$$

$$\left[-\frac{\hbar^2}{2m}\frac{m\omega_0}{\hbar}\frac{d^2}{d\xi^2}+\frac{1}{2}m\omega_0^2\frac{\hbar}{m\omega_0}\xi^2\right]\psi(\xi)=\hbar\omega_0\varepsilon\,\psi(\xi)$$

Quantum harmonic oscillator

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}+\frac{1}{2}m\omega_0^2 z^2\right]\psi(z)=E\psi(z) \qquad \varepsilon=\frac{E}{\hbar\omega_0} \qquad \xi=\sqrt{\frac{m\omega_0}{\hbar}}z$$

$$\left[-\frac{\hbar^2}{2m}\frac{m\omega_0}{\hbar}\frac{d^2}{d\xi^2}+\frac{1}{2}m\omega_0^2\frac{\hbar}{m\omega_0}\xi^2\right]\psi(\xi)=\hbar\omega_0\varepsilon\,\psi(\xi) \quad \Rightarrow \quad \left[\frac{d^2}{d\xi^2}-\xi^2\right]\psi(\xi)=-2\varepsilon\,\psi(\xi)$$

Tożsamości:

$$\left[\left(\frac{d}{d\xi}-\xi\right)\left(\frac{d}{d\xi}+\xi\right)\right]\psi_0(\xi)=\ldots$$

$$\left[\left(\frac{d}{d\xi}-\xi\right)\left(\frac{d}{d\xi}+\xi\right)-\left(\frac{d}{d\xi}+\xi\right)\left(\frac{d}{d\xi}-\xi\right)\right]\psi_0(\xi)=\ldots$$

Blackborad

Quantum harmonic oscillator

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}+\frac{1}{2}m\omega_0^2 z^2\right]\psi(z)=E\psi(z) \qquad \varepsilon=\frac{E}{\hbar\omega_0} \qquad \xi=\sqrt{\frac{m\omega_0}{\hbar}}z$$

$$\left[-\frac{\hbar^2}{2m}\frac{m\omega_0}{\hbar}\frac{d^2}{d\xi^2}+\frac{1}{2}m\omega_0^2\frac{\hbar}{m\omega_0}\xi^2\right]\psi(\xi)=\hbar\omega_0\varepsilon\,\psi(\xi) \quad \Rightarrow \quad \left[\frac{d^2}{d\xi^2}-\xi^2\right]\psi(\xi)=-2\varepsilon\,\psi(\xi)$$

Tożsamości:

$$\left[\left(\frac{d}{d\xi}-\xi\right)\left(\frac{d}{d\xi}+\xi\right)\right]\psi_0(\xi)=(-2\varepsilon_0+1)\psi_0(\xi)$$

$$\left[\left(\frac{d}{d\xi}-\xi\right)\left(\frac{d}{d\xi}+\xi\right)-\left(\frac{d}{d\xi}+\xi\right)\left(\frac{d}{d\xi}-\xi\right)\right]\psi_0(\xi)=2\psi_0(\xi)$$

Quantum harmonic oscillator

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}+\frac{1}{2}m\omega_0^2 z^2\right]\psi(z)=E\psi(z) \qquad \varepsilon=\frac{E}{\hbar\omega_0} \qquad \xi=\sqrt{\frac{m\omega_0}{\hbar}}\,z$$

$$\left[-\frac{\hbar^2}{2m}\frac{m\omega_0}{\hbar}\frac{d^2}{d\xi^2}+\frac{1}{2}m\omega_0^2\frac{\hbar}{m\omega_0}\xi^2\right]\psi(\xi)=\hbar\omega_0\varepsilon\,\psi(\xi) \quad \Rightarrow \quad \left[\frac{d^2}{d\xi^2}-\xi^2\right]\psi(\xi)=-2\varepsilon\,\psi(\xi)$$

Tożsamości:

$$\left[\left(\frac{d}{d\xi}-\xi\right)\left(\frac{d}{d\xi}+\xi\right)\right]\psi_0(\xi)=(-2\varepsilon_0+1\,)\psi_0(\xi)$$

$$\left[\left(\frac{d}{d\xi}-\xi\right)\left(\frac{d}{d\xi}+\xi\right)-\left(\frac{d}{d\xi}+\xi\right)\left(\frac{d}{d\xi}-\xi\right)\right]\psi_0(\xi)=2\psi_0(\xi)$$

$$(-2\varepsilon_0+1\,)\psi_0(\xi)-\left(\frac{d}{d\xi}+\xi\right)\left(\frac{d}{d\xi}-\xi\right)\psi_0(\xi)=2\psi_0(\xi)$$

Quantum harmonic oscillator

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}+\frac{1}{2}m\omega_0^2 z^2\right]\psi(z)=E\psi(z) \qquad \varepsilon=\frac{E}{\hbar\omega_0} \qquad \xi=\sqrt{\frac{m\omega_0}{\hbar}}z$$

$$(-2\varepsilon_0+1)\psi_0(\xi)-\left(\frac{d}{d\xi}+\xi\right)\left(\frac{d}{d\xi}-\xi\right)\psi_0(\xi)=2\psi_0(\xi)$$

$$\left(\frac{d}{d\xi}+\xi\right)\left(\frac{d}{d\xi}-\xi\right)\psi_0(\xi)=(-2\varepsilon_0-1)\psi_0(\xi) \qquad \left(\frac{d}{d\xi}-\xi\right)/\ldots$$

$$\left(\frac{d}{d\xi}-\xi\right)\left(\frac{d}{d\xi}+\xi\right)\left(\frac{d}{d\xi}-\xi\right)\psi_0(\xi)=(-2\varepsilon_0-1)\left(\frac{d}{d\xi}-\xi\right)\psi_0(\xi)$$

Quantum harmonic oscillator

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}+\frac{1}{2}m\omega_0^2 z^2\right]\psi(z)=E\psi(z) \qquad \varepsilon=\frac{E}{\hbar\omega_0} \qquad \xi=\sqrt{\frac{m\omega_0}{\hbar}}z$$

$$(-2\varepsilon_0+1)\psi_0(\xi)-\left(\frac{d}{d\xi}+\xi\right)\left(\frac{d}{d\xi}-\xi\right)\psi_0(\xi)=2\psi_0(\xi)$$

$$\left(\frac{d}{d\xi}+\xi\right)\left(\frac{d}{d\xi}-\xi\right)\psi_0(\xi)=(-2\varepsilon_0-1)\psi_0(\xi) \qquad \left(\frac{d}{d\xi}-\xi\right)/\ldots$$

$$\left(\frac{d}{d\xi}-\xi\right)\left(\frac{d}{d\xi}+\xi\right)\boxed{\left(\frac{d}{d\xi}-\xi\right)\psi_0(\xi)}=(-2\varepsilon_0-1)\boxed{\left(\frac{d}{d\xi}-\xi\right)\psi_0(\xi)}$$

$$\boxed{\left(\frac{d}{d\xi}-\xi\right)\psi_0(\xi)}=\psi_1(\xi)$$

$$\left(\frac{d}{d\xi}-\xi\right)\left(\frac{d}{d\xi}+\xi\right)\psi_1(\xi)=(-2\varepsilon_0-1)\psi_1(\xi)=(-2\varepsilon_1+1)\psi_1(\xi)$$

$$\varepsilon_1=\varepsilon_0+1$$

Quantum harmonic oscillator

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}+\frac{1}{2}m\omega_0^2 z^2\right]\psi(z)=E\psi(z) \qquad \varepsilon=\frac{E}{\hbar\omega_0} \qquad \xi=\sqrt{\frac{m\omega_0}{\hbar}}\,z$$

$$\left(\frac{d}{d\xi}-\xi\right)^n\psi_0(\xi)=\psi_n(\xi) \qquad \varepsilon_n=\varepsilon_0+n$$

$$\left(\frac{d}{d\xi}+\xi\right)^n\psi_0(\xi)=\psi_{-n}(\xi) \qquad \varepsilon_{-n}=\varepsilon_0-n$$

$$\left(\frac{d}{d\xi}-\xi\right)\psi_0(\xi)=\psi_1(\xi)$$

$$\left(\frac{d}{d\xi}-\xi\right)\left(\frac{d}{d\xi}+\xi\right)\psi_1(\xi)=(-2\varepsilon_0-1)\psi_1(\xi)=(-2\varepsilon_1+1)\psi_1(\xi)$$

$$\varepsilon_1=\varepsilon_0+1$$

Quantum harmonic oscillator

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}+\frac{1}{2}m\omega_0^2z^2\right]\psi(z)=E\psi(z) \qquad \varepsilon=\frac{E}{\hbar\omega_0} \qquad \xi=\sqrt{\frac{m\omega_0}{\hbar}}z$$

$$\left(\frac{d}{d\xi}-\xi\right)^n\psi_0(\xi)=\psi_n(\xi) \qquad \varepsilon_n=\varepsilon_0+n$$

Potencjał jest ograniczony z dołu

$$\left(\frac{d}{d\xi}+\xi\right)^n\psi_0(\xi)=\psi_{-n}(\xi) \qquad \varepsilon_{-n}=\varepsilon_0-n$$

We now choose $\psi_0(\xi)$:

$$\psi_{-1}(\xi)=0$$

$$\left(\frac{d}{d\xi}+\xi\right)\psi_0(\xi)=\psi_{-1}(\xi)=0$$

$$\psi_0(\xi)=\mathrm{A}\,\mathrm{e}^{-\frac{\xi^2}{2}}$$

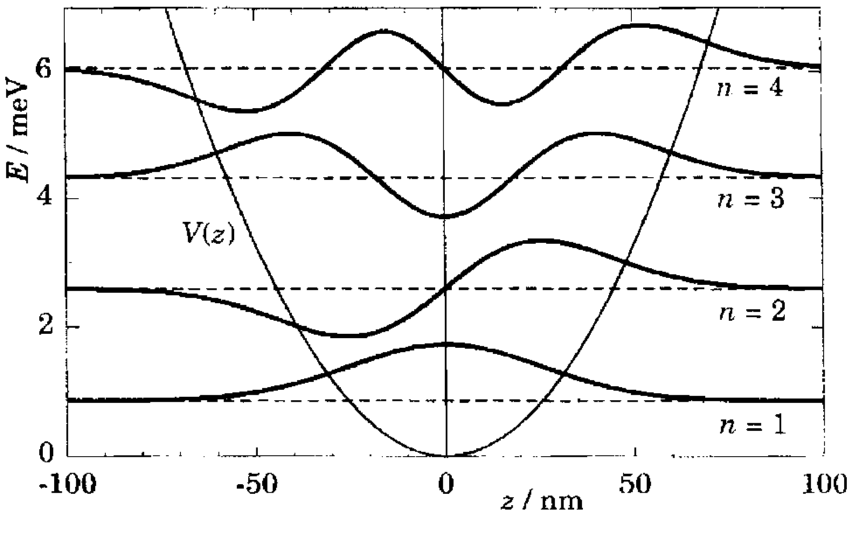


FIGURE 4.4. Potential well $V(z)$, energy levels, and wave functions of a harmonic oscillator. The potential is generated by a magnetic field of 1 T acting on electrons in GaAs.

Quantum harmonic oscillator

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}+\frac{1}{2}m\omega_0^2 z^2\right]\psi(z)=E\psi(z) \qquad \varepsilon=\frac{E}{\hbar\omega_0} \qquad \xi=\sqrt{\frac{m\omega_0}{\hbar}}z$$

$$\left(\frac{d}{d\xi}-\xi\right)\boxed{\left(\frac{d}{d\xi}+\xi\right)\mathrm{A}\,\mathrm{e}^{-\frac{\xi^2}{2}} = 0} = (-2\varepsilon_0+1)\mathrm{A}\,\mathrm{e}^{-\frac{\xi^2}{2}}=0 \quad \Rightarrow -2\varepsilon_0+1=0 \Rightarrow \varepsilon_0=\frac{1}{2}$$

$$\varepsilon_n = n+\frac{1}{2}$$

We now choose $\psi_0(\xi)$:

$$\psi_{-1}(\xi)=0$$

$$\left(\frac{d}{d\xi}+\xi\right)\psi_0(\xi)=\psi_{-1}(\xi)=0$$

$$\psi_0(\xi)=\mathrm{A}\,\mathrm{e}^{-\frac{\xi^2}{2}}$$

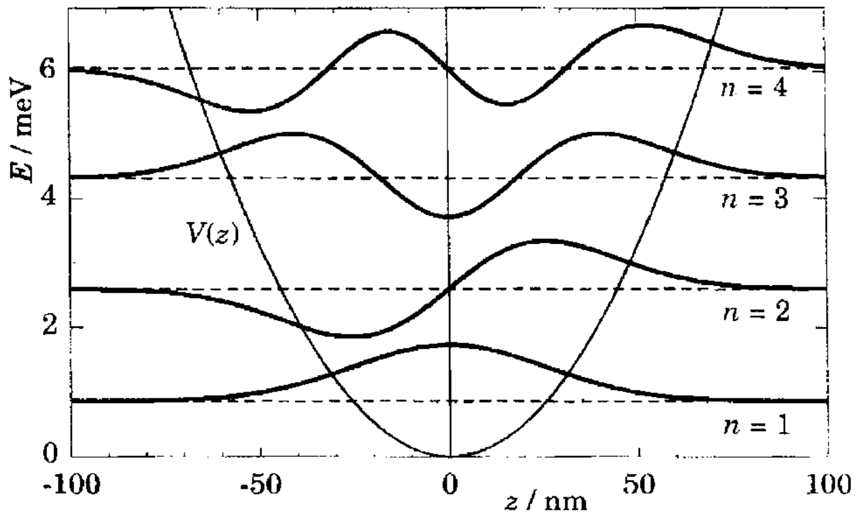


FIGURE 4.4. Potential well $V(z)$, energy levels, and wave functions of a harmonic oscillator. The potential is generated by a magnetic field of 1 T acting on electrons in GaAs.

Quantum harmonic oscillator

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}+\frac{1}{2}m\omega_0^2 z^2\right]\psi(z)=E\psi(z) \qquad \varepsilon=\frac{E}{\hbar\omega_0} \qquad \xi=\sqrt{\frac{m\omega_0}{\hbar}}z$$

$$\left(\frac{d}{d\xi}-\xi\right)\left(\frac{d}{d\xi}+\xi\right)\mathrm{A}\,\mathrm{e}^{-\frac{\xi^2}{2}}=(-2\varepsilon_0+1)\mathrm{A}\,\mathrm{e}^{-\frac{\xi^2}{2}}=0 \quad \Rightarrow -2\varepsilon_0+1=0 \;\Rightarrow\; \varepsilon_0=\frac{1}{2}$$

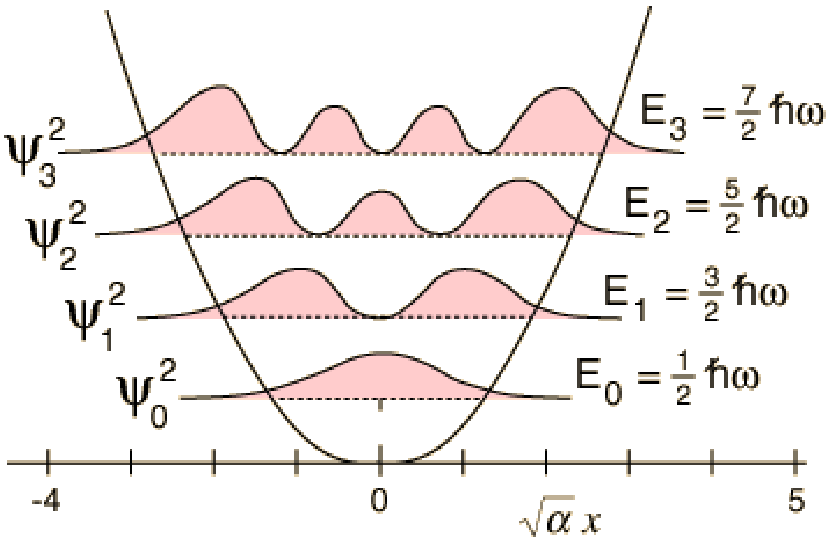
$$\varepsilon_n=n+\frac{1}{2}$$

$$E_n=\hbar\omega_0\left(n+\frac{1}{2}\right)$$

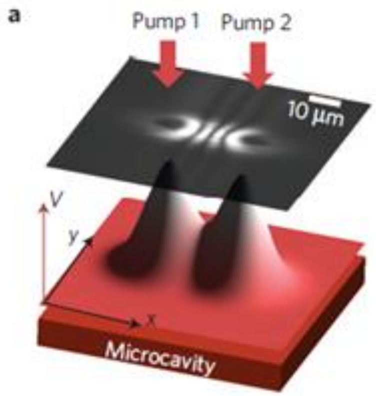
$$\psi_n(z)=A_n H_n\left(\sqrt{\frac{m\omega_0}{\hbar}}z\right)\exp\left(-\frac{m\omega_0}{2\hbar}z^2\right)$$

H_n - Hermite's polynomials

$$A_n=\left(2^n n!\,\sqrt{\frac{\pi\hbar}{m\omega}}\right)^{-1/2}$$



Quantum harmonic oscillator



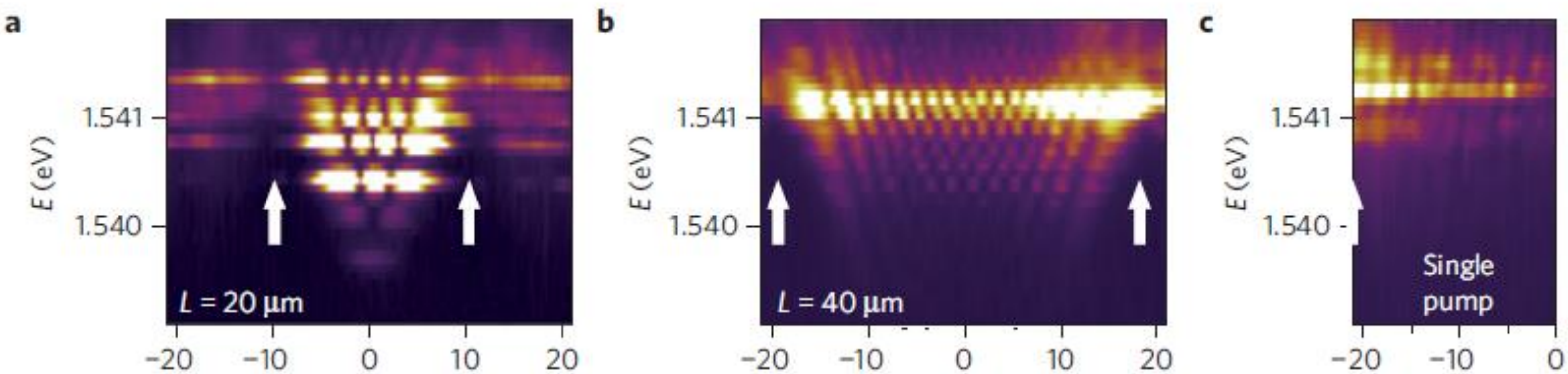
LETTERS

PUBLISHED ONLINE: 10 JANUARY 2012 | DOI: 10.1038/NPHYS2182

nature physics

Sculpting oscillators with light within a nonlinear quantum fluid

G. Tosi[1,2], G. Christmann[1], N. G. Berloff[3], P. Tsotsis[4], T. Gao[4,5], Z. Hatzopoulos[5,6], P. G. Savvidis[4,5] and J. J. Baumberg[1]*



Nat. Phys. 8, 190, (2012)

Quantum harmonic oscillator

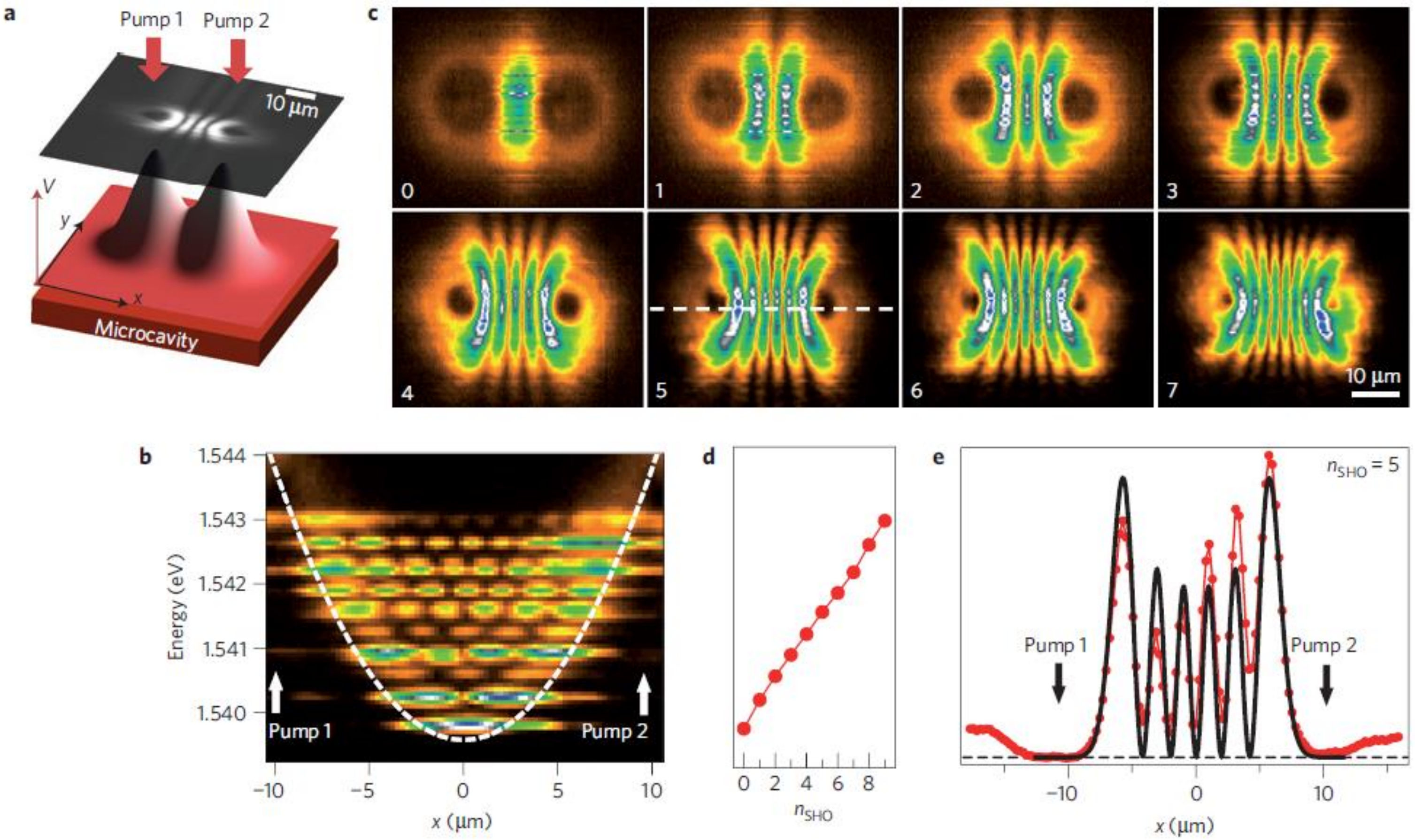


Figure 1 | Spatially mapped polariton-condensate wavefunctions. a, Experimental scheme with two 1 μm-diameter pump spots separated by 20 μm focused on the planar microcavity. The effective potential V (red) produces multiple condensates (grey image shows $n_{SHO} = 3$ mode). **b**, Real-space spectra along line between pump spots. **c**, Tomographic images of polariton emission (repulsive potential seen as dark circles around pump spots). Labelled according to n_{SHO} assigned from **d**. **d**, Extracted mode energies versus quantum number. **e**, Hermite-Gaussian fit of $\psi_{SHO}^{n=5}(x)$ to image cross-section, dashed in **c**.

Nat. Phys. 8, 190, (2012)

Coulomb potential in 2D

Coulomb potential in 2D

FIRST:

Coulomb potential in 3D in the semiconductor of dielectric constant ε_r, effective mass m^*:

$$V(r) = -\frac{e^2}{4\pi\varepsilon_r\varepsilon_0}\frac{1}{r}$$

$$Ry = \left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \frac{m}{2\hbar^2} = \frac{\hbar^2}{2ma_B^2} = \frac{1}{2}\frac{e^2}{4\pi\varepsilon_0 a_B} = 13.6 \text{ eV}$$

$$a_B = \frac{4\pi\varepsilon_0\hbar^2}{m_0 e^2} = 0.5 \text{ Å}$$

$$E_n = -Ry\frac{1}{n^2}$$

$$E_n = -\left(\frac{m^*}{m_0}\right)\frac{1}{\varepsilon_r^2}Ry\frac{1}{n^2}$$

$$a_B^* = \frac{4\pi\varepsilon_r\varepsilon_0\hbar^2}{m_0 e^2}\left(\frac{m_0}{m^*}\right) = a_B\varepsilon_r\left(\frac{m_0}{m^*}\right)$$

Coulomb potential in 2D

Electric charge moving in a plane in Coulomb potential. NOTE: this is not Gauss law in 2D (for which the relationship is like ($\ln r$)) but the „hydrogen atom" trapped in 2D.

$$V(r) = -\frac{1}{r}$$

$$\begin{aligned} x &= r\sin\phi \\ y &= r\cos\phi \end{aligned}$$

$$\begin{aligned} r &= \sqrt{x^2+y^2}, \quad r \in [0,\infty) \\ \phi &= \tan^{-1}\frac{x}{y}, \quad \phi \in [0, 2\pi]. \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial x} &= \sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial y} &= \cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi}. \end{aligned}$$

Laplacian:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$$

Problem Hamiltonian in 2D:

$$(-\frac{1}{2}\nabla^2 + V)\psi(\mathbf{r}) = -\frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right)\psi(\mathbf{r}) + V(r)\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

Coulomb potential in 2D

Electric charge moving in a plane in Coulomb potential

$$(-\frac{1}{2}\nabla^2 + V)\psi(\mathbf{r}) = -\frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right)\psi(\mathbf{r}) + V(r)\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

$$-\frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{2}{r}\right)\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

$$\Psi(\mathbf{r}) = R(r)\Phi(\phi)$$

Angular momentum magnitude: $\hat{L}^2_{2D} = -\frac{\partial^2}{\partial \phi^2}$

Angular momentum projection $\hat{L}_z = -i\frac{\partial}{\partial \phi}$

$$\frac{r^2}{R(r)}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{2}{r} + 2E\right)R(r) = -\frac{1}{\Phi(\phi)}\frac{\partial^2}{\partial \phi^2}\Phi(\phi) = m^2$$ (a number)

$$\Phi(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi}$$

From physical reasons $e^{im\phi} = e^{im(\phi+2\pi)}$ thus $m = 0, \pm 1, \pm 2, \pm 3$ ...

Coulomb potential in 2D

$$\Psi(\mathbf{r}) = R(r)\Phi(\phi)$$

Radial therm:

$$-\frac{1}{2}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} + \frac{2}{r}\right)R(r) = ER(r)$$

O! joj-joj-joj! (some substitutions, derivations nad equations):

$$R_{n,m}(\rho) = e^{-\frac{\rho}{2}} \sum_{j=0}^{N(n)} a_0 \frac{|m| + j - n}{j(2|m| + j)} \rho^{|m|+j}$$

Finally:

$$Ry^* = \left(\frac{e^2}{4\pi\varepsilon_r\varepsilon_0}\right)^2 \frac{m^*}{2\hbar^2} = \frac{1}{2}\frac{e^2}{4\pi\varepsilon_0\varepsilon_r a_B^*} = \left(\frac{m^*}{m_0}\right)\frac{Ry}{\varepsilon_r^2}$$

$$a_B^* = \varepsilon_r \left(\frac{m_0}{m^*}\right)$$

$$E_n = -\frac{Ry^*}{\left(n - \frac{1}{2}\right)^2}$$

What is the binding energy in 2D?

Coulomb potential in 2D

$\Psi(\mathbf{r}) = R(r)\Phi(\phi)$

Radial therm:

$$-\frac{1}{2}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} + \frac{2}{r}\right)R(r) = ER(r)$$

O! joj-joj-joj! (some substitutions, derivations nad equations):

$$R_{n,m}(\rho) = e^{-\frac{\rho}{2}} \sum_{j=0}^{N(n)} a_0 \frac{|m| + j - n}{j(2|m| + j)} \rho^{|m|+j}$$

Finally:

$$Ry^* = \left(\frac{e^2}{4\pi\varepsilon_r\varepsilon_0}\right)^2 \frac{m^*}{2\hbar^2} = \frac{1}{2}\frac{e^2}{4\pi\varepsilon_0\varepsilon_r a_B^*} = \left(\frac{m^*}{m_0}\right)\frac{Ry}{\varepsilon_r^2}$$

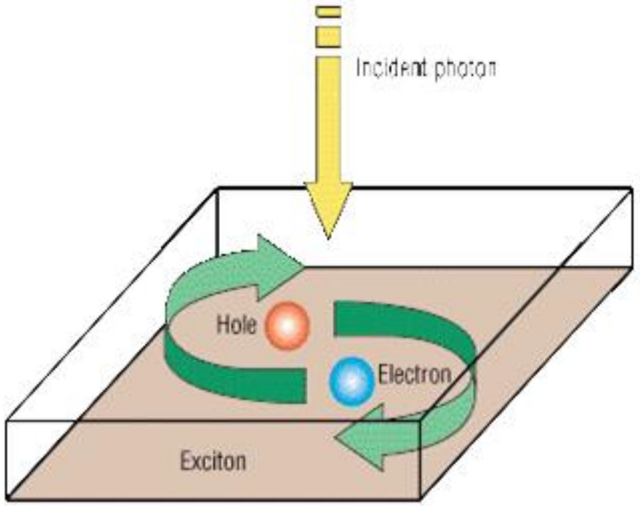
$$a_B^* = a_B\varepsilon_r\left(\frac{m_0}{m^*}\right)$$

$$E_n = -\frac{Ry^*}{\left(n - \frac{1}{2}\right)^2}$$

For Hydrogen $Ry = 13.6$ eV and $a_B = 0.053$ nm

For GaAs semiconductor $Ry^* \approx 5$ meV and $a_B^* \approx 10$ nm

Coulomb potential in 2D



Finally:

$$Ry^* = \left(\frac{e^2}{4\pi\varepsilon_r\varepsilon_0}\right)^2 \frac{m^*}{2\hbar^2} = \frac{1}{2}\frac{e^2}{4\pi\varepsilon_0\varepsilon_r a_B^*} = \left(\frac{m^*}{m_0}\right)\frac{Ry}{\varepsilon_r^2}$$

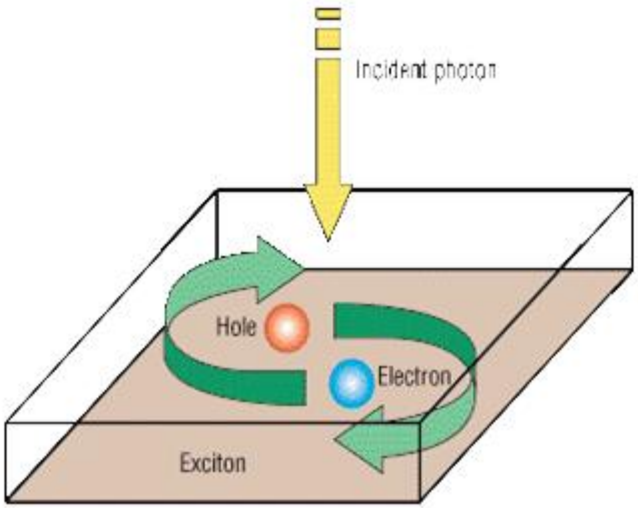
$$a_B^* = \varepsilon_r\left(\frac{m_0}{m^*}\right)$$

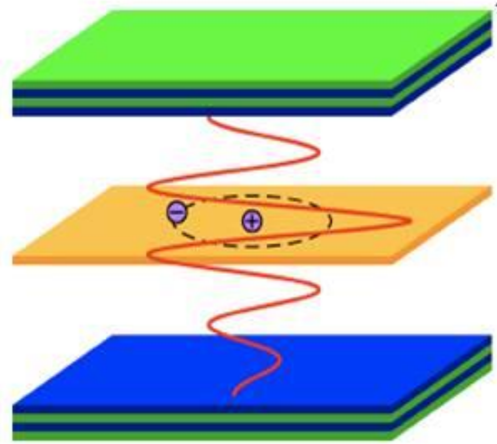
$$E_n = -\frac{Ry^*}{\left(n - \frac{1}{2}\right)^2}$$

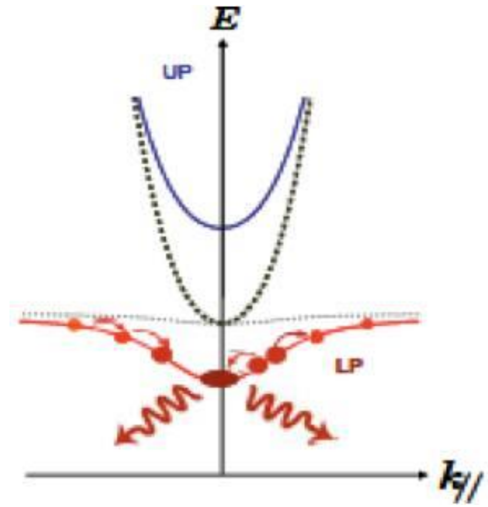
For Hydrogen $Ry = 13.6$ eV and $a_B = 0.053$ nm

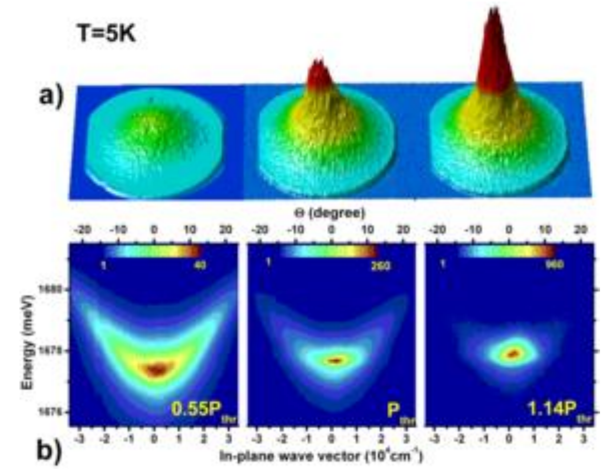
For GaAs semiconductor $Ry^* \approx 5$ meV and $a_B^* \approx 10$ nm

Polaritons









Dr Barbara Piętka – soon!

http://www.stanford.edu/group/yamamotogroup/research/EP/EP_main.html

Quantum wells – 2D, 1D, 0D

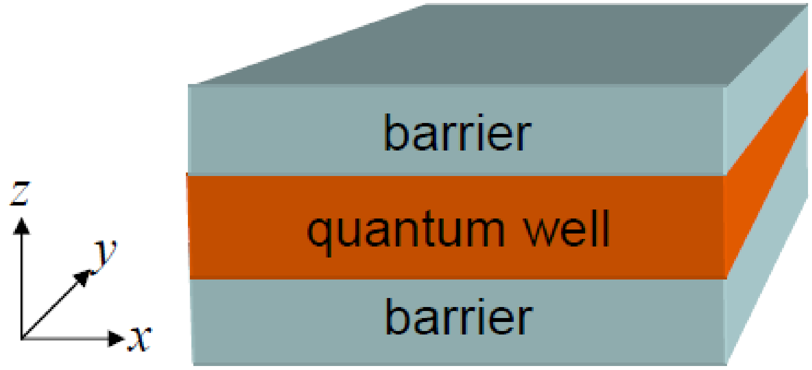
131

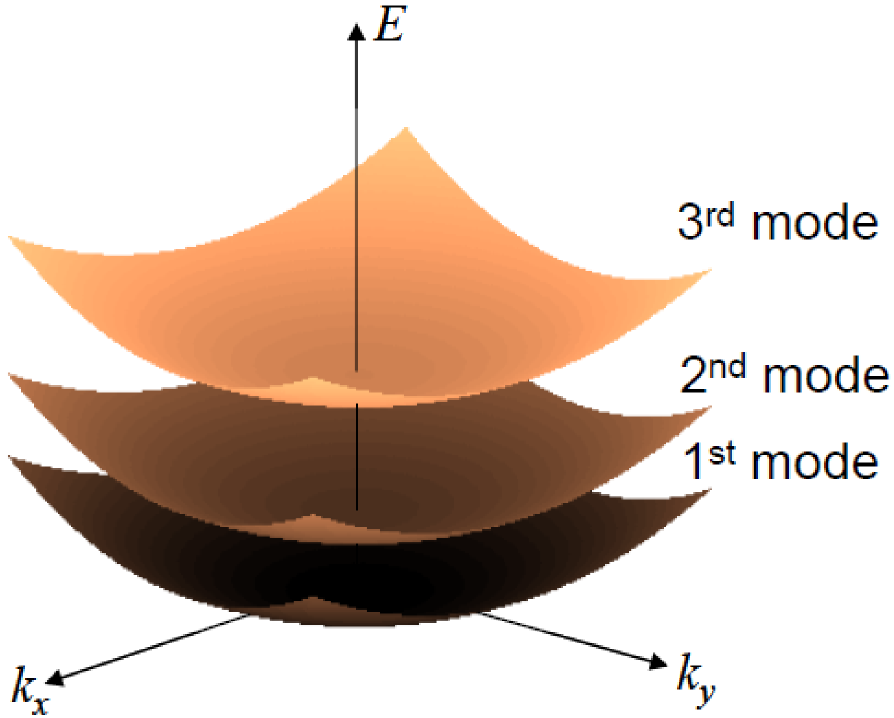
Low dimensional structures

$$\psi_{k_x,k_y,n}(x,y,z) = \exp(ik_x x)\exp(ik_y y)\,u_n(z) = \psi_{\boldsymbol{k},n}(\boldsymbol{r},z) = \exp(i\boldsymbol{k}\cdot\boldsymbol{r})\,u_n(z)$$

$$E_n(k_x,k_y) = \varepsilon_n + \frac{\hbar^2 k_x^2}{2m} + \frac{\hbar^2 k_y^2}{2m}$$

$$E_n(\boldsymbol{k}) = \varepsilon_n + \frac{\hbar^2 \boldsymbol{k}^2}{2m}$$





Marc Baldo MIT OpenCourseWare Publication May 2011

Low dimensional structures

The direct bandgap is required for optoelectronics

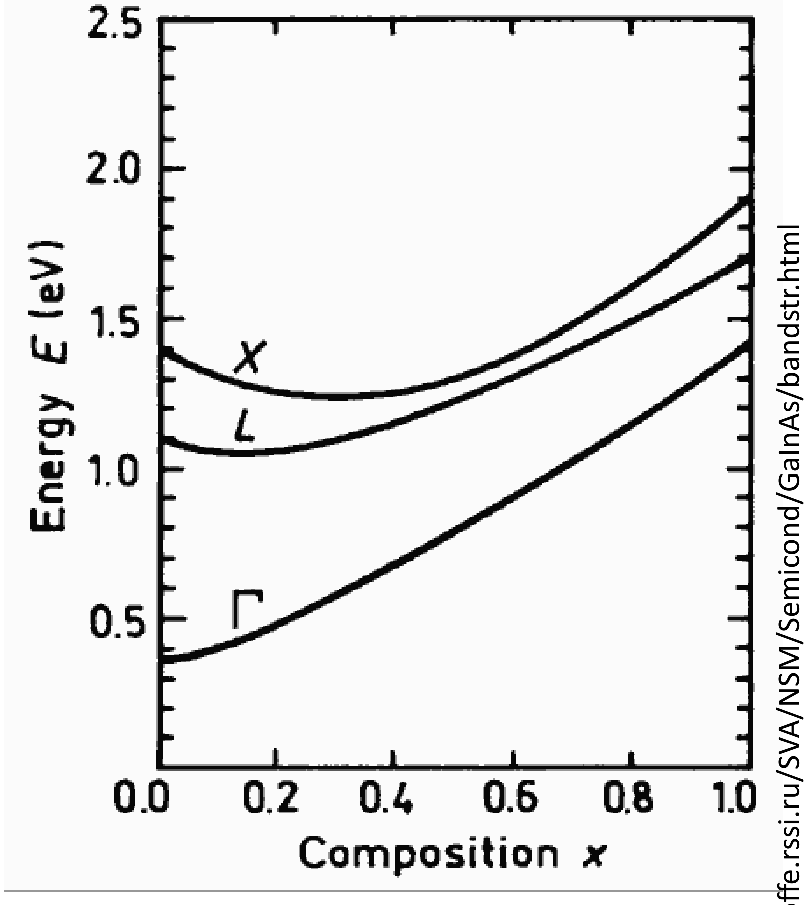
$$E_g^{InGaAs} = 0.4105 + 0.6337x + 0.475x^2 \text{ eV @ 2.0K}$$

$$\hbar\omega_n = \varepsilon_{e,n_e} - \varepsilon_{h,n_h} =$$

$$= E_g^{InGaAs} + \frac{\hbar^2\pi^2 n^2}{2m_0 a^2}\left(\frac{1}{m_e} + \frac{1}{m_h}\right) =$$

$$= E_g^{InGaAs} + \frac{\hbar^2\pi^2 n^2}{2m_0 m_{eh} a^2}$$

$$m_e = (0.023 - 0.037x + 0.003x^2)m_0$$
$$m_h = (0.41 - 0.1x)m_0$$



http://www.ioffe.rssi.ru/SVA/NSM/Semicond/GaInAs/bandstr.html

TENSION! - You can not choose only the thickness, an important factor for quantum wells and dots is the stress!

Low dimensional structures

The direct bandgap is required for optoelectronics

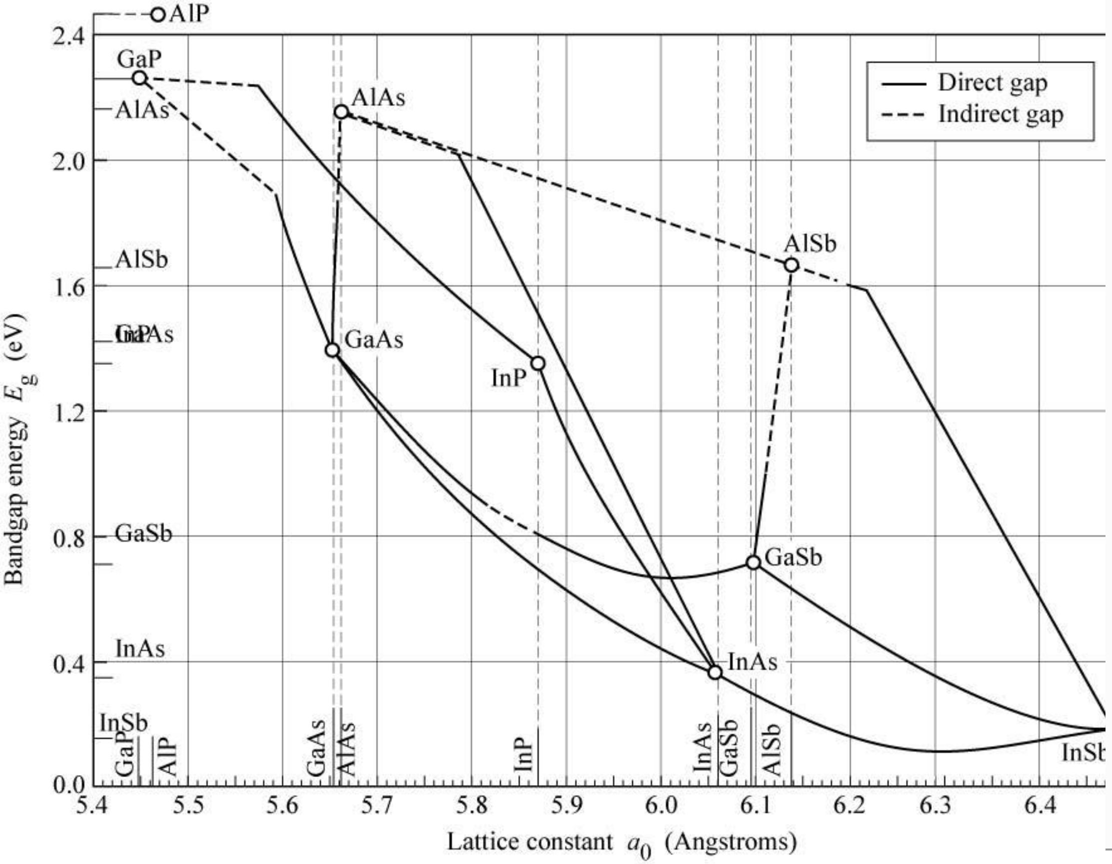
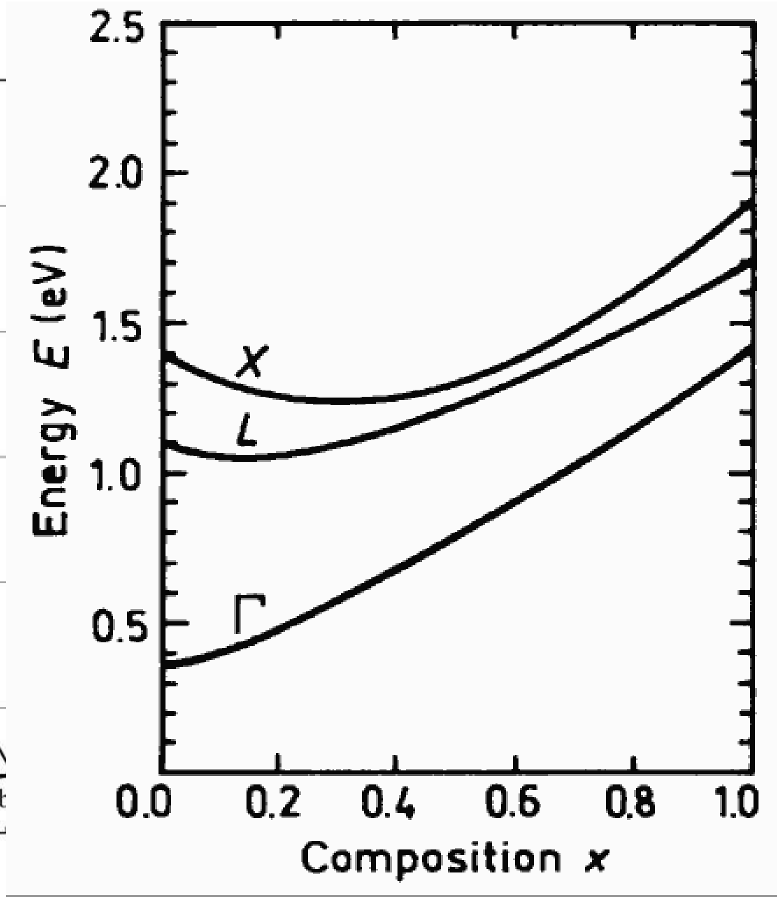


Fig. 12.6. Bandgap energy and lattice constant of various III–V semiconductors at room temperature (adopted from Tien, 1988).



$(0.4105+0.6337x+0.475x^2)$

http://www.ioffe.rssi.ru/SVA/NSM/Semicond/GaInAs/bandstr.html

Low dimensional structures

Full Hamiltonian in our universe has three spatial dimensions $(x, y, z, t) = (\vec{R}, t)$

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{R})\right]\psi(\vec{R}) = E\psi(\vec{R})$$

For $V(\vec{R}) = V(z)$ we obtain:

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + V(z)\right]\psi(x, y, z) = E\psi(x, y, z)$$

Along directions x and y we have uniform motion (*ruch swobodny*):

$$\psi(x, y, z) = \exp(ik_x x)\exp(ik_y y)\,u(z)$$

We can show (on the blackboard!), that final eigenenergies of the potential $V(z)$ are:

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2} + V(z)\right]u(z) = \varepsilon u(z)$$

$$\varepsilon = E - \frac{\hbar^2 k_x^2}{2m} - \frac{\hbar^2 k_y^2}{2m}$$

131

Low dimensional structures

$$\psi_{k_x,k_y,n}(x,y,z) = \exp(ik_x x)\exp(ik_y y)\,u_n(z) = \psi_{\boldsymbol{k},n}(\boldsymbol{r},z) = \exp(i\boldsymbol{k}\cdot\boldsymbol{r})\,u_n(z)$$

$$E_n(k_x,k_y) = \varepsilon_n + \frac{\hbar^2 k_x^2}{2m} + \frac{\hbar^2 k_y^2}{2m} \qquad E_n(\boldsymbol{k}) = \varepsilon_n + \frac{\hbar^2 \boldsymbol{k}^2}{2m}$$

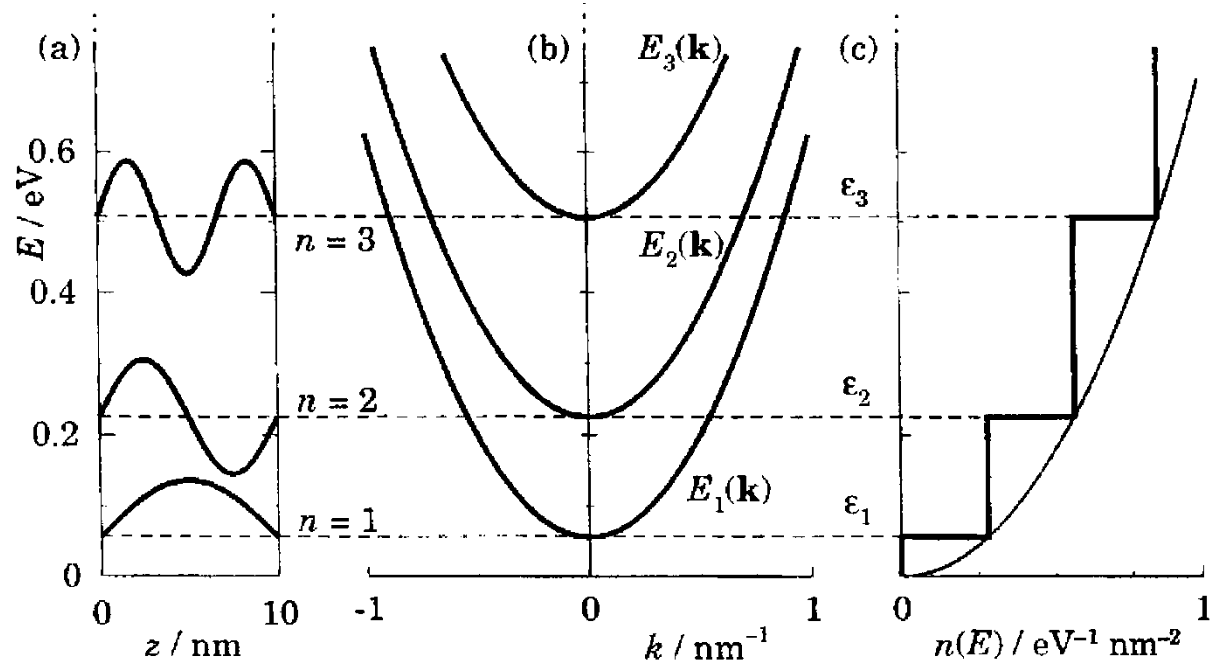


FIGURE 4.7. (a) Potential well with energy levels, (b) total energy including the transverse kinetic energy for each subband, and (c) steplike density of states of a quasi-two-dimensional system. The example is an infinitely deep square well in GaAs of width 10 nm. The thin curve in (c) is the parabolic density of states for unconfined three-dimensional electrons.

Low dimensional structures

$$\psi_{k_x,k_y,n}(x,y,z) = \exp(ik_x x)\exp(ik_y y)\,u_n(z) = \psi_{\boldsymbol{k},n}(\boldsymbol{r},z) = \exp(i\boldsymbol{k}\cdot\boldsymbol{r})\,u_n(z)$$

$$E_n(k_x,k_y) = \varepsilon_n + \frac{\hbar^2 k_x^2}{2m} + \frac{\hbar^2 k_y^2}{2m} \qquad E_n(\boldsymbol{k}) = \varepsilon_n + \frac{\hbar^2 \boldsymbol{k}^2}{2m}$$

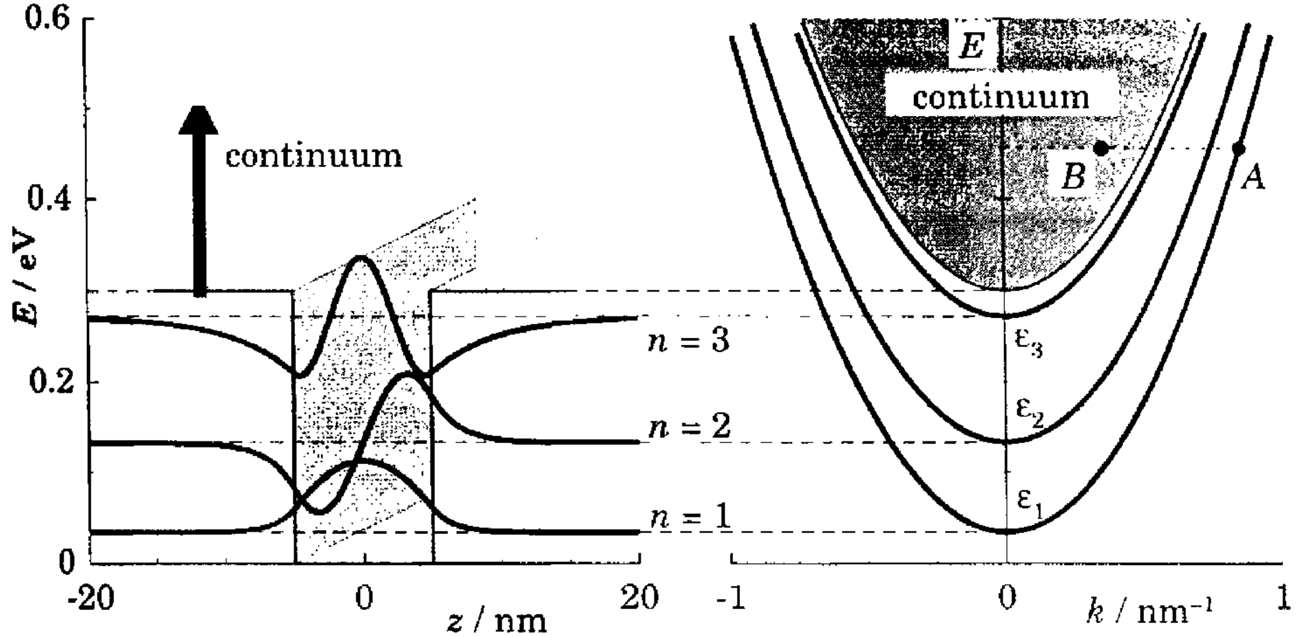
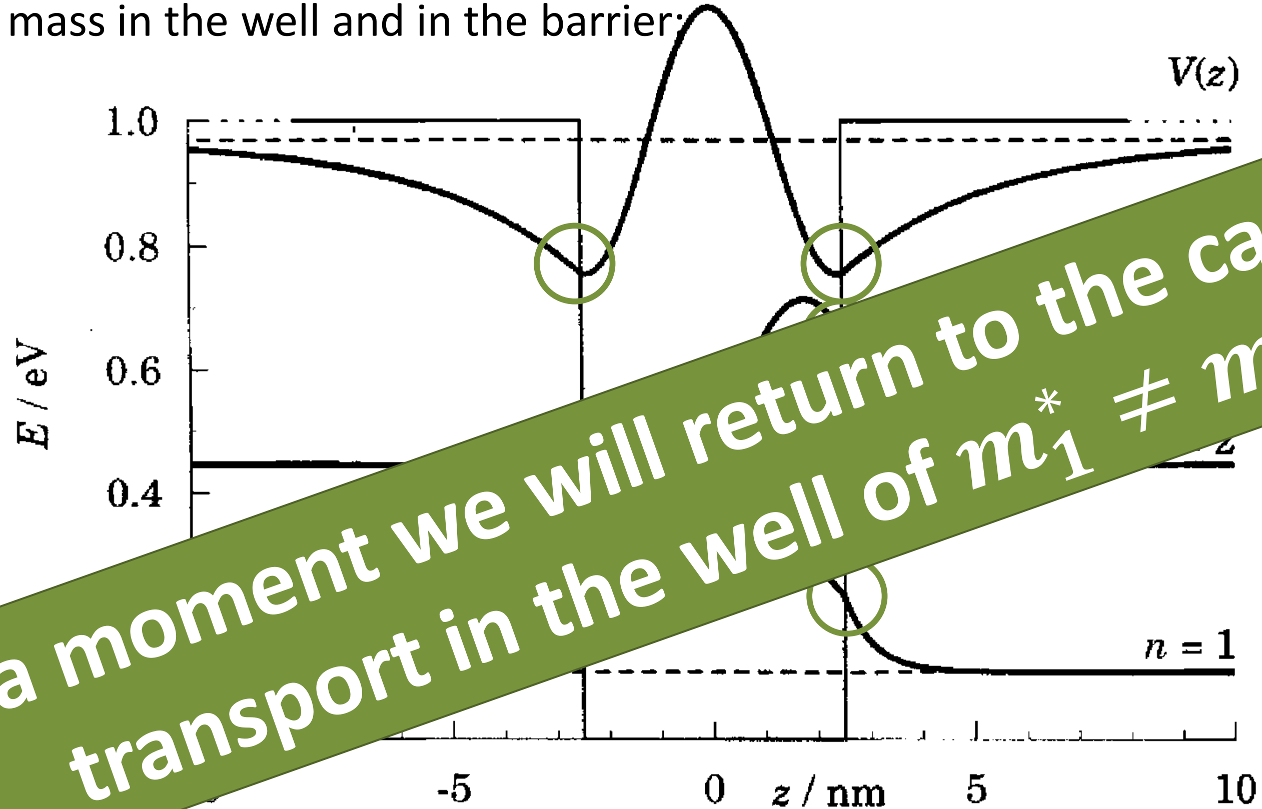


FIGURE 4.9. Quasi-two-dimensional system in a potential well of finite depth. Electrons with the same total energy can be bound in the well (*A*) or free (*B*).

Finite potential well – square well

THE DIFFERENT mass in the well and in the barrier:



In a moment we will return to the carrier transport in the well of $m_1^* \neq m_2^*$

RE 4.12. Finite square well of depth $V_0 = 1$ eV, width $a = 5$ nm along z, and effective masses $m_W = 0.067$ in the well and $m_B = 0.15$ in the barrier.

Low dimensional structures

Effective mass in the barrier m_B and in the well m_W

$$\left[-\frac{\hbar^2}{2m_0\, m_{W,B}}\,\nabla^2 + V(\vec{R})\right]\psi(\vec{R}) = E\psi(\vec{R})$$

For separated wave functions: $\psi(\vec{R}) = \psi_{\boldsymbol{k},n}(\boldsymbol{r}, z) = \exp(i\boldsymbol{k}\cdot\boldsymbol{r})\, u_n(z)$

$$\left[-\frac{\hbar^2}{2m_0\, m_W}\,\nabla^2 + E_W\right]\psi(\vec{R}) = E\psi(\vec{R})$$

$$\left[-\frac{\hbar^2}{2m_0\, m_B}\,\nabla^2 + E_B\right]\psi(\vec{R}) = E\psi(\vec{R})$$

We got (on the blackboard!):

$$\left[-\frac{\hbar^2}{2m_0\, m_W}\frac{d^2}{dz^2} + \frac{\hbar^2\boldsymbol{k}^2}{2m_0\, m_W} + E_W\right]u_n(z) = \varepsilon u_n(z)$$

$$\left[-\frac{\hbar^2}{2m_0\, m_B}\frac{d^2}{dz^2} + \frac{\hbar^2\boldsymbol{k}^2}{2m_0\, m_B} + E_B\right]u_n(z) = \varepsilon u_n(z)$$

Low dimensional structures

The particle moves in the well which potential depends on $\boldsymbol{k}$, in fact $k = |\boldsymbol{k}|$

$$\left[-\frac{\hbar^2}{2m_0\, m_W}\frac{d^2}{dz^2} + \frac{\hbar^2\boldsymbol{k}^2}{2m_0\, m_W} + E_W\right]u_n(z) = \varepsilon u_n(z)$$

$$\left[-\frac{\hbar^2}{2m_0\, m_B}\frac{d^2}{dz^2} + \frac{\hbar^2\boldsymbol{k}^2}{2m_0\, m_B} + E_B\right]u_n(z) = \varepsilon u_n(z)$$

$$V_0(k) = (E_B - E_W) + \frac{\hbar^2 k^2}{2m_0}\left(\frac{1}{m_B} - \frac{1}{m_W}\right)$$

The particle gains partially the effective mass of the barrier:

E.g. in GaAs-AlGaAs heterostructure $m_B > m_W$ thus the well gets „shallow"

$$E_n(k) = \varepsilon_n(k) + \frac{\hbar^2 k^2}{2m_0 m_W} \approx \varepsilon_n(k=0) + \frac{\hbar^2 k^2}{2m_0 m_{eff}}$$

energy of the bound state depends on k

$$m_{eff} \approx m_W P_W + m_B P_B$$

the probability of finding a particle

Low dimensional structures

The particle moves in the well which potential depends on $\boldsymbol{k}$, in fact $k = |\boldsymbol{k}|$

$$\left[-\frac{\hbar^2}{2m_0\, m_W}\frac{d^2}{dz^2} + \frac{\hbar^2 \boldsymbol{k}^2}{2m_0\, m_W} + E_W\right] u_n(z) = \varepsilon u_n(z)$$

$$\left[-\frac{\hbar^2}{2m_0\, m_B}\frac{d^2}{dz^2} + \frac{\hbar^2 \boldsymbol{k}^2}{2m_0\, m_B} + E_B\right] u_n(z) = \varepsilon u_n(z)$$

$$V_0(k) = (E_B - E_W) + \frac{\hbar^2 k^2}{2m_0}\left(\frac{1}{m_B} - \frac{1}{m_W}\right)$$

TABLE 4.2 Dependence on transverse wave vector $\mathbf{k}_\perp$ of the energies of the states bound in a well 5 nm wide and 1 eV deep, with effective mass $m_W = 0.067$ inside the well and $m_B = 0.15$ outside.

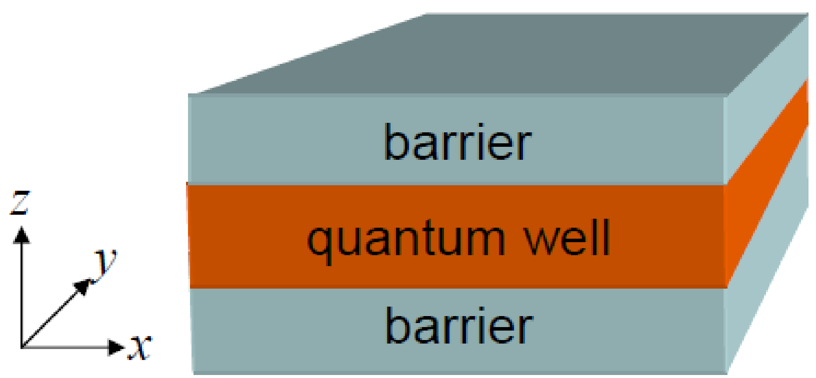
k (nm^{-1})	$\frac{\hbar^2 k^2}{2m_0 m_W}$ (eV)	$\frac{\hbar^2 k^2}{2m_0 m_B}$ (eV)	$V_0(k)$ (eV)	ε_1 (eV)	ε_2 (eV)	ε_3 (eV)	m_{eff}
0.0	0.000	0.000	1.000	0.108	0.446	0.969	0.067
0.5	0.142	0.064	0.921	0.106	0.435	0.919	0.069
1.0	0.570	0.254	0.685	0.096	0.397	—	0.076

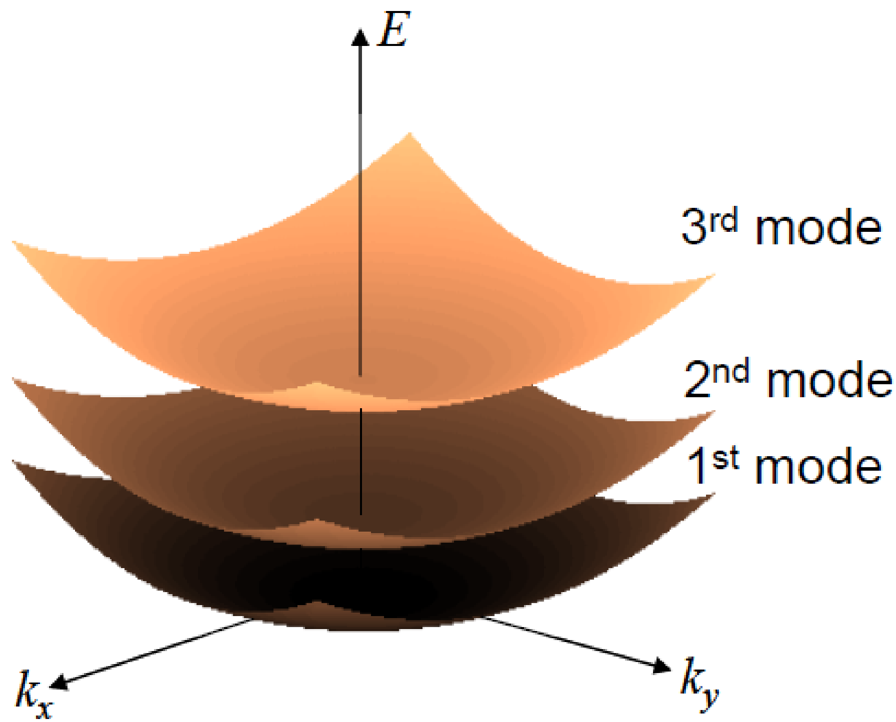
E.g. in GaAs-AlGaAs heterostructure $m_B > m_W$ thus the well gets „shallow”

Low dimensional structures

The particle moves in the well which potential depends on $\boldsymbol{k}$, in fact $k = |\boldsymbol{k}|$

$$\left[-\frac{\hbar^2}{2\ldots}\frac{d^2}{d\ldots} + \frac{\hbar^2\boldsymbol{k}^2}{2\ldots} + E_W\right]u_n(z) = \varepsilon u_n(z)$$





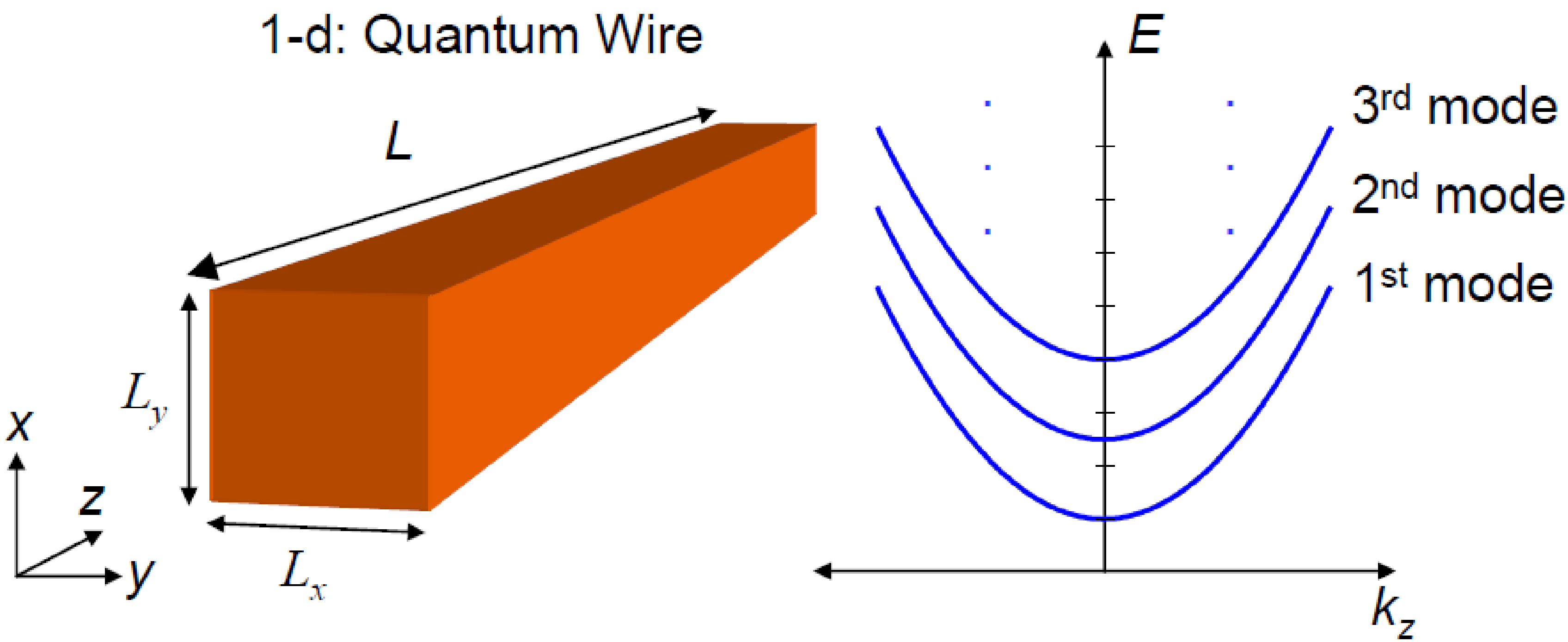
Marc Baldo MIT OpenCourseWare Publication May 2011

0.0	0.000	0.000	1.000	0.108	0.446	0.969	0.057
0.5	0.142	0.064	0.921	0.106	0.435	0.919	0.069
1.0	0.570	0.254	0.685	0.096	0.397	—	0.076

Quantum wire

$$\psi_{k_x,m,n}(x,y,z) = u_{m,n}(x,y)\exp(ik_z z) = \text{albo np.} = u_{n,l}(r,\theta)\exp(ik_z z)$$

$$E_n(k_x,k_y) = \varepsilon_{m,n} + \frac{\hbar^2 k_z^2}{2m}$$



Marc Baldo MIT OpenCourseWare Publication May 2011

Quantum wire

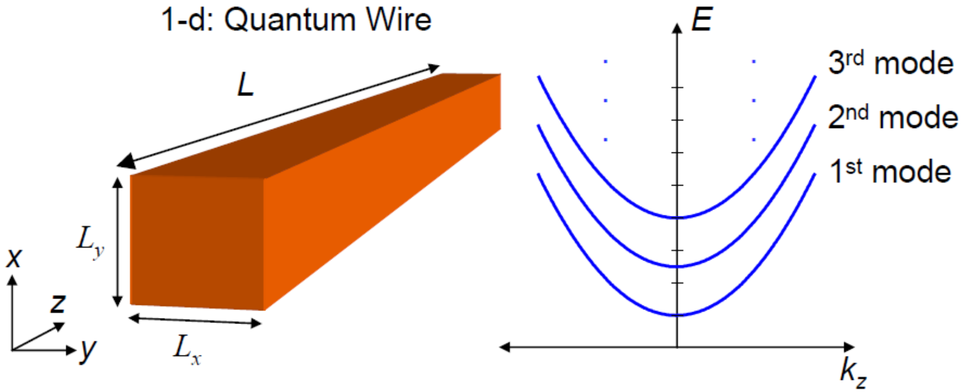
$$\psi_{k_x,m,n}(x,y,z) = u_{m,n}(x,y)\exp(ik_z z) = \text{albo np.} = u_{n,l}(r,\theta)\exp(ik_z z)$$

$$E_n(k_x,k_y) = \varepsilon_{m,n} + \frac{\hbar^2 k_z^2}{2m}$$

Square quantum well 2D $L_x L_y$, infinite potential:

$$\psi_{k_x,m,n}(x,y,z) = u_{m,n}(x,y)\exp(ik_z z) = \exp(ik_m x)\exp(ik_n y)\exp(ik_z z)$$

With boundary conditions $L_x k_m = n_x\pi$ and $L_y k_n = n_y\pi$ (dicrete spectrum)



Marc Baldo MIT OpenCourseWare Publication May 2011

Quantum wire

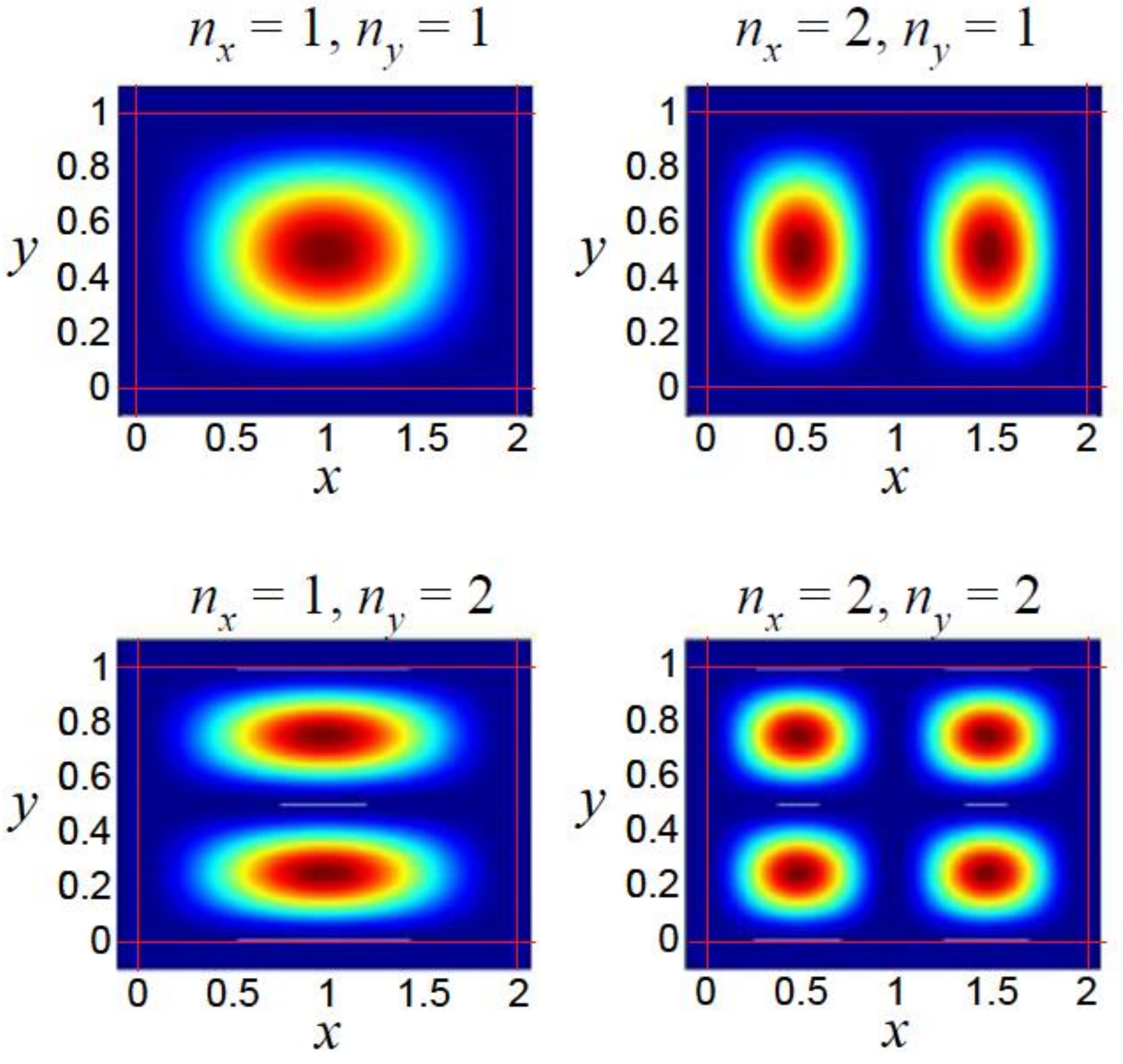


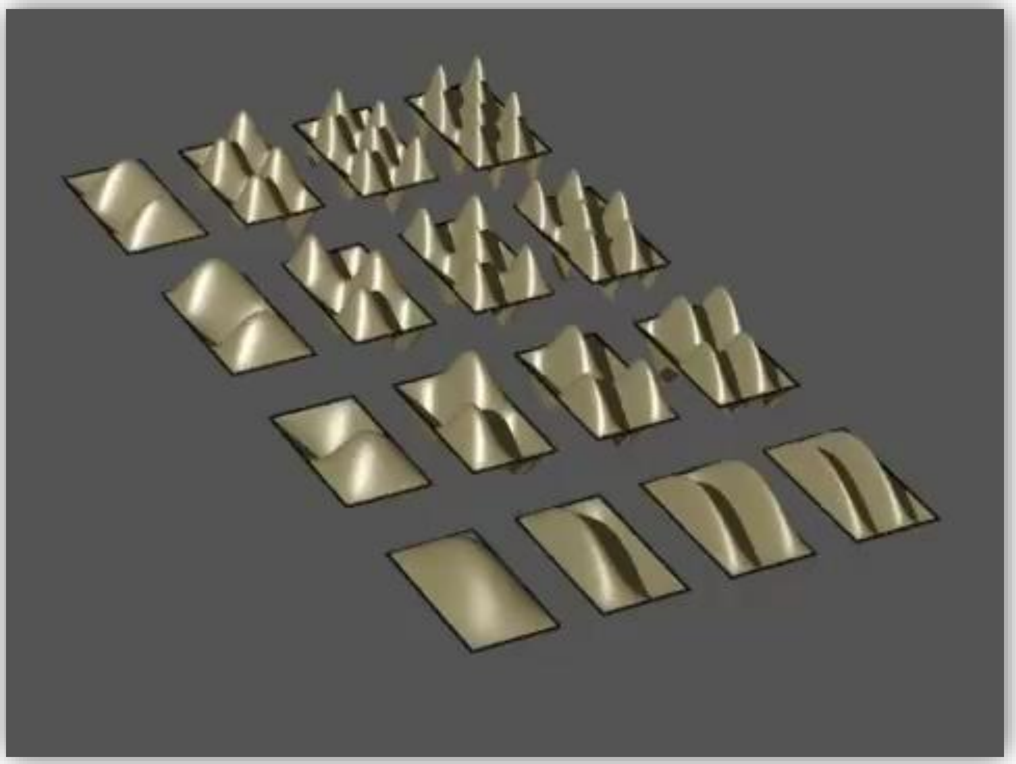
Fig. 2.13. The first four modes of the quantum wire. Since in this example, $L_x > L_y$ the $n_x = 2$, $n_y = 1$ mode has lower energy than the $n_x = 1$, $n_y = 2$ mode.

Marc Baldo MIT OpenCourseWare Publication May 2011

Quantum wire

Rectangular wire $(a \times b)$ – solutions like:

$$\varepsilon_{n_x,n_y} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2}\right)$$

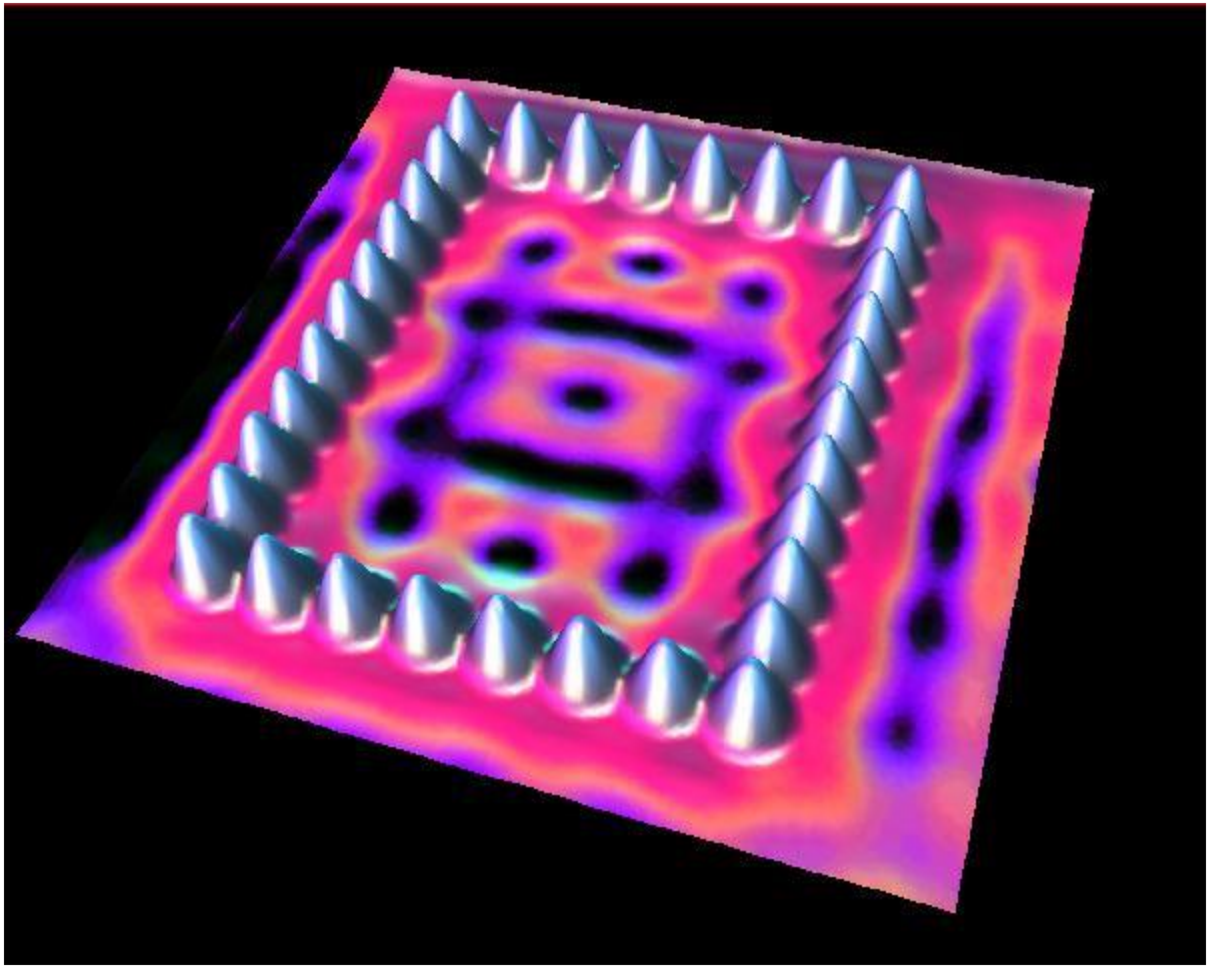


http://wn.com/2d_and_3d_standing_wave

Quantum wells 2D and 3D

Rectangular wire $(a \times b)$ – solutions like:

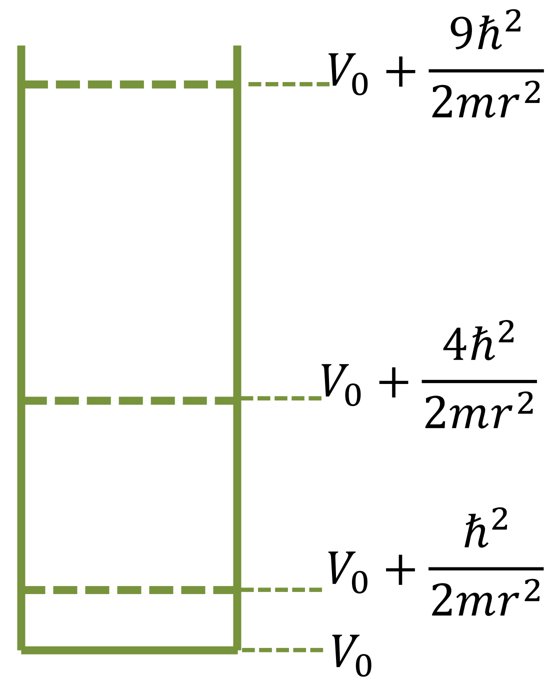
$$\varepsilon_{n_x,n_y} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2}\right)$$



http://www.almaden.ibm.com/vis/stm/images/stm14.jpg

Quantum wells 2D and 3D

Cylindrical well (with infinite walls)



$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + V_0\right)\psi(r,\theta) = E\psi(r,\theta)$$

$$\psi(r,\theta) = u(r)\exp(il\theta)$$

the depth of the potential depends on l^2

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\hbar^2 l^2}{2mr^2} + V_0\right]u(r) = Eu(r)$$

What gives solutions in the form Bessel functions

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} + [(kr)^2 - l^2]u = 0$$

$$J_l(kr) \sim \sqrt{\frac{2}{\pi kr}}\cos\left(kr - \frac{1}{2}l\pi - \frac{1}{4}\pi\right)$$

$$k = \sqrt{2m(E - V_0)}/\hbar$$

$$\phi_{nl}(r) \propto J_l\left(\frac{j_{l,n} r}{a}\right)\exp(il\theta)$$

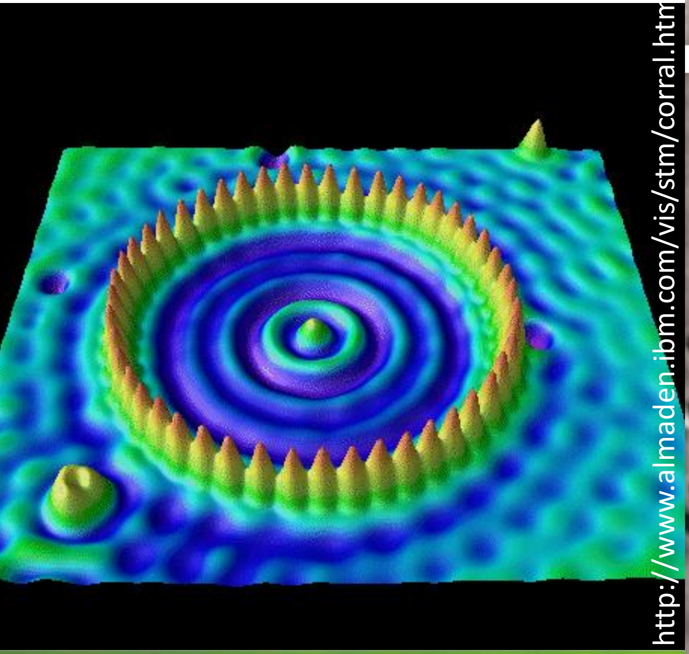
$$\varepsilon_{nl} = \frac{\hbar^2 j_{l,n}^2}{2ma}$$

Zeros of the Besseel function are $j_{l,n}$

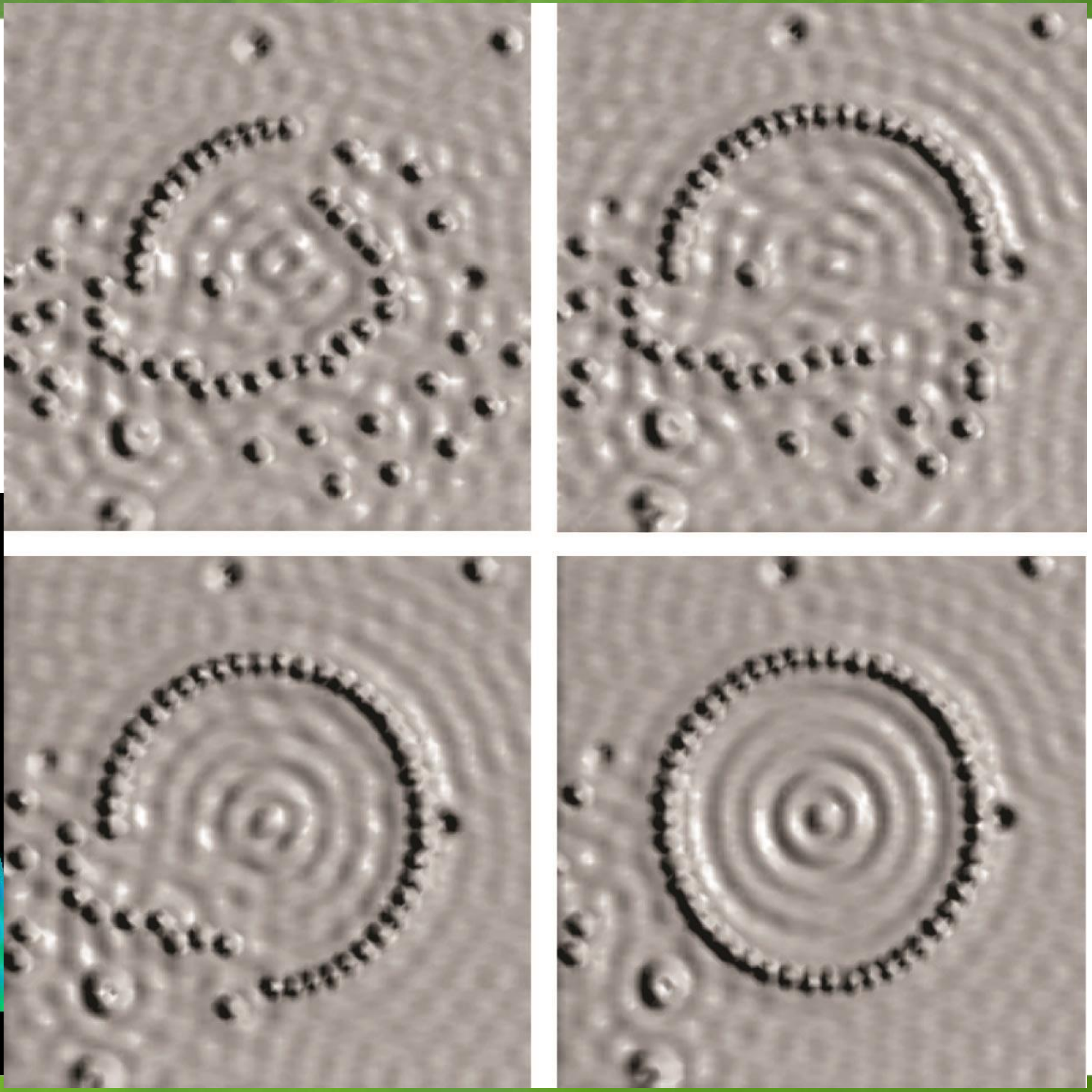
Quantum wells 2D and 3D

Cylindrical well

low temperature scanning tunneling microscope (STM)



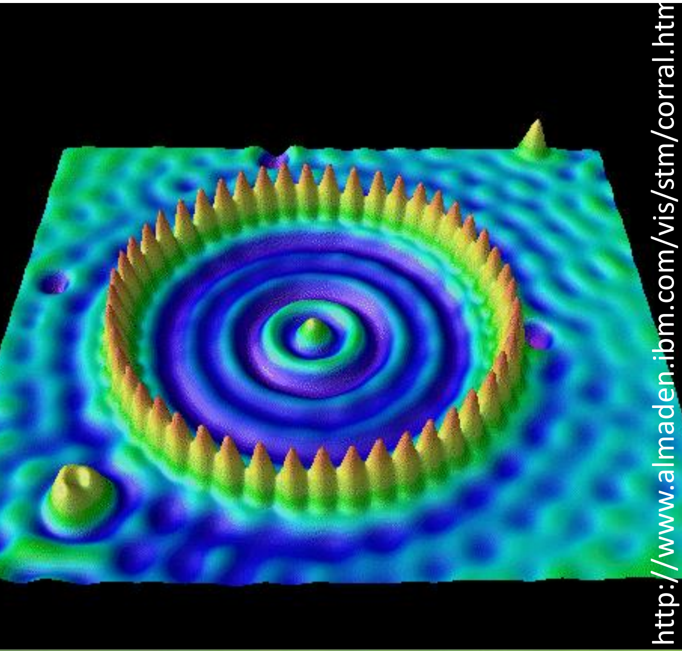
http://www.almaden.ibm.com/vis/stm/corral.html#stm16



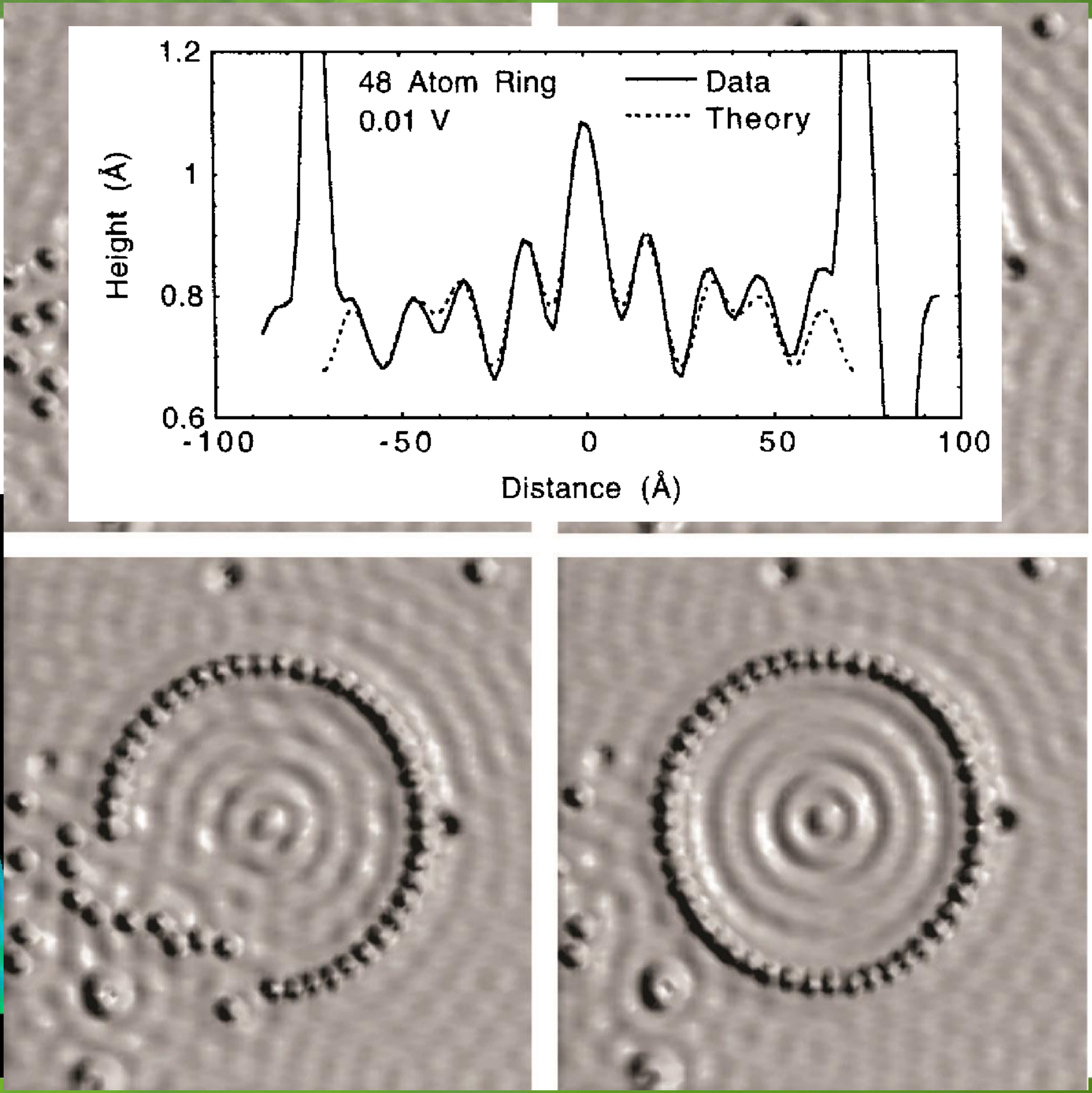
Quantum wells 2D and 3D

Cylindrical well

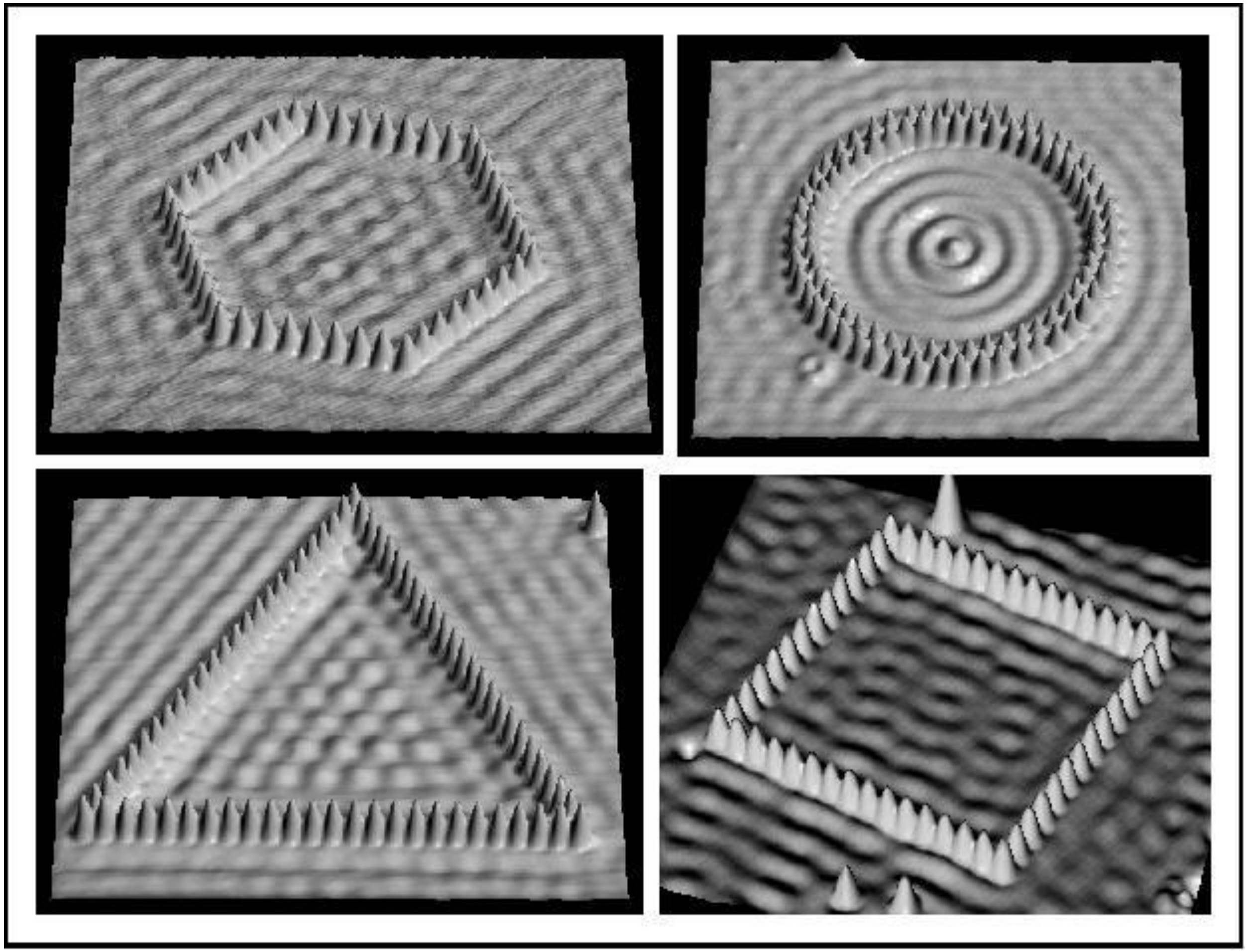
low temperature scanning tunneling microscope (STM)



http://www.almaden.ibm.com/vis/stm/corral.html#stm16



Quantum wells 2D and 3D



http://www.almaden.ibm.com/vis/stm/images/stm17.jpg